JOURNAL
de Théorie des Nombres
de BORDEAUX
anciennement Séminaire de Théorie des Nombres de Bordeaux

Haden SPENCE

Ax–Lindemann and André–Oort for a Nonholomorphic Modular Function

Tome 30, n° 3 (2018), p. 743-779.

<http://jtnb.cedram.org/item?id=JTNB_2018__30_3_743_0>

Journal de Théorie des Nombres
de Bordeaux **30** (2018), 743–779

Ax–Lindemann and André–Oort for a Nonholomorphic Modular Function

par Haden SPENCE

Résumé. Le cas modulaire de la Conjecture d'André–Oort est un théorème démontré par André et Pila, qui concerne la fonction modulaire bien connue j. Je décris deux autres classes « non classiques » de la fonction modulaire, à savoir les fonctions quasimodulaires (QM) et presque holomorphes modulaires (AHM). Celles-ci sont peut-être moins connues que j, mais divers auteurs, y compris Masser, Shimura et Zagier, les ont étudiées. Il suffit de se concentrer sur une fonction QM précise χ et sa fonction AHM duale χ^*, car celles-ci (avec j) engendrent les corps concernés. Après avoir discuté certaines des propriétés de ces fonctions, je montre par la suite quelques résultats de type Ax–Lindemann sur χ et χ^*. Je les combine ensuite avec une méthode ordinaire de o-minimalité et de comptage de points pour démontrer le résultat central de l'article; une analogique naturelle de la conjecture d'André–Oort modulaire qui s'applique à la fonction χ^*.

Abstract. The modular case of the André–Oort Conjecture is a theorem of André and Pila, having at its heart the well-known modular function j. I give an overview of two other "nonclassical" classes of modular function, namely the quasimodular (QM) and almost holomorphic modular (AHM) functions. These are perhaps less well-known than j, but have been studied by various authors including for example Masser, Shimura and Zagier. It turns out to be sufficient to focus on a particular QM function χ and its dual AHM function χ^*, since these (together with j) generate the relevant fields. After discussing some of the properties of these functions, I go on to prove some Ax–Lindemann results about χ and χ^*. I then combine these with a fairly standard method

---

Manuscrit reçu le 1er mars 2017, révisé le 9 juillet 2017, accepté le 7 octobre 2017.

2010 *Mathematics Subject Classification.* 11G18, 03C64.

Mots-clefs. Definable set, rational point, André–Oort conjecture, nonholomorphic modular function, Pila–Wilkie Theorem.

To my supervisor, Jonathan Pila, an enormous thank you; without Jonathan's unfailing support and guidance I would be utterly lost. Jonathan has also provided many very helpful suggestions regarding the content and structure of this document specifically. Thanks also go to Alan Lauder, whose suggestion for a miniproject got me interested in nonclassical modular functions in the first place, and to my father Derek, for proofreading and commenting on various early versions of this document. I am pleased to thank the referee for their very thorough reading of this paper, including many helpful comments, corrections and suggestions, and Zoë Sandford for the French translation. This work was supported by the Engineering and Physical Sciences Research Council.

of o-minimality and point counting to prove the central result of the paper; a natural analogue of the modular André–Oort conjecture for the function χ^*.

1. Introduction

Let $\mathcal{H} = \{\tau \in \mathbb{C} : \operatorname{Im}\tau > 0\}$ be the complex upper half plane. We begin with the classical j-function, mapping from $\mathcal{H}$ to $\mathbb{C}$, which is well-known to be a *modular function*. It is also transcendental, of course, but nonetheless has rich and beautiful arithmetic properties. For any quadratic point $\tau \in \mathcal{H}$, the number $j(\tau)$ is algebraic over $\mathbb{Q}$. Such a point $j(\tau)$ is called a *special point* or *singular modulus*. The singular moduli are precisely the j-invariants of elliptic curves with complex multiplication. By a classical theorem of Schneider [16], the imaginary quadratic numbers are the only algebraic elements of $\mathcal{H}$ whose image under j is also algebraic.

The special points of j turn out to be a particular instance of a more general phenomenon. A relation between coordinates in $\mathcal{H}$ is called a *geodesic relation* if it is of the form $\tau = c$ for some constant c, or $\tau_1 = g\tau_2$ for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$. For each $N \in \mathbb{N}$, there is a polynomial $\Phi_N \in \mathbb{Z}[X,Y]$ with the property that

$$\Phi_N(j(\tau), j(g\tau)) = 0,$$

for all $\tau \in \mathcal{H}$ and any $g \in \mathrm{GL}_2^+(\mathbb{Q})$ which, when written as a primitive integer matrix, has determinant N. So we see that geodesic relations between coordinates $\tau_i \in \mathcal{H}$ induce algebraic relations between their images $j(\tau_i) \in \mathbb{C}$. In fact, geodesic relations are the only algebraic relations in $\mathcal{H}$ that induce algebraic relations on the j-side. This fact, known as the Ax–Lindemann theorem for j, was proven by Pila in [12]; we will discuss it further later.

We call subvarieties of $\mathbb{C}^n$ which arise in this way *j-special subvarieties*. So a j-special subvariety of $\mathbb{C}^n$ is (an irreducible component of) a variety cut out by some equations of the form $\Phi_N(z_i, z_j) = 0$ and $z_i = j(\tau_i)$, for various values of N and singular moduli $j(\tau_i)$. In general, a *j-special point* is a zero-dimensional j-special subvariety, that is, an n-tuple $(j(\tau_1), \dots, j(\tau_n))$ where every τ_i is a quadratic point.

The j-special subvarieties of $\mathbb{C}^n$ are rather sparse; given a random variety $V \subseteq \mathbb{C}^n$, we would not expect many j-special subvarieties to be contained within it. Hence the following finiteness result, proven by Pila in [12]. We call the result Modular André–Oort, since it is a special case of the full André–Oort Conjecture, a statement about general Shimura varieties. The full André–Oort conjecture is known under GRH by work of Edixhoven, Klingler, Ullmo and Yafaev (see for instance [5], [6], [7] and [19]), and is known unconditionally for $\mathcal{A}_g$, the moduli space of principally polarised abelian varieties of genus g; a result of Tsimerman, Pila et al. [15], [18].

In turn, André–Oort is a special case of the far-reaching Zilber–Pink conjecture, so Modular André–Oort forms only a small part of a much larger picture. There is a variety of literature on these topics; good starting points include surveys by Pila [14] and Zannier [21].

Theorem 1.1 (André/Pila, Modular André–Oort)**.** *Let V be a subvariety of $\mathbb{C}^n$. Then V contains only finitely many* maximal *j-special subvarieties.*

The "maximal" is certainly necessary; in general, a positive-dimensional j-special variety will always contain infinitely many proper j-special subvarieties.

The purpose of this paper is to prove an analogue of this result in a slightly different setting. We will be discussing what happens when j is replaced (or supplemented) by certain "modular-like" functions, which, while not modular functions in the strict sense, exhibit many of the same properties. We will be focusing on two classes of near-modular function: the *quasimodular functions* and the *almost holomorphic modular functions*. Specifically, we will look at a quasimodular function χ and a related almost holomorphic modular function χ^*, defined as

$$\chi = 1728 \cdot \frac{E_2 E_4 E_6}{E_4^3 - E_6^2}, \qquad \chi^* = 1728 \cdot \frac{E_2^* E_4 E_6}{E_4^3 - E_6^2},$$

where E_k is the usual kth Eisenstein series and

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi \operatorname{Im} \tau}.$$

See Section 2 for details about the properties of χ and χ^*. One crucial fact is the existence of modular polynomials $\Psi_N \in \mathbb{Q}[X, Y, Z]$, having the property that

$$\Psi_N(\chi^*(g\tau), j(\tau), \chi^*(\tau)) = 0$$

for suitable $g \in \mathrm{GL}_2^+(\mathbb{Q})$ (compare with the classical modular polynomials Φ_N). In Section 3 we construct these Ψ_N, using them and the Φ_N to construct what we call "(j, χ^*)-special" varieties, directly analogous to the "j-special" varieties discussed above. The (j, χ^*)-special varieties exist only inside even Cartesian powers of $\mathbb{C}$; we consider $\mathbb{C}^{2n}$ as the Zariski closure of $\pi(\mathcal{H}^n)$, where

$$\pi : \mathcal{H}^n \to \mathbb{C}^{2n}$$

is defined by

$$\pi(\tau_1, \ldots, \tau_n) = (j(\tau_1), \chi^*(\tau_1), \ldots, j(\tau_n), \chi^*(\tau_n)).$$

The central theorem of this paper is the analogue of 1.1 in this setting:

Theorem 5.8 (André–Oort for (j, χ^*))**.** *Let $V \subseteq \mathbb{C}^{2n}$ be a variety. Then V contains only finitely many maximal (j, χ^*)-special subvarieties.*

The proof is quite similar to that of 1.1, following a standard strategy of o-minimality and point-counting developed by Pila and Zannier. The majority of the novelty in its proof lies in the following "Ax–Lindemann type" result. Loosely, it says that all the algebraic sets $S \subseteq \mathcal{H}^n$ with $\pi(S) \subseteq V$ are accounted for by the "weakly $\mathcal{H}$-special varieties". These are defined in Section 3; put simply, they are subvarieties of $\mathcal{H}^n$ cut out by geodesic relations.

Corollary 4.13. *Let V be an irreducible subvariety of $\mathbb{C}^{2n}$ and let $\mathcal{Z} = \pi^{-1}(V) \subseteq \mathcal{H}^n$. Then $\mathcal{Z}^{alg}$ is just the union of the weakly $\mathcal{H}$-special subvarieties of $\mathcal{Z}$.*

Here, $\mathcal{Z}^{\mathrm{alg}}$ is defined as the union of all connected, positive-dimensional, real semialgebraic subsets of $\mathcal{Z}$.

It is important to emphasise the difficulties that lie in the proof of 4.13. Traditional Ax–Lindemann results have always relied heavily on the holomorphicity of the functions involved. Since χ^* is not holomorphic, a lot of technical trickery is required to reach 4.13. By contrast, the argument needed to get from 4.13 to 5.8, done in Section 5, is a standard "point-counting" argument applying the Pila–Wilkie theorem: a well known result from the theory of o-minimal structures.

