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Near Counterexamples to Weil's Converse Theorem

par Raphael S. STEINER

Résumé. Nous démontrons que dans le théorème réciproque de Weil les équations fonctionnelles pour les caractères de Dirichlet modulo $\leq \sqrt{\frac{p-24}{3}}$ ne suffisent pas à assurer la modularité relativement au groupe $\Gamma_0(p)$, où p est un nombre premier.

Abstract. We show that in Weil's converse theorem the functional equations of multiplicative twists of the first $\sqrt{\frac{p-24}{3}}$ moduli are not sufficient to conclude modularity for the group $\Gamma_0(p)$, where p is a prime number.

1. Introduction

Let $\mathrm{SL}_2(\mathbb{R})$ act on the upper half-plane $\mathbb{H}$ by Möbius transformations. That is, we define as usual

$$\gamma z = \frac{az+b}{cz+d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

For such a matrix, we further define the cocycle $j(\gamma, z) = cz + d$ and an action on the holomorphic functions $f : \mathbb{H} \to \mathbb{C}$ on the upper half-plane as $(f|_k\gamma)(z) = j(\gamma, z)^{-k} f(\gamma z)$. For our convenience, let us denote the following special matrices:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad W_N = \begin{pmatrix} 0 & -N^{-\frac{1}{2}} \\ N^{\frac{1}{2}} & 0 \end{pmatrix}.$$

Let $(a_m)_{m\in\mathbb{N}}, (b_m)_{m\in\mathbb{N}}$ be two complex sequences that satisfy $|a_m|, |b_m| = O(m^\sigma)$ for some $\sigma > 0$, where we follow Landau's big O convention. Associated to these sequences, we define the two holomorphic functions on the upper half-plane:

$$f(z) = \sum_{m\geq 1} a_m e(mz) \text{ and } g(z) = \sum_{m\geq 1} b_m e(mz),$$

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where $e(z) = e^{2\pi i z}$. Let us further define their associated Dirichlet series and completions thereof:

$$L(f,s) = \sum_{m\geq 1} a_m m^{-s}, \qquad L(g,s) = \sum_{m\geq 1} b_m m^{-s},$$
$$\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s), \quad \Lambda(g,s) = (2\pi)^{-s}\Gamma(s)L(g,s),$$

where $\Gamma(s)$ is the Euler gamma function. These are *a priori* only defined for $\Re(s) > \sigma + 1$. For $k \in \mathbb{Z}$, Hecke [2] proved the equivalence of the following two statements.

(1) f, g are cusp forms of weight k for $\mathrm{SL}_2(\mathbb{Z})$, and $(f|_k S)(z) = g(z)$.
(2) The functions $\Lambda(f,s)$ and $\Lambda(g,s)$ admit a holomorphic continuation to the whole complex plane, are bounded in any vertical strip, and satisfy the functional equation

$$\Lambda(f,s) = i^k \Lambda(g, k-s).$$

An equivalent statement for

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, N|c \right\}$$

is much harder. Especially, the implication going from the functional equations to proving that f and g must be cusp forms requires further information. Weil [7] resolved this issue by assuming additional functional equations coming from multiplicative twists. To this end, define for a primitive character ψ modulo q with $(q,N) = 1$ the Dirichlet series and their respective completions

$$L(f,\psi,s) = \sum_{m\geq 1} a_m \psi(m) m^{-s}, \qquad L(g,\psi,s) = \sum_{m\geq 1} b_m \psi(m) m^{-s},$$
$$\Lambda(f,\psi,s) = (2\pi)^{-s}\Gamma(s)L(f,\psi,s), \quad \Lambda(g,\psi,s) = (2\pi)^{-s}\Gamma(s)L(g,\psi,s).$$

