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A variational open image theorem in positive characteristic

par Gebhard BÖCKLE, Wojciech GAJDA et Sebastian PETERSEN

Résumé. Nous démontrons un théorème de l'image adélique ouverte variationnel pour l'action du groupe de Galois sur la cohomologie d'un S-schéma propre et lisse, où S est une variété lisse sur un corps de type fini sur $\mathbb{F}_p$. Notre outil clé est un résultat récent de Cadoret, Hui et Tamagawa.

Abstract. We prove a variational open adelic image theorem for the Galois action on the cohomology of smooth proper S-schemes where S is a smooth variety over a finitely generated field of positive characteristic. A central tool is a recent result of Cadoret, Hui and Tamagawa.

Introduction

Let k be a finitely generated infinite field of characteristic $p > 0$, S a smooth geometrically connected k-variety of positive dimension and $f : \mathscr{X} \to S$ a smooth proper morphism of schemes. Let $K = k(S)$ be the function field of S and let $X/k(S)$ be the generic fibre of $\mathscr{X}$. Fix $j \in \mathbb{N}$. For every prime number $\ell \neq p$ we define $V_\ell := H^j(X_{\overline{K}}, \mathbb{Q}_\ell)$ and let

$$\rho_{\ell^\infty} : \pi_1(S) \to \underline{\mathrm{GL}}_{V_\ell}(\mathbb{Q}_\ell)$$

be the representation of $\pi_1(S)$ on the $\mathbb{Q}_\ell$-vector space V_ℓ. We write

$$\rho : \pi_1(S) \to \prod_{\ell \neq p} \underline{\mathrm{GL}}_{V_\ell}(\mathbb{Q}_\ell)$$

for the induced adelic representation $\prod_{\ell \neq p} \rho_{\ell^\infty}$.

For every point $s \in S$ with residue field $k(s)$ we denote by

$$s_* : \mathrm{Gal}(k(s)) \to \pi_1(S)$$

the homomorphism induced by s (well-defined up to conjugation), and for any group homomorphism $\tau : \pi_1(S) \to H$, we define $\tau_s := \tau \circ s_* :$

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$\mathrm{Gal}(k(s)) \to H$ as the *specialization of* τ *at* s. Note that $\rho_{\ell^\infty,s}$ is isomorphic to the representation of $\mathrm{Gal}(k(s))$ on $H^j(X_{s,\overline{k(s)}}, \mathbb{Q}_\ell)$ where $X_s = \mathscr{X} \times_S \mathrm{Spec}(k(s))$ is the special fibre of $\mathscr{X}$ in s, see [17, VI. Cor. 4.2].

The aim of this paper is to study the variation of the monodromy groups $\rho_{\ell^\infty,s}(\mathrm{Gal}(k(s)))$ (resp. $\rho_s(\mathrm{Gal}(k(s)))$) for closed points $s \in S$ in comparing them to the corresponding monodromy group $\rho_{\ell^\infty}(\pi_1(S))$ (resp. $\rho(\pi_1(S))$) of the generic point of S.

For every prime number $\ell \neq p$ let $\underline{G}(\rho_{\ell^\infty,s})$ (resp. $\underline{G}(\rho_{\ell^\infty})$) be the connected component of the Zariski closure of $\rho_{\ell^\infty,s}(\mathrm{Gal}(k(s)))$ (resp. of $\rho_{\ell^\infty}(\pi_1(S))$) in $\underline{\mathrm{GL}}_{V_\ell}/\mathbb{Q}_\ell$, and define

$$S^{\mathrm{gen}}(\rho_{\ell^\infty}) = \{s \in S \text{ a closed point} : \underline{G}(\rho_{\ell^\infty,s}) = \underline{G}(\rho_{\ell^\infty})\}$$

Being in $S^{\mathrm{gen}}(\rho_{\ell^\infty})$ is a priori weaker than being ℓ-Galois generic in the sense of Cadoret–Kret (see [6, 3.1], [4, 1.5.3] and [18, §6]). By Theorem A (c), however, the notions are equivalent.

The following result is the main theorem of the present work.

Theorem A (see Proposition 2.7, Lemma 2.1, Theorem 3.5)**.**

(a) *The sets* $S^{\mathrm{gen}}(\rho_{\ell^\infty})$ *are independent of* ℓ. *Let* $S^{\mathrm{gen}}(\mathscr{X}/S) := S^{\mathrm{gen}}(\rho_{\ell^\infty})$ *for any* $\ell \neq p$.

(b) *The set* $S^{\mathrm{gen}}(\mathscr{X}/S)$ *is Zariski dense in* S, *and in particular it is infinite.*

(c) *The group* $\rho_s(\mathrm{Gal}(k(s)))$ *is open in* $\rho(\pi_1(S))$ *for every* $s \in S^{\mathrm{gen}}(\mathscr{X}/S)$.

The above result relies on a similar result that holds if one replaces S by its base change $S_{\overline{\mathbb{F}}_p k}$ under $k \to \overline{\mathbb{F}}_p k$. This base change allows us to apply standard tools to derive (a) and (b), and recent results from [5] by Cadoret, Hui and Tamagawa to deduce (c). Given these results, the proof of Theorem A is rather elementary.

In the case where k is a finitely generated field of zero characteristic, Cadoret established a theorem analogous to our main theorem in the case where $\mathscr{X}/S$ is an abelian scheme (cf. [4]). Her theorem offers a strong tool to reduce the proof of conjectures about Galois representations attached to abelian varieties over finitely generated fields of zero characteristic to the number field case. Similarly our theorem can be used in order to reduce the proof of conjectures about smooth proper varieties over finitely generated fields of positive characteristic to the case where the ground field is a global function field.