Along the way to our nonholomorphic Ax–Lindemann result, it turns out that we need an analogous Ax–Lindemann result for the quasimodular function χ. Such a result is of course interesting in its own right. Since the quasimodular functions are holomorphic, not much is required for this. We simply take an Ax–Lindemann result of Pila [13], concerning j and its derivatives, and strengthen it slightly[1].

The plan for this paper is as follows. In Section 2, we discuss some of the basic background of quasimodular and almost holomorphic modular forms and functions. In Section 3, we discuss the special sets and special points of χ^* and χ, which is of course crucial to any André–Oort statement. In Section 4, the largest section of the paper, we prove the required Ax–Lindemann results, before concluding in Section 5 with the proof of Theorem 5.8.

Note. This work was undertaken during the course of the author's DPhil studies at the University of Oxford, and much of it is intended to appear in the author's DPhil thesis.

[1]In the presence of an Ax–Lindemann theorem for χ, it is reasonable to ask whether there is also an André–Oort result in that setting. In fact it is not even obvious that we can formulate such a result. The function χ, unlike χ^*, does not take algebraic values at quadratic points, so there is no clear notion of what a "(j, χ)-special" variety should be.

2. Quasimodular and Almost Holomorphic Modular Functions

Let us begin by recalling some basic background about modular functions and Eisenstein series.

Definition 2.1. A modular function is a map $f : \mathcal{H} \to \mathbb{C}$ with the following properties:

- For any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and any $\tau \in \mathcal{H}$, we have $f(\gamma \cdot \tau) = f(\tau)$. Here, as usual, elements of the group $\mathrm{SL}_2(\mathbb{R})$ act on $\mathcal{H}$ via Möbius transformations,
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$
- f is meromorphic on $\mathcal{H}$.
- f is "meromorphic at ∞". That is, the Fourier expansion of f,
$$f(\tau) = \sum_{k=-\infty}^{\infty} c_k \exp(2k\pi i\tau),$$
has only finitely many negative terms.

One way to construct modular functions is through Eisenstein series. The kth Eisenstein series E_k is a function from $\mathcal{H}$ to $\mathbb{C}$, defined as
$$E_k(\tau) = \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z} \\ (m,n)=1}} \frac{1}{(m\tau + n)^k}.$$
For even $k \geq 4$, it is easy to see that E_k converges absolutely, defining a holomorphic function, and further that
$$E_k(\gamma\tau) = (c\tau + d)^k E_k(\tau),$$
where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. (For odd k, of course, the sum vanishes.) So in particular the function
$$1728 \cdot \frac{E_4^3}{E_4^3 - E_6^2}$$
is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$; it turns out to be a modular function. In fact this is simply the definition of the j-function. It is well-known that the denominator
$$\frac{1}{1728}(E_4^3 - E_6^2),$$
which is known as the discriminant function and denoted Δ, does not vanish anywhere on $\mathcal{H}$, so j is holomorphic on all of $\mathcal{H}$. It turns out that j is really the only modular function we need to worry about, since the field of modular functions is just $\mathbb{C}(j)$.

Note. In the remainder of this section we will quite freely use facts proven in Zagier's excellent paper [20, p. 18–22, 48–49, 58–60].

So far we have only used the absolutely convergent Eisenstein series, namely those E_k with $k \geq 4$. The Eisenstein series E_2 does not converge absolutely, but by taking the terms of the sum in a suitable order, it does define a holomorphic function E_2. It does not have the same transformation properties as the other E_k, but rather satisfies

$$E_2(\gamma\tau) = (c\tau + d)^2 E_2(\tau) - \frac{6i}{\pi} c(c\tau + d).$$

Hence one can see that the modified function

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi \operatorname{Im} \tau}$$

has the usual weight 2 transformation law, that is

$$E_2^*(\gamma\tau) = (c\tau + d)^2 E_2^*(\tau).$$

The functions E_2 and E_2^* are the prototype examples of, respectively, quasimodular forms and almost holomorphic modular forms.

Definition 2.2. A function $f : \mathcal{H} \to \mathbb{C}$ is an *almost holomorphic modular form* of weight k if:

- $f(\tau)$ can be written as a polynomial in $(\operatorname{Im} \tau)^{-1}$, with coefficients which are holomorphic functions, bounded as $\operatorname{Im} \tau \to \infty$.
- f satisfies the weight k transformation law:

$$f(\gamma\tau) = (c\tau + d)^k f(\tau).$$

Definition 2.3. A function $f : \mathcal{H} \to \mathbb{C}$ is a *quasimodular form* of weight k if it arises as the constant term (with respect to $(\operatorname{Im} \tau)^{-1}$) of an almost holomorphic modular form of weight k. Equivalently:

- $f(\tau)$ is a holomorphic function, bounded as $\operatorname{Im} \tau \to \infty$.
- f satisfies the modified transformation law:

$$\frac{f(\gamma\tau)}{(c\tau + d)^k} = f(\tau) + \sum_{r=1}^{p} f_r(\tau) \left(\frac{c}{c\tau + d} \right)^r,$$

for some holomorphic functions f_r, bounded as $\operatorname{Im} \tau \to \infty$.

The graded algebra of almost holomorphic modular forms is generated over $\mathbb{C}$ by E_2^*, E_4 and E_6. The graded algebra of quasimodular forms, similarly, is generated by E_2, E_4 and E_6. In fact, these two graded algebras are isomorphic to each other via the map sending E_2^* to E_2 and fixing E_4, E_6. One can see this map as that sending an almost holomorphic modular form to its constant coefficient.

For proofs of the various assertions made above, as well as more details about quasimodular and almost holomorphic modular forms in general, see [20, p. 58–60]. For this paper, we are more interested in quasimodular and almost holomorphic modular *functions*.

Definition 2.4. An almost holomorphic modular (or AHM) function is a quotient of almost holomorphic modular forms of the same weight.

A quasimodular (or QM) function is a quotient of quasimodular forms of the same weight.

The space of AHM functions and the space of QM functions are both obviously fields. We will write F^* for the field of AHM functions, and $\tilde{F}$ for the field of QM functions. Each contains the field of classical modular functions. These have been studied in a few places before, perhaps most notably by Masser in [9, Appendix A]. Masser works with an AHM function he calls ψ, defined by $E_2^* E_4/E_6$. This function has a singularity at i, so we work instead with a related function that has no singularities.

Define:

$$f = \frac{E_4 E_6}{\Delta}, \qquad \chi = E_2 f, \qquad \chi^* = E_2^* f,$$

where Δ is again the discriminant function $(E_4^3 - E_6^2)/1728$. The function f is then a meromorphic modular form of weight -2. Since Δ does not vanish, none of these three functions have singularities inside $\mathcal{H}$.

Further, χ^* is an AHM function and χ is a QM function. The function χ is holomorphic on $\mathcal{H}$, but of course χ^* is only real analytic. We note for future use that

$$\chi^*(\tau) = \chi(\tau) - \frac{3}{\pi \operatorname{Im} \tau} f(\tau),$$

and (by the transformation properties of E_2) that

$$\chi(\gamma\tau) = \chi(\tau) - \frac{6i}{\pi}\frac{c}{c\tau + d} f(\tau),$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

Proposition 2.5. *The fields F^* and $\tilde{F}$ are characterised by:*

$$F^* = \mathbb{C}(j, \chi^*), \qquad \tilde{F} = \mathbb{C}(j, \chi).$$

Moreover, F^* and $\tilde{F}$ are isomorphic via the map fixing j and sending χ^* to χ.

Proof Sketch. Zagier proves in [20, Proposition 20, p. 59] that the graded algebras of QM and AHM forms are generated by E_4, E_6 and (respectively) E_2 or E_2^*. Given that fact, it is a simple exercise to write down a generating set for the "monomial quotients" of QM and AHM forms, and see that they are all expressible as rational functions of j and χ or χ^*.

The isomorphism of fields is induced directly by the isomorphism between the graded algebras of QM and AHM forms. □

The following will also be of use.

Theorem 2.6. *The functions j, χ and f are algebraically independent over $\mathbb{C}$.*

Proof. Follows easily from the standard fact that j, j' and j'' are algebraically independent functions over $\mathbb{C}$. See for instance Zagier [20, p. 49]. □

Our intent is to discuss the special sets corresponding to the functions χ^* and χ. Such things do exist; they are the subject of the next section.

3. Special Sets

3.1. New Modular Polynomials. Our discussion of special sets begins with the following proposition involving the construction of some modular polynomials for χ^*. Although this follows fairly easily from facts known about j and its derivatives, together with the upcoming Lemma 3.2, the explicit existence of these polynomials seems not to have been noted before. The construction is very similar to the standard construction of the usual modular polynomials; we follow Zagier [20, Proposition 23, p. 68–69] closely.

Proposition 3.1. *For a positive integer N, let M_N be the set of primitive integer matrices $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with determinant N. For each such N, there is a nonzero polynomial $\Psi_N \in \mathbb{Q}[X,Y,Z]$, irreducible over $\mathbb{C}$, such that*

$$\Psi_N(\chi^*(g\tau), j(\tau), \chi^*(\tau)) = 0$$

for each $g \in M_N$ and all $\tau \in \mathcal{H}$.

Proof. The set

$$D_N = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{N}, ad = N, 0 \le b < d, \gcd(a,b,d) = 1 \right\}$$

is a full set of representatives for M_N under the action of $\mathrm{SL}_2(\mathbb{Z})$. That is, for all $g \in M_N$ there is some $g' \in D_N$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma g' = g$. (This is a standard fact; see for instance Lang [8] or Diamond/Shurman [1, Exercise 1.2.11].)

We will consider a polynomial in X, defined by

$$\prod_{g \in D_N} (X - \chi^*(g\tau)). \tag{3.1}$$

Clearly (for each τ) this is 0 if and only if X is $\chi^*(h\tau)$, for some $h \in D_N$. Thanks to the invariance of χ^* under $\mathrm{SL}_2(\mathbb{Z})$, this holds if and only if X is $\chi^*(h\tau)$ for some $h \in M_N$.

Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. For each $g \in D_N$, we have $g \cdot \gamma = \gamma' \cdot h$, for some other $\gamma' \in \mathrm{SL}_2(\mathbb{Z})$ and some $h \in D_N$. So by the invariance of χ^*, we have

$$\chi^*(g \cdot \gamma\tau) = \chi^*(\gamma' \cdot h\tau) = \chi^*(h\tau).$$

Thus the map $\tau \mapsto \gamma\tau$ induces a permutation of the set

$$S_N = \{\chi^*(g\tau) : g \in D_N\}.$$

In fact, the described action of $\mathrm{SL}_2(\mathbb{Z})$ on S_N is transitive. Indeed, any $g \in D_N$ can be written as

$$g = \gamma h \gamma', \qquad \gamma, \gamma' \in \mathrm{GL}_2(\mathbb{Z}),$$

with h in Smith Normal Form, meaning it is a diagonal matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$, with $A|D$ (see for instance [4, Exercise 19, p. 470]). Further, h must be primitive since g is, whence $A = 1$ and $D = N$. By replacing γ, γ' by $\gamma \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \gamma'$ if necessary, we can ensure that they are in fact elements of $\mathrm{SL}_2(\mathbb{Z})$. The claimed transitivity follows immediately[2].

Each coefficient of X in the polynomial (3.1) is a symmetric polynomial in the functions $\chi^*(g\tau)$, $g \in D_N$, so each coefficient must be invariant under $\tau \mapsto \gamma\tau$. Moreover, if $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in D_N$, then

$$\mathrm{Im}(g\tau) = \frac{a \,\mathrm{Im}\,\tau}{d}.$$

Hence each coefficient is a polynomial in $1/\mathrm{Im}\,\tau$ with coefficients which are meromorphic functions on $\mathcal{H}$. Since they are also $\mathrm{SL}_2(\mathbb{Z})$-invariant, each coefficient is therefore an element of the field of AHM functions $F^* = \mathbb{C}(j, \chi^*)$, so can be written as a quotient of complex polynomials in j and χ^*. In each such rational function, we can replace instances of j and χ^* with variables Y and Z. If we do this for each coefficient, we get a polynomial

$$\Psi_N^0(X, Y, Z) \in \mathbb{C}(Y, Z)[X]$$

with $\Psi_N^0(X, j(\tau), \chi^*(\tau)) = 0$ if and only if $X = \chi^*(g\tau)$ for some $g \in M_N$.

Next, note that

$$\chi^*(x + iy) = \frac{E_2 E_4 E_6}{\Delta} - \frac{3}{\pi y} \cdot \frac{E_4 E_6}{\Delta}.$$

Each of the Eisenstein series and Δ has a power series expansion in $q = e^{2\pi i z}$, with integer coefficients. The coefficient of the leading term in each case is 1; the coefficients of the q-expansions of E_2, E_4 and E_6 are given, for example, in [20, p. 17 and 19], and the q-expansion of Δ is easily calculated from those.

Hence $\chi^*(x + iy)$ is a polynomial in $3/\pi y$ with coefficients that are Laurent series in q with integer coefficients and leading term q^{-1}. The function j also has a q-expansion, which is just an integer Laurent series in q, again with leading term q^{-1}. We will use this to show that Ψ_N^0 is defined over $\mathbb{Q}$.

---

[2] I thank David Speyer for showing me the proof of this fact, which is taken as read in many texts.

We have (writing $y = \operatorname{Im}\tau$)

$$\Psi_N^0(X, j(\tau), \chi^*(\tau)) = \prod_{\substack{ad=N\\ d>0}} \prod_{\substack{0\le b<d\\ (a,b,d)=1}} \left(X - \chi^*\left(\frac{a\tau+b}{d}\right)\right)$$
$$= \prod_{\substack{ad=N\\ d>0}} \prod_{\substack{0\le b<d\\ (a,b,d)=1}} \left(X - \sum_{n=-1}^{\infty} c_n \zeta_d^{nb} q^{na/d} + \frac{d}{a}\frac{3}{\pi y}\sum_{n=-1}^{\infty} c'_n \zeta_d^{nb} q^{na/d}\right),$$

where $\zeta_d = e^{2\pi i/d}$ and $c_n, c'_n \in \mathbb{Z}$. The inner product is a polynomial in $3/\pi y$, with coefficients which are Laurent series in $q^{1/d}$ with coefficients from $\mathbb{Z}\left[\frac{d}{a}, \zeta_d\right]$, and leading term no smaller than $-a/d$. But it is 1-periodic, so the fractional powers of q must cancel out. Further, the coefficients in the resulting q-expansions must be in $\mathbb{Z}[\frac{d}{a}]$, since every Galois conjugation $\zeta_d \mapsto \zeta_d^r$, where $r \in (\mathbb{Z}/d\mathbb{Z})^*$, fixes the inner product; the numbers b and rb range over the same set.

So each coefficient f_k of X^k in Ψ_N^0 is a polynomial in $3/\pi y$ with coefficients which are rational Laurent series in q. Each coefficient is also equal to a quotient of polynomials p_k and q_k in j and χ^*, thus

$$p_k(j, \chi^*) = f_k \cdot q_k(j, \chi^*).$$

If we compare the coefficients of $(3/\pi y)^k$ on each side, we get various equalities between q-expansions. The coefficients of those q-expansions are $\mathbb{Q}$-linear in the coefficients of p_k and q_k. So we get a homogeneous system of $\mathbb{Q}$-linear equations holding for the coefficients of p_k and q_k. This system certainly has a solution since p_k and q_k exist. By basic linear algebra, the solution can be chosen to be rational up to scaling, ie. p_k and q_k are in $\lambda\mathbb{Q}[Y, Z]$, for some λ. In particular, p_k/q_k can be rewritten as a quotient of rational polynomials.

Thus $\Psi_N^0 \in \mathbb{Q}(Y, Z)[X]$. Finally, since, as noted earlier, $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on S_N, no subproduct of

$$\prod_{g\in D_N} (X - \chi^* \circ g) = \Psi_N^0(X, j, \chi^*) \in F^*[X]$$

can have coefficients that are $\mathrm{SL}_2(\mathbb{Z})$-invariant. Hence $\Psi_N^0(X, j, \chi^*)$ is irreducible over F^*. In particular, $\Psi_N^0(X, Y, Z)$ is irreducible over $\mathbb{C}(Y, Z)$ as a polynomial in X. It is also monic in X, so if we clear the denominators in Y and Z exactly, we get an irreducible polynomial $\Psi_N \in \mathbb{Q}[X, Y, Z]$ having the required properties. $\square$

In the above, we have made essential use of the fact that M_N is represented (up to the action of $\mathrm{SL}_2(\mathbb{Z})$) by the finitely many *upper triangular* matrices in D_N. Since χ^* is $\mathrm{SL}_2(\mathbb{Z})$-invariant, it is enough that the relation

$$\Psi_N(\chi^*(g\tau), j(\tau), \chi^*(\tau)) = 0$$

holds for $g \in D_N$; that implies the relation for all of M_N. This is not the case for the QM function χ, which only exhibits nice properties with respect to upper triangular elements of $\mathrm{GL}_2^+(\mathbb{Q})$. The best we can do is the following.

Lemma 3.2. *Let $g_1 \dots, g_k$ be* upper triangular *primitive integer matrices with positive determinant, and consider the two fields:*

$$A = F^*(j \circ g_1, \dots, j \circ g_k, \chi^* \circ g_1, \dots, \chi^* \circ g_k)$$

and

$$B = \tilde{F}(j \circ g_1, \dots, j \circ g_k, \chi \circ g_1, \dots, \chi \circ g_k),$$

considered as fields of real analytic functions, defined locally. Then A and B are isomorphic via the map

$$\chi^* \mapsto \chi, \qquad \chi^* \circ g_i \mapsto \chi \circ g_i,$$

fixing j and all of the $j \circ g_i$.

Proof. The map is clearly a well-defined bijection. If some $\chi^* \circ g_i$ and $j \circ g_i$ satisfy a polynomial equation $p(\chi^* \circ g_1, j \circ g_1, \dots, \chi^* \circ g_k, j \circ g_k) = 0$, then (by comparing growth rates) every coefficient of $1/\operatorname{Im}\tau$ on the left hand side must vanish. In particular, the constant term $p(\chi \circ g_1, j \circ g_1, \dots, \chi \circ g_k, j \circ g_k)$ must vanish. That is, the same polynomial equation holds for the $\chi \circ g_i$ and $j \circ g_i$, so the map is indeed an isomorphism. ☐

It follows that

$$\Psi_N(\chi(g\tau), j(\tau), \chi(\tau)) = 0,$$

for all *upper-triangular* primitive integer matrices of determinant N. The relation fails for any matrix which is not upper triangular; simply look at the transformation law satisfied by χ.

The existence of the polynomials Ψ_N allows us to say the same thing about χ^* that is true of j, namely: *geodesic relations between coordinates $\tau_i \in \mathcal{H}$ induce algebraic relations between their images $j(\tau_i), \chi^*(\tau_i) \in \mathbb{C}$.* Similarly, we can say of χ that *upper triangular geodesic relations* induce algebraic relations. Hence, just as for j, we can talk about the special varieties of $\mathbb{C}^n$ corresponding to χ^* and χ.

3.2. Special Subvarieties. We will discuss various related types of special subvariety:

- $\mathcal{H}$-special and weakly $\mathcal{H}$-special subvarieties of $\mathcal{H}^n$.
- j-special and weakly j-special subvarieties of $\mathbb{C}^n$.
- (j, χ^*)-special, weakly (j, χ^*)-special and weakly (j, χ)-special subvarieties of $\mathbb{C}^{2n}$.

- χ^*-special, weakly χ^*-special and weakly χ-special subvarieties of $\mathbb{C}^n$.

We shall see that the weakly χ^*-special and weakly χ-special subvarieties turn out to be very similar objects. However, the naive approach towards producing a "truly χ-special" subvariety does not appear to work. The same goes for (j,χ)-special subvarieties.

3.2.1. *$\mathcal{H}$-special and j-special Varieties.* We will start with the $\mathcal{H}$-special subvarieties.

Definition 3.3. Let $n \in \mathbb{N}$.

