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par Tomos PARRY

Résumé. Une formule asymptotique pour la variance des nombres entiers sans facteur carré dans une progression arithmétique de raison donnée a été trouvée par Nunes dans [9]. Pour l'un des termes d'erreur, nous donnons la meilleure amélioration que l'on puisse espérer d'avoir.

Abstract. An asymptotic formula for the variance of squarefree numbers in arithmetic progressions of given modulus was obtained by Nunes, see [9]. We improve one of the error terms as far as one would expect to be able to go.

1. Introduction

For $k \geq 2$ let

$$\mathcal{S} = \{n \in \mathbb{N} \,|\, \text{there is no prime } p \text{ with } p^k | n\}$$

be the set of k-free numbers. The k-frees are a classically studied sequence which, for large k, resemble the natural numbers whilst, for decreasing k, they have more and more divisibility constraints imposed on them. Consequently they may be thought of as approximating to the primes, which are of course what we are most interested in.

There is a suitable approximation, $\eta(q,a)$, which will be defined precisely soon enough, to the count of k-frees in arithmetic progressions, and the natural question is then on the extent to which the error

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S} \\ n \equiv a \bmod (q)}} 1 - x\eta(q,a)$$

is small. Just as is the case for primes, we are particularly interested in this error since it is connected to the distribution of the zeros of the Riemann zeta function; the particular case of the k-frees can be seen in e.g. the second

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page of [10]. In view of this, one might be inclined to conjecture that the above error is

$$\approx \left(\frac{x}{q}\right)^{1/2k}. \tag{1.1}$$

For the case of the squarefrees ($k = 2$) this was conjectured by Montgomery (see [2]), and a result of Hooley (see [4]), says the error is then

$$\ll \left(\frac{x}{q}\right)^{1/2} + q^{1/2+\epsilon}.$$

Under current knowledge of the zeros of the Riemann zeta function it is difficult to get errors better than $\approx (x/q)^{1/k}$. This is parallel to what happens with the primes so just as in that case, we look at the *average* behaviour instead and consider, for example, the quantity

$$\sum_{a=1}^{q}\left(\sum_{\substack{n\le x,\, n\in\mathcal{S}\\ n\equiv a \bmod (q)}} 1 - x\eta(q,a)\right)^2,$$

which is a variance for k-free numbers in arithmetic progressions when averaging over a (complete) residue system modulo q. In accordance with (1.1) one would like to establish that this is

$$\approx q\left(\frac{x}{q}\right)^{1/k}.$$

This was managed a few years ago by Nunes, see [9], who found an asymptotic formula for this type of variance. For the squarefree numbers one obtains an asymptotic formula up to an error essentially

$$\ll \left(\frac{x}{q}\right)^{1/3} + \frac{x^{5/3}}{q} \tag{1.2}$$

by an elementary argument. Indeed Nunes obtains a better second bound by employing the square sieve, and according to [6] an even better bound is possible. Nunes' result is concerned with averaging just over the reduced residues, not a complete residue set, and we will briefly come back to this at the end of the introduction. Before Nunes' result only upper bound results were recorded (see [5] and the references therein), although some of these are stronger in the range where the above asymptotic formula doesn't hold and are concerned with more general sequences than the k-free numbers.

The above elementary argument alluded to is valid for general k, and we will carry it out in this paper for completeness.

Theorem 1. *Let $k \geq 2$ and denote by $\mathcal{S}$ the set of k-free numbers. For $q, a \in \mathbb{N}$ and $x > 0$ define*

$$\eta(q,a) = \sum_{\substack{d=1 \\ (q,d^k)|a}}^{\infty} \frac{\mu(d)}{[q,d^k]}, \qquad E_x(q,a) = \sum_{\substack{n \leq x \\ n \in \mathcal{S} \\ n \equiv a(q)}} 1 - x\eta(q,a)$$

and

$$V_x(q) = \sum_{a=1}^{q} |E_x(q,a)|^2.$$

Define

$$C_k = \frac{2\zeta(1/k-1)}{1/k-1} \prod_p \left(1 - \frac{p^k + 2p(p-1)}{p^{k+2}}\right)$$

and

$$f_k(q) = C_k \prod_{p|q} \frac{1 - 2/p^k + (q,p^k)^{1/k-1}/p}{1 + 1/p - 2/p^k}.$$

For $1 \leq q \leq x$ we have for every $\epsilon > 0$

$$V_x(q) = q\left(\frac{x}{q}\right)^{1/k} f_k(q) + \mathcal{O}_{k,\epsilon}\left(x^{\epsilon}\left(q\left(\frac{x}{q}\right)^{1/(k+1)} + \frac{x^{1+2/(k+1)}}{q}\right)\right).$$

However the main result of this paper is the following.

Theorem 2. *Suppose we are in the setting of Theorem 1. Then for $k = 2$*

$$V_x(q) = q\left(\frac{x}{q}\right)^{1/2} f_2(q) + \mathcal{O}_{\epsilon}\left(x^{\epsilon}\left(q\left(\frac{x}{q}\right)^{1/4} + \frac{x^{7/5}}{q}\left(q^{1/5} + \left(\frac{x}{q}\right)^{1/5}\right) + x^{3/4}\right)\right).$$

If q is squarefree we may replace the middle error term through $x^{3/2}/q$.

The main feature is the improvement in the first error term exponent from $1/3$ to $1/4$, what is most likely optimal. This may catch the attention of any analytic number theorist, since they will of course be aware of the state of the error term in the famous Dirichlet divisor problem. The second feature is that the second error also improves on the second error in [9] as well as that claimed in [6].

For $x/q \to \infty$, Theorem 1 provides an asymptotic formula once q is larger than $x^{\frac{k^2+2k-1}{(k+1)(2k-1)}}$. Theorem 2 provides one once q is larger than $x^{9/13}$ or $x^{2/3}$ in the squarefree case.

We should mention a very recent work [3] of Ofir Gorodetsky, Kaisa Matomäki, Maksym Radziwiłł and Brad Rodgers, in which the range of validity for the asymptotic formula extends to q as small as $x^{5/11}$, which far surpasses what we have here. On the other hand their results are proven only for prime q and their error term which corresponds to our first error term, not being their prime concern but being our main concern, is far weaker than what we have. We should in particular mention that we found our argument for Lemma 3.6 after looking over [3].

The question of whether one averages over a complete set of residues, as here, or over a reduced one, as in [3] and [9], is, most likely harmless but still remains to be seen. The squarefree numbers are, in contrast to the primes, not better understood outside the reduced residue classes, and therefore it isn't clear, at least to the author, neither why averaging over a reduced system is guaranteed to be just as insightful as averaging over a complete system, nor that averaging over a complete system is no harder. For example, in the reduced residue case, it seems Theorem 2 holds without the squarefree condition (see Lemma 3.8).

Our argument for Theorem 2 results in replacing the exponent $1/(k+1)$ in the first error term in Theorem 1 with $2k/(9k-2)$, and so is inferior once $k \geq 4$, but we could have stated a result for cube-free numbers with exponent $6/25$.

The paper is structured as follows. In Section 2 we carry out the necessary technical work to establish Theorem 1. This is routine and is simply repeating the outline of the argument in [9] for general k. In Section 3 we carry out the necessary technical work to establish Theorem 2. Here our argument is less routine and requires exploiting cancellation in the integrals arising from applying Perron's formula. In Section 4 we prove our theorems.

A number of helpful comments were provided by a referee to this article.

Notation. Throughout this paper we are only concerned with "power-savings", so that factors involving x^ϵ, where ϵ as usual denotes an arbitrarily small positive quantity, and $\log x$ are of no importance. Consequently in various proofs we will often write

$$f(X_1, \dots, X_n) \ll g(X_1, \dots, X_n)$$

where we really mean

$$f(X_1, \dots, X_n) \ll_\epsilon |X_1 \cdots X_n|^\epsilon g(X_1, \dots, X_n).$$

However, in the statements of lemmas we will always include these factors explicitly if present.

2. Lemmas for Theorem 1

For $\alpha \in \mathbb{N}$ define $\widehat{\alpha}$ to be the smallest multiple of k which is $\geq \alpha$. If

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$$

define

$$r_k(n) = p_1^{\widehat{\alpha_1}/k} \cdots p_r^{\widehat{\alpha_r}/k}$$

so that

$$\text{for } d, n \in \mathbb{N}, \qquad d|n^k \iff r_k(d)|n \tag{2.1}$$

and

$$r_k(n)^k = n \prod_{p|n} p_1^{\widehat{\alpha_1}-\alpha_1} \cdots p_r^{\widehat{\alpha_r}-\alpha_r}. \tag{2.2}$$

We will use this function at various points throughout this work, as well as the well-known convolution formula

$$\sum_{d^k|n} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is } k\text{-free} \\ 0 & \text{if not} \end{cases} \tag{2.3}$$

for the k-free numbers.

A large but necessary part of any dispersion argument is calculating the constants associated with large terms, which in number theory often amounts to computing Euler products. This is our first lemma.

Lemma 2.1. *For $N \in \mathbb{N}$ denote by $K(N)$ its squarefree part. For $w, L \in \mathbb{C}$ with $w \neq 0$ and $L \neq 1$ define*

$$\mathcal{Z}_w = \prod_p \Big(1 - p^w\Big), \qquad \mathcal{Z}_w(N) = \prod_{p|N} \Big(1 - p^w\Big),$$

$$\Delta_w(N) = \mathcal{Z}_w(N)\mathcal{Z}_{2w}^{-1}(N), \qquad \theta_w(N) = \prod_{p|N} \left(1 - \frac{2\Delta_w(p)}{p^k}\right)^{-1},$$

$$c_w = \prod_p \left(1 - \frac{2}{p^k\,(1+p^w)}\right), \qquad \alpha = \prod_p \left(1 - \frac{2}{p^k} + \frac{1}{p^{2k}}\right),$$

$$F_w(N) = \sum_{st|N} \mu(t)\mu_{2k+1}(st)\mu\left(K(st)\right) K(st)^{2w} r_k\left(N/s\right)^w \mathcal{Z}_{2w}^{-1}\left(K(st)\right),$$

$$Z_L(N) = F_{k(L-1)}(N) \sum_{Qh=N} \frac{\mu(h)}{Q^L}$$

and

$$\mathcal{U}_L(N) = \prod_{p|N} \left(1 - p^{k(L-1)} \cdot \frac{1-(N,p^k)^{-L}}{1+p^{k(L-1)} - 2/p^k}\right).$$

(A) *For $Q \in \mathbb{N}$ and $w \in \mathbb{C}\backslash\{0\}$*

$$\sum_{\substack{h,d,d'=1 \\ (h,dd')=1 \\ (Q,hdd')=1}}^{\infty} \frac{\mu(h)\mu(d)\mu(d')\Delta_w(hdd')}{h^{2k}d^k d'^k} = c_w\theta_w(Q).$$

(B) *For Q a power of a prime p and $L \in \mathbb{C}\backslash\{1\}$*

$$\sum_{N|Q} Z_L(N)\theta_{k(L-1)}(N) = \mathcal{U}_L(Q).$$

(C) *Let $\eta(q,a)$ be as in Theorem 1. For any $q, n \in \mathbb{N}$*

$$\eta(q,n) \ll_\epsilon q^{\epsilon-1}$$

and

$$\sum_{a=1}^{q} \eta(q,a)^2 = \frac{\alpha\mathcal{U}_{-1}(q)}{q}.$$

Proof. (A). Write $S_w(Q)$ for the sum in question. We have

$$S_w(Q) = \sum_{\substack{h=1\\(h,Q)=1}}^{\infty} \frac{\mu(h)\Delta_w(h)}{h^{2k}} \sum_{\substack{D=1\\(D,hQ)=1}}^{\infty} \frac{\Delta_w(D)}{D^k} \sum_{dd'=D} \mu(d)\mu(d')$$

and the D sum here is

$$\prod_p \left(1 - \frac{2\Delta_w(p)}{p^k} + \frac{\Delta_w(p^2)}{p^{2k}}\right) \prod_{p|hQ} \left(1 - \frac{2\Delta_w(p)}{p^k} + \frac{\Delta_w(p^2)}{p^{2k}}\right)^{-1} =: AX(hQ),$$

and since

$$\sum_{\substack{h=1\\(h,Q)=1}}^{\infty} \frac{\mu(h)\Delta_w(h)X(h)}{h^{2k}} = \prod_p \left(1 - \frac{\Delta_w(p)X(p)}{p^{2k}}\right) \prod_{p|Q} \left(1 - \frac{\Delta_w(p)X(p)}{p^{2k}}\right)^{-1}$$
$$=: BY(Q)$$

we therefore conclude that

$$\begin{aligned}
S_w(Q) &= ABX(Q)Y(Q) \\
&= \prod_p \left(1 - \frac{2\Delta_w(p)}{p^k} + \frac{\Delta_w(p^2)}{p^{2k}}\right)\left(1 - \frac{\Delta_w(p)X(p)}{p^{2k}}\right) \\
&\quad \times \prod_{p|Q} \left(1 - \frac{2\Delta_w(p)}{p^k} + \frac{\Delta_w(p^2)}{p^{2k}}\right)^{-1}\left(1 - \frac{\Delta_w(p)X(p)}{p^{2k}}\right)^{-1}.
\end{aligned} \tag{2.4}$$

Since $\Delta_w(p^2) = \Delta_w(p)$ we see that

$$\left(1 - \frac{2\Delta_w(p)}{p^k} + \frac{\Delta_w(p^2)}{p^{2k}}\right)\left(1 - \frac{\Delta_w(p)X(p)}{p^{2k}}\right) = 1 - \frac{2\Delta_w(p)}{p^k}$$

and the claim follows from (2.4).