The results from [5] also give one a rather precise conceptual description of $\rho(\pi_1(S_{\overline{\mathbb{F}}_p k}))$, as we shall explain in Section 4, and as is presumably well-known to the authors of [5]: Let $\underline{D}_{\ell^\infty}$ denote the connected component of the Zariski closure of $\rho_{\ell^\infty}(\pi_1(S_{\overline{\mathbb{F}}_p k}))$, or, equivalently, the derived group of $\underline{G}(\rho_{\ell^\infty})$ (see Theorem 1.4). In Theorem 4.1 we prove that for $s \in S^{\mathrm{gen}}(\mathscr{X}/S)$ the group $\rho_s(\mathrm{Gal}(\overline{\mathbb{F}}_p k(s)))$ generates a special adelic subgroup in $\prod_{\ell \neq p} \underline{D}^{\mathrm{sc}}_{\ell^\infty}(\mathbb{Q}_\ell)$ in the sense of Hui and Larsen [12], where $\underline{D}^{\mathrm{sc}}_{\ell^\infty} \to \underline{D}_{\ell^\infty}$ denotes the simply connected cover.

After submitting our manuscript to ArXiv we were informed by Anna Cadoret that she has a manuscript [3] with similar results, now also available on her homepage.

Notation

For a field k we denote by $\overline{k}$ an algebraic closure and by $\mathrm{Gal}(k)$ the absolute Galois group of k. For a k-variety S we denote by $k(S)$ its function field, by $|S|$ its set of closed points, and by $\pi_1(S)$ the étale fundamental group of S with base point the geometric generic point $\mathrm{Spec}(\overline{k(S)}) \to S$. If $\mathrm{char}\, k = p$, let $S_{\overline{\mathbb{F}}_p k}$ denote the base change of S under $k \to \overline{\mathbb{F}}_p k$.

Suppose that V is a finite-dimensional $\mathbb{Q}_\ell$-vector space, Π is a profinite group and $\rho : \Pi \to \underline{\mathrm{GL}}_V(\mathbb{Q}_\ell)$ is a continuous homomorphism. We denote by $\underline{G}(\rho)$ the connected component of the Zariski closure of $\rho(\Pi)$ in $\underline{\mathrm{GL}}_V$. Then $\underline{G}(\rho)$ is an algebraic group over $\mathbb{Q}_\ell$. It is reductive if ρ is semisimple (see [16, 22.138]). We write $\rho|H$ for the restriction of ρ to a subgroup H of Π, and we denote by Π^+ the closed subgroup of Π generated by its pro-ℓ Sylow subgroups.

If $\Pi = \pi_1(S)$, then we define the *set of Galois generic points with respect to* ρ as

$$S^{\mathrm{gen}}(\rho) := \{s \in |S| : \underline{G}(\rho) = \underline{G}(\rho_s)\}.$$

Let K be a finitely generated field of characteristic $p > 0$. By $\mathbb{L}'$ we denote the set of all prime numbers $\ell \neq p$. We call a family $(\rho_{\ell^\infty} : \mathrm{Gal}(K) \to \underline{\mathrm{GL}}_{V_\ell}(\mathbb{Q}_\ell))_{\ell \in \mathbb{L}'}$ of continuous homomorphisms, where the V_ℓ are finite-dimensional $\mathbb{Q}_\ell$-vector spaces, a *strictly compatible system over K pure of weight j* if there exists a smooth $\mathbb{F}_p$-variety T with $\mathbb{F}_p(T) = K$ such that the following properties (i) and (ii) hold:

(i) ρ_{ℓ^∞} factors through $\pi_1(T)$ for every $\ell \in \mathbb{L}'$.

(ii) For every $t \in |T|$, denoting by $\mathrm{Fr}_t \in \mathrm{Gal}(\mathbb{F}_p(t))$ the arithmetic Frobenius $x \mapsto x^{\frac{1}{|\mathbb{F}_p(t)|}}$, the characteristic polynomial of $\rho_{\ell^\infty,t}(\mathrm{Fr}_t)$ has coefficients in $\mathbb{Z}$, it is independent of ℓ, and its roots all have absolute value $|\mathbb{F}_p(t)|^{\frac{j}{2}}$.

1. Preliminaries

In this section we collect basic results, mostly not due to the present authors, for use in later sections. Let K be a finitely generated infinite field of characteristic $p > 0$ and X/K a smooth proper scheme. Fix $j \in \mathbb{N}$ and define for $\ell \in \mathbb{L}'$

$$\begin{aligned} V_\ell(X) &:= H^j(X_{\overline{K}}, \mathbb{Q}_\ell) \\ \text{and} \quad T_\ell(X) &:= H^j(X_{\overline{K}}, \mathbb{Z}_\ell)/(\text{Torsion}). \end{aligned}$$

Then $V_\ell(X)$ is a finitely generated $\mathbb{Q}_\ell$-vector space, $T_\ell(X)$ is a finitely generated free $\mathbb{Z}_\ell$-module and $V_\ell(X) = T_\ell(X) \otimes \mathbb{Q}_\ell$ for all $\ell \in \mathbb{L}'$. Let ρ_{ℓ^∞} be the representation of $\mathrm{Gal}(K)$ on $V_\ell(X)$.

By Deligne's theorem on the Weil conjectures (cf. [8, Thm. 1.6]), standard spreading-out principles, and proper-smooth base change (cf. [17, VI. Cor. 4.2]), one has the following results.

Theorem 1.1 (Deligne)**.** *The family of representations $(\rho_{\ell^\infty})_{\ell\in\mathbb{L}'}$ is a strictly compatible system over K pure of weight j.*

Theorem 1.2 (Deligne, Grothendieck)**.**

(1) *The restriction $\rho_{\ell^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K)$ is semisimple.*
(2) *The group $\underline{G}(\rho_{\ell^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K))$ is semisimple.*

Proof. Part (1) is [9, Cor. 3.4.13] and due to Deligne. Part (2) is attributed by Deligne to Grothendieck and given in [9, Cor. 1.3.9]. □

For a linear algebraic group $\underline{G}$ defined over a field, we denote by $\underline{\mathscr{D}G}$ its derived group.