Let $S_0 \cup S_1 \cup \cdots \cup S_k$ be a partition of $\{1,\dots,n\}$, where $k \geq 0$ and $S_i \neq \emptyset$ for $i > 0$. For each $s \in S_0$, choose any point $q_s \in \mathcal{H}$. For each $i > 0$, let s_i be the least element of S_i and for each $s_i \neq s \in S_i$ choose a geodesic matrix $g_{i,s} \in \mathrm{GL}_2^+(\mathbb{Q})$. A *weakly $\mathcal{H}$-special* subvariety of $\mathcal{H}^n$ is a set of the form

$$\big\{(\tau_1,\dots,\tau_n) \in \mathcal{H}^n : \tau_s = q_s \text{ for } s \in S_0, \tau_s = g_{i,s}\tau_{s_i} \text{ for } s \in S_i, \\ s \neq s_i, i = 1,\dots,k\big\},$$

for some given data S_i, q_s, $g_{i,s}$.

A weakly $\mathcal{H}$-special subvariety is *$\mathcal{H}$-special* if the constant factors q_s are imaginary quadratic numbers for all $s \in S_0$.

Remark 3.4. What we call a "weakly $\mathcal{H}$-special variety" is elsewhere in the literature referred to as a "geodesic variety". We have chosen our terminology differently here so that it meshes as closely as possible with the terminology we use for j-special varieties and so forth.

This definition may look more complex than it actually is. Put more loosely, a weakly $\mathcal{H}$-special variety is simply one defined by some number of equations of the form $\tau_i = q_i$ or $\tau_i = g_{ij}\tau_j$, with q_i constants and $g_{ij} \in \mathrm{GL}_2^+(\mathbb{Q})$. If it happens that the q_i are quadratic points then the variety is $\mathcal{H}$-special.

This theme continues for all the types of special variety we will define here; a special variety will be defined by some combination of:

- Equations requiring some coordinate to be constant (perhaps a "special" constant).
- Equations coming from the modular polynomials.

These other types of special variety will all arise as (the Zariski closures of) the images of $\mathcal{H}$-special varieties under various maps. The easiest to deal with are the well-known j-special varieties.

Definition 3.5. Let $n \in \mathbb{N}$ and let $S_0 \cup S_1 \cup \cdots \cup S_k$ be a partition of $\{1,\dots,n\}$, where $k \geq 0$ and $S_i \neq \emptyset$ for $i > 0$. For each $s \in S_0$, choose a point $j_s \in \mathbb{C}$. For each $i > 0$, let s_i be the least element of S_i and for each

$s_i \neq s \in S_i$ choose a positive integer $N_{i,s}$. A weakly j-special subvariety of $\mathbb{C}^n$ is an irreducible component of a subvariety of the form

$$\{(z_1, \dots, z_n) \in \mathbb{C}^n : z_s = j_s \text{ for } s \in S_0,$$
$$\Phi_{N_{i,s}}(z_{s_i}, z_s) = 0 \text{ for } s \in S_i, s \neq s_i, i = 1, \dots, k\}$$

for some given data S_i, j_s, $N_{i,s}$.

A weakly j-special variety is j-special if all of the constant factors j_s are singular moduli, ie. $j_s = j(\tau_s)$ for some quadratic $\tau_s \in \mathcal{H}$.

The j-special (resp. weakly j-special) varieties of $\mathbb{C}^n$ are precisely those varieties that arise as the image, under the map

$$(\tau_1, \dots, \tau_n) \mapsto (j(\tau_1), \dots, j(\tau_n)),$$

of an $\mathcal{H}$-special (resp. weakly $\mathcal{H}$-special) subvariety of $\mathcal{H}^n$. These are the special varieties discussed in 1.1.

3.2.2. ***(j, χ^*)-special Varieties.*** The (j, χ^*)-special subvarieties are slightly more intricate. We start in the simplest positive dimensional case, considering the variety

$$V_N' = \{(W, X, Y, Z) \in \mathbb{C}^4 : \Phi_N(W, Y) = 0,$$
$$\Psi_N(X, Y, Z) = 0, \Psi_N(Z, W, X) = 0\} \subseteq \mathbb{C}^4.$$

By counting equations, $\dim_{\mathbb{C}} V_N'$ is at most 2. In fact, $\dim_{\mathbb{C}} V_N' = 2$. To see this, note that V_N' contains the set

$$S_g = \{(j(\tau), \chi^*(\tau), j(g\tau), \chi^*(g\tau)) : \tau \in \mathcal{H}\}$$

for any $g \in M_N$. Since j and χ^* are algebraically independent, S_g cannot be contained in any algebraic curve; hence $\dim_{\mathbb{C}} V_N' > 1$.

I believe that the variety V_N' is always irreducible. I have calculated the first few modular polynomials Ψ_N to see in each case that V_N' is irreducible, but so far have been unable to prove that this is the case for all N. I leave this as an open problem, which fortunately has no impact whatsoever on the wider picture: by real analytic continuation, V_N' has an irreducible component containing S_g. Call this component V_N; it is still 2-dimensional. Since it contains S_g, the variety V_N in fact contains all the S_g, $g \in M_N$, by modularity of j and χ^*. Moreover, by 3.2, V_N contains

$$S_g' = \{(j(\tau), \chi(\tau), j(g\tau), \chi(g\tau)) : \tau \in \mathcal{H}\}$$

for any upper triangular $g \in M_N$. These V_N will form the building blocks of (j, χ^*)-special varieties.

Definition 3.6. Let $n \in \mathbb{N}$ and let $S_0 \cup S_1 \cup \dots \cup S_k$ be a partition of $\{1, \dots, n\}$, where $k \geq 0$ and $S_i \neq \emptyset$ for $i > 0$. For each $s \in S_0$, choose $\tau_s \in \mathcal{H}$ and let $(j_s, c_s) = (j(\tau_s), \chi^*(\tau_s)) \in \mathbb{C}^2$. For each $i > 0$, let s_i be the least element of S_i and for each $s_i \neq s \in S_i$ choose a positive integer $N_{i,s}$. A weakly (j, χ^*)-special subvariety of $\mathbb{C}^{2n}$ is an irreducible component of a subvariety of the form

$$\begin{aligned}\{(w_1, z_1, \dots, w_n, z_n) \in \mathbb{C}^{2n} : (w_s, z_s) = (j_s, c_s) \text{ for } s \in S_0,\\ (w_s, z_s, w_{s_i}, z_{s_i}) \in V_{N_{i,s}} \text{ for } s \in S_i,\\ s \neq s_i, i = 1, \dots, k\},\end{aligned}$$

for some given data S_i, (j_s, c_s), $N_{i,s}$.

A weakly (j, χ^*)-special variety is (j, χ^*)-special if every constant factor (j_s, c_s) is of the form $(j(\tau_s), \chi^*(\tau_s))$ for some quadratic point $\tau_s \in \mathcal{H}$.

Every weakly (j, χ^*)-special variety arises as the Zariski closure of the image of a weakly $\mathcal{H}$-special variety under the map

$$(\tau_1, \dots, \tau_n) \mapsto (j(\tau_1), \chi^*(\tau_1), \dots, j(\tau_n), \chi^*(\tau_n)).$$

One can see from the polynomials Ψ_N that $\chi^*(\tau)$ is an algebraic number whenever $\tau \in \mathcal{H}$ is quadratic; this also follows from the work of Masser [9, Appendix A]. This is why (j, χ^*)-special varieties are called such; all their constant factors are special algebraic numbers.

The weakly (j, χ)-special varieties differ from (j, χ^*)-special varieties only in their constant factors. The definition is identical, except that the constant factors (j_s, c_s) are chosen to be of the form $(j(\tau), \chi(\tau))$. Unlike in the AHM case, these (j, χ)-special varieties do not arise as the Zariski closure of any arbitrary $\mathcal{H}$-special set.

Definition 3.7. A geodesic upper triangular (or GUT) variety is a weakly $\mathcal{H}$-special variety for which all of the relations $g_{i,s}$ are upper triangular matrices.

Since χ behaves nicely only under upper-triangular matrices, one can see that weakly (j, χ)-special varieties arise only from GUT varieties. A weakly (j, χ)-special variety is the Zariski closure of the image of a GUT variety under the map

$$(\tau_1, \dots, \tau_n) \mapsto (j(\tau_1), \chi(\tau_1), \dots, j(\tau_n), \chi(\tau_n)).$$

Remark 3.8. The polynomials Ψ_N only work for χ and $\chi \circ g$ when all of the relevant matrices are upper triangular. As a result, numbers $\chi(\tau)$ with τ quadratic are *not* algebraic in general. Diaz has proven and conjectured various results about these points and their transcendence properties in his paper [2]. So χ seems not to have any points that we could reasonably call special points, other than perhaps the $\mathrm{SL}_2(\mathbb{Z})$-translates of i and $e^{2\pi i/3}$, where χ vanishes.

This is why we have not attempted to define a notion of (j, χ)-special variety; the naive approach does not seem to produce a correct definition and it is not immediately clear whether a correct such notion exists. This is also why, in 3.7, we have not defined any notion of "$\mathcal{H}$-special GUT variety". We might, for instance, have required all of the constant factors to be quadratic, or to be special in some other way, but this seems unlikely to produce a meaningful outcome since χ apparently has no special points.

3.2.3. *χ^*-special Varieties.* The final special varieties we are interested in are the χ^*-special varieties. The idea is that, while any individual polynomial Ψ_N introduces a dependence on a j-coordinate, multiple different relations induced by Ψ_N can cancel each other out and introduce a relation that exists between the $\chi^* \circ g_i$ alone, not involving j.

Unfortunately, it seems difficult to isolate the specific polynomials that arise in this process. So the easiest definition to use for χ^*-special varieties is the following:

Definition 3.9. A (weakly) χ^*-special variety is an irreducible component of the Zariski closure of the projection of a (weakly) (j, χ^*)-special variety onto the coordinates corresponding to χ^*-variables.

Equivalently, a χ^*-special (resp. weakly χ^*-special) variety is an irreducible component of the Zariski closure of the image of an $\mathcal{H}$-special (resp. weakly $\mathcal{H}$-special) set under the map

$$(\tau_1, \ldots, \tau_n) \mapsto (\chi^*(\tau_1), \ldots, \chi^*(\tau_n)).$$

As before, the weakly χ-special varieties differ from the weakly χ^*-special varieties only in their constant factors, and there is no obvious concept of a χ-special variety.

In what follows, we prove a few results about the possible shape of weakly χ^*-special varieties.

Proposition 3.10. *Let $N \geq 2$ and let $S = \chi^{-1}\{0\}$. Then there is an upper triangular $g \in M_N$ such that the set*

$$\{\chi(gs) : s \in S\}$$

is infinite.