(B). Take $N \in \mathbb{N}$ a power of a prime p and write $w = k(L-1)$. Write

$$g_w(N) = \mu\left(K(N)\right) K(N)^{2w} \mathcal{Z}_{2w}^{-1}(N); \quad \text{note that} \quad g_w(p) = \frac{p^{2w}}{p^{2w}-1}.$$

For $d^{2k}|N$ we have $r_k(N/d^{2k}) = d^{-2}r_k(N)$ so

$$\begin{aligned}
F_w(N) &= r_k(N)^w - g_w(p)r_k(N)^w + g_w(p)\sum_{st|N}\mu(t)\mu_{2k+1}(st)r\left(N/s\right)^w \\
&= r_k(N)^w - g_w(p)r_k(N)^w + g_w(p)r\left(\frac{N}{(N,p^{2k})}\right)^w \\
&= \begin{cases} r_k(N)^w(1-g_w(p)) + g_w(p) & \text{if } N|p^{2k} \\ r_k(N)^w\left(1-g_w(p)+p^{-2w}g_w(p)\right) & \text{if } p^{2k+1}|N \end{cases} \\
&= \chi_{2k+1}(N)\Big(r_k(N)^w(1-g_w(p)) + g_w(p)\Big)
\end{aligned}$$

and, since

$$\sum_{Hh=N}\frac{\mu(h)}{H^L} = \frac{p^L\left(p^{-L}-1\right)}{N^L} =: \frac{p^L a_L(p)}{N^L},$$

we see that

$$\begin{aligned}
Z_L(N) &= \frac{\chi_{2k+1}(N)}{N^L}\left(p^L a_L(p)r_k(N)^w\Big(1-g_w(p)\Big) + p^L a_L(p)g_w(p)\right) \\
&=: \frac{\chi_{2k+1}(N)}{N^L}\left(p^L A_L(p)r_k(N)^w + p^L B_L(p)\right) \\
&=: f_L(N)
\end{aligned}$$

so that

$$\sum_{N|Q} Z_L(N)\theta_w(N) = 1 + \theta_w(p)\sum_{p|N|Q} f_L(N). \tag{2.5}$$

Note that

$$A_L(p) = \frac{p^{-2w}\left(p^{-L}-1\right)}{p^{-2w}-1}, \quad B_L(p) = -\frac{p^{-L}-1}{p^{-2w}-1}.$$

Write $p^D = (Q, p^{2k})$. We have $r_k(N)^w = p^w$ for $p|N|p^k$ and $r_k(N)^w = p^{2w}$ for $p^{k+1}|N|p^{2k}$ so that, if $D \geq k+1$,

$$\begin{aligned}
\sum_{p|N|Q} f_L(N) &= p^L A_L(p)\sum_{p|N|p^k}\frac{r_k(N)^w}{N^L} + p^L A_L(p)\sum_{p^{k+1}|N|p^D}\frac{r_k(N)^w}{N^L} \\
&\quad + p^L B_L(p)\sum_{p|N|p^D}\frac{1}{N^L} \\
&= p^w A_L(p)\sum_{j=0}^{k-1}\frac{1}{p^{jL}} + p^{2w-kL}A_L(p)\sum_{j=0}^{D-k-1}\frac{1}{p^{jL}} + B_L(p)\sum_{j=0}^{D-1}\frac{1}{p^{jL}}
\end{aligned}$$

$$= \frac{1}{p^{-2w}-1}\left(p^{-w}\left(p^{-kL}-1\right)+\frac{p^{-L(D-k)}-1}{p^{kL}}-\left(p^{-DL}-1\right)\right)$$
$$= -\frac{\mathcal{Z}_{-kL}(p)\mathcal{Z}_{-w}(p)}{\mathcal{Z}_{-2w}(p)}.$$

If $D \leq k$ the same argument goes through (with this time the first sum running up to p^D and no middle sum) to give

$$\sum_{p|N|Q} f_L(n) = -\frac{\mathcal{Z}_{-DL}(p)\mathcal{Z}_{-w}(p)}{\mathcal{Z}_{-2w}(p)}$$

so we can deduce

$$\sum_{p|N|Q} f_L(n) = -\frac{\mathcal{Z}_{-L\min(D,k)}(p)\mathcal{Z}_{-w}(p)}{\mathcal{Z}_{-2w}(p)}$$
$$= -\frac{p^w\mathcal{Z}_{-L\min(D,k)}(p)\mathcal{Z}_{w}(p)}{\mathcal{Z}_{2w}(p)}$$

so that (2.5) implies

$$\sum_{N|Q} Z_L(N)\theta_w(N) = 1 - p^w \cdot \frac{\mathcal{Z}_{-L\min(D,k)}(p)\mathcal{Z}_{w}(p)}{\left(1-2\Delta_w(p)/p^k\right)\mathcal{Z}_{2w}(p)}$$
$$= 1 - p^w \cdot \frac{\mathcal{Z}_{-L\min(D,k)}(p)}{1+p^w-2/p^k}.$$

(C). Arranging the d according to the value of $D = (q, d^k)$ and using (2.1) we have

$$\eta(q,a) \leq \frac{1}{q}\sum_{D|q,a} D \sum_{\substack{d=1 \\ D|d^k}}^{\infty} \frac{|\mu(d)|}{d^k}$$
$$\ll \frac{1}{q}\sum_{D|q,a} \frac{D}{r_k(D)^k}.$$

From (2.2) we see that the summand here is ≤ 1 and the first claim follows. We have

$$
\begin{aligned}
\sum_{a=1}^{q}\eta(q,a)^2 &= \sum_{d,d'=1}^{\infty}\frac{\mu(d)\mu(d')}{[q,d^k][q,d'^k]}\sum_{\substack{a=1\\(q,d^k),(q,d'^k)\mid a}}^{q} 1 \\
&= q\sum_{d,d'=1}^{\infty}\frac{\mu(d)\mu(d')}{[q,d^k][q,d'^k][(q,d^k),(q,d'^k)]} \\
&= \frac{1}{q}\sum_{d,d'=1}^{\infty}\frac{\mu(d)\mu(d')(q,d^k,d'^k)}{d^k d'^k} \\
&= \frac{1}{q}\sum_{N=1}^{\infty}\frac{1}{N^k}\sum_{dd'=N}\mu(d)\mu(d')(q,d^k,d'^k) \\
&=: \frac{1}{q}\sum_{N=1}^{\infty}\frac{b_q(N)}{N^k}.
\end{aligned}
\tag{2.6}
$$

Clearly $b_q(N)$ is multiplicative and simple calculations show

$$
\begin{aligned}
b_q(p) &= -2,\\
b_q(p^2) &= (q,p^k)
\end{aligned}
$$

and $b_q(p^t)=0$ for $t\ge 3$. Consequently

$$
\begin{aligned}
\sum_{N=1}^{\infty}\frac{b_q(N)}{N^k} &= \prod_p\left(1-\frac{2}{p^k}+\frac{(q,p^k)}{p^{2k}}\right)\\
&= \prod_p\left(1-\frac{2}{p^k}+\frac{1}{p^{2k}}\right)\prod_{p\mid q}\frac{1-2/p^k+(q,p^k)/p^{2k}}{1-2/p^k+1/p^{2k}}\\
&= \alpha U_{-1}(q)
\end{aligned}
$$

which with (2.6) is the second claim. $\square$

We will need to evaluate precisely a sum of type $\Sigma_{d^k n\le X}(X-d^k n)$. One option is through the Euler–Maclaurin summation formula, which gets us Theorem 1, and another is through Perron's formula, which gets us Theorem 2. The next lemma contains the main work in using the Euler–Maclaurin summation formula.

For $X > 0$ define $B_1(X)$ and $B_2^*(X)$ as in [12, (2.19)–(2.22)]. Display (6.7) of that paper says $|B_2^*(X)| \ll X$ so that

$$\sum_{d,d'=1}^{\infty} \frac{(d^k, d'^k)}{(q, d^k, d'^k)[d,d']^k} B_k^*\left(\frac{x(q,d^k,d'^k)}{q(d^k,d'^k)}\right) \ll_{x,q} \sum_{d,d'=1}^{\infty} \frac{1}{[d,d']^k}$$
$$= \sum_{D=1}^{\infty} D^k \sum_{\substack{d,d'=1\\(d,d')=D}}^{\infty} \frac{1}{d^k d'^k}$$
$$\ll 1.$$

From the line preceeding (6.7) of [12] we have

$$B_2^*(\alpha) = \int_0^\alpha B_1(\beta)\,\mathrm{d}\beta. \tag{2.7}$$

The function $B_1(X)$ is exactly $\chi(v)$ in display (2.2) of [9], so by Lemma 4.1 of that paper we have

$$\int_0^X \frac{B_1(X)\,\mathrm{d}\beta}{\beta^{1/k}} = \frac{\zeta(1/k-1)}{1/k-1} + \mathcal{O}\left(\frac{1}{X^{1/k}}\right). \tag{2.8}$$

Lemma 2.2. *For $X > 0$ define $B_2^*(X)$ as in the discussion above. For $d, d' \in \mathbb{N}$ write $N = (d, d')$ and define*

$$\mathcal{B}_q^*(x) = \sum_{d,d'=1}^{\infty} \frac{\mu(d)\mu(d')}{(q,N^k)}\left(\frac{N^2}{dd'}\right)^k B_2^*\left(\frac{x(q,N^k)}{qN^k}\right)$$

and, for $L \le 0$,

$$\mathcal{B}_q^L = \sum_{d,d'=1}^{\infty} \frac{\mu(d)\mu(d')}{(q,N^k)^L}\left(\frac{N^{L+1}}{dd'}\right)^k,$$

which, by the discussion above, are both absolutely convergent. Then

$$\mathcal{B}_q^{-1} = \alpha\mathcal{U}_{-1}(q),$$

$$\mathcal{B}_q^0 = \frac{\mathcal{U}_0(q)}{\zeta(k)}$$

and

$$2\mathcal{B}_q^*(X) = C_k X^{1/k}\mathcal{U}_{1-1/k}(q) + \mathcal{O}_\epsilon\left(q^\epsilon X^{1/(k+1)}\right),$$

where $\mathcal{U}_L(q)$ and α are as given in Lemma 2.1 and C_k as in Theorem 1.

Proof. Throughout this proof we will use

$$\sum_{d|n}\mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1 \end{cases}$$

in the form

$$\sum_{\substack{a,b\\(a,b)=1}} = \sum_h \mu(h) \sum_{\substack{a,b\\h|a\\h|b}}$$

as well as use the vanishing of μ on squares in the form

$$\sum_{a,b} \mu(ab) = \sum_{\substack{a,b\\(a,b)=1}} \mu(a)\mu(b)$$

regularly without comment. Define $r_k(D)$ as at the start of this section and define

$$w_T(D) = \frac{\phi(T)}{T r_k(D)}. \tag{2.9}$$

Define $\mathcal{Z}_w(N)$ as in Lemma 2.1 so that

$$w_T(D) = \mathcal{Z}_{-1}(T) r_k(D)^{-1} \tag{2.10}$$

and so that (2.1) implies

$$\sum_{\substack{N\le Z\\(N,T)=1\\D|N^k}} 1 = Z w_T(D) \begin{Bmatrix} 1 & \text{if } (D,T)=1 \\ 0 & \text{if } (D,T)>1 \end{Bmatrix} + \mathcal{O}(1). \tag{2.11}$$

Arranging the N according to the value of $Q = (q, N^k)$ and using (2.3), we see that

$$\begin{aligned} &\sum_{\substack{N=1\\(N,T)=1}}^{\infty} \frac{\mu(N)^2}{(q,N^k)} B_2^*\left(\frac{x(q,N^k)}{N^k}\right) \\ &\qquad = \sum_{Qh|q} \frac{\mu(h)}{Q} \sum_{\substack{d=1\\(d,T)=1}}^{\infty} \mu(d) \mathcal{N}^*_{Qh/(Qh,d^{2k}),T}\left(\frac{xQ}{d^{2k}}\right) \end{aligned} \tag{2.12}$$

where

$$\mathcal{N}^*_{D,T}(X) = \sum_{\substack{N=1\\(N,T)=1\\D|N^k}}^{\infty} B_2^*\left(X/N^k\right),$$

and

$$\sum_{\substack{N=1\\(N,T)=1}}^{\infty}\frac{\mu(N)^2N^{k(L-1)}}{(q,N^k)^L} = \sum_{Qh|q}\frac{\mu(h)}{Q^L}\sum_{\substack{d=1\\(d,T)=1}}^{\infty}d^{2k(L-1)}\mu(d)\mathcal{N}^L_{Qh/(Qh,d^{2k}),T} \tag{2.13}$$

where

$$\mathcal{N}^L_{D,T} = \sum_{\substack{N=1\\(N,T)=1\\D|N^k}}^{\infty}N^{k(L-1)}.$$

Write

$$\mathcal{Z} = \frac{\zeta(1/k-1)}{1/k-1}.$$

From (2.7), (2.11) and (2.8) we have, for $(D,T)=1$ and a parameter $1\le y\le X$ to be chosen,

$$\begin{aligned}\sum_{\substack{N>y\\(N,T)=1\\D|N^k}}B_2^*\left(X/N^k\right) &= \int_0^{X/y^k}B_1(\beta)\left(\sum_{\substack{y<N\le(X/\beta)^{1/k}\\(N,T)=1,\, D|N^k}}1\right)\mathrm{d}\beta\\ &= w_T(D)X^{1/k}\int_0^{X/y^k}\frac{B_1(\beta)}{\beta^{1/k}}\,\mathrm{d}\beta\\ &\qquad - y\underbrace{w_T(D)}_{\ll 1}\underbrace{\int_0^{X/y^k}B_1(\beta)\mathrm{d}\beta}_{\ll 1}+\mathcal{O}\left(\int_0^{X/y^k}\underbrace{|B_1(\beta)|}_{\ll 1}\,\mathrm{d}\beta\right)\\ &= \mathcal{Z}w_T(D)X^{1/k}+\mathcal{O}\left(y+\frac{X}{y^k}\right)\end{aligned}$$

and if $(D,T)>1$ the same argument gives the same conclusion but with no main term, so

$$\begin{aligned}\mathcal{N}^*_{D,T}(X) &= \mathcal{Z}w_T(D)X^{1/k}\begin{Bmatrix}1 & \text{if } (D,T)=1\\ 0 & \text{if } (D,T)>1\end{Bmatrix}+\mathcal{O}\left(y+\frac{X}{y^k}\right)\\ &= \mathcal{Z}\mathcal{Z}_{-1}(T)r_k(D)^{-1}X^{1/k}\begin{Bmatrix}1 & \text{if } (D,T)=1\\ 0 & \text{if } (D,T)>1\end{Bmatrix}+\mathcal{O}\left(X^{1/(k+1)}\right)\end{aligned}$$

on choosing $y = X^{1/(k+1)}$ and using (2.10). Recall the definitions of $\mathcal{Z}_w$ and $\mathcal{Z}_w(N)$ from Lemma 2.1. From (2.1) we have

$$\begin{aligned}\mathcal{N}_{D,T}^{L} &= r_k(D)^{k(L-1)} \sum_{\substack{N=1\\(N,T)=1}}^{\infty} N^{k(L-1)} \begin{Bmatrix} 1 & \text{if } (D,T)=1 \\ 0 & \text{if } (D,T)>1 \end{Bmatrix} \\ &= \mathcal{Z}_{k(L-1)}^{-1}\mathcal{Z}_{k(L-1)}(T) r_k(D)^{k(L-1)} \begin{Bmatrix} 1 & \text{if } (D,T)=1 \\ 0 & \text{if } (D,T)>1 \end{Bmatrix}.\end{aligned}$$