Corollary 1.3. *We have $\underline{\mathscr{D}G}(\rho_{\ell^\infty}) = \underline{G}(\rho_{\ell^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K))$.*

Proof. Clearly $\rho_{\ell^\infty}(\mathrm{Gal}(K))$ normalizes $\rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K))$. This is preserved under closures, so that $\underline{G}(\rho_{\ell^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K))$ is a normal subgroup of $\underline{G}(\rho_{\ell^\infty})$. In particular the quotient $\underline{Q} := \underline{G}(\rho_{\ell^\infty})/\underline{G}(\rho_{\ell^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K))$ is a connected linear algebraic group. Now $\underline{Q}$ contains as a Zariski dense subset a finite index subgroup of $\rho_{\ell^\infty}(\mathrm{Gal}(K))/\rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K))$, and the latter is a quotient of $\mathrm{Gal}(\overline{\mathbb{F}}_p K/K) \cong \hat{\mathbb{Z}}$ and thus $\underline{Q}$ is abelian. From the universal property of the derived group, we deduce $\underline{\mathscr{D}G}(\rho_{\ell^\infty}) \subset \underline{G}(\rho_{\ell^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K))$. By Theorem 1.2 (2), we have $\underline{\mathscr{D}G}(\rho_{\ell^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K)) = \underline{G}(\rho_{\ell^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K))$, and so also $\underline{G}(\rho_{\ell^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K)) \subset \underline{\mathscr{D}G}(\rho_{\ell^\infty})$. □

The following combines results from [2], [7], [19] and [21], and extends a result from [15].

Theorem 1.4. *There exists a finite Galois extension K_{ind}/K with the following properties.*

(1) *For all $\ell \in \mathbb{L}'$, one has $\rho_{\ell^\infty}(\mathrm{Gal}(K_{\mathrm{ind}})) \subset \underline{G}(\rho_{\ell^\infty})$.*
(2) *One has $\rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K_{\mathrm{ind}})) = \rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K_{\mathrm{ind}}))^+$ for all $\ell \gg 0$ in $\mathbb{L}'$.*

(3) *If* $\rho : \mathrm{Gal}(K) \to \prod_{\ell \in \mathbb{L}'} \underline{\mathrm{GL}}_{T_\ell(X)}(\mathbb{Z}_\ell)$ *is the homomorphism induced by* $\prod_{\ell \in \mathbb{L}'} \rho_{\ell^\infty}$*, then*

$$\rho(\mathrm{Gal}(\overline{\mathbb{F}}_p K_{\mathrm{ind}})) = \prod_{\ell \in \mathbb{L}'} \rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K_{\mathrm{ind}})).$$

(4) *For all* $\ell \in \mathbb{L}'$ *the group* $\rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}_p} K_{\mathrm{ind}}))$ *is an open subgroup of* $\underline{G}(\rho_{\ell^\infty} | \mathrm{Gal}(\overline{\mathbb{F}_p} K_{\mathrm{ind}}))$.

(5) *If* $K = k(S)$ *for a smooth* k*-variety* S *such that* ρ *factors via* $\pi_1(S)$*, then one can further require that there exists a connected finite étale cover* S_{ind} *of* S *such that* $K_{\mathrm{ind}} = k(S_{\mathrm{ind}})$.

Proof. The existence of a finite Galois extension K_{ind}/K such that (1) holds true is due to Serre (see [21, 2nd letter]); see also [14, Prop. 6.14]. It follows from [2, Thm. 7.7], or alternatively from [7], that after replacing K_{ind} by a larger finite Galois extension of K also (2) and (3) hold. If K is a global field, then (4) follows from [15, 2.2] by Larsen and Pink. We now explain an independent proof of (4) for any finitely generated K.

By [2, Def. 5.1. and Cor. 7.4] and by (1)–(3), there exists a finite Galois extension K_{ind} of K satisfying the properties in (1), (2) and (3), and a smooth $\mathbb{F}_p$-variety V such that $\mathbb{F}_p(V) = K_{\mathrm{ind}}$ and such that the restriction $\rho_{\ell^\infty} | \mathrm{Gal}(K_{\mathrm{ind}})$ factors through the fundamental group $\pi_1(V)$ and such that for each $\ell \in \mathbb{L}'$ the restriction $\rho_{\ell^\infty} | \mathrm{Gal}(K_{\mathrm{ind}})$ is ℓ-tame (cf. [2, Def. 4.2]). After making an alteration on V and replacing K_{ind} accordingly we can assume that V admits a smooth compactification $\overline{V}$ such that $\overline{V} \setminus V$ is a normal crossing divisor (cf. [1, Thm. 1.2]). It follows that $\rho_{\ell^\infty} | \mathrm{Gal}(K_{\mathrm{ind}})$ factors through the tame fundamental group $\pi_1^t(V)$ for every $\ell \in \mathbb{L}'$, where $\pi_1^t(V)$ is defined as in [13]. Consider the homomorphism

$$\overline{\rho} : \pi_1^t(V) \to \prod_{\ell \in \mathbb{L}'} (\underline{G}(\rho_{\ell^\infty}) / \underline{\mathscr{D}G}(\rho_{\ell^\infty}))(\mathbb{Q}_\ell)$$

induced by the ρ_{ℓ^∞}. Because $\overline{\rho}$ has an abelian image, it factors through the abelianization $\pi_1^{t,\mathrm{ab}}(V)$ of $\pi_1^t(V)$. Let $\mathbb{F}'$ be the largest algebraic extension of $\mathbb{F}_p$ in K_{ind}. Then $\mathbb{F}'/\mathbb{F}_p$ is finite, $V/\mathbb{F}'$ is geometrically connected and by [13, Thm. 7.5] the kernel $\mathrm{Ker}(\pi_1^{t,\mathrm{ab}}(V) \to \mathrm{Gal}(\mathbb{F}'))$ is finite, and thus $\overline{\rho}(\mathrm{Gal}(\overline{\mathbb{F}}_p K_{\mathrm{ind}}))$ is finite. Therefore we can replace K_{ind} by a finite Galois extension of K, so that $\overline{\rho}(\mathrm{Gal}(\overline{\mathbb{F}}_p K_{\mathrm{ind}})) = \{e\}$. By Corollary 1.3, this implies the containment $\rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K_{\mathrm{ind}})) \subset \underline{\mathscr{D}G}(\rho_{\ell^\infty})$ asserted in (4) for every $\ell \in \mathbb{L}'$. The openness claimed in (4) follows from [19, Prop. 2 and its Cor.].