Let M be a fixed positive integer. Assume for every primitive character ψ modulo q with $(q, MN) = 1$ the holomorphic continuation of $\Lambda(f,\psi,s)$, $\Lambda(g,\psi,s)$ to the whole complex plane and boundedness in any vertical strip. Furthermore, assume all the functional equations

$$\Lambda(f,\psi,s) = i^k \chi(q)\psi(N)\frac{\tau(\psi)^2}{q}(Nq^2)^{\frac{k}{2}-s}\Lambda(g,\overline{\psi},k-s),$$

for some fixed Dirichlet character χ of modulus N, where $\tau(\psi)$ is the Gauss sum of ψ. In this case, Weil [7] was able to prove that f, respectively g, are cusp forms of weight k for $\Gamma_0(N)$ with Nebencharakter χ, respectively $\overline{\chi}$, and $(f|_k W_N)(z) = g(z)$. Khoai [3] later refined the number of functional equations of twists required. He proved that the primitive twists ψ of modulus $q < N^2$ with $(q,N) = 1$ suffice to come to the same conclusion. Moreover, if $N = p^r$ is a power of a prime, then already $q < N$ suffices. In this paper, we are able to show that for $N = p$, a prime, it is not sufficient to only assume the functional equations of the primitive multiplicative twists of modulus at most $(\frac{1}{\sqrt{3}} + o(1))\sqrt{p}$.

Theorem 1.1. *Let $p \geq 29$ be a prime, χ an even Dirichlet character modulo p and $k \geq 16$ an even integer. There exist two complex sequences $(a_m)_{m\in\mathbb{N}}, (b_m)_{m\in\mathbb{N}}$ with $|a_m|, |b_m| = O(m^\sigma)$ for some $\sigma > 0$ which satisfy the following two properties:*

(1) *Both $f(z) = \sum_{m\geq 1} a_m e(mz)$ and $g(z) = \sum_{m\geq 1} b_m e(mz)$ are not elements of $\bigcup_{l,N} S_l(\Gamma(N))$, that is to say f and g are not classical cusp forms of any weight for any congruence subgroup.*

(2) *For any primitive character ψ modulo q with $(q, p) = 1$, the functions $\Lambda(f, \psi, s), \Lambda(g, \psi, s)$ can be holomorphically continued to the whole complex plane, are bounded in any vertical strip. If $q \leq \sqrt{\frac{p-24}{3}}$, they furthermore satisfy the functional equation*

$$\Lambda(f, \psi, s) = i^k \chi(q)\psi(p)\frac{\tau(\psi)^2}{q}(pq^2)^{\frac{k}{2}-s}\Lambda(g, \overline{\psi}, k-s). \tag{1.1}$$

In the context of converse theorems for classical modular forms, it is more natural to look at additive twists rather than multiplicative ones. For this purpose, we define for $(a, q) = 1$:

$$L\left(f, \tfrac{a}{q}, s\right) = \sum_{m\geq 1} a_m e\left(\frac{am}{q}\right) m^{-s}, \qquad L\left(g, \tfrac{a}{q}, s\right) = \sum_{m\geq 1} b_m e\left(\frac{am}{q}\right) m^{-s},$$

$$\Lambda\left(f, \tfrac{a}{q}, s\right) = (2\pi)^{-s}\Gamma(s) L\left(f, \tfrac{a}{q}, s\right), \quad \Lambda\left(g, \tfrac{a}{q}, s\right) = (2\pi)^{-s}\Gamma(s) L\left(g, \tfrac{a}{q}, s\right).$$

For additive twists, we are able to give the following converse theorem for prime level p.

Theorem 1.2. *Let $p > 3$ be a prime. There exists a (computable) set $\mathcal{Q}$ of $2\lfloor \frac{p}{12} \rfloor + 3$ natural numbers q, which satisfy $1 \leq q \leq p-2$, such that for every integer $k > 0$ and Dirichlet character χ modulo p with $\chi(-1) = (-1)^k$ the following holds. Given any two complex sequences $(a_m)_{m\in\mathbb{N}}$ and $(b_m)_{m\in\mathbb{N}}$ with $|a_m|, |b_m| = O(m^\sigma)$ for some $\sigma > 0$, then the following two statements are equivalent:*