These expressions for $\mathcal{N}_{D,T}^{*}(X)$ and $\mathcal{N}_{D,T}^{L}$ mean that (2.12) and (2.13) become

$$\begin{aligned}&\sum_{\substack{N=1\\(N,T)=1}}^{\infty} \frac{\mu(N)^2}{(q,N^k)} B_2^{*}\left(\frac{x(q,N^k)}{N^k}\right) \\ &\quad = \mathcal{Z}\mathcal{Z}_{-1}(T) x^{1/k} \sum_{\substack{Qh \mid q\\(Qh,T)=1}} \frac{\mu(h)}{Q^{1-1/k}} \sum_{\substack{d=1\\(d,T)=1}}^{\infty} \frac{\mu(d)}{d^2} r_k\left(\frac{Qh}{(Qh,d^{2k})}\right)^{-1} \\ &\qquad + \mathcal{O}\left(x^{1/(k+1)} \sum_{Qh \mid q} \frac{1}{Q^{1-1/(k+1)}} \sum_{d=1}^{\infty} \frac{1}{d^{2k/(k+1)}}\right) \\ &\quad =: \mathcal{Z}\mathcal{Z}_{-1}(T) x^{1/k} \sum_{\substack{Qh \mid q\\(Qh,T)=1}} \frac{\mu(h)\mathcal{D}_T^{-1}(Qh)}{Q^{1-1/k}} + \mathcal{O}\left(q^{\epsilon} x^{1/(k+1)}\right)\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}&\sum_{\substack{N=1\\(N,T)=1}}^{\infty} \frac{\mu(N)^2 N^{k(L-1)}}{(q,N^k)^L} \\ &\quad = \mathcal{Z}_{k(L-1)}^{-1}\mathcal{Z}_{k(L-1)}(T) \\ &\qquad \times \sum_{\substack{Qh \mid q\\(Qh,T)=1}} \frac{\mu(h)}{Q^L} \sum_{\substack{d=1\\(d,T)=1}}^{\infty} d^{2k(L-1)}\mu(d) r_k\left(\frac{Qh}{(Qh,d^{2k})}\right)^{k(L-1)} \\ &\quad =: \mathcal{Z}_{k(L-1)}^{-1}\mathcal{Z}_{k(L-1)}(T) \sum_{\substack{Qh \mid q\\(Qh,T)=1}} \frac{\mu(h)\mathcal{D}_T^{k(L-1)}(Qh)}{Q^L},\end{aligned} \tag{2.15}$$

where

$$D_T^{w}(N) = \sum_{\substack{d=1\\(d,T)=1}}^{\infty} d^{2w}\mu(d) r_k\left(\frac{N}{(N,d^{2k})}\right)^{w}.$$

Arranging the d according to the value of $s=(N,d^{2k})$ we see that

$$\mathcal{D}_T^w(N)=\sum_{st|N}\mu(t)r_k\,(N/s)^w\sum_{\substack{d=1\\(d,T)=1\\st|d^{2k}}}^{\infty}d^{2w}\mu(d) \tag{2.16}$$

$$=:\sum_{st|N}\mu(t)r_k\,(N/s)^w\,\mathcal{E}_T^w(st).$$

For $N\in\mathbb{N}$ denote by $K(N)$ the squarefree part of N. For $(T,N)=1$ and $M|N$ we have

$$\mathcal{E}_T^w(M)=\mu_{2k+1}(M)\sum_{\substack{d=1\\(d,T)=1\\K(M)|d}}^{\infty}d^{2w}\mu(d) \tag{2.17}$$

$$=K(M)^{2w}\mu_{2k+1}(M)\mu\left(K(M)\right)\sum_{\substack{d=1\\(d,TK(M))=1}}^{\infty}d^{2w}\mu(d)$$

$$=\mathcal{Z}_{2w}\mathcal{Z}_{2w}^{-1}(T)\mu_{2k+1}(M)\mu\left(K(M)\right)K(M)^{2w}\mathcal{Z}_{2w}^{-1}\left(K(M)\right)$$

so from (2.16) we have, for $(T,N)=1$,

$$\mathcal{D}_T^w(N)=\mathcal{Z}_{2w}\mathcal{Z}_{2w}^{-1}(T)F_w(N),$$

where $F_w(N)$ is as given in Lemma 2.1. Define $\Delta_w(N)$ and $Z_L(N)$ as in Lemma 2.1. Then the last equality means that (2.14) and (2.15) become

$$\sum_{\substack{N=1\\(N,T)=1}}^{\infty}\frac{\mu(N)^2}{(q,N^k)}B_2^*\left(\frac{x(q,N^k)}{N^k}\right) \tag{2.18}$$

$$=\mathcal{Z}\mathcal{Z}_{-2}x^{1/k}\Delta_{-1}(T)\sum_{\substack{M|q\\(M,T)=1}}Z_{1-1/k}(M)+\mathcal{O}\left(q^{\epsilon}x^{1/(k+1)}\right)$$

and

$$\sum_{\substack{N=1\\(N,T)=1}}^{\infty}\frac{\mu(N)^2N^{k(L-1)}}{(q,N^k)^L} \tag{2.19}$$

$$=\mathcal{Z}_{k(L-1)}^{-1}\mathcal{Z}_{2k(L-1)}\Delta_{k(L-1)}(T)\sum_{\substack{M|q\\(M,T)=1}}Z_L(M).$$

Arranging the d, d' according to the value $N = (d, d')$ we have

$$\mathcal{B}_q^*(x) = \sum_{\substack{h,d,d'=1 \\ (h,dd')=1}}^{\infty} \frac{\mu(h)\mu(d)\mu(d')}{h^{2k}d^k d'^k} \sum_{\substack{N=1 \\ (N,dd'h)=1}}^{\infty} \frac{\mu(N)^2}{(q,N^k)} B_2^* \left(\frac{x(q,N^k)}{qN^k} \right)$$

and

$$\mathcal{B}_q^L = \sum_{\substack{h,d,d'=1 \\ (h,dd')=1}}^{\infty} \frac{\mu(h)\mu(d)\mu(d')}{h^{2k}d^k d'^k} \sum_{\substack{N=1 \\ (N,dd'h)=1}}^{\infty} \frac{\mu(N)^2 N^{k(L-1)}}{(q,N^k)^L}.$$

Using (2.18) and (2.19) these equalities imply

$$\mathcal{B}_q^*(x) = \mathcal{Z}\mathcal{Z}_{-2} x^{1/k} \sum_{M|q} Z^{1-1/k}(M) \sum_{\substack{h,d,d'=1 \\ (h,dd')=1 \\ (M,dd'h)=1}}^{\infty} \frac{\mu(h)\mu(d)\mu(d')\Delta_{-1}(hdd')}{h^{2k}d^k d'^k} + \mathcal{O}\left(q^{\epsilon} x^{1/(k+1)}\right)$$

and

$$\mathcal{B}_q^L = \mathcal{Z}_{k(L-1)}^{-1} \mathcal{Z}_{2k(L-1)} \sum_{M|q} Z_L(M) \sum_{\substack{h,d,d'=1 \\ (h,dd')=1 \\ (M,dd'h)=1}}^{\infty} \frac{\mu(h)\mu(d)\mu(d')\Delta_{k(L-1)}(hdd')}{h^{2k}d^k d'^k}.$$

From Lemma 2.1 (A) and (B) it now follows that

$$\mathcal{B}_q^*(x) = \mathcal{Z}\mathcal{Z}_{-2} c_{-1} x^{1/k} \mathcal{U}_{1-1/k}(q) + \mathcal{O}\left(q^{\epsilon} x^{1/(k+1)}\right)$$

and

$$\mathcal{B}_q^L = \mathcal{Z}_{k(L-1)}^{-1} \mathcal{Z}_{2k(L-1)} c_{k(L-1)} \mathcal{U}_L(q),$$

where c_w is as given in Lemma 2.1. Since

$$\mathcal{Z}_{-2} c_{-1} = \prod_p \left(1 - \frac{p^k + 2p(p-1)}{p^{k+2}}\right)$$

and

$$\mathcal{Z}_w^{-1} \mathcal{Z}_{2w} c_w = \prod_p \left(1 + p^w - \frac{2}{p^k}\right).$$

we are finished. □

We will need to bound a tail end of a series. For the elementary error term $x^{1+2/(k+1)+\epsilon}/q$ in Theorem 1 the following lemma suffices.

Lemma 2.3. *For any $X, Y \geq 1$ and $N \in \mathbb{Z}$*

$$\sum_{[d,d'] \leq Y} 1 \ll_\epsilon Y^{1+\epsilon},$$

$$\sum_{[d,d'] > Y} \frac{1}{[d^k, {d'}^k]} \ll_\epsilon Y^{1-k+\epsilon}$$

and

$$\sum_{\substack{d,d \\ [d,d'] > Y}} \sum_{\substack{n \leq X \\ n \equiv 0 (d^k) \\ n \equiv N ({d'}^k)}} 1 \ll_\epsilon XY^{1-k+\epsilon} + X^{2/(k+1)}(X^\epsilon + N^\epsilon).$$

Proof. Since

$$\sum_{[d,d'] = n} 1 \ll n^\epsilon$$

the first two claims are clear. Let $Z > 0$ be a parameter. We have with a divisor estimate for the d'

$$\begin{aligned} \sum_{\substack{d,d' \\ d > Z}} \sum_{\substack{n \leq X \\ n \equiv 0 (d^k) \\ n \equiv N ({d'}^k)}} 1 &\ll (X^\epsilon + N^\epsilon) \sum_{\substack{d^k \leq X \\ d > Z}} \sum_{\substack{n \leq X \\ n \equiv 0 (d^k)}} 1 \\ &\ll X(X^\epsilon + N^\epsilon) \sum_{d > Z} \frac{1}{d^k} \\ &\ll XZ^{1-k}(X^\epsilon + N^\epsilon) \end{aligned}$$

and similarly for the terms with $d' > Z$. On the other hand the second claim implies

$$\begin{aligned} \sum_{\substack{d,d' \leq Z \\ [d,d'] > Y}} \sum_{\substack{n \leq X \\ n \equiv 0 (d^k) \\ n \equiv N ({d'}^k)}} 1 &\ll \sum_{\substack{d,d' \leq Z \\ [d,d'] > Y}} \left(\frac{X}{[d^k, {d'}^k]} + 1 \right) \\ &\ll XY^{1-k+\epsilon} + Z^2 \end{aligned}$$

and therefore

$$\sum_{[d,d'] > Y} \sum_{\substack{n \leq X \\ n \equiv 0 (d^k) \\ n \equiv N ({d'}^k)}} 1 \ll XY^{1-k+\epsilon} + Z^2 + XZ^{1-k}(X^\epsilon + N^\epsilon)$$

which gives the last claim on choosing $Z = X^{1/(k+1)}$. $\square$

3. Lemmas for Theorem 2

For this section we fix some $0 < \delta < 1/2k$ and all $\mathcal{O}$ constants are allowed to depend on this δ. For $\mathfrak{Re}(s) > 1$ define

$$\mathcal{F}(s) = \sum_{d,d'=1}^{\infty} \frac{\mu(d)\mu(d')}{[d^k, d'^k][q, (d^k, d'^k)]^s},$$

absolutely convergent since the summands are bounded by

$$\frac{1}{[d^k, d'^k](d^k, d'^k)} \ll \frac{1}{d^k d'^k}.$$

For $\mathfrak{Re}(s) \geq -1 + \delta$ define

$$\mathcal{F}^*(s) = \prod_{p|q} \frac{1 + (q, p^k)^s / p^{k(1+s)}}{1 + 1/p^{k(1+s)}} \prod_p \left(1 - \frac{2}{p^k \left(1 + (q, p^k)^s / p^{k(1+s)}\right)}\right).$$

Since for $\mathfrak{Re}(s) \geq -1 + \delta$

$$\left|(q, p^k)^s / p^{k(1+s)}\right| \leq \begin{cases} 1/p^k & \text{for } \mathfrak{Re}(s) \geq 0 \\ 1/p^{k\delta} & \text{for } \mathfrak{Re}(s) < 0 \end{cases} \tag{3.1}$$
$$\ll 1$$

and therefore

$$1 + (q, p^k)^s / p^{k(1+s)} \geq 1 - 1/2^{k\delta} \gg 1$$

we see that each Euler factor of the infinite product in $\mathcal{F}^*(s)$ is of the form

$$1 + \mathcal{O}\left(1/p^k\right)$$

and therefore this product converges absolutely and is $\ll 1$ for $\mathfrak{Re}(s) \geq -1 + \delta$, so holomorphic. Since for $\mathfrak{Re}(s) \geq -1 + \delta$

$$\frac{1}{p^{k(1+s)}} \geq \frac{1}{p^{k\delta}}$$

and therefore

$$1 + 1/p^{k(1+s)} \geq 1 - 1/2^{k\delta} \gg 1$$

we see from (3.1) that each factor in the finite product in $\mathcal{F}^*(s)$ is $\ll 1$ for $\mathfrak{Re}(s) \geq -1 + \delta$, and so the whole product is $\ll q^\epsilon$, and is obviously holomorphic. We conclude that $\mathcal{F}^*(s)$ is holomorphic and $\ll q^\epsilon$ for $\mathfrak{Re}(s) \geq -1 + \delta$.

We first obtain an analytic continuation for $\mathcal{F}(s)$.

Lemma 3.1. *Let $\mathcal{F}(s)$ and $\mathcal{F}^*(s)$ be as above. If $\mathfrak{Re}(s) > 1$ then*

$$\mathcal{F}(s) = \frac{\zeta\left(k(s+1)\right) \mathcal{F}^*(s)}{q^s \zeta\left(2k(s+1)\right)}.$$

Proof. We have

$$\sum_{d,d'} \frac{\mu(d)\mu(d')(d^k,d'^k)^{1-s}(q,d^k,d'^k)^s}{d^k d'^k} \tag{3.2}$$

$$= \sum_{N=1}^{\infty} \frac{1}{N^k} \sum_{dd'=N} \mu(d)\mu(d')(d^k,d'^k)^{1-s}(q,d^k,d'^k)^s$$

$$=: \sum_{N=1}^{\infty} \frac{a_q(N)}{N^k}.$$

Clearly $a_q(N)$ is multiplcative and simple calculations show

$$a_q(p) = -2,$$
$$a_q(p^2) = p^{k(1-s)}(q,p^k)^s$$

and $a_q(p^t) = 0$ for $t \geq 3$. Consequently ((3.1) ensuring no problems with zeros of denominators)

$$\begin{aligned}
\sum_{N=1}^{\infty} \frac{a_q(N)}{N^k} &= \prod_p \left(1 - \frac{2}{p^k} + \frac{(q,p^k)^s}{p^{k(1+s)}}\right) \\
&= \prod_p \left(1 + \frac{(q,p^k)^s}{p^{k(1+s)}}\right) \prod_p \left(1 - \frac{2}{p^k\left(1+(q,p^k)^s/p^{k(1+s)}\right)}\right) \\
&= \prod_p \left(1 + \frac{1}{p^{k(1+s)}}\right) \prod_{p|q} \frac{1+(q,p^k)^s/p^{k(1+s)}}{1+1/p^{k(1+s)}} \\
&\qquad \times \prod_p \left(1 - \frac{2}{p^k\left(1+(q,p^k)^s/p^{k(1+s)}\right)}\right) \\
&= \frac{\zeta\left(k(1+s)\right)\mathcal{F}^*(s)}{\zeta\left(2k(1+s)\right)}
\end{aligned} \tag{3.3}$$

so that (3.2) becomes

$$\sum_{d,d'} \frac{\mu(d)\mu(d')(d^k,d'^k)^{1-s}(q,d^k,d'^k)^s}{d^k d'^k} = \frac{\zeta\left(k(1+s)\right)\mathcal{F}^*(s)}{\zeta\left(2k(1+s)\right)}$$

and the claim follows. □

To exploit cancellation when integrating $\mathcal{F}^*(s)$ we need to write $\mathcal{F}^*(s)$ as a Dirichlet series.