Concerning (5), let S_{ind} be the connected finite étale cover of S corresponding to the image of $\mathrm{Gal}(K_{\mathrm{ind}})$ of $\pi_1(S)$. Then (1)–(5) hold if we replace K_{ind} by $k(S_{\mathrm{ind}})$. $\square$

2. The sets of Galois generic points

In addition to the data introduced above let k be an *infinite* finitely generated field and S/k a smooth geometrically connected variety with function field K. Assume that X extends to a smooth proper scheme $\mathscr{X}$ over S. Then ρ_{ℓ^∞} factors through $\pi_1(S)$. As recalled in the introduction, for every $s \in |S|$ the representation $\rho_{\ell^\infty,s}$ is isomorphic to the representation of $\mathrm{Gal}(k(s))$ on $H^j(X_{s,\overline{k(s)}}, \mathbb{Q}_\ell)$ where $X_s = \mathscr{X} \times_S \mathrm{Spec}(k(s))$ is the special fibre of $\mathscr{X}$ in s. For our applications below, note that the results of Section 1 also apply to $((\rho_{\ell^\infty,s})_{\ell\in\mathbb{L}'}, k(s))$ instead of $((\rho_{\ell^\infty})_{\ell\in\mathbb{L}'}, K)$.

In this section we group together various results about the sets $S^{\mathrm{gen}}(\rho_{\ell^\infty})$ and some consequences. The following result is due to Serre (cf. [20, §10.6]). We outline the argument.

Lemma 2.1. *For any $\ell \in \mathbb{L}'$ the set $S^{\mathrm{gen}}(\rho_{\ell^\infty})$ is Zariski-dense in S.*

Proof. Let ℓ be in $\mathbb{L}'$ and let Φ_ℓ be the Frattini subgroup of $G := \rho_{\ell^\infty}(\pi_1(S))$, i.e., the intersection of all maximal closed subgroups of G. Clearly G is a compact subgroup of $\underline{\mathrm{GL}}_{V_\ell}(\mathbb{Q}_\ell)$, and so by [10, Thm. 8.33, p. 201] there is an open pro-ℓ subgroup of G of finite rank. We deduce from [20, §10.6 Prop.] that Φ_ℓ is open in G.

Consider now the composite homomorphism:

$$\overline{\rho}_\ell\colon\ \mathrm{Gal}(K) \xrightarrow{\rho_{\ell^\infty}} G \to G/\Phi_\ell.$$

Both ρ_{ℓ^∞} and $\overline{\rho}_\ell$ factor via $\pi_1(S)$, and by the universal property of the Frattini group, we have

$$\{s \in |S| : \rho_{\ell^\infty,s}(\mathrm{Gal}(k(s))) = G\} = \{s \in |S| : \overline{\rho}_{\ell,s}(\mathrm{Gal}(k(s))) = G/\Phi_\ell\}. \tag{2.1}$$

Let M denote the right hand set. Because G/Φ_ℓ is finite and k is Hilbertian, the set M is Zariski dense in S. This completes the proof, because $S^{\mathrm{gen}}(\rho_{\ell^\infty})$ contains the left hand side of (2.1). $\square$

For the rest of this paper we define, for $s \in |S|$ and $\ell \in \mathbb{L}'$, the semisimple groups

$$\underline{D}_{\ell^\infty} := \underline{G}(\rho_{\ell^\infty}|\,\mathrm{Gal}(\overline{\mathbb{F}}_p K)) \text{ and } \underline{D}_{\ell^\infty,s} := \underline{G}(\rho_{\ell^\infty,s}|\,\mathrm{Gal}(\overline{\mathbb{F}}_p k(s)))$$

over $\mathbb{Q}_\ell$ and recall from Corollary 1.3 that $\underline{D}_{\ell^\infty} = \underline{\mathscr{D}G}(\rho_{\ell^\infty})$ and $\underline{D}_{\ell^\infty,s} = \underline{\mathscr{D}G}(\rho_{\ell^\infty,s})$.

Because of Theorem 1.1, the following result follows from [15, Thm. 2.4].

Theorem 2.2 (Larsen and Pink)**.** *If $\mathrm{trdeg}_{\mathbb{F}_p} K = 1$, then the functions $\ell \mapsto \dim \underline{D}_{\ell^\infty}$ and $\ell \mapsto \dim \underline{D}_{\ell^\infty,s}$ on $\mathbb{L}'$ are both constant.*

As an application of Lemma 2.1 we extend Theorem 2.2 to all fields K considered here.

Corollary 2.3. *The functions $\ell \mapsto \dim \underline{D}_{\ell^\infty}$ and $\ell \mapsto \dim \underline{D}_{\ell^\infty,s}$ on $\mathbb{L}'$ are both constant.*

Proof. Note that it suffices to prove the assertion on $\ell \mapsto \dim \underline{D}_{\ell^\infty}$, since for the second assertion one may take $k(s)$ for K and $\rho_{\ell^\infty,s}$ for ρ_{ℓ^∞}. Let now K be a finitely generated field over $\mathbb{F}_p$. We choose a subfield $\kappa \subset K$ such that $\mathrm{trdeg}_{\mathbb{F}_p} \kappa = 1$ and K/κ is a regular extension of fields. Next we choose a geometrically connected smooth κ-variety B with $\kappa(B) = K$ and a smooth proper morphism $\mathscr{X}_B \to B$ with generic fibre X/K. Let $\ell_0 \in \mathbb{L}'$ be such that $\dim(\underline{D}_{\ell_0^\infty}) = \max_\ell \dim(\underline{D}_{\ell^\infty})$. By Lemma 2.1 there exists a point $b \in B^{\mathrm{gen}}(\rho_{\ell_0^\infty})$; note that $\mathrm{trdeg}_{\mathbb{F}_p} \kappa(b) = 1$. Then for any $\ell \in \mathbb{L}'$ we have

$$\dim(\underline{D}_{\ell^\infty}) \geq \dim(\underline{\mathscr{D}G}(\rho_{\ell^\infty,b})) \overset{\text{Thm. 2.2}}{=} \dim(\underline{\mathscr{D}G}(\rho_{\ell_0^\infty,b})) \overset{\text{choice of } b}{=} \dim(\underline{D}_{\ell_0^\infty}),$$

and it follows from the choice of ℓ_0 that $\ell \mapsto \dim \underline{D}_{\ell^\infty}$ is constant. □