(1) *$f(z) = \sum_{m\geq 1} a_m e(mz)$ is a cusp form of weight k for $\Gamma_0(p)$ with Nebencharakter χ and $(f|_k W_p)(z) = g(z) = \sum_{m\geq 1} b_m e(mz)$.*

(2) *For every $q \in \mathcal{Q}$, the functions $\Lambda(f, \frac{-1}{q}, s)$ and $\Lambda(g, \frac{(qq_\star+1)/p}{q}, s)$, where $1 \leq q_\star \leq p$ and satisfies $qq_\star \equiv -1 \bmod (p)$, can be holomorphically continued to the whole complex plane and are bounded in every vertical strip, and satisfy the functional equation*

$$\Lambda\left(f, \frac{-1}{q}, s\right) = i^k \chi(q)(pq^2)^{\frac{k}{2}-s}\Lambda\left(g, \frac{(qq_\star+1)/p}{q}, k-s\right). \tag{1.2}$$

Remark 1.3. The number of functional equations of additive twists to go back from (2) to (1) in Theorem 1.2 is essentially optimal as one can construct counterexamples similar to the proof of Theorem 1.1 if one assumes at most $2\lfloor \frac{p}{12} \rfloor - 2$ additive twists.

Note that this further strengthens Khoai's result, since, by using Gauss sums, one may reduce down to about $p/6$ moduli for which one needs to consider the multiplicative twists.

The proofs of both theorems rely on Hecke's converse theorem [2], which in its most general form, proven by Bochner [1], establishes an equivalence of a functional equation and a modular relation. Subsequently, Theorem 1.2 follows from a result of Rademacher [5] on the generators of $\Gamma_0(p)$. The proof of Theorem 1.1, however, relies on the interesting observation that the modularity of $|f|$ for a finite index subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$, i.e. $|(f|_k\gamma)| = |f|, \forall\, \gamma \in \Gamma$, does not always imply the modularity (in the classical sense) of f on some congruence subgroup $\Gamma(N)$, despite this being the case for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, where it would follow that f is modular on $\Gamma(12)$. Essentially, this was shown by van Lint [4] as a consequence of the fact that $\Gamma(12)$ is contained in the commutator subgroup of $\mathrm{SL}_2(\mathbb{Z})$. In general, it is no longer true that the commutator subgroup of a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup. In fact, the abelianisation of the group $\Gamma_0(p)$ has rank approximately $p/6$ (see Corollary 2.2). It is this large rank which will give us plenty of freedom to construct a multiplier system υ of infinite order satisfying certain equations stemming from the imposed functional equations of multiplicative twists. Such a multiplier system is of course never trivial on any congruence subgroup and therefore cannot agree with a multiplier system that comes from a Dirichlet character. Hence, any non-trivial cusp form with respect to the constructed multiplier system υ gives rise to the counterexample in Theorem 1.1.

2. Notation and Preliminaries

Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that contains $-I$. We call a group character $\upsilon : \Gamma \to S^1$ that satisfies $\upsilon(-I) = (-1)^k$ a *multiplier system* of weight k for Γ. We call a holomorphic function f on the upper half-plane *modular* of weight k for Γ with respect to υ if it satisfies $(f|_k\gamma)(z) = \upsilon(\gamma)f(z)$ for every $\gamma \in \Gamma$. Such a function f admits an expansion of the shape

$$(f|_k\tau)(z) = \sum_{m=-\infty}^{\infty} a_m e\left(\frac{m+\kappa_\tau}{n_\tau} z\right)$$

for every $\tau \in \mathrm{SL}_2(\mathbb{Z})$, where n_τ is the *cusp width* of the cusp $\tau\infty$ and κ_τ is the *cusp parameter* at the cusp $\tau\infty$. They are both independent of the choice of representative of $\tau\infty \mod(\Gamma)$. The former is characterised by being the smallest natural number n such that $\tau T^n \tau^{-1} \in \Gamma$ and the latter is characterised by $e(\kappa_\tau) = \upsilon(\tau T^{n_\tau} \tau^{-1})$ and $\kappa_\tau \in [0, 1)$. We say f is a *modular form* of weight k for Γ with respect to υ if we can restrict the summation to $m + \kappa_\tau \geq 0$ for every $\tau \in \mathrm{SL}_2(\mathbb{Z})$. Moreover, we say f is a *cusp form* of weight k for Γ with respect to υ if the summation can be restricted to $m + \kappa_\tau > 0$ for every $\tau \in \mathrm{SL}_2(\mathbb{Z})$. We refer the interested reader to [6] for a more detailed treatment on modular forms (of arbitrary real weight) with respect to an arbitrary multiplier system.