Lemma 3.2. *Suppose q has ω distinct prime factors $p_1,\dots,p_\omega$ and let $\mathcal{F}^*(s)$ be as given at the start of this section. Then:*

for each $n \in \mathbb{N}$ and each $l_1,\dots,l_\omega,l'_1,\dots,l'_\omega \geq 0$

there are $\lambda_n, W_n, C_{\mathbf{l},\mathbf{l}'}, Z_{\mathbf{l},\mathbf{l}'} \in \mathbb{R}$ with $W_n, Z_{\mathbf{l},\mathbf{l}'} > 0$

such that

$$\mathcal{F}^*(s) = \sum_{\substack{l_1,\dots,l_\omega \geq 0 \\ l'_1,\dots,l'_\omega \geq 0}} \sum_{n=1}^{\infty} C_{\mathbf{l},\mathbf{l}'} Z^s_{\mathbf{l},\mathbf{l}'} \lambda_n W_n^{1+s}$$

for $\mathfrak{Re}(s) \geq -1+\delta$*. Moreover for these* s

$$\sum_{\substack{l_1,\dots,l_\omega \geq 0 \\ l'_1,\dots,l'_\omega \geq 0}} \sum_{n=1}^{\infty} \left| C_{\mathbf{l},\mathbf{l}'} Z^s_{\mathbf{l},\mathbf{l}'} \lambda_n W_n^{1+s} \right| \ll \log(q+1).$$

Proof. From (3.1) we have $|(q,p^k)^s/p^{k(1+s)}| < 1$ and therefore

$$\begin{aligned} (3.4) \quad \prod_p \left(1 - \frac{2}{p^k\left(1+(q,p^k)^s/p^{k(1+s)}\right)}\right) &= \prod_p \left(1 - \frac{2}{p^k} \sum_{t\geq 1} \left(\frac{-(q,p^k)^s}{p^{k(1+s)}}\right)^{t-1}\right) \\ &= \sum_{n=1}^{\infty} f_s^*(n) \end{aligned}$$

where $f_s^*(n)$ is the multiplicative function given on prime powers by

$$f_s^*(p^t) = -\frac{2}{p^k}\left(\frac{-(q,p^k)^s}{p^{k(1+s)}}\right)^{t-1}.$$

For any $n \in \mathbb{N}$ and prime $p|n$ define $t = t(p)$ through $p^t||n$. Then

$$\begin{aligned} (3.5) \qquad f_s^*(n) &= \prod_{p|n} \left(-\frac{2}{p^k}\right)\left(\frac{-(q,p^k)^s}{p^{k(1+s)}}\right)^{t-1} \\ &= \left(\prod_{p|n}(-1)^{t-1}\right)\left(\prod_{p|n}\frac{-2}{p^k}\right) \\ &\quad \times \left(\prod_{p|n}(q,p^k)^{-(t-1)}\right)\left(\prod_{p|n}\frac{(q,p^k)^{(t-1)(1+s)}}{p^{(t-1)k(1+s)}}\right). \end{aligned}$$

If we now define

$$\lambda_n = \left(\prod_{p|n}(-1)^{t-1}\right)\left(\prod_{p|n}\frac{-2}{p^k}\right)\left(\prod_{p|n}(q,p^k)^{1-t}\right)$$

and

$$W_n = \prod_{p|n}\frac{(q,p^k)^{t-1}}{p^{(t-1)k}}$$

then (3.5) becomes

$$f^*(n) = \lambda_n W_n^{1+s}$$

so (3.4) becomes

$$\prod_p \left(1 - \frac{2}{p^k\left(1+(q,p^k)^s/p^{k(1+s)}\right)}\right) = \sum_{n=1}^{\infty} \lambda_n W_n^{1+s}. \tag{3.6}$$

Just as (3.4) is true so is

$$\sum_{n=1}^{\infty} |f_s^*(n)| = \prod_p \left(1 - \frac{2}{p^k}\sum_{t\geq 1}\left|\left(\frac{-(q,p^k)^s}{p^{k(1+s)}}\right)^{t-1}\right|\right). \tag{3.7}$$

The t sum here is from (3.1)

$$\ll \sum_{t\geq 1}\left(\frac{1}{p^{k\delta}}\right)^{t-1} = \frac{1}{1-1/p^{k\delta}} \ll 1$$

so the Euler product in (3.7) is $\ll 1$ and therefore

$$\sum_{n=1}^{\infty} |f^*(n)| \ll 1, \quad \text{for } -1+\delta \leq \mathfrak{Re}(s) \leq 0. \tag{3.8}$$

We have for $\mathfrak{Re}(s) \geq -1+\delta$

$$\frac{1}{1+1/p^{k(1+s)}} = \sum_{l\geq 0}\left(\frac{-1}{p^{k(1+s)}}\right)^l = \sum_{l\geq 0}\frac{C_p(l)}{p^{k(1+s)l}} \tag{3.9}$$

for some $C_p(l)$ with

$$\sum_{l\geq 0}\left|\frac{C_p(l)}{p^{k(1+s)l}}\right| \ll \sum_{l\geq 0}\left(\frac{1}{p^{k\delta}}\right)^l \ll 1 \tag{3.10}$$

as well as

$$1 + \frac{(q,p^k)^s}{p^{k(1+s)}} = \sum_{l'\geq 0}\frac{C'_p(l')(q,p^k)^{sl'}}{p^{k(1+s)l'}} \tag{3.11}$$

for some $C'_p(l')$ with

$$\sum_{l'\geq 0}\left|\frac{C'_p(l')(q,p^k)^{sl'}}{p^{k(1+s)l'}}\right| \ll 1+1 \tag{3.12}$$

from (3.1). From (3.9), (3.10), (3.11) and (3.12) there are for each prime p and $l, l' \in \mathbb{N}$ some $C_p(l), C'_p(l')$ for which

$$\frac{1+(q,p^k)^s/p^{k(1+s)}}{1+1/p^{k(1+s)}} = \sum_{l,l'\geq 0}\frac{C_p(l)C'_p(l')(q,p^k)^{sl'}}{p^{k(1+s)(l+l')}}$$

and

$$\sum_{l,l'\geq 0}\left|\frac{C_p(l)C'_p(l')(q,p^k)^{sl'}}{p^{k(1+s)(l+l')}}\right| \ll 1.$$

Consequently

$$\prod_{p|q} \frac{1+(q,p^k)^s/p^{k(1+s)}}{1+1/p^{k(1+s)}}$$
$$= \sum_{\substack{l_1,\dots,l_\omega\ge 0\\ l'_1,\dots,l'_\omega\ge 0}} \frac{C_{p_1}(l_1)C'_{p_1}(l'_1)\cdots C_{p_\omega}(l_\omega)C'_{p_\omega}(l'_\omega)(q,p_1^k)^{sl'_1}\cdots(q,p_\omega^k)^{sl'_\omega}}{p_1^{k(1+s)(l_1+l'_1)}\cdots p_\omega^{k(1+s)(l_\omega+l'_\omega)}}$$

and, for some $A>0$,

$$\sum_{\substack{l_1,\dots,l_\omega\ge 0\\ l'_1,\dots,l'_\omega\ge 0}} \left|\frac{C_{p_1}(l_1)C'_{p_1}(l'_1)\cdots C_{p_\omega}(l_\omega)C'_{p_\omega}(l'_\omega)(q,p_1^k)^{sl'_1}\cdots(q,p_\omega^k)^{sl'_\omega}}{p_1^{k(1+s)(l_1+l'_1)}\cdots p_\omega^{k(1+s)(l_\omega+l'_\omega)}}\right|$$
$$\le A^\omega \ll \log(q+1).$$

If we now define

$$C^*_{\mathbf{l},\mathbf{l}'} = \prod_{i=1}^{\omega} C_{p_i}(l_i)C'_{p_i}(l'_i), \qquad W_{\mathbf{l},\mathbf{l}'} = \left(\prod_{i=1}^{\omega} p_i^{l_i+l'_i}\right)^k, \qquad C_{\mathbf{l},\mathbf{l}'} = \frac{C^*_{\mathbf{l},\mathbf{l}'}}{W_{\mathbf{l},\mathbf{l}'}},$$
$$D_{\mathbf{l}'} = \prod_{i=1}^{\omega}(q,p_i^k)^{l'_i}, \quad \text{and} \quad Z_{\mathbf{l},\mathbf{l}'} = \frac{D_{\mathbf{l}'}}{W_{\mathbf{l},\mathbf{l}'}}$$

then

$$\prod_{p|q} \frac{1+(q,p^k)^s/p^{k(1+s)}}{1+1/p^{k(1+s)}} = \sum_{\substack{l_1,\dots,l_\omega\ge 0\\ l'_1,\dots,l'_\omega\ge 0}} C_{\mathbf{l},\mathbf{l}'}Z^s_{\mathbf{l},\mathbf{l}'} \tag{3.13}$$

with

$$\sum_{\substack{l_1,\dots,l_\omega\ge 0\\ l'_1,\dots,l'_\omega\ge 0}} |C_{\mathbf{l},\mathbf{l}'}Z^s_{\mathbf{l},\mathbf{l}'}| \ll \log(q+1). \tag{3.14}$$

The first claim now follows from (3.6) and (3.13), and the boundedness claim from (3.8) and (3.14). □

As mentioned before Lemma 2.2, we will have to evaluate precisely a sum of type $\Sigma_{d^kn\le X}(X-d^kn)$. That lemma contained the work necessary for the elementary argument and consequently Theorem 1, whilst for Theorem 2 we use Perron's formula. We weren't able to find a quantative version for Perron's formula with Cesáro weights in the literature so we produce one here.

Lemma 3.3. *Let $c>1$, let*

$$\mathcal{A}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be absolutely convergent for $\mathfrak{Re}(s) > c$*, and let*

$$A(Q) = \max_{Q/2 \leq n \leq 3Q/2} |a_n|.$$

Then for $T \geq 1$ *and non-integer* $Q > 0$

$$\sum_{n \leq Q} a_n \Big(Q - n\Big) = \frac{1}{2\pi i} \int_{c \pm iT} \frac{\mathcal{A}(s) Q^{s+1}}{s(s+1)} \, \mathrm{d}s$$
$$+ \mathcal{O}\left(\frac{QA(Q)2^c}{T} \left(1 + \frac{Q\,(\log Q + 1)}{T} \right) + \frac{Q^{c+1}}{T^2} \sum_{n=1}^{\infty} \frac{|a_n|}{n^c} \right).$$

In particular if $c = 1 + \mathcal{O}\left(1/\log Q\right)$ *then*

$$\sum_{n \leq Q} \Big(Q - n\Big) = \frac{1}{2\pi i} \int_{c \pm iT} \frac{\zeta(s) Q^{s+1}}{s(s+1)} \, \mathrm{d}s + \mathcal{O}\left(Q^{\epsilon} \left(\frac{Q}{T} + \frac{Q^2}{T^2} \right) \right).$$

Proof. Take $X > 0$ and define

$$\delta(X) = \begin{cases} 0 & \text{if } 0 < X < 1 \\ X - 1 & \text{if } X > 1 \end{cases}$$

and

$$I_X(T) = \frac{1}{2\pi i} \int_{c \pm iT} \frac{X^{s+1}}{s(s+1)} \, \mathrm{d}s.$$

We first prove

$$|I_X(T) - \delta(X)| \ll \frac{X^{c+1}}{T} \min\left\{1, \frac{1}{T|\log X|}\right\}. \tag{3.15}$$

Suppose first $0 < X < 1$ so that for $R > c$ we have $X^R \ll X^c \ll 1$. The integrand is holomorphic to the right of 0 so for $R > c$

$$\begin{aligned} 2\pi i I_X(T) &= -\left(\int_{c+iT}^{R+iT} + \int_{R+iT}^{R-iT} + \int_{R-iT}^{c-iT} \right) \frac{X^{s+1}}{s(s+1)} \, \mathrm{d}s \\ &\ll \frac{1}{T^2} \int_c^R X^{\sigma+1} \mathrm{d}\sigma + \frac{X^{R+1}}{R^2} \int_{\pm T} \mathrm{d}t \\ &\ll \frac{X^{c+1} + X^{R+1}}{T^2 |\log X|} + \frac{X^{R+1} T}{R^2} \ll \frac{X^{c+1}}{T^2 |\log X|} \end{aligned}$$

with $R \to \infty$. Suppose now that $X > 1$ so that for $R < -1$ we have $X^R \ll 1$ we have $X^{s+1} \ll 1$. The integrand is holomorphic except for at two places, so for $R < -1$

$$\begin{aligned}2\pi i I_X(T) &= \mathrm{Res}_{s=0}\left(\frac{X^{s+1}}{s(s+1)}\right) + \mathrm{Res}_{s=-1}\left(\frac{X^{s+1}}{s(s+1)}\right)\\ &\qquad - \left(\int_{c+iT}^{R+iT} + \int_{R+iT}^{R-iT} + \int_{R-iT}^{c-iT}\right)\frac{X^{s+1}}{s(s+1)}\,\mathrm{d}s\\ &= X - 1 + \mathcal{O}\left(\frac{1}{T^2}\int_c^R X^{\sigma+1}\mathrm{d}\sigma + \frac{X^{R+1}}{R^2}\int_{\pm T}\mathrm{d}t\right)\\ &= X - 1 + \mathcal{O}\left(\frac{1}{T^2}\int_c^\infty X^{\sigma+1}\mathrm{d}\sigma + \frac{X^{R+1}T}{|R|^2}\right)\\ &= X - 1 + \mathcal{O}\left(\frac{X^{c+1}+X^{R+1}}{T^2|\log X|} + \frac{X^{R+1}T}{|R|^2}\right)\\ &= X - 1 + \mathcal{O}\left(\frac{X^{c+1}}{T^2|\log X|}\right)\end{aligned}$$

so we can conclude that for all $X > 0$ the second bound in (3.15) is clear; now for the first bound. If $0 < X < 1$ and if $\mathcal{C}$ is the arc of the circle going clockwise from $c+iT$ to $c-iT$ (so a circle of radius $\sqrt{T^2+c^2} > T$) then on $\mathcal{C}$ we have $X^s \ll X^c$ on $\mathcal{C}$). Noting again that the integrand is holomorphic to the right of 0 we have

$$\begin{aligned}2\pi i I_X(T) &= -\int_{\mathcal{C}}\frac{X^{s+1}}{s(s+1)}\,\mathrm{d}s\\ &\ll X^{c+1}\int_{\mathcal{C}}\frac{1}{|s|\cdot|s+1|}\,\mathrm{d}s \ll \frac{X^{c+1}}{T}.\end{aligned}$$

If $X > 1$ the remaining part of the circle should be taken as the contour so that $X^s \ll X^c$ holds on the contour, and this gives a similar result. We conclude that the first bound in (3.15) also holds for any value of $X > 0$ and so the proof of (3.15) is complete.