Proof. For any $\tau \in \mathcal{H}$ which is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to i, the Eisenstein series E_6 is equal to 0. In particular, $\mathrm{SL}_2(\mathbb{Z}) \cdot i \subseteq S$. So we only need to show that (for some g) $\chi(g(\gamma \cdot i))$ takes infinitely many values as γ varies. This is easy to see simply by considering matrices of the form

$$g = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \in D_N, \qquad \gamma_n = \begin{pmatrix} 1 & -1 \\ 1 - nN & nN \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Then we get

$$g \cdot \gamma_n = \begin{pmatrix} N & -1 \\ 1 - nN & n \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix},$$

so using the transformation law for χ, we have

$$\chi(g(\gamma_n \cdot \tau)) = \chi(\tau/N) - \frac{6i}{\pi}\frac{1-nN}{(1-nN)(\tau/N)+n}f(\tau/N),$$

where $f = E_4E_6/\Delta$. Setting $\tau = i$, the above expression clearly takes infinitely many values as n varies, provided that $f(i/N) \neq 0$, which is clear since the only zeros of E_4 and E_6 are $\mathrm{SL}_2(\mathbb{Z})$-equivalent to i or ρ. □

Corollary 3.11. *The modular polynomial $\Psi_N(X,Y,Z)$ is nonconstant in Y for all $N \geq 2$.*

Proof. Otherwise $\Psi_N = \Psi_N(X,Z)$, and then by 3.10, $\Psi_N(0,Z)$ has infinitely many solutions, and so is identically zero. Since Ψ_N is irreducible, this would mean that $\Psi_N(X,Z)$ is a constant multiple of X, which is clearly false. □

Remark 3.12. The modular polynomial $\Psi_1(X,Y,Z)$ is just $X-Z$. By the above it is the only modular polynomial which does not depend on Y.

Corollary 3.13. *Let $n \geq 1$. For each $1 \leq i \leq n$, let g_i be an* upper triangular *primitive integer matrix with positive determinant N_i. Suppose that not all the N_i are equal to 1. Then the Zariski closure of the set*

$$S = \{(\chi(\tau), \chi(g_1\tau), \ldots, \chi(g_n\tau)) : \tau \in \mathcal{H}\}$$

has complex dimension exactly 2. Similarly, the Zariski closure of

$$S^* = \{(\chi^*(\tau), \chi^*(g_1\tau), \ldots, \chi^*(g_n\tau)) : \tau \in \mathcal{H}\}$$

has complex dimension exactly 2.

Proof. Consider the (j,χ^*)-special variety $W \subseteq \mathbb{C}^{2n+2}$, defined by

$$W = \{(X_0, \ldots, X_n, Y_0, \ldots, Y_n) : (X_0, Y_0, X_i, Y_i) \in V_{N_i}, 1 \leq i \leq n\}.$$

W is a 2-dimensional variety and contains the sets

$$\{(j(\tau), j(g_1\tau), \ldots, j(g_n\tau), \chi^*(\tau), \chi^*(g_1\tau), \ldots, \chi^*(g_n\tau))\}$$

and

$$\{(j(\tau), j(g_1\tau), \ldots, j(g_n\tau), \chi(\tau), \chi(g_1\tau), \ldots, \chi(g_n\tau))\}.$$

So the sets S and S^* are each contained in the projection of W onto the Y_i-coordinates (which correspond to χ^*-variables). Since $\dim W = 2$, the projection certainly has dimension at most 2. So we need to show that S is not contained in a curve, which is immediate from 3.10 since not all the N_i are equal to 1. By 3.2, S^* cannot be contained in any curve either. □

Corollary 3.14. *The only positive-dimensional, proper weakly χ^*-special (or indeed weakly χ-special) subvarieties of $\mathbb{C}^2$ are the diagonal $X = Y$ and the horizontal and vertical lines.*

Proof. Immediate. □

4. Ax–Lindemann Theorems

4.1. The Pila–Wilkie Counting Theorem. In the remainder of this document we will use, several times, the theory of o-minimal structures, a branch of model theory. The study of o-minimal structures has been used to great success by Habegger, Masser, Pila, Tsimerman, Zannier and many others to work with problems in functional transcendence, diophantine geometry and other areas. Readers unfamiliar with the topic can see the excellent book by van den Dries [3] or surveys by Pila [14].

The crucial theorem we need for the purposes of this article is the following.

Theorem 4.1 (Pila–Wilkie Counting Theorem)**.** *Let $Z \subseteq \mathbb{R}^n$ be a definable set in an o-minimal structure* $(\mathbb{R}, \{+, \cdot, \dots\}, \{<, \dots\})$.

For each $\epsilon > 0$ and each $k \in \mathbb{N}$ there is a constant $c(Z, k, \epsilon)$, such that for every $T \in \mathbb{N}$, we have

$$\#\Big\{(x_1, \dots, x_n) \in \overline{\mathbb{Q}}^n \cap Z \setminus Z^{alg} : \max_i [\mathbb{Q}(x_i) : \mathbb{Q}] \leq k, \\ \max_i \operatorname{Ht}(x_i) \leq T\Big\} \leq c(Z, k, \epsilon) T^{\epsilon}.$$

This may require some explanation. The set Z here is supposed to be "definable in an o-minimal structure". For details about what this means, one can see the surveys cited above. It is enough to know that there is a certain class of subsets of $\mathbb{C}^n = \mathbb{R}^{2n}$ which will be called "definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$," or just "definable". Crucially, the graphs of the functions j, χ and χ^*, restricted to any $\mathrm{SL}_2(\mathbb{Z})$-translate of the standard fundamental domain

$$\mathbb{D} = \left\{\tau \in \mathcal{H} : -\frac{1}{2} < \operatorname{Re} \tau \leq \frac{1}{2}, |\tau| > 1\right\},$$

are all definable sets. This follows from the fact that each of the functions can be written as a sum of convergent q-expansions, but can also be seen using a result of Peterzil and Starchenko [10] about the definability of the Weierstrass $\wp$-function, going via the theory of elliptic curves.

Consequently, for any variety $V \subseteq \mathbb{C}^{kn}$, the restricted preimage

$$\pi^{-1}(V) \cap \mathbb{D}^n$$

is a definable set whenever $\pi : \mathcal{H}^n \to \mathbb{C}^{kn}$ is some combination of the maps j, χ and χ^*. We will be applying 4.1 to sets of precisely this form.

Given a definable set Z, we can consider what is called the "transcendental part of Z", $Z \setminus Z^{\mathrm{alg}}$, where Z^{alg} is the union of all connected, positive-dimensional, real semialgebraic subsets of Z. Pila–Wilkie tells us that the number of algebraic points in $Z \setminus Z^{\mathrm{alg}}$, of degree less than some fixed k and height at most T, grows more slowly than any positive power of T. Hence, if we can prove that a given definable set Z contains "too many" algebraic points of a given height and degree, then Z must contain an arc of a real algebraic curve.

If $Z = \pi^{-1}(V) \cap \mathbb{D}^n$ as above, our next task is to find out which real algebraic curves can exist within such preimages. For this we need what is known as an Ax–Lindemann-type result; such results are the topic of this section.

4.2. Ax–Lindemann for j. In the classical setting, Pila proved the upcoming result in his paper towards André–Oort, [12]. It is called the Ax–Lindemann theorem for j. Of great interest in its own right, it is also vital to the proof of André–Oort-type results, via Pila–Wilkie. Before we can state it, we will need the following definition:

Definition 4.2. Consider some subset $Z \subseteq \mathcal{H}^n$. A *complex algebraic component* A of Z is a connected component of a set of the form

$$W \cap \mathcal{H}^n,$$

for W an irreducible subvariety of $\mathbb{C}^n$, with the property that $A \subseteq Z$.

Theorem 4.3 (Pila, "Ax–Lindemann for j"). *Let $V \subseteq \mathbb{C}^n$ be a variety. Define a map $\pi : \mathcal{H}^n \to \mathbb{C}^n$ by*

$$\pi(\tau_1, \dots, \tau_n) = (j(\tau_1), \dots, j(\tau_n)),$$

and let $\mathcal{Z} = \pi^{-1}(V)$.

A maximal complex algebraic component of $\mathcal{Z}$ is weakly $\mathcal{H}$-special.

As we noted in the introduction, this is loosely saying: "the only complex algebraic relations between coordinates in $\mathcal{H}$ that induce algebraic relations between their j-images in $\mathbb{C}^n$ are the geodesic relations."

For our purposes, the Ax–Lindemann theorem for j also tells us the following.

Corollary 4.4. *Let V, π and $\mathcal{Z}$ be as in 4.3. Then $\mathcal{Z}^{alg}$ is simply the union of all positive-dimensional weakly $\mathcal{H}$-special subvarieties of $\mathcal{Z}$.*

To go from 4.3 to 4.4 one just uses the holomorphicity of j. A real semialgebraic arc in $\mathcal{Z}$ is contained in a complex algebraic component of $\mathcal{Z}$ by analytic continuation.

4.3. Quasimodular Ax–Lindemann. For the QM function χ, a good portion of the work on Ax–Lindemann results is already done for us. The upcoming result is due to Pila, in [13]. To state it, we will need a definition.

Definition 4.5. Let $\tau_1, \dots, \tau_n$ be elements of some algebraic function field $\mathbb{C}(W)$. Then $\tau_1, \dots, \tau_n$ are called *geodesically dependent* if either:

- For some $g \in \mathrm{GL}_2^+(\mathbb{Q})$ and some i, j, we have $\tau_i = g\tau_j$ whenever τ_i, τ_j take values in $\mathcal{H}$, *or*
- At least one of the τ_i is constant.

Otherwise, the τ_i are called *geodesically independent.*

Theorem 4.6 (Pila, Ax–Lindemann with Derivatives)**.** *Suppose that* $\mathbb{C}(W)$ *is an algebraic function field and that*

$$\tau_1, \dots, \tau_n \in \mathbb{C}(W)$$

take values in $\mathcal{H}$ *at some* $P \in W$*, and are geodesically independent. Then the 3n functions*

$$j(\tau_1), \dots, j(\tau_n), \qquad j'(\tau_1), \dots, j'(\tau_n), \qquad j''(\tau_1), \dots, j''(\tau_n)$$

(considered as functions on W *locally near* P*) are algebraically independent over* $\mathbb{C}(W)$*.*

For our purposes, we need a slightly stronger formulation of this result.