Let $\Gamma = \Gamma_0(p)$, where $p > 3$ is a prime. The multiplier systems of weight k for $\Gamma_0(p)$ include all Dirichlet characters $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \to S^1$ with $\chi(-1) = (-1)^k$; they are given by

$$\upsilon_\chi(\gamma) = \chi(d) \text{ if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p),$$

but, in fact, there are many more multiplier systems. They are in one-to-one correspondence with group homomorphisms

$$\upsilon : \Gamma_0(p) \big/ [\Gamma_0(p), \Gamma_0(p)] \to S^1$$

satisfying $\upsilon(-I) = (-1)^k$. When k is even, we can characterise this quotient completely by a proposition of Rademacher [5].

Proposition 2.1 (Rademacher)**.** *Let $p > 3$ be a prime. Then, we have*

$$\Gamma_0(p) \big/ \pm I \cong F_{l-2a-2b} * (\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z})^a * (\mathbb{Z}/3\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})^b\,,$$

where $l = 2\lfloor \frac{p}{12} \rfloor + 3$; $a = 0$, unless $p \equiv 1 \bmod(4)$ in which case $a = 1$; $b = 0$, unless $p \equiv 1 \bmod(3)$ in which case $b = 1$; and $F_{l-2a-2b}$ is the free group with $l - 2a - 2b$ generators. The isomorphism stems from a set of free generators of $\Gamma_0(p)/\{\pm I\}$, which are given by S and some matrices of the shape

$$V_q = \begin{pmatrix} -q_\star & -1 \\ qq_\star + 1 & q \end{pmatrix},$$

with $2 \leq q \leq p - 2$, where $q_\star$ is the integer such that $1 \leq q^\star \leq p$ and $qq_\star \equiv -1 \bmod(p)$.

Corollary 2.2. *Let $p > 3$ be a prime. Then, we have*

$$\Gamma_0(p) \big/ \pm[\Gamma_0(p), \Gamma_0(p)] \cong \mathbb{Z}^{l-2a-2b} \times (\mathbb{Z}/2\mathbb{Z})^{2a} \times (\mathbb{Z}/3\mathbb{Z})^{2b}\,,$$

where $l = 2\lfloor \frac{p}{12} \rfloor + 3$; $a = 0$, unless $p \equiv 1 \bmod(4)$ in which case $a = 1$; and $b = 0$, unless $p \equiv 1 \bmod(3)$ in which case $b = 1$.

Thus, we get plenty of freedom when p becomes large, which we may use to satisfy certain equations.

3. Proof of the Theorems

Let $f(z)=\sum_{m>0} a_m e(mz)$ be a cusp form of integer weight $k>0$ with respect to a multiplier system υ on $\Gamma_0(p)$ with cusp parameters $\kappa_I=\kappa_S=0$ and let us denote

$$g(z)=(f|_k W_p)(z)=\sum_{m>0} b_m e(mz),$$

which is another cusp form with respect to a conjugated multiplier system of υ. Let $q\in\mathbb{N}$ with $q\neq p$ and $0\leq a<q$ with $(a,q)=1$. Let B,D be two integers that satisfy the relation $qD+apB=1$, which admits a solution by Bézout. We have the matrix identity

$$\begin{pmatrix}1 & \frac{a}{q}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}D & a\\ -pB & q\end{pmatrix} W_p \begin{pmatrix}p^{\frac{1}{2}}B & p^{-\frac{1}{2}}q^{-1}\\ -p^{\frac{1}{2}}q & 0\end{pmatrix}$$

and therefore we get the identity

$$\left(f\Big|_k \begin{pmatrix}1 & \frac{a}{q}\\ 0 & 1\end{pmatrix}\right)(z)=\upsilon\left(\begin{pmatrix}D & a\\ -pB & q\end{pmatrix}\right)\left(g\Big|_k \begin{pmatrix}p^{\frac{1}{2}}B & p^{-\frac{1}{2}}q^{-1}\\ -p^{\frac{1}{2}}q & 0\end{pmatrix}\right)(z). \tag{3.1}$$