By (3.15) (and absolute convergence)

$$\begin{aligned}&\int_{c\pm iT}\frac{\mathcal{A}(s)Q^{s+1}}{s(s+1)}\,\mathrm{d}s\\ &\quad= \sum_{n=1}^{\infty} a_n n\int_{c\pm iT}\frac{1}{s(s+1)}\left(\frac{Q}{n}\right)^{s+1}\mathrm{d}s\\ &\quad= \sum_{n=1}^{\infty} a_n n\delta(Q/n) + \mathcal{O}\left(\frac{Q^{c+1}}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^c}\min\left\{1, \frac{1}{T|\log(Q/n)|}\right\}\right).\end{aligned}\tag{3.16}$$

For $Q/2 \le n \le 3Q/2$

$$|\log(Q/n)| = \left|\log\left(1 + \frac{n-Q}{Q}\right)\right| \gg \frac{|n-Q|}{n} \ge \Big\lfloor |n-Q| \Big\rfloor \Big/ n.$$

Therefore

$$\sum_{Q/2 \le n \le 3Q/2} \frac{|a_n|}{n^c} \min\left\{1, \frac{1}{T|\log(Q/n)|}\right\}$$
$$\le A(Q)\left(\frac{1}{(Q/2)^c} + \frac{2\,(Q/2)^{1-c}}{T} \sum_{h \le Q/2+1} \frac{1}{h}\right)$$
$$\ll Q^{-c} A(Q) 2^c \left(1 + \frac{Q\,(\log Q + 1)}{T}\right)$$

and if n is not in this range then $|\log(Q/n)| \gg 1$ so we deduce

$$\frac{Q^{c+1}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^c} \min\left\{1, \frac{1}{T|\log(Q/n)|}\right\}$$
$$\ll \frac{QA(Q)2^c}{T}\left(1 + \frac{Q\,(\log Q + 1)}{T}\right) + \frac{Q^{c+1}}{T^2} \sum_{n=1}^{\infty} \frac{|a_n|}{n^c}.$$

Therefore the error term in (3.16) is of the right order of magnitude and of course the main term is

$$\sum_{n \le Q} a_n \Big(Q - n\Big)$$

so the main claim is proven. For the "in particular claim" the main claim implies an error term

$$Q^{\epsilon}\left(\frac{Q}{T} + \frac{Q^2}{T^2} + \frac{Q^{c+1}\zeta(c)}{T^2}\right);$$

now use $\zeta(c) \ll 1/(c-1) \ll \log Q$ and $Q^c \ll Q$. □

We need bounds for the integrals arising from Perron's formula that go beyond taking absolute values.

Lemma 3.4. *Take $Q > 0$, $L \ge 2$ and $\Delta \in [1/2k, 1/k)$. Let*

$$R_1 = -1 + \Delta \quad \textit{and} \quad R_2 = \Delta k.$$

Then

$$\int_1^L \frac{\zeta(R_1 + it)\zeta(R_2 + itk)Q^{it}}{t^2}\, \mathrm{d}t \ll L^{1/4 - 1/2k} \log L.$$

Proof. Take $s = \sigma + it \in \mathbb{C}$ with $t \geq 1$ and take two parameters $N, M \gg 1$ with $NM = t/2\pi$. Let

$$\chi(s) = \frac{2^{s-1}\pi^s \sec(s\pi/2)}{\Gamma(s)}.$$

By formula (4.12.3) of [11] (the definition of $\chi(s)$ comes just before) we have for $-1 \leq \sigma \leq 1$

$$\begin{aligned} \chi(s) &= \left(\frac{t}{2\pi}\right)^{1/2-\sigma-it} e^{i(t+\pi/4)} \left(1 + \mathcal{O}\left(\frac{1}{t}\right)\right) \\ &= \left(\frac{t}{2\pi}\right)^{1/2-\sigma-it} e^{i(t+\pi/4)} + \mathcal{O}\left(\frac{1}{t^{1/2+\sigma}}\right) \end{aligned} \tag{3.17}$$

so that

$$\begin{aligned} &\chi(R_2 + itk) \sum_{n \leq M} \frac{1}{n^{1-R_2-itk}} \\ &\qquad = \left(\frac{t}{2\pi}\right)^{1/2-R_2-itk} e^{i(tk+\pi/4)} \sum_{n \leq M} \frac{1}{n^{1-R_2-itk}} + \mathcal{O}\left(\frac{M^{R_2}}{t^{1/2+R_2}}\right) \end{aligned}$$

so by the approximate functional equation [11, (4.12.4)]

$$\begin{aligned} \zeta(R_2+itk) &= \sum_{n \leq N} \frac{1}{n^{R_2+itk}} + \chi(R_2 + itk) \sum_{n \leq M} \frac{1}{n^{1-R_2-itk}} \\ &\quad + \mathcal{O}\left(N^{-R_2} + t^{1/2-R_2} M^{R_2-1}\right) \\ &= \sum_{n \leq N} \frac{1}{n^{R_2+itk}} + \left(\frac{t}{2\pi}\right)^{1/2-R_2-itk} e^{i(tk+\pi/4)} \sum_{n \leq M} \frac{1}{n^{1-R_2-it}} \\ &\quad + \mathcal{O}\left(\left(\frac{M}{t}\right)^{R_2} \left(1 + \frac{t^{1/2}}{M}\right)\right). \end{aligned} \tag{3.18}$$

From the functional equation (this just preceeds formula (4.12.1) of [11]) and (3.17) we have

$$\begin{aligned} \zeta(R_1 + it) &= \left(\left(\frac{t}{2\pi}\right)^{1/2-R_1-it} e^{i(t+\pi/4)} + \mathcal{O}\left(\frac{1}{t^{1/2+R_1}}\right)\right) \zeta(1 - R_1 - it) \\ &= \left(\frac{t}{2\pi}\right)^{1/2-R_1-it} e^{i(t+\pi/4)} \zeta(1 - R_1 - it) + \mathcal{O}\left(\frac{1}{t^{1/2+R_1}}\right) \end{aligned}$$

so that with (3.18) we get

$$
\begin{aligned}
&\zeta(R_1+it)\zeta(R_2+itk)\\
&\quad=\left(\frac{t}{2\pi}\right)^{1/2-R_1-it} e^{i(t+\pi/4)}\zeta(1-R_1-it)\sum_{n\le N}\frac{1}{n^{R_2+itk}}\\
&\qquad+\left(\frac{t}{2\pi}\right)^{1-R_1-R_2-it(k+1)} e^{i(t(k+1)+\pi/4)}\zeta(1-R_1-it)\sum_{n\le M}\frac{1}{n^{1-R_2-itk}}\\
&\qquad+\mathcal{O}\left(t^{1/2-R_1}|\zeta(1-R_1-it)|\left(\frac{M}{t}\right)^{R_2}\left(1+\frac{t^{1/2}}{M}\right)\right.\\
&\qquad\qquad\left.+\frac{(t/M)^{1-R_2}+t^{1/2-R_2}M^{R_2}}{t^{1/2+R_1}}+\frac{1}{t^{1/2+R_1}}\left(\frac{M}{t}\right)^{R_2}\left(1+\frac{t^{1/2}}{M}\right)\right)\\
&\quad=: M_1(t)+M_2(t)+\mathcal{O}\left(t^{1/2-R_1}\left(\frac{M}{t}\right)^{R_2}\left(1+\frac{t^{1/2}}{M}\right)\right).
\end{aligned}
$$

Write $N=t^{1/A}$ and $M=t^{1/B}$ so the above reads

$$
\begin{aligned}
\zeta(R_1+it)\zeta(R_2+itk)&\\
&= M_1(t)+M_2(t)+\mathcal{O}\left(t^{1/2-R_1+R_2/B-R_2}\left(1+t^{1/2-1/B}\right)\right).
\end{aligned}\tag{3.19}
$$

For some constant C

$$
\begin{aligned}
M_1(t)Q^{it}&=Ct^{1/2-R_1}\sum_{n\le N}\sum_{m=1}^{\infty}\frac{e^{it(-\log t+1-\log n^k+\log m+\log Q)}}{n^{R_2}m^{1-R_1}}\\
&=Ct^{1/2-R_1}\sum_{\substack{n^A\le t\\ 2\pi nM\le t}}\sum_{m=1}^{\infty}\frac{e\big(f_{mQ/n^k}(t)\big)}{n^{R_2}m^{1-R_1}}
\end{aligned}
$$

where

$$
f_X(t)=\frac{t(-\log t+1+\log X)}{2\pi}
$$

and the two summation conditions on n are equivalent. So for any $T\ge 1$ (and absolute convergence)

$$
\begin{aligned}
\int_T^{2T}&\frac{M_1(t)Q^{it}}{t^2}\,\mathrm{d}t\\
&=C\sum_{m=1}^{\infty}\frac{1}{m^{1-R_1}}\sum_{n^A\le 2T}\frac{1}{n^{R_2}}\int_{\max(2\pi nM,T)}^{2T}\frac{e\big(f_{mQ/n^k}(t)\big)}{t^{3/2+R_1}}\,\mathrm{d}t.
\end{aligned}\tag{3.20}
$$

We now bound this oscillatory integral. We have

$$
2\pi f_X'(t)=-\log t+\log X.\tag{3.21}
$$

Suppose first that T is large and $0 < X \ll 1$. For $\max(2\pi nM, T) < t < 2T$ we have from (3.21)

$$f_X'(t) \gg 1$$

and

$$t^{3/2+R_1} \gg T^{3/2+R_1}$$

so from Lemma 4.3 of [11]

$$\int_{\max(2\pi nM,T)}^{2T} \frac{e\left(f_X(t)\right)}{t^{3/2+R_1}}\,\mathrm{d}t \ll \frac{1}{T^{3/2+R_1}}, \qquad \text{if } 0 < X \ll 1. \tag{3.22}$$

Suppose now that X is large. Since from (3.21)

$$\begin{aligned} f_X'(t) &\gg |\log(t/X)| \\ &= |\log\left(1 + (t-X)/X\right)| \\ &\gg \begin{cases} |t-X|/X & \text{if } t \in (X/2, 3X/2) \\ 1 & \text{if not} \end{cases} \\ &\gg \begin{cases} 1/\sqrt{X} & \text{if } t \in (X/2, X-\sqrt{X}) \cup (X+\sqrt{X}, 3X/2) \\ 1 & \text{if } t \notin (X/2, 3X/2) \end{cases} \end{aligned}$$

and since for $t > T$

$$t^{3/2+R_1} \gg T^{3/2+R_1} \tag{3.23}$$

we have from Lemma 4.3 of [11]

$$\begin{aligned} &\int_{\max(2\pi nM,T)}^{2T} \frac{e\left(f_X(t)\right)}{t^{3/2+R_1}}\,\mathrm{d}t \\ &\qquad = \int_{\substack{\max(2\pi nM,T) \\ t \notin (X-\sqrt{X}, X+\sqrt{X})}}^{2T} + \int_{\substack{\max(2\pi nM,T) \\ t \in (X-\sqrt{X}, X+\sqrt{X})}}^{2T} \\ &\qquad \ll \begin{cases} \sqrt{X}/T^{3/2+R_1} & \text{if } (T,2T) \cap (X/2, 3X/2) \neq \emptyset \\ 1/T^{3/2+R_1} & \text{if } (T,2T) \subseteq (1,\infty) \backslash (X/2, 3X/2) \end{cases} \\ &\qquad \ll \frac{1}{T^{1+R_1}}, \end{aligned}$$

where we have used a trivial bound for the second integral. Therefore from (3.22)

$$\int_{\max(2\pi nM,T)}^{2T} \frac{e(f_X(t))}{t^{3/2+R_1}}\,\mathrm{d}t \ll \frac{1}{T^{1+R_1}}$$

holds in fact for all $X > 0$, and indeed all $T > 0$, being trivial for T not large. We deduce from (3.20) that for any $T > 0$

$$\int_T^{2T} \frac{M_1(t)Q^{it}}{t^2}\,\mathrm{d}t \ll \frac{1}{T^{1+R_1}} \sum_{m=1}^{\infty} \frac{1}{m^{1-R_1}} \sum_{n^A \le T} \frac{1}{n^{R_2}} \ll \frac{1}{T^{1+R_1}} \left(T^{1/A}\right)^{1-R_2} \ll T^{1/A-1-R_1-R_2/A}. \tag{3.24}$$

Similarly we have (for a slightly different f)

$$\int_T^{2T} \frac{M_2(t)Q^{it}}{t^2}\,\mathrm{d}t = C \sum_{m=1}^{\infty} \frac{1}{m^{1-R_1}} \sum_{n^B \le T} \frac{1}{n^{1-R_2}} \int_{\max(nN,T)}^{2T} \frac{e\left(f(t)\right)}{t^{1+R_1+R_2}}\,\mathrm{d}t$$

where the oscillatroy integral is

$$\ll \frac{1}{T^{1/2+R_1+R_2}}$$

so that

$$\int_T^{2T} \frac{M_2(t)Q^{it}}{t^2}\,\mathrm{d}t \ll \frac{1}{T^{1/2+R_1+R_2}} \sum_{m=1}^{\infty} \frac{1}{m^{1-R_1}} \sum_{n^B \le T} \frac{1}{n^{1-R_2}} \ll \frac{1}{T^{1/2+R_1+R_2}} \left(T^{1/B}\right)^{R_2} \ll T^{R_2/B-1/2-R_1-R_2}. \tag{3.25}$$

Note that

$$-1/2 - R_1 - R_2/2 = 1/2 - \Delta - \Delta k/2 \le 1/2 - \Delta - 1/4 \tag{3.26}$$

so taking $A = B = 2$ we see from (3.24) and (3.25)

$$\int_T^{2T} \frac{\left(M_1(t) + M_2(t)\right)Q^{it}}{t^2}\,\mathrm{d}t \ll T^{-1/2-R_1-R_2/2} \ll T^{1/4-\Delta}$$

for any $T > 0$ and so we conclude

$$\int_1^{L} \frac{\left(M_1(t) + M_2(t)\right)Q^{it}}{t^2}\,\mathrm{d}t \ll L^{1/4-\Delta}\log L$$

and so from (3.19) and (3.26)

$$\int_1^{L} \frac{\zeta(R_1+it)\zeta(R_2+itk)Q^{it}}{t^2}\,\mathrm{d}t \ll L^{1/4-\Delta}\log L + \int_1^{L} t^{-3/2-R_1-R_2/2}\,\mathrm{d}t \ll L^{1/4-\Delta}\log L. \qquad \square$$

We summarise the result of the last two lemmas.