Theorem 4.7 (Ax–Lindemann with Derivatives, Stronger Form)**.** *Let* F *be an irreducible polynomial in* $3n+1$ *variables over* $\mathbb{C}$*. Let* $A \subseteq \mathcal{H}^n$ *be a complex algebraic component and let* G *be the smallest weakly* $\mathcal{H}$*-special variety containing* A*. Suppose that* G *is a GUT variety and that*

$$F(\tau_1, j(\tau_1), j'(\tau_1), j''(\tau_1), \dots, j(\tau_n), j'(\tau_n), j''(\tau_n)) = 0$$

for all $(\tau_1, \dots, \tau_n) \in A$*. Then in fact this holds for all* $(\tau_1, \dots, \tau_n) \in G$*.*

Proof. We will work by induction on n. The case $n = 1$ is immediate.

By definition, the algebraic component A is a connected component of some variety $W \subseteq \mathbb{C}^n$. Treating $\tau_1, \dots, \tau_n$ as the coordinate functions on W, the hypotheses of the theorem imply that

$$j(\tau_1), j'(\tau_1), j''(\tau_1), \dots, j(\tau_n), j'(\tau_n), j''(\tau_n),$$

treated as functions locally near some $P \in A$, are algebraically dependent over $\mathbb{C}(W)$, whence Theorem 4.6 tells us that the τ_i are geodesically dependent.

By induction, we may assume that no τ_i is constant on A. Hence there are $1 \leq i, j \leq n$ and $g \in \mathrm{GL}_2^+(\mathbb{Q})$ such that $\tau_i = g\tau_j$ on A. Since this is a symmetric condition, we may assume that $i \neq 1$. Then without loss of generality, $i = n$.

Since G is a GUT variety, g is upper triangular. Hence there are algebraic functions ϕ_1, ϕ_2, ϕ_3 (induced by the modular polynomials and their derivatives) such that:

$$j(\tau_i) = \phi_1(j(\tau_j)), \tag{4.1}$$

$$j'(\tau_i) = \phi_2(j(\tau_j), j'(\tau_j)), \tag{4.2}$$

and

$$j''(\tau_i) = \phi_3(j(\tau_j), j'(\tau_j), j''(\tau_j)). \tag{4.3}$$

Substituting this into F yields

$$F[\tau_1, j(\tau_1), j'(\tau_1), j''(\tau_1), \dots, j(\tau_{n-1}), j'(\tau_{n-1}), j''(\tau_{n-1}),$$
$$\phi_1(j(\tau_j)), \phi_2(j(\tau_j), j'(\tau_j)), \phi_3(j(\tau_j), j'(\tau_j), j''(\tau_j))] = 0$$

whenever $(\tau_1, \dots, \tau_{n-1}, g\tau_j) \in A$. We can then rewrite this as

$$\sigma(\tau_1, j(\tau_1), j'(\tau_1), j''(\tau_1), \dots, j(\tau_{n-1}), j'(\tau_{n-1}), j''(\tau_{n-1})) = 0,$$

for some algebraic function σ. This will hold for all $(\tau_1, \dots, \tau_{n-1}) \in A'$, where A' is the projection of A onto the first $n-1$ coordinates.

It is possible that σ is the zero function. If so, then working backwards we see that F vanishes whenever (4.1), (4.2) and (4.3) hold. In particular, F vanishes whenever $\tau_i = g\tau_j$. Hence it must vanish on G, as required.

If $\sigma \neq 0$, we have more work to do. There is an irreducible polynomial p_σ such that

$$p_\sigma(\sigma(\mathbf{X}), \mathbf{X}) = 0$$

for all $\mathbf{X}$. In particular,

$$p_\sigma(0, \tau_1, j(\tau_1), j'(\tau_1), j''(\tau_1), \dots, j(\tau_{n-1}), j'(\tau_{n-1}), j''(\tau_{n-1})) = 0 \tag{4.4}$$

for all $(\tau_1, \dots, \tau_{n-1}) \in A'$. Note that $p_\sigma(0, \mathbf{X})$ is not the zero polynomial.

We can now appeal to induction to see that (4.4) holds for all

$$(\tau_1, \dots, \tau_{n-1}) \in G',$$

where G' is the projection of G onto its first $n-1$ coordinates. Putting it in different terms: 0 is a root of

$$p_\sigma(X, \tau_1, j(\tau_1), \dots, j''(\tau_{n-1})) \tag{4.5}$$

whenever $(\tau_1, \dots, \tau_{n-1}) \in G'$. We can choose a point $\mathbf{p} \in A'$, a G'-open neighbourhood V of $\mathbf{p}$ and a complex-open neighbourhood W of 0 such that: for all $\mathbf{q} \in V$, the only root of (4.5) within W is the root 0. However, $\sigma(\tau_1, j(\tau_1), \dots, j''(\tau_{n-1}))$ is always a root of (4.5). So for all $(\tau_1, \dots, \tau_{n-1}) \in V$, we must have

$$\sigma(\tau_1, j(\tau_1), \dots, j''(\tau_{n-1})) = 0.$$

By analytic continuation, this holds for all $(\tau_1, \dots, \tau_{n-1}) \in G'$. Recalling the definition of σ, we get that

$$F[\tau_1, j(\tau_1), j'(\tau_1), j''(\tau_1), \dots, j(\tau_{n-1}), j'(\tau_{n-1}), j''(\tau_{n-1}),$$
$$\phi_1(j(\tau_j)), \phi_2(j(\tau_j), j'(\tau_j)), \phi_3(j(\tau_j), j'(\tau_j), j''(\tau_j))] = 0$$

whenever $(\tau_1, \ldots, \tau_{n-1}) \in G'$. Hence

$$F(\tau_1, j(\tau_1), \ldots, j''(\tau_{n-1}), j(g\tau_j), j'(g\tau_j), j''(g\tau_j)) = 0$$

for all $(\tau_1, \ldots, \tau_{n-1}) \in G'$. In other words

$$F(\tau_1, j(\tau_1), \ldots, j''(\tau_n)) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in G$, as required. □

For our purposes, we need a version of this result that discusses j and χ, rather than the derivatives of j, hence the following corollary.

Corollary 4.8. *Let F be an irreducible polynomial in $3n+1$ variables over $\mathbb{C}$. Let $A \subseteq \mathcal{H}^n$ be a complex algebraic component and let G be the smallest weakly $\mathcal{H}$-special variety containing A. Suppose that G is a GUT variety and that*

$$F(\tau_1, j(\tau_1), \chi(\tau_1), f(\tau_1), \ldots, j(\tau_n), \chi(\tau_n), f(\tau_n)) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in A$. Then in fact this holds for all $(\tau_1, \ldots, \tau_n) \in G$. (Recall that f is the function E_4E_6/Δ, which arises in the transformation law for χ and as the coefficient of $1/\operatorname{Im}\tau$ in χ^*.)

Proof. Follows easily from 4.7, using the fact that $j, \chi, f \in \mathbb{C}(j, j', j'')$ and that $j(\tau), \chi(\tau), f(\tau)$ are algebraically independent functions over $\mathbb{C}(\tau)$. □

4.4. Almost Holomorphic Ax–Lindemann. In the classical situation, as we see above, the holomorphicity of the functions involved allows us to "complexify the parameter" to produce a complex algebraic set from a real algebraic one. Since χ^* is not holomorphic, there is substantial difficulty in attempting to complexify the parameter in the same way. While a real algebraic arc in $\mathcal{H}^n$ is certainly contained in a complex algebraic component of $\mathcal{H}^n$, there is no guarantee that this algebraic component remains within the preimage of the given variety V. Fortunately, the simple shape of χ^* allows us to use some tricks to get around this problem. This subsection is dedicated to proving the desired Ax–Lindemann results for χ^*. This is a crucial step towards our central André–Oort result for χ^*; most of the novelty in our proof of 5.8 lies in this nonholomorphic Ax–Lindemann result.

As we mentioned in Section 1, we will be discussing a map $\pi : \mathcal{H}^n \to \mathbb{C}^{2n}$, defined by

$$\pi(\tau_1, \ldots, \tau_n) = (j(\tau_1), \chi^*(\tau_1), \ldots, j(\tau_n), \chi^*(\tau_n)).$$

Theorem 4.9 (AHM Ax–Lindemann)**.** *Let S be an arc of a real algebraic curve in $\mathcal{H}^n$ and suppose that $S \subseteq \pi^{-1}(V)$, where V is some irreducible variety in $\mathbb{C}^{2n}$. Then S is contained in a weakly $\mathcal{H}$-special variety G with $G \subseteq \pi^{-1}(V)$.*

The proof of this is necessarily rather technical, so for ease of reading we have broken it into various smaller chunks. The plan is as follows. Firstly, we deal with the case in which the imaginary part of every complex coordinate is constant on the arc S. This is the content of Lemma 4.10 and Corollary 4.11. With this done, we can assume that the imaginary part of at least one coordinate (say τ_1) is nonconstant on S. Hence we can parametrise S in terms of the imaginary part of τ_1.

Using this parametrisation, we will show that a particular algebraic function ϕ in the variables $\operatorname{Im}\tau_1$, $j(\tau_i)$, $\chi(\tau_i)$, $f(\tau_i)$ vanishes on S. If ϕ takes a very specific shape, we can conclude via 4.8. Otherwise, we will see that

$$\operatorname{Im}\tau_1 = \psi(j(\tau_1),\dots,j(\tau_n),\chi(\tau_1),\dots,\chi(\tau_n),f(\tau_1),\dots,f(\tau_n))$$

on S, for some algebraic function ψ. In this situation, Lemma 4.12 shows that $\operatorname{Im}\tau_1$ must be constant on S after all, which is a contradiction.

Lemma 4.10. *Let $G \subseteq \mathcal{H}^n$ be a GUT variety, let F be a polynomial in $2n$ variables, and let $c_1,\dots,c_n$ be real constants. Suppose that*

$$F\left(j(\tau_1),\chi(\tau_1)-\frac{3}{\pi c_1}f(\tau_1),\dots,j(\tau_n),\chi(\tau_n)-\frac{3}{\pi c_n}f(\tau_n)\right)=0$$

for all $(\tau_1,\dots,\tau_n)\in G$. Then

$$F\left(j(\tau_1),\chi^*(\tau_1),\dots,j(\tau_n),\chi^*(\tau_n)\right)=0$$

for all $(\tau_1,\dots,\tau_n)\in G$.