By definition, this is just

$$\begin{aligned}\sum_{m>0} a_m e\left(\frac{am}{q}\right)e(mz)\\ = (-1)^k\upsilon\left(\begin{pmatrix}D & a\\ -pB & q\end{pmatrix}\right)p^{-\frac{k}{2}}q^{-k}z^{-k}\sum_{m>0} b_m e\left(-\frac{Bm}{q}\right)e\left(-\frac{m}{pq^2 z}\right).\end{aligned} \tag{3.2}$$

Now, $f|_k\left(\begin{smallmatrix}1 & a/q\\ 0 & 1\end{smallmatrix}\right)$ and $g|_k\left(\begin{smallmatrix}1 & -B/q\\ 0 & 1\end{smallmatrix}\right)$ are modular forms on $\Gamma(pq^2)$ for some multiplier system, respectively. Thus, by Hecke's original proof, $\Lambda(f,\frac{a}{q},s)$ and $\Lambda(g,\frac{-B}{q},s)$ possess a holomorphic continuation to the whole complex plane and are bounded in every vertical strip. Furthermore, by Bochner [1, Theorem 4], the modular relation (3.2) is equivalent to the functional equation

$$\Lambda\left(f,\tfrac{a}{q},s\right)=i^k\upsilon\left(\begin{pmatrix}D & a\\ -pB & q\end{pmatrix}\right)(pq^2)^{\frac{k}{2}-s}\Lambda\left(g,\tfrac{-B}{q},k-s\right). \tag{3.3}$$

Setting $\upsilon=\upsilon_\chi$ shows the direction (1) $\Rightarrow$ (2) in Theorem 1.2. For the reverse direction, we need to fix our set $\mathcal{Q}$. It shall consist of 1 and those q for which V_q is needed to generate $\Gamma_0(p)/\pm I$ in Proposition 2.1. We'll make use of the equivalence of (3.3) and (3.1), where $\upsilon(\left(\begin{smallmatrix}D & a\\ -pB & q\end{smallmatrix}\right))$ is to be regarded as any fixed constant. First, we set $(a,q,B,D)=(-1,1,-1,1-p)$ and make use of the functional equation (1.2) for $1\in\mathcal{Q}$. This yields

$$\left(f\Big|_k \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\right)(z)=\chi(1)\left(g\Big|_k \begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix}W_p\right)(z)$$

or equivalently $f(z) = (g|_k W_p)(z)$ and consequently $(f|_k W_p)(z) = g(z)$ after applying $|_k W_p$ to both sides. With this extra information, we further find that (3.1) is equivalent to

$$\begin{aligned}\left(f\Big|_k \begin{pmatrix} D & a \\ -pB & q \end{pmatrix}\right)(z) &= \upsilon\left(\begin{pmatrix} D & a \\ -pB & q \end{pmatrix}\right)(g|_k W_p)(z) \\ &= \upsilon\left(\begin{pmatrix} D & a \\ -pB & q \end{pmatrix}\right) f(z).\end{aligned}$$

By using this with $(a, q, B, D) = (1, q, -(qq_\star + 1)/p, -q_\star)$, where $q \in \mathcal{Q}$, and the functional equation (1.2) for q, we find

$$\left(f\Big|_k \begin{pmatrix} -q_\star & 1 \\ qq_\star + 1 & q \end{pmatrix}\right)(z) = \chi(q) f(z).$$

By combining this with Proposition 2.1 and the trivial facts that $(f|_k T)(z) = f(z)$ and $(f|_k -I)(z) = (-1)^k f(z) = \chi(-1) f(z)$, we find that f is modular of weight k for $\Gamma_0(p)$ with Nebencharakter χ. The fact that f is indeed a cusp form can be seen from the expansions at the cusps ∞ and 0 which are given by the definitions of f, respectively g.