Remark 2.4. In the above proof, for the reduction from K to transcendence degree 1, one could also use results on "space filling curves", as for instance [11, Rem. 2.18 (ii)], cf. [7, Ex. 3.1].

We also need an analog of [6, 3.2.3] for S^{gen}, as defined here.

Lemma 2.5.

(1) *For $\bar{s} \in |S_{\overline{\mathbb{F}}_p k}|$ denote by $\bar{s}_S$ the closed point of S under $\bar{s}$. Then*

$$S^{\mathrm{gen}}(\rho_{\ell^\infty}) = \{\bar{s}_S \mid \bar{s} \in S^{\mathrm{gen}}_{\overline{\mathbb{F}}_p k}(\rho_{\ell^\infty}|\pi_1(S_{\overline{\mathbb{F}}_p k}))\}.$$

(2) *If S' is a finite étale cover of S and for $s' \in |S'|$ denote by s'_S the closed point of S under s'. Then*

$$S^{\mathrm{gen}}(\rho_{\ell^\infty}) = \{s'_S \mid s' \in (S')^{\mathrm{gen}}(\rho_{\ell^\infty}|\pi_1(S'))\}.$$

Proof. We only prove (1), the proof of (2) being elementary. There is a bijection between points in $|S|$ and orbits under $\mathrm{Gal}(\overline{\mathbb{F}}_p k/k)$ in $|S_{\overline{\mathbb{F}}_p k}|$. So let s be in $|S|$ and denote by $\bar{s}$ a point in $|S_{\overline{\mathbb{F}}_p k}|$ above it. Consider the commutative diagram

$$\begin{array}{ccc} \underline{D}_{\ell^\infty,s} & \hookrightarrow & \underline{G}(\rho_{\ell^\infty,s}) \\ \big\downarrow & & \big\downarrow \\ \underline{D}_{\ell^\infty} & \hookrightarrow & \underline{G}(\rho_{\ell^\infty}). \end{array}$$

If s is Galois generic, then the right vertical inclusion is an isomorphism. Hence by Corollary 1.3 the same holds for the left inclusion, and this means that $\bar{s}$ is Galois generic. Conversely, let $\bar{s}$ be Galois generic, so that the left vertical inclusion is an isomorphism, and we have an induced monomorphism

$$\iota_s \colon \underline{G}(\rho_{\ell^\infty,s})/\underline{D}_{\ell^\infty,s} \hookrightarrow \underline{G}(\rho_{\ell^\infty})/\underline{D}_{\ell^\infty}$$

of commutative algebraic groups (see the proof of Theorem 1.4). Now the image of some open subgroup of $\mathrm{Gal}(\overline{\mathbb{F}}_p k/k)$ is Zariski dense in $\underline{G}(\rho_{\ell^\infty})/\underline{D}_{\ell^\infty}$, and moreover there exists a finite extension $k'/k(s)$ such that the image of $\mathrm{Gal}(\overline{\mathbb{F}}_p k'/k')$ is Zariski dense in $\underline{G}(\rho_{\ell^\infty,s})/\underline{D}_{\ell^\infty,s}$. But clearly

$$\mathrm{Gal}(\overline{\mathbb{F}}_p k'/k') \hookrightarrow \mathrm{Gal}(\overline{\mathbb{F}}_p k/k)$$

is of finite index, and thus the map ι_s is an isomorphism, and it follows that s is Galois generic. $\square$

Remark 2.6. Let ℓ be in $\mathbb{L}'$ and denote by $\rho_{\ell^\infty}^{\mathrm{ss}}$ the semisimplification of ρ_{ℓ^∞}. Then from Lemma 2.5, Theorem 1.2 and Theorem 1.4 it is immediate that $S^{\mathrm{gen}}(\rho_{\ell^\infty}^{\mathrm{ss}}) = S^{\mathrm{gen}}(\rho_{\ell^\infty})$.

Proposition 2.7. *For any two primes $\ell_1, \ell_2 \in \mathbb{L}'$ we have $S^{\mathrm{gen}}(\rho_{\ell_1^\infty}) = S^{\mathrm{gen}}(\rho_{\ell_2^\infty})$.*

Proof. By Lemma 2.5 it suffices to show

$$S^{\mathrm{gen}}_{\overline{\mathbb{F}}_p k}(\rho_{\ell_1^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K)) = S^{\mathrm{gen}}_{\overline{\mathbb{F}}_p k}(\rho_{\ell_2^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K)). \tag{2.2}$$