Proof. By induction on n we may assume that no coordinate is constant on G. So up to permutation of coordinates, we have

$$\begin{aligned}G=\{(\tau_1,g_{1,1}\tau_1,\dots,g_{1,k_1}\tau_1,\tau_2,g_{2,1}\tau_2,\dots,g_{2,k_2}\tau_2,\dots\\ \dots,\tau_r,g_{r,1}\tau_r,\dots,g_{r,k_r}\tau_r):\tau_1,\dots,\tau_r\in\mathcal{H}\},\end{aligned}$$

for some upper triangular matrices $g_{i,j}$. Hence

$$\begin{aligned}F\Big[&j(\tau_1),\chi(\tau_1)-\frac{3}{\pi d_1}f(\tau_1),\dots\\ &\dots,j(g_{1,k_1}\tau_1),\chi(g_{1,k_1}\tau_1)-\frac{3}{\pi d_{1,k_1}}f(g_{1,k_1}\tau_1),\\ &\dots,j(\tau_r),\chi(\tau_r)-\frac{3}{\pi d_r}f(\tau_r),\\ &\dots,j(g_{r,k_r}\tau_r),\chi(g_{r,k_r}\tau_r)-\frac{3}{\pi d_{r,k_r}}f(g_{r,k_r}\tau_r)\Big]=0,\end{aligned}\tag{4.6}$$

for some suitable relabelling d_i, $d_{i,j}$ of the constants c_i.

All of the $g_{i,j}$ are upper triangular matrices in $\mathrm{GL}_2^+(\mathbb{Q})$, so let us consider a general upper triangular matrix $g=\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)$. Let $A=\gcd(b,d)$ and $D=$

ad/A. Let k, m be integers such that $mb + kd = A$. For all integers t, we have

$$\begin{pmatrix} b/A & -k+tb \\ d/A & m+td \end{pmatrix} \cdot \begin{pmatrix} A & -ma \\ 0 & D \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & tD \end{pmatrix}.$$

The leftmost matrix is an element of $\mathrm{SL}_2(\mathbb{Z})$. The matrix $\left(\begin{smallmatrix} A & -ma \\ 0 & D \end{smallmatrix}\right)$ has the same determinant as g; we shall call this matrix h. Note (taking $t = 0$ above) that

$$\begin{pmatrix} b/A & -k \\ d/A & m \end{pmatrix} h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = g.$$

From these matrix equations and the transformation properties of j, χ and f, we can easily see that

$$\begin{aligned} j\left(g\begin{pmatrix} 0 & -1 \\ 1 & tD \end{pmatrix}\tau\right) &= j(h\tau), && \text{for all } t. \\ \chi\left(g\begin{pmatrix} 0 & -1 \\ 1 & tD \end{pmatrix}\tau\right) &\to \chi(h\tau) && \text{as } t \to \infty. \\ f\left(g\begin{pmatrix} 0 & -1 \\ 1 & tD \end{pmatrix}\tau\right) &\to 0 && \text{as } t \to \infty. \end{aligned}$$

Also,

$$\begin{aligned} j\left(\begin{pmatrix} 0 & -1 \\ 1 & tD \end{pmatrix}\tau\right) &= j(\tau). \\ \chi\left(\begin{pmatrix} 0 & -1 \\ 1 & tD \end{pmatrix}\tau\right) &\to \chi(\tau) && \text{as } t \to \infty. \\ f\left(\begin{pmatrix} 0 & -1 \\ 1 & tD \end{pmatrix}\tau\right) &\to 0 && \text{as } t \to \infty. \end{aligned}$$

Now, equation (4.6) holds for all $\tau_1, \ldots, \tau_r \in \mathcal{H}$. Hence we can replace each τ_i in (4.6) by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & tD_i \end{smallmatrix}\right)\tau_i$, for suitable fixed D_i and arbitrary t. Letting t tend to infinity we see by continuity of F that

$$\begin{aligned} F\Big[& j(\tau_1), \chi(\tau_1), \ldots, j(h_{1,k_1}\tau_1), \chi(h_{1,k_1}\tau_1), \ldots, \\ & j(\tau_r), \chi(\tau_r), \ldots, j(h_{r,k_r}\tau_r), \chi(h_{r,k_r}\tau_r)\Big] = 0 \end{aligned}$$

for all $\tau_i \in \mathcal{H}$ and certain upper triangular matrices $h_{j,k}$. By 3.2 (the isomorphism between upper triangular extensions of the fields of QM/AHM functions), we therefore have

$$F\Big[j(\tau_1), \chi^*(\tau_1), \dots, j(h_{1,k_1}\tau_1), \chi^*(h_{1,k_1}\tau_1), \dots, \\ j(\tau_r), \chi^*(\tau_r), \dots, j(h_{r,k_r}\tau_r), \chi^*(h_{r,k_r}\tau_r)\Big] = 0 \tag{4.7}$$

The matrices $h_{j,k}$ each have the same relation to $g_{j,k}$ as h does to g in the calculation above. In particular, there is $\gamma_{j,k} \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\gamma_{j,k} h_{j,k} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = g_{j,k}.$$

So we can replace each τ_i in (4.7) by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\tau_i$ and use the modularity of j and χ^* to see that

$$F\Big[j(\tau_1), \chi^*(\tau_1), \dots, j(g_{1,k_1}\tau_1), \chi^*(g_{1,k_1}\tau_1), \dots, \\ j(\tau_r), \chi^*(\tau_r), \dots, j(g_{r,k_r}\tau_r), \chi^*(g_{r,k_r}\tau_r)\Big] = 0.$$

This says precisely that

$$F(j(\tau_1), \chi^*(\tau_1), \dots, j(\tau_n), \chi^*(\tau_n)) = 0$$

for all $(\tau_1, \dots, \tau_n) \in G$. □

Corollary 4.11. *Let S and V be as in 4.9. Suppose that the imaginary part of every complex coordinate is constant on S. Then S is contained in a weakly $\mathcal{H}$-special variety G with $G \subseteq \pi^{-1}(V)$.*

Proof. By induction on n, we may assume that no complex coordinate is constant on S. So consider the smallest weakly $\mathcal{H}$-special variety containing S, which we will call G. Since no complex coordinate is constant on S, the same is true of G. We want to show $G \subseteq \pi^{-1}(V)$.

Consider some coordinate τ_i on $S \subseteq G$. It takes the form $\tau_i = x_i + ic_i$. Suppose that on G, there is some τ_j, $j \neq i$ which is related to τ_i by some matrix g which fails to be upper triangular. Then on S, we have $\tau_j = g(x_i + ic_i)$. Since τ_i is nonconstant on S, x_i must vary, which then forces $\operatorname{Im}\tau_j$ to vary since g is not upper triangular. This is a contradiction. So G is a GUT variety.

Now pick any of the irreducible polynomials F which define V. We have

$$F\left(j(\tau_1), \chi(\tau_1) - \frac{3}{\pi c_1} f(\tau_1), \dots, j(\tau_n), \chi(\tau_n) - \frac{3}{\pi c_n} f(\tau_n)\right) = 0 \tag{4.8}$$

for all $(\tau_1, \dots, \tau_n) \in S$ and for real constants $c_i = \operatorname{Im}\tau_i$.

Let us parametrise S in terms of some real parameter t, as the image of a map $t \mapsto (\tau_1(t), \dots, \tau_n(t))$ around $t = 0$. Without loss of generality, suppose that τ_1 is nonconstant, so that all of the other functions τ_i are algebraic over τ_1. The functions τ_i may then be extended to complex t in some complex neighbourhood of 0. The image of this complex neighbourhood under the map then necessarily lives in some irreducible complex algebraic curve C. Since (4.8) holds on $S \subseteq C$ and all of the functions arising in (4.8) are complex analytic, it follows that (4.8) holds on the whole of C. (This method of complexifying the parameter will arise several times; compare with, for instance, [11, Lemma 2.1].)

So we get that (4.8) holds on some complex algebraic component A containing S. Define G to be the smallest weakly $\mathcal{H}$-special variety containing A. As previously, we may assume that G is a GUT variety. Hence we can apply 4.8 to see that

$$F\left(j(\tau_1), \chi(\tau_1) - \frac{3}{\pi c_1} f(\tau_1), \ldots, j(\tau_n), \chi(\tau_n) - \frac{3}{\pi c_n} f(\tau_n)\right) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in G$. By Lemma 4.10, we then have

$$F\left(j(\sigma_1), \chi^*(\sigma_1), \ldots, j(\sigma_n), \chi^*(\sigma_n)\right) = 0$$

for all $(\sigma_1, \ldots, \sigma_n) \in G$. This holds for all of the defining polynomials of V, hence $S \subseteq G \subseteq \pi^{-1}(V)$ as required. $\square$

Lemma 4.12. *Let S be an arc of a real algebraic curve in $\mathcal{H}^n$ and let ψ be an algebraic function in $3n$ variables. Suppose that*

$$\operatorname{Im}\tau_1 = \psi(j(\tau_1), \ldots, j(\tau_n), \chi(\tau_1), \ldots, \chi(\tau_n), f(\tau_1), \ldots f(\tau_n))$$

for all $(\tau_1, \ldots, \tau_n) \in S$.

Let G be the smallest weakly $\mathcal{H}$-special variety containing S, and suppose that G is a GUT variety. Then $\operatorname{Im}\tau_1$ is constant on S.

Notation. The tuple

$$(j(\tau_1), \ldots, j(\tau_n), \chi(\tau_1), \ldots, \chi(\tau_n), f(\tau_1), \ldots f(\tau_n))$$

will arise often in what follows, so we abbreviate it as $\tilde{\pi}(\tau_1, \ldots, \tau_n)$. We will also abbreviate $y = \operatorname{Im}\tau_1$ throughout. So the first hypothesis of the Lemma may be written as

$$y = \psi(\tilde{\pi}(\tau_1, \ldots, \tau_n)).$$

Proof of Lemma 4.12. Suppose for a contradiction that $y = \operatorname{Im}\tau_1$ is nonconstant on S. Then we can parametrise S in terms of y, yielding

$$S = \{(x(y) + iy, u_2(y) + iv_2(y), \ldots, u_n(y) + iv_n(y)) : y \in U\}$$

for some open set $U \subseteq \mathbb{R}$ and algebraic functions x, u_i, v_i, real-valued on U.