We move onto the proof of Theorem 1.1. Let k now be even and υ, f and g be defined as in the beginning of this section. The restriction that $\kappa_I = \kappa_S = 0$ implies that the dependence on B in both

$$\upsilon\left(\begin{pmatrix} D & a \\ -pB & q \end{pmatrix}\right) \text{ and } \Lambda\left(g, \tfrac{-B}{q}, s\right)$$

is only on $B \bmod (q)$ (and thus only depends on q and $a \bmod q$). We further make the assumption that

$$\upsilon\left(\begin{pmatrix} D & a \\ -pB & q \end{pmatrix}\right) = \chi(q), \quad \text{for } (a, q) = 1 \text{ and } 1 \le q \le Q, \tag{3.4}$$

where χ is a fixed Dirichlet character modulo p. That is to say we want υ to pretend to be the Dirichlet character χ for small values of q.

Now, let ψ be a primitive Dirichlet character modulo q with $(q,p)=1$ and $1 \le q \le Q$. Then, we have

$$\begin{aligned}\Lambda(f,\psi,s) &= \frac{1}{\tau(\overline{\psi})}\sideset{}{'}\sum_{a \bmod (q)} \overline{\psi}(a)\Lambda\left(f,\tfrac{a}{q},s\right)\\ &= \frac{i^k\chi(q)}{\tau(\overline{\psi})}(pq^2)^{\frac{k}{2}-s}\sideset{}{'}\sum_{a \bmod (q)} \overline{\psi}(a)\Lambda\left(g,\tfrac{-\overline{ap}}{q},k-s\right)\\ &= \frac{i^k\chi(q)\psi(-p)}{\tau(\overline{\psi})}(pq^2)^{\frac{k}{2}-s}\sideset{}{'}\sum_{a \bmod (q)} \psi(-\overline{ap})\Lambda\left(g,\tfrac{-\overline{ap}}{q},k-s\right)\\ &= i^k\chi(q)\psi(p)\frac{\tau(\psi)^2}{q}(pq^2)^{\frac{k}{2}-s}\Lambda(g,\overline{\psi},k-s),\end{aligned}$$

where the prime in the summation denotes that we are only summing over $(a,q)=1$ and $\overline{a}$ denotes the multiplicative inverse of $a \bmod (q)$. This is the functional equation we would expect for classical modular forms with Nebencharakter χ. In order to prove Theorem 1.1, we shall construct such a multiplier system υ of infinite order as such a multiplier system cannot be trivial on any congruence subgroup. We further have to show that there is a non-zero cusp form with respect to that multiplier system for some weight k.

Our assumptions on our multiplier system, that is (3.4) and $\kappa_I = \kappa_S = 0$, form a system of linear equations with at most $2 + Q^2/2$ equations in $\log \upsilon(\gamma)$, where γ runs over the generators in Corollary 2.2. Moreover, this system of equations admits a solution, namely, υ_χ. Thus, if $2+Q^2/2+5 \le 2\lfloor \frac{p}{12} \rfloor + 3$ or $Q \le \sqrt{(p-24)/3}$ we find that the kernel has dimension at least 5. Thus, we can find an element υ' of infinite order in the kernel which satisfies $\log \upsilon'(\gamma) = 0$ for any generator γ of finite order and hence $\upsilon = \upsilon'\upsilon_\chi$ is a multiplier system which satisfies our requirements. The construction of a non-zero cusp form is now straightforward. As in [6, Theorem 5.1.2], one can construct Eisenstein series for $k \ge 4$, which are non-zero, say for example $G_I(z;0;\Gamma_0(p),k,\upsilon)$, and multiply it with the cusp form $\Delta(z)$ of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$ to get a cusp form of weight $k+12$ with respect to υ for $\Gamma_0(p)$, which concludes the proof of Theorem 1.1.

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