For this let $\bar{s}$ be in $|S_{\overline{\mathbb{F}}_p k}|$. Observe that for any $\ell \in \mathbb{L}'$ we have the obvious assertion (a_ℓ) that $\underline{D}_{\ell^\infty,s} \hookrightarrow \underline{D}_{\ell^\infty}$ is an inclusion of connected semisimple groups and from Corollary 2.3 the assertion (2) that both functions $\ell \mapsto \dim(\underline{D}_{\ell^\infty})$ and $\ell \mapsto \dim(\underline{D}_{\ell^\infty,s})$ on $\mathbb{L}'$ are constant. From these one deduces the following chain of equivalences

$$\begin{aligned}
\bar{s} \in S^{\mathrm{gen}}_{\overline{\mathbb{F}}_p k}(\rho_{\ell_1^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K)) &\overset{(a_{\ell_1})}{\Longleftrightarrow} \dim \underline{D}_{\ell_1^\infty,s} = \dim \underline{D}_{\ell_1^\infty} \\
&\overset{(b)}{\Longleftrightarrow} \dim \underline{D}_{\ell_2^\infty,s} = \dim \underline{D}_{\ell_2^\infty} \\
&\overset{(a_{\ell_2})}{\Longleftrightarrow} \bar{s} \in S^{\mathrm{gen}}_{\overline{\mathbb{F}}_p k}(\rho_{\ell_2^\infty} | \mathrm{Gal}(\overline{\mathbb{F}}_p K)).
\end{aligned}$$

$\square$

We define $S^{\mathrm{gen}}(\mathscr{X}/S) = S^{\mathrm{gen}}(\rho_{\ell^\infty})$ for any $\ell \in \mathbb{L}'$, and we call $S^{\mathrm{gen}}(\mathscr{X}/S)$ the *set of Galois generic points of $\mathscr{X}/S$.*

3. Adelic openness

Lemma 3.1. *If Theorem A* (c) *holds over K_{ind} (see Theorem 1.4), then it holds over K.*

Proof. Let S_{ind} be the cover of S given in Theorem 1.4 (5). Let s be in $|S|$ and let $s' \in |S_{\mathrm{ind}}|$ be above s. Then by Lemma 2.5 (2), we have $s \in S^{\mathrm{gen}}(\rho_{\ell^\infty})$ if and only if $s' \in S^{\mathrm{gen}}(\rho_{\ell^\infty}|\pi_1(S_{\mathrm{ind}}))$. Proposition 2.7 therefore implies that $s \in S^{\mathrm{gen}}(\mathscr{X}/S)$ if and only if $s' \in S^{\mathrm{gen}}(\mathscr{X} \times_S S_{\mathrm{ind}}/S_{\mathrm{ind}})$. Now suppose that Theorem A (c) holds over K_{ind}. Then $\rho_{s'}(\mathrm{Gal}(k(s')))$ is open in $\rho(\pi_1(S_{\mathrm{ind}}))$ for every $s' \in S^{\mathrm{gen}}(\mathscr{X} \times_S S_{\mathrm{ind}}/S_{\mathrm{ind}})$, and since

$\rho_{s'}(\mathrm{Gal}(k(s'))) \subset \rho_s(\mathrm{Gal}(k(s)))$ and $\rho(\pi_1(S_{\mathrm{ind}})) \subset \rho(\pi_1(S))$ are open, the lemma is proved. $\square$

For the remainder of this section, we shall assume $K = K_{\mathrm{ind}}$, cf. Theorem 1.4.

Lemma 3.2. *Let s be in $S^{\mathrm{gen}}(\mathscr{X}/S)$. Then for all $\ell \in \mathbb{L}'$ the group $\rho_{\ell^\infty,s}(\mathrm{Gal}(\overline{\mathbb{F}}_p k(s)))$ is open in $\rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K))$.*

Proof. Because $K = K_{\mathrm{ind}}$, the group $\rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K))$ lies in $\underline{D}_{\ell^\infty}(\mathbb{Q}_\ell)$, and hence so does $\rho_{\ell^\infty,s}(\mathrm{Gal}(\overline{\mathbb{F}}_p k(s))) \subset \rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K))$. By our choice of s we have $\underline{D}_{\ell^\infty} = \underline{D}_{\ell^\infty,s}$, and thus by Theorem 1.4(4), both $\rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K))$ and $\rho_{\ell^\infty,s}(\mathrm{Gal}(\overline{\mathbb{F}}_p k(s)))$ are open in $\underline{D}_{\ell^\infty}(\mathbb{Q}_\ell)$. This implies the asserted openness and completes the proof. $\square$

Note that ρ_{ℓ^∞} has its image in $\underline{\mathrm{GL}}_{T_\ell(X)}(\mathbb{Z}_\ell)$. Let $\underline{\mathscr{D}}_{\ell^\infty}/\mathbb{Z}_\ell$ be the Zariski closure of $\underline{D}_{\ell^\infty}$ in $\underline{\mathrm{GL}}_{T_\ell(X)}$. The following result is powered by two theorems from a recent paper of Cadoret, Hui and Tamagawa (cf. [5]).

Theorem 3.3.

(1) *For all $\ell \gg 0$ the group scheme $\underline{\mathscr{D}}_{\ell^\infty}/\mathbb{Z}_\ell$ is semisimple.*
(2) *For every $s \in S^{\mathrm{gen}}(\mathscr{X}/S)$ we have the equalities*

$$\rho_{\ell^\infty,s}(\mathrm{Gal}(\overline{\mathbb{F}}_p k(s)))^+ = \rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K)) = \underline{\mathscr{D}}_{\ell^\infty}(\mathbb{Z}_\ell)^+$$

for all $\ell \gg 0$ (depending on s).

Proof. Part (1) is immediate from [5, Thm. 1.2] and [5, Cor. 7.5]. For part (2), let $s \in S^{\mathrm{gen}}(\mathscr{X}/S)$, so that $\underline{D}_{\ell^\infty,s} = \underline{D}_{\ell^\infty}$. Then $\underline{\mathscr{D}}_{\ell^\infty}$ is also the Zariski closure of $\rho_{\ell^\infty,s}(\mathrm{Gal}(\overline{\mathbb{F}}_p k(s))$ in $\underline{\mathrm{GL}}_{T_\ell(X)}$. We now apply [5, 7.3] twice in order to get

$$\begin{aligned} &\underline{\mathscr{D}}_{\ell^\infty}(\mathbb{Z}_\ell)^+ = \rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K))^+ \ = \ \rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p K)) \\ \text{and}\quad &\underline{\mathscr{D}}_{\ell^\infty}(\mathbb{Z}_\ell)^+ = \rho_{\ell^\infty}(\mathrm{Gal}(\overline{\mathbb{F}}_p k(s)))^+. \end{aligned}$$ $\square$

Corollary 3.4. *Consider the adelic representation*

$$\rho : \mathrm{Gal}(K) \to \prod_{\ell \in \mathbb{L}'} \underline{\mathrm{GL}}_{T_\ell(A)}(\mathbb{Z}_\ell).$$

For every $s \in S^{\mathrm{gen}}(\mathscr{X}/S)$ the group $\rho_s(\mathrm{Gal}(\overline{\mathbb{F}}_p k(s)))$ is an open subgroup of $\rho(\mathrm{Gal}(\overline{\mathbb{F}}_p K))$.