Lemma 3.5. *Let $\mathcal{F}^*(s)$ be given as at the start of this section. For $X, T \geq 1$ and $c > 1$ we have, for some $\alpha, \beta, \gamma \in \mathbb{C}$,*

$$\frac{1}{2\pi i}\int_{c\pm iT} \frac{\zeta(s)\zeta\left(k(s+1)\right)\mathcal{F}^*(s)X^{s+1}}{s(s+1)\zeta\left(2k(s+1)\right)}\,\mathrm{d}s$$
$$= \alpha X^2 + \beta X + \gamma X^{1/k} + \mathcal{O}\left((qXT)^\epsilon\left(T^{1/4}\left(\frac{X}{T}\right)^{1/2k} + \frac{X^{c+1}}{T^2} + 1\right)\right).$$

Proof. For $s \in \mathbb{C}$ write always $s = \sigma + it$ for $\sigma, t \in \mathbb{R}$ and let

$$\mathcal{I}(s) = \frac{\zeta(s)\zeta\left(k(s+1)\right)\mathcal{F}^*(s)}{\zeta\left(2k(s+1)\right)}. \tag{3.27}$$

Let $R_1 = -1 + 1/2k + \tau$ for some $0 < \tau < 1/k$, and remember that we have fixed at the very beginning a small $\delta \in (0, 1/2k)$. We have already established (just before Lemma 3.1) that $\mathcal{F}^*(s) \ll q^\epsilon$ for $\sigma \geq -1 + \delta$, therefore

$$\mathcal{I}(s) \ll \frac{q^\epsilon|\zeta(s)\zeta\left(k(s+1)\right)|}{|\zeta\left(2k(s+1)\right)|} \quad \text{for } \sigma \geq R_1. \tag{3.28}$$

On $\mathfrak{Re}(s) \geq -1 + \delta$ we know by the comments before Lemma 3.1 that $\mathcal{I}(s)$ is holomorphic except for simple poles at $s = 1$ and $s = -1 + 1/k$ so by the Residue Theorem

$$\int_{c\pm iT} \frac{\mathcal{I}(s)X^{s+1}}{s(s+1)}\,\mathrm{d}s = 2\pi i\Big(\alpha X^2 + \beta X + \gamma X^{1/k}\Big) - \left(\int_{c+iT}^{R_1+iT} + \int_{R_1+iT}^{R_1-iT} + \int_{R_1-iT}^{c-iT}\right)\frac{\mathcal{I}(s)X^{s+1}}{s(s+1)}\,\mathrm{d}s \tag{3.29}$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$. It is standard that for $t \geq 1$

$$\zeta(s) \ll t^\epsilon \begin{cases} t^{1/2-\sigma} & \text{for } \sigma \leq 0 \\ \max\{1, t^{1/2-\sigma/2}\} & \text{for } \sigma \geq 0 \\ t^{1/4} & \text{for } \sigma \geq 1/2 \end{cases}$$

and

$$\zeta(\sigma) \ll \begin{cases} 1 & \text{for } \sigma \geq 2k \\ 1/|\sigma - 1| & \text{for } 1 \leq \sigma \leq 2; \end{cases}$$

we will now use these bounds freely without comment. If $0 \leq \sigma \leq 2$ and $t \geq 1$ we have

$$\zeta(s) \ll t^\epsilon \max\Big\{1, t^{1/2-\sigma/2}\Big\}, \qquad \zeta\left(k(s+1)\right) \ll 1$$

and

$$\frac{1}{\zeta\left(2k(s+1)\right)} \ll \zeta\left(2k(\sigma+1)\right) \ll 1,$$

so from (3.28)

$$\mathcal{I}(s) \ll q^\epsilon t^\epsilon \max\left\{1, t^{1/2-\sigma/2}\right\}$$

and therefore

$$\int_{iT}^{c+iT} \frac{\mathcal{I}(s)X^{s+1}}{s(s+1)}\,\mathrm{d}s \ll q^\epsilon T^\epsilon\left(\frac{X}{T^{3/2}}+\frac{X^{c+1}}{T^2}\right). \tag{3.30}$$

If $R_1 \le \sigma \le 0$ then for $t \ge 1$

$$\zeta(s) \ll t^{1/2-\sigma+\epsilon},$$
$$\zeta\left(k(s+1)\right) \ll t^{1/4+\epsilon}$$

and

$$\frac{1}{\zeta\left(2k(s+1)\right)} \ll \zeta\left(2k(\sigma+1)\right) \ll \frac{1}{|2k(\sigma+1)-1|} \ll \frac{1}{\tau},$$

so from (3.28)

$$\mathcal{I}(s) \ll \frac{q^\epsilon t^{1-\sigma}}{\tau} \tag{3.31}$$

and therefore

$$\int_{R_1+iT}^{iT} \frac{\mathcal{I}(s)X^{s+1}}{s(s+1)}\,\mathrm{d}s \ll \frac{q^\epsilon}{\tau}\left(\frac{X^{R_1+1}}{T^{1+R_1}}+\frac{X}{T}\right) \ll \frac{q^\epsilon}{\tau}\left(1+\frac{X}{T}\right). \tag{3.32}$$

From (3.30) and (3.32) we have

$$\left(\int_{c+iT}^{R_1+iT}+\int_{R_1-iT}^{c-iT}\right)\frac{\mathcal{I}(s)X^{s+1}}{s(s+1)}\,\mathrm{d}s \ll \frac{q^\epsilon T^\epsilon}{\tau}\left(1+\frac{X^{c+1}}{T^2}\right) \tag{3.33}$$

a similar argument for the second integral obviously valid. We now turn to the vertical contribution in (3.29). Denote by ω the number of prime factors of q. For given integers $n, l_1, \ldots, l_\omega, l_1', \ldots, l_\omega'$ write $\mathbf{n} = (n, l_1, \ldots, l_\omega, l_1', \ldots, l_\omega')$. Denote by $\mathcal{N}$ the set of all n for which all the n, l_i, l_i' are ≥ 0. Let $W_n, Z_{\mathbf{l},\mathbf{l}'}$ be as in Lemma 3.2. Then that lemma says that for given $\mathbf{n} \in \mathcal{N}$ there are $a_\mathbf{n} = a_\mathbf{n}(\sigma) \in \mathbb{R}$ such that for $\sigma \ge -1+\delta$

$$\mathcal{F}^*(s) = \sum_{\mathbf{n}\in\mathcal{N}} a_\mathbf{n}\left(W_n Z_{\mathbf{l},\mathbf{l}'}\right)^{it}$$

and

$$\sum_{\mathbf{n}\in\mathcal{N}} |a_\mathbf{n}| \ll q^\epsilon. \tag{3.34}$$

Therefore

$$\frac{\mathcal{F}^*(R_1+it)X^{it}}{\zeta\left(2k(R_1+it+1)\right)} = \sum_{m=1}^{\infty}\sum_{\mathbf{n}\in\mathcal{N}} \frac{\mu(m)a_\mathbf{n}}{m^{2k(R_1+1)}}\left(\frac{XW_nZ_{\mathbf{l},\mathbf{l}'}}{m^{2k}}\right)^{it},$$

so from (3.27), Lemma 3.4 and (3.34)

$$\tag{3.35} \begin{aligned} \int_1^T \frac{\mathcal{I}(R_1+it)X^{it}}{t^2}\,\mathrm{d}t &\ll T^{1/4-1/2k}\log T \sum_{m=1}^{\infty}\sum_{\mathbf{n}\in\mathcal{N}} \left|\frac{\mu(m)a_{\mathbf{n}}}{m^{2k(R_1+1)}}\right| \\ &\ll T^{1/4-1/2k}(\log T)\zeta(1+2k\tau) \\ &\ll \frac{T^{1/4-1/2k}\log T}{\tau}. \end{aligned}$$

We clearly have for $\sigma \geq -1+\delta$

$$\mathcal{I}(s) \ll q^{\epsilon} \begin{cases} 1 & \text{for } 0 \leq t \leq 1 \\ t^{7/4} & \text{for } t \geq 1 \end{cases}$$

and for $t \geq 1$ we have

$$\frac{1}{s(s+1)} = \frac{1}{t^2} + \mathcal{O}\left(\frac{1}{t^3}\right),$$

therefore from (3.35)

$$\begin{aligned} &\int_{R_1}^{R_1+iT} \frac{\mathcal{I}(s)X^{s+1}}{s(s+1)}\,\mathrm{d}s \\ &\quad = X^{R_1+1}\int_1^T \frac{\mathcal{I}(R_1+it)X^{it}}{t^2}\,\mathrm{d}s \\ &\qquad + \mathcal{O}\left(X^{R_1+1}\int_{\substack{R_1 \\ t\geq 1}}^{R_1+i\infty} \frac{|\mathcal{I}(s)|}{t^3}\,\mathrm{d}s + X^{R_1+1}\int_{R_1}^{R_1+i} \frac{|\mathcal{I}(s)|}{|s(s+1)|}\,\mathrm{d}s\right) \\ &\quad \ll \frac{X^{R_1+1}T^{1/4-1/2k}\log T}{\tau} + q^{\epsilon}X^{R_1+1} \\ &\quad = \frac{X^{\tau}T^{1/4+\epsilon}}{\tau}\left(\frac{X}{T}\right)^{1/2k} q^{\epsilon}. \end{aligned}$$

A similar bound obviously holding for t negative we conclude

$$\tag{3.36} \int_{R_1+iT}^{R_1-iT} \frac{\mathcal{I}(s)X^{s+1}}{s(s+1)}\,\mathrm{d}s \ll \frac{X^{\tau}T^{1/4+\epsilon}}{\tau}\left(\frac{X}{T}\right)^{1/2k} q^{\epsilon}.$$

Putting this, (3.33) and (3.36) into (3.29) gives the claim but with error

$$\begin{aligned} &\ll \frac{q^{\epsilon}T^{\epsilon}}{\tau}\left(X^{\tau}T^{1/4}\left(\frac{X}{T}\right)^{1/2k} + 1 + \frac{X^{c+1}}{T^2}\right) \\ &\ll \log X q^{\epsilon}T^{\epsilon}\left(T^{1/4}\left(\frac{X}{T}\right)^{1/2k} + 1 + \frac{X^{c+1}}{T^2}\right) \end{aligned}$$

on taking $\tau = 1/\log X$, so long as X is large. This is the claim for large X. If X is not large then the claim is trivial, the integrand being trivially $\ll t^{\epsilon-2}$ for $\sigma = c$. □

Obtaining a good second error in Theorem 2 amounts to counting solutions to $d^k n \equiv d'^k n'(q)$ with $d^k n, d'^k n' \leq x$ and $D < d \leq 2D$ and $D' < d' \leq 2D'$. If D, D' have some size the following lemma is useful.

Lemma 3.6. *For $x, D, D' > 0$ and $q \in \mathbb{N}$*

$$\sum_{\substack{d^2 n, d'^2 n' \leq x \\ d^2 n \equiv d'^2 n'(q) \\ d < D \leq 2D \\ d' < D' \leq 2D'}} 1 \ll_\epsilon \frac{x}{q^{1-\epsilon}(DD')^{1/2}} \left(q + \frac{x}{D}\right)^{1/2} \left(q + \frac{x}{D'}\right)^{1/2}.$$

Proof. Let $k \geq 2$ until told otherwise. We will denote the conditions $d^k n$, $d'^k n' \leq x$ by $\mathbf{d}^k \mathbf{n} \leq x$ and the conditions $D < d \leq 2D$ and $D' < d' \leq 2D'$ by $\mathbf{d} \sim \mathbf{D}$. Arranging the sum in question according to the value of $h = (d^k n, q) = (d'^k n', q)$ and using

$$\log X \geq \begin{cases} 1 & \text{for } X > e \\ 0 & \text{for } X > 1 \end{cases}$$

we have

$$\sum_{\substack{\mathbf{d}^k \mathbf{n} \leq x \\ d^k n \equiv d'^k n'(q) \\ \mathbf{d} \sim \mathbf{D}}} 1 \leq \sum_{h | q} \sum_{\substack{\mathbf{d}^k \mathbf{n} \leq xe \\ d^k n/h \equiv d'^k n'/h \, (q/h) \\ \mathbf{d} \sim \mathbf{D} \\ h | d^k n, d'^k n' \\ (d^k n/h, q/h) = (d'^k n'/h, q/h) = 1}} \log\left(\frac{xe}{d^k n}\right) \log\left(\frac{xe}{d'^k n}\right).$$

Denote by Σ_χ a sum running over the Dirichet characters modulo q/h. From orthogonality and then the Cauchy–Schwarz Inequality the inner sum is

$$\begin{aligned} &\leq \frac{1}{\phi(q/h)} \left(\sum_\chi \left| \sum_{\substack{d^k n \leq ex \\ d \sim D, h | d^k n}} \chi(d^k n/h) \log\left(\frac{xe}{d^k n}\right) \right|^2 \right)^{1/2} \\ &\quad \times \left(\sum_\chi \left| \sum_{\substack{d^k n \leq ex \\ d \sim D', h | d^k n}} \chi(d^k n/h) \log\left(\frac{xe}{d^k n}\right) \right|^2 \right)^{1/2} \\ &=: \frac{1}{\phi(q/h)} \left(\sum_\chi |X_D(\chi)|^2 \right)^{1/2} \left(\sum_\chi |X_{D'}(\chi)|^2 \right)^{1/2} \end{aligned}$$

and so to prove the lemma it is enough to prove, for any $h|q$,

$$\sum_{\chi} |X_D(\chi)|^2 \ll \frac{xq^\epsilon}{hD}\left(q+\frac{x}{D}\right). \tag{3.37}$$