Proof. Let $G = \mathrm{Gal}(\overline{\mathbb{F}_p}K)$ and $G_s = \mathrm{Gal}(\overline{\mathbb{F}_p}k(s))$. Then

$$\rho(G) = \prod_{\ell \in \mathbb{L}'} \rho_{\ell^\infty}(G)$$

by Theorem 1.4(3) because $K = K_{\mathrm{ind}}$. Furthermore, again by Theorem 1.4(3), there exists an open normal subgroup $H_s = \mathrm{Gal}(\overline{\mathbb{F}}_p k(s)_{\mathrm{ind}})$ of G_s such that

$$\rho_s(H_s) = \prod_{\ell \in \mathbb{L}'} \rho_{\ell^\infty,s}(H_s).$$

For every prime number $\ell \in \mathbb{L}'$ the group $\rho_{\ell^\infty,s}(H_s)$ is open in $\rho_{\ell^\infty,s}(G_s)$ (because H_s is open in G_s), and $\rho_{\ell^\infty,s}(G_s)$ is open in $\rho_{\ell^\infty}(G)$ by Lemma 3.2. It follows that $\rho_{\ell^\infty,s}(H_s)$ is open in $\rho_{\ell^\infty}(G)$ for all $\ell \in \mathbb{L}'$.

By Theorem 3.3 we have $\rho_{\ell^\infty,s}(G_s)^+ = \rho_{\ell^\infty}(G)$ for all $\ell \gg 0$ in $\mathbb{L}'$ because $K = K_{\mathrm{ind}}$. From our construction of H_s via Theorem 1.4, we deduce $\rho_{\ell^\infty,s}(H_s) = \rho_{\ell^\infty,s}(G_s)^+$ for all $\ell \gg 0$ in $\mathbb{L}'$, and hence $\rho_{\ell^\infty,s}(H_s) = \rho_{\ell^\infty}(G)$ for these ℓ. By the definition of the product topology, the group $\rho_s(H_s)$ is open in $\rho(G)$. As $\rho_s(H_s) \subset \rho_s(G_s) \subset \rho(G)$, the assertion follows. □

Theorem 3.5. *For $s \in S^{\mathrm{gen}}(\mathscr{X}/S)$ the group $\rho_s(\mathrm{Gal}(k(s)))$ is open in $\rho(\mathrm{Gal}(K))$.*

Proof. Let $\bar{s} \in |S_{\overline{\mathbb{F}}_p k}|$ be above $s \in S^{\mathrm{gen}}(\mathscr{X}/S)$, and consider the following commutative diagram with exact rows, where $\overline{\rho}$ is induced from ρ:

$$\begin{tikzcd}
1 \arrow[r] & \rho(\pi_1(S_{\overline{\mathbb{F}}_p k})) \arrow[r] & \rho(\pi_1(S)) \arrow[r] & \rho(\pi_1(S))/\rho(\pi_1(S_{\overline{\mathbb{F}}_p k})) \arrow[r] & 1 \\
1 \arrow[r] & \pi_1(S_{\overline{\mathbb{F}}_p k}) \arrow[r] \arrow[u, "\rho"] & \pi_1(S) \arrow[r] \arrow[u, "\rho"] & \mathrm{Gal}(\overline{\mathbb{F}_p}k/k) \arrow[r] \arrow[u, "\overline{\rho}"] & 1 \\
1 \arrow[r] & \mathrm{Gal}(\overline{\mathbb{F}_p}k(s)) \arrow[r] \arrow[u, "\bar{s}_*"] & \mathrm{Gal}(k(s)) \arrow[r] \arrow[u, "s_*"] & \mathrm{Gal}(\overline{\mathbb{F}_p}k(s)/k(s)) \arrow[r] \arrow[u, "i"] & 1
\end{tikzcd}$$

Now $\rho_s(\mathrm{Gal}(\overline{\mathbb{F}_p}k(s)))$ is open in $\rho(\pi_1(S_{\overline{\mathbb{F}}_p k}))$ by Corollary 3.4. Furthermore $\overline{\rho}(\mathrm{Gal}(\overline{\mathbb{F}_p}k(s)/k(s)))$ is open in $\rho(\pi_1(S))/\rho(\pi_1(S_{\overline{\mathbb{F}}_p k}))$, because $k(s)/k$ is finite and $\overline{\rho}$ is surjective. It follows that $\rho_s(\mathrm{Gal}(k(s)))$ is open in $\rho(\mathrm{Gal}(K))$. □

4. Largeness in the sense of Hui–Larsen

Throughout this section we assume $K = K_{\mathrm{ind}}$ and fix a Galois generic point $s \in S^{\mathrm{gen}}(\mathscr{X}/S)$. Consider the restricted direct product

$$\underline{D}_{\mathbb{A}} := \prod\nolimits'_{\ell \in \mathbb{L}'} \underline{D}_{\ell^\infty}(\mathbb{Q}_\ell)$$

with respect to the compact open subgroups $\underline{\mathscr{D}}_{\ell^\infty}(\mathbb{Z}_\ell) \subset D_{\ell^\infty}(\mathbb{Q}_\ell)$ for $\ell \gg 0$ from Theorem 3.3, so that

$$\Gamma_s := \rho_s(\mathrm{Gal}(\overline{\mathbb{F}}_p k(s))) \subset \Gamma := \rho(\mathrm{Gal}(\overline{\mathbb{F}}_p K)) \subset \underline{D}_{\mathbb{A}}.$$

It is tempting to expect that Γ_s is open in $\underline{D}_{\mathbb{A}}$. But by Theorem 3.3, this adelic openness statement can only hold if almost all $\underline{\mathscr{D}}_{\ell^\infty}$ are simply connected since only then $\underline{\mathscr{D}}_{\ell^\infty}(\mathbb{Z}_\ell)^+ = \underline{\mathscr{D}}_{\ell^\infty}(\mathbb{Z}_\ell)$. In [12], Hui and Larsen suggest a reformulation that allows one to state a meaningful adelic openness conjecture, [12, Conj. 1.3], which they formulate in the case that K is a number field. Below we prove the analogue of their conjecture for compatible systems arising in the cohomology of a smooth projective variety over a finite type base in positive characteristic.