We will avoid writing q^ϵ factors and assume they are included in the $\ll$ notation. The Dirichet series of the sequence

$$f(N) := \left(\sum_{\substack{d^k n = N\\ d\sim D}} 1\right)\chi\left(N/h\right)\cdot \begin{cases}1 & \text{if } h|N\\ 0 & \text{if not}\end{cases}$$

is absolutely convergent for $\sigma>1$ and is there equal to

$$\begin{aligned}\sum_{\substack{d,n=1\\ d\sim D\\ h|d^k n}}^{\infty} \frac{\chi(d^k n/h)}{(d^k n)^s} &= \sum_{d\sim D}\frac{1}{d^{ks}} \sum_{\substack{n=1\\ h/(d^k,h)|n}}^{\infty} \frac{\chi(d^k n/h)}{n^s}\\ &= \frac{L(s,\chi)}{h^s}\sum_{d\sim D}\frac{(d^k,h)^s\chi(d^k/(d^k,h))}{d^{ks}}\\ &=: T(s,\chi)L(s,\chi).\end{aligned}$$

Therefore by (5.20) and (5.22) of [8]

$$X_D(\chi) = \int_{2\pm i\infty} \frac{T(s,\chi)L(s,\chi)\,(ex)^s}{s^2}\,\mathrm{d}s. \tag{3.38}$$

Define $r_k(N)$ as at the start of Section 2. From (2.1) and from (2.2), which says that $r_k(H)^k/H$ is an integer,

$$\begin{aligned}T(s,\chi) &= \frac{1}{h^s}\sum_{H|h} H^s \sum_{\substack{d\sim D\\ (d^k,h)=H}} \frac{\chi(d^k/H)}{d^{ks}}\\ &= \frac{1}{h^s}\sum_{\substack{H|h\\ (r_k(H)^k/H,h/H)=1}} \frac{H^s\chi(r_k(H)^k/H)}{r_k(H)^{ks}} \sum_{\substack{d\sim D/r_k(H)\\ (d^k,h/H)=1}} \frac{\chi(d^k)}{d^{ks}}.\end{aligned}$$

In particular

$$\begin{aligned}T(1,\chi_0) &\le \frac{1}{h}\sum_{H|h}\frac{H}{r_k(H)^k}\sum_{d\sim D/r_k(H)}\frac{1}{d^k}\\ &\ll \frac{D^{1-k}}{h}\sum_{H|h}\frac{H}{r_k(H)}\end{aligned} \tag{3.39}$$

and for any $t \in \mathbb{R}$

$$
\begin{aligned}
T(1/2+it,\chi) &\ll \frac{1}{h^{1/2}} \sum_{H|h} \frac{H^{1/2}}{r_k(H)^{k/2}} \left| \sum_{\substack{d\sim D/r_k(H)\\ (d^k,h/H)=1}} \frac{\chi(d^k)}{d^{k/2}d^{kit}} \right| \\
&=: \frac{1}{h^{1/2}} \sum_{H|h} \frac{H^{1/2}\,|S_H(it,\chi)|}{r_k(H)^{k/2}}
\end{aligned}
$$

so that, from the Cauchy–Schwarz Inequality,

$$
|T(s,\chi)|^2 \ll \frac{1}{h} \sum_{H|h} \frac{H\,|S_H(it,\chi)|^2}{r_k(H)^k} \quad \text{for } s=1/2+it, \quad t\in\mathbb{R}. \tag{3.40}
$$

If $\chi = \chi_0$ then the integrand in (3.38) has a simple pole at $s = 1$ with residue

$$
exT(1,\chi)\,\mathrm{Res}_{s=1}\, L(s,\chi_0) \ll \frac{xD^{1-k}}{h} \sum_{H|h} \frac{H}{r_k(H)}
$$

but is otherwise homorphic for $\sigma > 0$; if $\chi \neq \chi_0$ the integrand is holomorphic throughout $\sigma > 0$. Therefore by absolute convergence of the integral and by the Residue Theorem

$$
\begin{aligned}
X_D(\chi) &= \int_{1/2\pm i\infty} \frac{T(s,\chi)L(s,\chi)\,(ex)^s}{s^2}\,\mathrm{d}s \\
&\quad + \mathcal{O}\left(\frac{xD^{1-k}}{h} \sum_{H|h} \frac{H}{r_k(H)} \begin{Bmatrix} 1 & \text{if } \chi=\chi_0 \\ 0 & \text{if } \chi\neq\chi_0 \end{Bmatrix} \right) \\
&=: Y(\chi) + \mathcal{O}\left(Z(\chi)\right).
\end{aligned} \tag{3.41}
$$

It follows from classical results that

$$
\int_{1/2\pm\infty} \frac{|L(s,\chi)|^2}{|s|^2}\,\mathrm{d}s \quad \text{is absolutely convergent and} \ll 1
$$

so from (3.40) and the Cauchy–Schwarz Inequality

$$
\sum_{\chi} |Y(\chi)|^2 \ll \frac{x}{h} \sum_{H|h} \frac{H}{r_k(H)^k} \sum_{\chi} \int_{1/2\pm i\infty} \frac{|S_H(it,\chi)|^2}{|s|^2}\,\mathrm{d}s. \tag{3.42}
$$

If we put

$$
a_n = \begin{cases} 1/n^{1/2} & \text{if } n \text{ is a } k\text{-th power, } n^{1/k}\sim D/r_k(H) \text{ and } (n,h/H)=1 \\ 0 & \text{if not} \end{cases}
$$

then Theorem 6.4 of [7] says that for any interval I of length $T > 0$ on the vertical line $1/2 \pm i\infty$ we have

$$\begin{aligned}\sum_{\chi} \int_I |S_H(it,\chi)|^2 \,\mathrm{d}s &\ll \left(\frac{qT}{h} + \frac{D}{r_k(H)}\right) \sum_{d\sim D/r_k(H)} \frac{1}{d^k} \\ &\ll D^{1-k} r_k(H)^k \left(\frac{qT}{h} + \frac{D}{r_k(H)^2}\right)\end{aligned}$$

so (3.42) becomes

$$\sum_{\chi} |Y(\chi)|^2 \ll \frac{xD^{1-k}}{h} \sum_{H|h} H \left(\frac{q}{h} + \frac{D}{r_k(H)^2}\right).$$

From (3.41)

$$\sum_{\chi} |Z(\chi)|^2 \ll \frac{x^2 D^{2-2k}}{h^2} \sum_{H,H'|h} \frac{HH'}{r_k(H) r_k(H')}.$$

From now on suppose that $k = 2$. In that case (2.2) says that $r_k(H)/H^{1/2} \geq 1$ so the last two equations with (3.41) imply

$$\sum_{\chi} |X_D(\chi)|^2 \ll \frac{xD^{1-k}}{h} \Big(q + D + xD^{1-k}\Big).$$

which is (3.37) since we can assume w.l.o.g. that $D^k \leq x$. □

We will also need to count solutions to $d^k n \equiv a(q)$ with $d^k n \leq x$, for fixed a and q.

Lemma 3.7. *Take $a, q \in \mathbb{N}$ and $x, R > 0$. Then*

$$\sum_{\substack{d^2 n \leq x \\ d^2 n \equiv a(q) \\ d > R}} 1 \ll_\epsilon q^\epsilon \left(\frac{x(a,q)}{qR^2} + \sqrt{\frac{x}{q}} + \sqrt{q}\right).$$

Proof. In [1, p. 283] our lemma is proven under the assumption that $(a, q) = 1$; we will deduce the case of general (q, a) from this. Write $\mathcal{D} = (q, a)$, $a' = a/\mathcal{D}$ and $q' = q/\mathcal{D}$. Then

$$\begin{aligned}\sum_{\substack{d^2 n \leq x \\ d^2 n \equiv a(q) \\ d > R}} 1 &= \sum_{d > R} \sum_{\substack{n \leq x/d^2 \\ d^2 n/\mathcal{D} \equiv a'(q') \\ \mathcal{D} | d^2 n}} 1 \\ &= \sum_{h|\mathcal{D}} \sum_{\substack{d > R \\ (d^2, \mathcal{D}) = h}} \sum_{\substack{n \leq xh/d^2\mathcal{D} \\ d^2 n/h \equiv a'(q')}} 1.\end{aligned} \tag{3.43}$$

For $N \in \mathbb{N}$ define $r_k(N)$ as at the start of Section 2, recall (2.1) and note that (2.2) says that $H := r_2(h)^2/h$ is an integer. Then

$$\sum_{\substack{d>R\\(d^2,\mathcal{D})=h}} \sum_{\substack{n\le xh/d^2\mathcal{D}\\ d^2n/h\equiv a'(q')}} 1 \le \sum_{d>R/r_2(h)} \sum_{\substack{n\le x/Hd^2\mathcal{D}\\ Hd^2n\equiv a'(q')}} 1$$

so from (3.43)

$$\sum_{\substack{d^2n\le x\\ d^2n\equiv a(q)\\ d>R}} 1 \le \sum_{\substack{h|\mathcal{D}\\ (H,Q')=1}} \sum_{\substack{d^2n\le x/H\mathcal{D}\\ d^2n\equiv \overline{H}a'(q')\\ d>R/r_2(h)}} 1.$$

Using the known version of the lemma this is

$$\ll \sum_{h|\mathcal{D}} \left(\frac{xr_2(h)^2}{H\mathcal{D}q'R^2} + \sqrt{\frac{x}{H\mathcal{D}q'}} + \sqrt{q'} \right)$$

and the result follows. □

Now we deal with the case that one of the D, D' is small.

Lemma 3.8. *For $x, D, D' > 0$ and $q \in \mathbb{N}$*

$$\sum_{\substack{d^2n,d'^2n'\le x\\ d^2n\equiv d'^2n'(q)\\ d<D\le 2D\\ d'<D'\le 2D'}} 1 \ll_\epsilon x^\epsilon \left(\frac{x^2}{qD^2D'} \begin{Bmatrix} 1 & \textit{if } q \textit{ is squarefree} \\ \sqrt{q} & \textit{otherwise} \end{Bmatrix} + \frac{x^{3/2}}{q} + \frac{xD'^2}{q} + \frac{x}{\max\{D, D'\}} \right)$$

and similarly with D and D' switched on the RHS.

Proof. We will denote the conditions $d^2n, d'^2n' \le x$ by $\mathbf{d}^2\mathbf{n} \le x$ and the conditions $D < d \le 2D$ and $D' < d' \le 2D'$ by $\mathbf{d} \sim \mathbf{D}$. Again we will let x^ϵ factors be absorbed in the $\ll$ notation; note that then for any $n \in \mathbb{N}$ and $X > 0$ with $X, n \ll x^{\mathcal{O}(1)}$

$$\sum_{0\le |l|\le X} (l, n) \ll X.$$

We can of course assume w.l.o.g. that $D^2 \le x$, but we can also assume that $q \le x$, since otherwise the sum in question is

$$\ll x^\epsilon \sum_{\substack{dn^2\le x\\ d>D}} 1 \ll \frac{x}{D}.$$

From Lemma 3.7 and the last sentence

$$\sum_{\substack{\mathbf{d}^2\mathbf{n}\le x\\ d^2n\equiv d'^2n'(q)\\ \mathbf{d}\sim\mathbf{D}}} 1 \le \sum_{0\le|l|\le x/q}\sum_{d'\sim D'}\sum_{\substack{d^2n\le x\\ d^2n\equiv ql(d'^2)\\ d>D}} 1 \tag{3.44}$$

$$\ll \sum_{0\le|l|\le x/q}\sum_{d'\sim D'}\left(\frac{x(ql,d'^2)}{d'^2D^2}+\frac{\sqrt{x}}{d'}+d'\right)$$

$$\ll \frac{x^2}{qD^2}\sum_{d'\sim D'}\frac{(q,d'^2)}{d'^2}+\frac{x^{3/2}}{q}+\frac{xD'^2}{q}.$$

Define $r_2(N)$ as the start of Section 2. Note that (2.2) says that $r_2(d)^2\ge d$ in general whilst $r_2(d)=d$ for squarefree d. From (2.1)

$$\sum_{n>D'}\frac{(q,n^2)}{n^2}\le\sum_{d|q}d\sum_{\substack{n>D'\\ d|n^2}}\frac{1}{n^2}\ll\frac{1}{D'}\sum_{d|q}\frac{d}{r_2(d)}\ll\frac{1}{D'}\left\{\begin{matrix}1 & \text{if } q \text{ is squarefree}\\ \sqrt{q} & \text{otherwise}\end{matrix}\right\}$$

and now the lemma follows from (3.44). □

4. Proofs of Theorems 1 and 2

Let $1\le q\le x$ be given, and let $\eta(q,a)$, $V_x(q)$ and $\mathcal{S}$ be as in Theorem 1. For the rest of the paper, x^ϵ bounds will be contained in the $\ll,\mathcal{O}$ notation. Opening the square we have

$$V_x(q)=\sum_{\substack{n,n'\le x\\ n,n'\in\mathcal{S}\\ n\equiv n'(q)}}1-2x\sum_{\substack{n\le x\\ n\in\mathcal{S}}}\eta(q,n)+x^2\sum_{a=1}^{q}\eta(q,a)^2 \tag{4.1}$$

$$=: A_x(q)-2xB_x(q)+x^2\sum_{a=1}^{q}\eta(q,a)^2.$$

From Lemma 2.1 (C) we have $\eta(q,d)\ll 1/q$ and of course $\eta(q,n)=\eta(q,(q,n))$. Therefore from Lemma 2.2 (ii) of [12] we have for some constants c_{dh},c_q and a new parameter $X\ge 1$

$$B_X(q)=\sum_{d|q}\eta(q,d)\sum_{\substack{n\le X\\ n\in\mathcal{S}\\ (n,q)=d}}1 \tag{4.2}$$

$$=\sum_{d|q}\eta(q,d)\sum_{h|q/d}\mu(h)\sum_{\substack{n\le X\\ n\in\mathcal{S}\\ dh|n}}1$$

$$
\begin{aligned}
&= X\sum_{d|q}\eta(q,d)\sum_{h|q/d}\mu(h)c_{dh}+\mathcal{O}\left(X^{1/k+\epsilon}\sum_{d|q}|\eta(q,d)|\sum_{h|q/d}|\mu(h)|\right)\\
&= Xc_q+\mathcal{O}\left(\frac{X^{1/k+\epsilon}}{q}\right)\\
&\sim Xc_q, \qquad \text{with } X\to\infty.
\end{aligned}
$$

But it is easy to establish

$$
\sum_{\substack{n\le X\\ n\in\mathcal{S}\\ n\equiv a(q)}} 1\sim X\eta(q,a), \qquad \text{with } X\to\infty,
$$

so that

$$
B_X(q)=\sum_{a=1}^{q}\eta(q,a)\sum_{\substack{n\le X\\ n\in\mathcal{S}\\ n\equiv a(q)}}1\sim X\sum_{a=1}^{q}\eta(q,a)^2, \qquad \text{with } X\to\infty,
$$

so (4.2) implies

$$
c_q=\sum_{a=1}^{q}\eta(q,a)^2
$$

and therefore the last but one line of (4.2) says

$$
B_x(q)=x\sum_{a=1}^{q}\eta(q,a)^2+\mathcal{O}\left(\frac{x^{1/k}}{q}\right). \tag{4.3}
$$