Let $p_{\ell^\infty} : \underline{D}^{\mathrm{sc}}_{\ell^\infty} \to \underline{D}_{\ell^\infty}$ (resp. $\underline{\mathscr{D}}^{\mathrm{sc}}_{\ell^\infty} \to \underline{\mathscr{D}}_{\ell^\infty}$) be the simply connected cover of the semisimple $\mathbb{Q}_\ell$-group $\underline{D}_{\ell^\infty}$ (resp. the $\mathbb{Z}_\ell$-group $\underline{\mathscr{D}}_{\ell^\infty}$). Since p_{ℓ^∞} is a central isogeny, the commutator morphism $\underline{D}^{\mathrm{sc}}_{\ell^\infty} \times \underline{D}^{\mathrm{sc}}_{\ell^\infty} \to \underline{D}^{\mathrm{sc}}_{\ell^\infty}, (x,y) \mapsto xyx^{-1}y^{-1}$ factors through a morphism $\kappa_{\ell^\infty} : \underline{D}_{\ell^\infty} \times \underline{D}_{\ell^\infty} \to \underline{D}^{\mathrm{sc}}_{\ell^\infty}$. Furthermore let

$$\underline{D}^{\mathrm{sc}}_{\mathbb{A}} := \prod\nolimits'_{\ell \in \mathbb{L}'} \underline{D}^{\mathrm{sc}}_{\ell^\infty}(\mathbb{Q}_\ell)$$

be the restricted direct product with respect to the $\underline{\mathscr{D}}^{\mathrm{sc}}_{\ell^\infty}(\mathbb{Z}_\ell)$ for $\ell \gg 0$, and let

$$\kappa : \underline{D}_{\mathbb{A}} \times \underline{D}_{\mathbb{A}} \to \underline{D}^{\mathrm{sc}}_{\mathbb{A}}$$

be the map derived from the κ_{ℓ^∞}. Because the groups $\underline{\mathscr{D}}^{\mathrm{sc}}_{\ell^\infty}(\mathbb{Z}_\ell)$ for $\ell \gg 0$ are hyperspecial maximal compact, as they are the $\mathbb{Z}_\ell$-points of a semisimple group scheme over $\mathbb{Z}_\ell$, the compact open subgroups of $\underline{D}^{\mathrm{sc}}_{\mathbb{A}}$ are precisely the *special adelic* groups as defined in [12, §2]. For a subset M of a group H and $u \in \mathbb{N}$ we define the set $M^u := \{s_1 \cdot \ldots \cdot s_u \mid s_1, \ldots, s_u \in M\}$.

Theorem 4.1 (Analog of [12, Conj. 1.3]). *Let M be in $\{\kappa(\Gamma_s, \Gamma_s), \kappa(\Gamma, \Gamma)\}$. Then the set M generates a compact open subgroup of $\underline{D}^{\mathrm{sc}}_{\mathbb{A}}$ which is equal to M^u for some $u \in \mathbb{N}$. Moreover M^2 contains a compact open subgroup of $\underline{D}^{\mathrm{sc}}_{\mathbb{A}}$.*

Proof. Denote by $\mathrm{pr}_{\ell^\infty} : \underline{D}_{\mathbb{A}} \to \underline{D}_{\ell^\infty}$ the projection on the ℓ-th factor of the product. Note that $\mathrm{pr}_{\ell^\infty}(\Gamma_s) = \rho_{\ell^\infty,s}(\mathrm{Gal}(\overline{\mathbb{F}}_p k(s)))$ is Zariski dense in $\underline{D}_{\ell^\infty}$ for each $\ell \in \mathbb{L}'$ because s is Galois generic. By [2, Thm. 1.2] there exists a finite extension $F/\overline{\mathbb{F}}_p k(s)$ such that

$$\rho_s(\mathrm{Gal}(F)) = \prod_{\ell \in \mathbb{L}'} \rho_{\ell^\infty,s}(\mathrm{Gal}(F)).$$

Thus, if we define $\Gamma_{\ell^\infty,s} := \rho_{\ell^\infty,s}(\mathrm{Gal}(F))$, then $\prod_{\ell \in \mathbb{L}'} \Gamma_{\ell^\infty,s} \subset \Gamma_s \subset \Gamma$. For each $\ell \in \mathbb{L}'$ the group $\Gamma_{\ell^\infty,s}$ is open in $\underline{D}_{\ell^\infty}(\mathbb{Q}_\ell)$ by Lemma 3.2 and Theorem 1.4(4). Moreover, as $\underline{\mathscr{D}}_{\ell^\infty}$ is semisimple and $\Gamma_{\ell^\infty,s} = \underline{\mathscr{D}}_{\ell^\infty}(\mathbb{Z}_\ell)^+$ for $\ell \gg 0$ (cf. Theorem 3.3), by [5, Cor. 8.2] we have $\mathrm{pr}_{\ell^\infty}^{-1}(\Gamma_{\ell^\infty,s}) = \underline{\mathscr{D}}^{\mathrm{sc}}_{\ell^\infty}(\mathbb{Z}_\ell)$ for $\ell \gg 0$. □

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