It is well known that

$$
\sum_{\substack{n\le x\\ n\in\mathcal{S}}}1=\frac{x}{\zeta(k)}+\mathcal{O}\left(x^{1/k}\right)
$$

therefore

$$
\begin{aligned}
A_x(q)&=2\sum_{\substack{n<n'\le x\\ n,n'\in\mathcal{S}\\ n\equiv n'(q)}}1+\sum_{\substack{n\le x\\ n\in\mathcal{S}}}1 \\
&=2\sum_{l\le x/q}\sum_{\substack{n,n'\le x\\ n,n'\in\mathcal{S}\\ n'-n=ql}}1+\frac{x}{\zeta(k)}+\mathcal{O}\left(x^{1/k}\right)\\
&=: 2C_x(q)+\frac{x}{\zeta(k)}+\mathcal{O}\left(x^{1/k}\right)
\end{aligned} \tag{4.4}
$$

so we deduce from (4.1) and (4.3)

$$V_x(q) = 2C_x(q) + \frac{x}{\zeta(k)} - x^2 \sum_{a=1}^{q} \eta(q,a)^2 + \mathcal{O}\left(\frac{x^{1+1/k}}{q}\right). \tag{4.5}$$

Using (2.3) we see that for some parameter $1 \le y \le x$ to be chosen

$$\begin{aligned}
\sum_{\substack{n,n'\le x\\ n,n'\in\mathcal{S}\\ n'-n=ql}} 1 &= \sum_{d,d'\le x} \mu(d)\mu(d') \sum_{\substack{n,n'\le x\\ n\equiv 0(d^k)\\ n'\equiv 0(d'^k)\\ n'-n=ql}} 1 \\
&= \sum_{d,d'\le x} \mu(d)\mu(d') \sum_{\substack{n\le x-ql\\ n\equiv 0(d^k)\\ n\equiv -ql(d'^k)}} 1 \\
&= \sum_{\substack{[d,d']\le y\\ (d^k,d'^k)\mid ql}} \mu(d)\mu(d')\left(\frac{x-ql}{[d^k,d'^k]} + \mathcal{O}(1)\right) + \mathcal{O}\left(\sum_{dd'>y} \sum_{\substack{n\le x\\ n\equiv 0(d^k)\\ n\equiv -ql(d'^k)}} 1\right) \\
&= (x-ql) \sum_{\substack{d,d'=1\\ (d^k,d'^k)\mid ql}}^{\infty} \frac{\mu(d)\mu(d')}{[d^k,d'^k]} + \mathcal{O}\left(\sum_{[d,d']\le y} 1\right) \\
&\quad + \mathcal{O}\left((x-ql)\sum_{[d,d']>y} \frac{1}{[d^k,d'^k]}\right) + \mathcal{O}\left(\sum_{dd'>y} \sum_{\substack{n\le x\\ n\equiv 0(d^k)\\ n\equiv -ql(d'^k)}} 1\right).
\end{aligned} \tag{4.6}$$

From Lemma 2.3 the first two error terms here

$$\ll y + xy^{1-k}$$

so that, writing $N = (d,d')$,

$$\begin{aligned}
C_x(q) = &\sum_{d,d'=1}^{\infty} \frac{\mu(d)\mu(d')}{[d^k,d'^k]} \sum_{\substack{l\le x/q\\ N^k\mid ql}} \left(x-ql\right) \\
&+ \mathcal{O}\left(\left(y+xy^{1-k}\right)\sum_{l\le x/q} 1 + \sum_{l\le x/q}\sum_{dd'>y} \sum_{\substack{n\le x\\ n\equiv 0(d^k)\\ n\equiv -ql(d'^k)}} 1\right)
\end{aligned} \tag{4.7}$$

$$= q \sum_{d,d'=1}^{\infty} \frac{\mu(d)\mu(d')N^{2k}}{d^k d'^k (q,N^k)} \sum_{l \le x/[q,N^k]} \left(\frac{x}{[q,N^k]} - l \right)$$

$$+ \mathcal{O}\left(\frac{x(y + xy^{1/k-1})}{q} + \sum_{\substack{d^k n, d'^k n' \le x \\ d^k n \equiv d'^k n' (q) \\ dd' > y}} 1 \right)$$

$$=: q\mathcal{J}_q(x) + \mathcal{O}\left(\frac{x(y + xy^{1-k})}{q} + \mathcal{E}_q(x) \right),$$

and so from (4.5)

$$V_x(q) = 2q\mathcal{J}_q(x) + \frac{x}{\zeta(k)} - x^2 \sum_{a=1}^{q} \eta(q,a)^2 + \mathcal{O}\left(\frac{x(y+xy^{1-k})}{q} + \mathcal{E}_q(x) \right). \tag{4.8}$$

Define $B_2^*(X)$ as in the discussion before Lemma 2.2; that is, define it as in [12]. From Lemma 2.16 of that paper

$$\sum_{n \le X} (X - n) = \frac{X^2}{2} - \frac{X}{2} - B_2^*(X)$$

so from Lemma 2.2

$$2\mathcal{J}_q(x) = \left(\frac{x}{q}\right)^2 \sum_{d,d'=1}^{\infty} \mu(d)\mu(d') \left(q, N^k\right) \left(\frac{1}{dd'}\right)^k - \frac{x}{q} \sum_{d,d'=1}^{\infty} \mu(d)\mu(d') \left(\frac{N}{dd'}\right)^k$$

$$- 2 \sum_{d,d'=1}^{\infty} \frac{\mu(d)\mu(d')}{(q,N^k)} \left(\frac{N^2}{dd'}\right)^k B_2^*\left(\frac{x}{[q,N^k]}\right)$$

$$= \alpha\mathcal{U}_{-1}(q)\left(\frac{x}{q}\right)^2 - \frac{x\mathcal{U}_0(q)}{\zeta(k)q} - C_k\mathcal{U}_{1-1/k}(q)\left(\frac{x}{q}\right)^{1/k} + \mathcal{O}\left(\left(\frac{x}{q}\right)^{1/(k+1)}\right),$$

where $\mathcal{U}_L(q)$ are as given in Lemma 2.1, and so from (4.8) we deduce

$$V_x(q) = \left(\frac{\alpha\mathcal{U}_{-1}(q)}{q} - \sum_{a=1}^{q} \eta(q,a)^2 \right) x^2$$

$$+ \left(1 - \mathcal{U}_0(q)\right) \frac{x}{\zeta(k)q} + C_k\mathcal{U}_{1-1/k}(q) \left(\frac{x}{q}\right)^{1/k}$$

$$+ \mathcal{O}\left(q\left(\frac{x}{q}\right)^{1/(k+1)} + \frac{x(y + xy^{1-k})}{q} + \mathcal{E}_q(x) \right).$$

Lemma 2.1 shows the x^2 terms cancel, so does the x term, and Lemma 2.3 (C) shows the error term to be $\ll x^{1+2/(k+1)}/q$ if we take $y \le x^{1/k}$. This proves Theorem 1, and we now turn to Theorem 2.

Assuming as we can that x is not an integer, write $Q = x/[q,(d^k, d'^k)]$ and let $c = 1 + 1/\log Q$. From Lemma 3.3 the inner sum in $\mathcal{J}_q(x)$ is for any $T \geq 1$

$$\frac{1}{2\pi i}\int_{c\pm iT}\frac{\zeta(s)}{s(s+1)}\left(\frac{x}{[q,(d^k,d'^k)]}\right)^{s+1}\mathrm{d}s \\ + \mathcal{O}\left(\frac{x}{[q,(d^k,d'^k)]T} + \left(\frac{x}{[q,(d^k,d'^k)]T}\right)^2\right)$$

so from Lemma 3.1 (and absolute convergence)

$$\begin{aligned}(4.9)\quad \mathcal{J}_q(x) &= \frac{1}{2\pi i}\int_{c\pm iT}\frac{\zeta(s)x^{s+1}}{s(s+1)}\left(\sum_{d,d'}\frac{\mu(d)\mu(d')}{[d^k,d'^k][q,(d^k,d'^k)]^s}\right)\mathrm{d}s \\ &\quad + \mathcal{O}\left(\sum_{d,d'}\frac{[q,(d^k,d'^k)]}{[d^k,d'^k]}\left(\frac{x}{[q,(d^k,d'^k)]T} + \frac{x^2}{[q,(d^k,d'^k)]^2T^2}\right)\right) \\ &= \frac{q}{2\pi i}\int_{c\pm iT}\frac{\zeta(s)\zeta\left(k(s+1)\right)\mathcal{F}^*(s)}{s(s+1)\zeta\left(2k(s+1)\right)}\left(\frac{x}{q}\right)^{s+1}\mathrm{d}s \\ &\quad + \mathcal{O}\left(\frac{x}{T}\sum_{d,d'=1}^{\infty}\frac{(d^k,d'^k)}{d^kd'^k} + \frac{x^2}{qT^2}\sum_{d,d'=1}^{\infty}\frac{(q,d^k,d'^k)}{d^kd'^k}\right).\end{aligned}$$

Suppose

$$(4.10)\qquad T \leq \frac{x}{q}.$$

It is straightforward to establish that for any $N \in \mathbb{N}$

$$\sum_{d,d'=1}^{\infty}\frac{(N,d^k,d'^k)}{d^kd'^k} \ll N^{\epsilon}$$

so that the error term in (4.9) is

$$\ll \frac{x}{T} + \frac{x^2}{qT^2} \ll \frac{x^2}{qT^2},$$

and from Lemma 3.5 the main term in (4.9) is

$$q\left(\alpha\left(\frac{x}{q}\right)^2 + \beta x + \gamma\left(\frac{x}{q}\right)^{1/k}\right) + \mathcal{O}\left(q\left(T^{1/4}\left(\frac{x}{qT}\right)^{1/2k} + \frac{x^{c+1}}{q^2T^2}\right)\right)$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$, and so we can conclude (remembering that $x^{c+1} \ll x^2$)

$$(4.11) \quad \mathcal{J}_q(x) = \frac{\alpha x^2}{q} + \beta x + \gamma q^{1-1/k} x^{1/k} + \mathcal{O}\left(q\left(T^{1/4}\left(\frac{x}{qT}\right)^{1/2k} + \left(\frac{x}{qT}\right)^2\right)\right).$$

Setting

$$T = \left(\frac{x}{q}\right)^V$$

where

$$V = \frac{2-1/2k}{9/4-1/2k} \qquad \text{(which is ≤ 1 in accordance with (4.10))}$$

then (4.11) becomes

$$\mathcal{J}_q(x) = \frac{\alpha x^2}{q} + \beta x + \gamma q^{1-1/k} x^{1/k} + \mathcal{O}\left(q\left(\frac{x}{q}\right)^{2/(9-2/k)}\right)$$

and so from (4.8)

$$(4.12) \quad V_x(q) = \left(\frac{\alpha}{q} - \sum_{a=1}^{q} \eta(q,a)^2\right) x^2 + \left(\frac{1}{\zeta(k)} + \beta\right) x + \gamma q^{1-1/k} x^{1/k} + \mathcal{O}\left(q\left(\frac{x}{q}\right)^{2/(9-2/k)} + \frac{x(y + xy^{1-k})}{q} + \mathcal{E}_q(x)\right).$$

Suppose from now on that $k = 2$. From Theorem 1

$$V_x(q) = f_2(q) q^{1/2} x^{1/2} + o(q^{1/2} x^{1/2})$$

once q is significantly larger than $x^{7/9}$, so the x^2 and x coefficient in (4.12) must vanish and the third must be $f_2(q)$, and therefore

$$(4.13) \quad V_x(q) = f_2(q) q^{1/2} x^{1/2} + \mathcal{O}\left(q\left(\frac{x}{q}\right)^{1/4} + \frac{x(y + x/y)}{q} + \mathcal{E}_q(x)\right).$$

We have (recall $\mathcal{E}_q(x)$ is given in (4.7))

$$(4.14) \quad \mathcal{E}_q(x) \ll \max_{\substack{D,D' \leq x \\ DD' > y}} \left\{ \sum_{\substack{d^2 n, d'^2 n' \leq x,\ d^2 n \equiv d'^2 n' (q) \\ D < d \leq 2D,\ D' < d' \leq 2D'}} 1 \right\} =: \max_{\substack{D,D' \leq x \\ DD' > y}} \{F(D,D')\}.$$

If one of the D, D' is $\leq \sqrt{y}$ then the other is $\geq \sqrt{y}$ and so Lemma 3.8 says

$$F(D,D') \ll \frac{x^2}{q y^{3/2}} \begin{Bmatrix} 1 & \text{if q is squarefree} \\ \sqrt{q} & \text{otherwise} \end{Bmatrix} + \frac{xy}{q} + \frac{x}{\sqrt{y}},$$

whilst if both are $\geq \sqrt{y}$ then Lemma 3.6 says that

$$F(D, D') \ll \frac{x}{q\sqrt{y}} \left(q + \frac{x}{\sqrt{y}}\right)^{1/2} \left(q + \frac{x}{\sqrt{y}}\right)^{1/2} \ll \frac{x}{\sqrt{y}} + \frac{x^2}{qy}.$$

Consequently (4.14) says that

$$\mathcal{E}_q(x) \ll \frac{x^2}{qy^{3/2}} \begin{Bmatrix} 1 & \text{if } q \text{ is squarefree} \\ \sqrt{q} & \text{otherwise} \end{Bmatrix} + \frac{xy}{q} + \frac{x}{\sqrt{y}} + \frac{x^2}{qy}$$

so the error term in (4.13) is

$$\ll q \left(\frac{x}{q}\right)^{1/4} + \frac{xy}{q} + \frac{x^2}{qy} + \frac{x^2}{qy^{3/2}} \begin{Bmatrix} 1 & \text{if } q \text{ is squarefree} \\ \sqrt{q} & \text{otherwise} \end{Bmatrix} + \frac{x}{\sqrt{y}}.$$

Recall y was the parameter introduced before the display (4.6). If q is squarefree we set $y = \sqrt{x}$ to deduce that the last error is

$$\ll q \left(\frac{x}{q}\right)^{1/4} + \frac{x^{3/2}}{q} + x^{3/4}$$

whilst if q is not squarefree we set $y = x^{2/5} q^{1/5}$ for an error (assume w.l.o.g that $q \geq \sqrt{x}$ so that $y \geq \sqrt{x}$)

$$q \left(\frac{x}{q}\right)^{1/4} + \frac{x^{7/5}}{q} \left(q^{1/5} + \left(\frac{x}{q}\right)^{1/5}\right) + x^{3/4}.$$

and we have Theorem 2.

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