Since S is an algebraic arc, we also have some polynomials a_i such that

$$a_i(x(y), y, u_2(y), v_2(y), \ldots, u_n(y), v_n(y)) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in S$. Noting that $\tau_1 = x(y) + iy$, and replacing instances of y with ψ, we get

$$\begin{aligned}(4.9)\quad a_i[\tau_1 - i\psi(\tilde{\pi}(\tau_1, \ldots, \tau_n)), \psi(\tilde{\pi}(\tau_1, \ldots, \tau_n)),\\ u_2(\psi(\tilde{\pi}(\tau_1, \ldots, \tau_n))), v_2(\psi(\tilde{\pi}(\tau_1, \ldots, \tau_n))), \ldots,\\ u_n(\psi(\tilde{\pi}(\tau_1, \ldots, \tau_n))), v_n(\psi(\tilde{\pi}(\tau_1, \ldots, \tau_n)))] = 0,\end{aligned}$$

for all $(\tau_1, \ldots, \tau_n) \in S$. We rewrite the left hand side of this equation as an algebraic function

$$\sigma(\tau_1, \tilde{\pi}(\tau_1, \ldots, \tau_n)).$$

Then there is an irreducible polynomial p_σ such that

$$p_\sigma(\sigma(T, \mathbf{J}, \mathbf{X}, \mathbf{F}), T, \mathbf{J}, \mathbf{X}, \mathbf{F}) = 0$$

identically. In particular, since σ vanishes on S, we have

$$p_\sigma(0, \tau_1, \tilde{\pi}(\tau_1, \ldots, \tau_n)) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in S$. By complexifying the parameter, as in Corollary 4.11 and [11, Lemma 2.1], this holds on a complex algebraic component A containing S. Now, the weakly special closure of A is the same as the weakly special closure of S, namely G. Since G is a GUT variety, we may therefore apply 4.8 to see that

$$p_\sigma(0, \tau_1, \tilde{\pi}(\tau_1, \ldots, \tau_n)) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in G$.

In other words, 0 is a root of

$$p_\sigma(X, \tau_1, \tilde{\pi}(\tau_1, \ldots, \tau_n)) \tag{4.10}$$

for all $(\tau_1, \ldots, \tau_n) \in G$. Since

$$p_\sigma(\sigma(T, \mathbf{J}, \mathbf{X}, \mathbf{F}), T, \mathbf{J}, \mathbf{X}, \mathbf{F}) = 0$$

identically, we know that

$$\sigma(\tau_1, \tilde{\pi}(\tau_1, \ldots, \tau_n))$$

is also root of (4.10) for all $(\tau_1, \ldots, \tau_n) \in G$.

We can pick a point $a \in S$, a G-open neighbourhood W of a, and a complex neighbourhood U of 0, such that: as $(\tau_1, \ldots, \tau_n)$ varies within W, the only root of (4.10) within U is 0 itself. However, as $(\tau_1, \ldots, \tau_n)$ varies in W, the function $\sigma(\tau_1, \tilde{\pi}(\tau_1, \ldots, \tau_n))$ remains a root of (4.10). Since σ vanishes on S, we can get it arbitrarily close to 0 within W. In particular, we can get $\sigma(\tau_1, \tilde{\pi}(\tau_1, \ldots, \tau_n))$ to lie within U. Since it is a root of (4.10), we must have

$$\sigma(\tau_1, \tilde{\pi}(\tau_1, \ldots, \tau_n)) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in W$. By analytic continuation, this holds everywhere on G, which says that (4.9) holds on G.

For notational simplicity, let us suppose that the coordinates which are related to τ_1 in G are the first k coordinates, that is:

$$G = \{(\tau_1, g_2\tau_1, \ldots, g_k\tau_1) : \tau_1 \in \mathcal{H}\} \times G',$$

for some other GUT variety G'. So, whenever

$$(\tau_1, g_2\tau_1, \ldots, g_k\tau_1, \tau_{k+1}, \ldots, \tau_n) \in G,$$

we also have

$$(\tau_1 + t, g_2(\tau_1 + t), \ldots, g_k(\tau_1 + t), \tau_{k+1}, \ldots \tau_n) \in G$$

for every $t \in \mathbb{Z}$. Since G is a GUT set, the g_i are upper triangular, so the numbers $g_i(\tau_1 + t)$, up to translation by an integer, take only finitely many values as t varies. In particular, since j, χ and f are periodic, each of the functions

$$j(g_i(\tau_1 + t)), \qquad \chi(g_i(\tau_1 + t)), \qquad f(g_i(\tau_1 + t))$$

takes only finitely many values as t varies. Hence

$$\psi(\tilde{\pi}(\tau_1 + t, g_2(\tau_1 + t), \ldots, g_k(\tau_1 + t), \tau_{k+1}, \ldots, \tau_n))$$

takes only finitely many values as $t \in \mathbb{Z}$ varies. If we plug this into (4.9), we see that

$$a_i(\tau_1 + t - ic, c, u_2(c), v_2(c), \ldots, v_n(c)) = 0$$

for some constant c and infinitely many distinct t. Thus a_i is independent of its first coordinate. Since this is true of all the a_i defining S, the only possibility for S is that it is the product of a horizontal line in the τ_1 plane and points in the other coordinates. So y is constant on S, which is a contradiction. $\square$

With all the above lemmas done, we may finally proceed to the body of the proof of 4.9.

Proof of 4.9. By induction on n, we may assume that no complex coordinate is constant on S. It might be, however, that the imaginary part of one or more coordinates is constant on S. If $\operatorname{Im}\tau_i$ is constant on S for every i, then we are in the situation of Lemma 4.11 so we conclude immediately. Hence we may assume without loss of generality that $\operatorname{Im}\tau_1$ is nonconstant on S.

Next, let G be the unique smallest weakly $\mathcal{H}$-special subvariety of $\mathcal{H}^n$ containing S. It is a standard fact (which we have used once already; see for instance Lang [8] or [1, Exercise 1.2.11]) that any $g \in \mathrm{GL}_2^+(\mathbb{Q})$ takes the form $\gamma \cdot h$ for some upper triangular $h \in \mathrm{GL}_2^+(\mathbb{Q})$ and some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Therefore there is some $\gamma \in \mathrm{SL}_2(\mathbb{Z})^n$ such that γG is a GUT variety. The subset $\gamma S \subseteq \gamma G$ is still a real semialgebraic arc. By the modularity of j and χ^*, $\gamma G \subseteq \pi^{-1}(V)$ if and only if $G \subseteq \pi^{-1}(V)$. So by working with γS we may assume without loss of generality that G is a GUT variety.

We will write $y = \operatorname{Im} \tau_1$ throughout, and retain the abbreviation

$$\tilde{\pi}(\tau_1, \ldots, \tau_n) = (j(\tau_1), \ldots, j(\tau_n), \chi(\tau_1), \ldots, \chi(\tau_n), f(\tau_1), \ldots, f(\tau_n)).$$

Since y is nonconstant, we can parametrise S as

$$S = \{(x(y) + iy, u_2(y) + iv_2(y), \ldots, u_n(y) + iv_n(y)) : y \in U\},$$

for some open $U \subseteq \mathbb{R}$ and algebraic functions x, u_i, v_i, real-valued on U.

Consider one of the polynomials F which defines V. We have

$$F\left(j(\tau_1), \chi(\tau_1) - \frac{3}{\pi y} f(\tau_1), \ldots, j(\tau_n), \chi(\tau_n) - \frac{3}{\pi v_n(y)} f(\tau_n)\right) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in S$. We can rewrite the left hand side of this equation as an algebraic function

$$\phi(y, \tilde{\pi}(\tau_1, \ldots, \tau_n)).$$

Since ϕ is an algebraic function, there is an irreducible polynomial p_ϕ with the property that

$$p_\phi(\phi(T, \mathbf{J}, \mathbf{X}, \mathbf{F}), T, \mathbf{J}, \mathbf{X}, \mathbf{F}) = 0$$

for all T, $\mathbf{J} = (J_1, \ldots, J_n)$, $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{F} = (F_1, \ldots, F_n)$. In particular, we have that

$$p_\phi(0, y, \tilde{\pi}(\tau_1, \ldots, \tau_n)) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in S$. So let us define

$$P(T, \mathbf{J}, \mathbf{X}, \mathbf{F}) = p_\phi(0, T, \mathbf{J}, \mathbf{X}, \mathbf{F}).$$

Note that P is not the zero polynomial, since p_ϕ is irreducible.

We are going to modify P as follows. Consider each coefficient of T^k in P separately. These are polynomials

$$C_k(\mathbf{J}, \mathbf{X}, \mathbf{F}).$$

For each k, if

$$C_k(\tilde{\pi}(\tau_1, \ldots, \tau_n)) = 0$$

for $(\tau_1, \ldots, \tau_n) \in S$, then remove this coefficient of T^k from the polynomial P. Having done this for each coefficient, we have a modified polynomial which we call $\tilde{P}$. Note that we still have

$$\tilde{P}(y, \tilde{\pi}(\tau_1, \ldots, \tau_n)) = 0$$

for $(\tau_1, \ldots, \tau_n) \in S$.

It is possible that $\tilde{P}$ is the zero polynomial. This happens if and only if every coefficient C_k has the property that

$$C_k(\tilde{\pi}(\tau_1, \ldots, \tau_n)) = 0 \tag{4.11}$$

for $(\tau_1, \ldots, \tau_n) \in S$. By complexifying the parameter, as in Corollary 4.11 and [11, Lemma 2.1], the equation (4.11) holds for $(\tau_1, \ldots, \tau_n) \in A$, where A

is the smallest complex algebraic component containing S. Now, the weakly $\mathcal{H}$-special closure of A is the same as the weakly special closure of S, which is the GUT variety G. Hence we can apply 4.8, to see that (4.11) holds for all $(\tau_1, \ldots, \tau_n) \in G$.

Since this holds for all C_k, we have

$$p_\phi(0, Y, \tilde{\pi}(\tau_1, \ldots, \tau_n)) = P(Y, \tilde{\pi}(\tau_1, \ldots, \tau_n)) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in G$ and all choices of Y. In other words, 0 is a root of

$$p_\phi(X, Y, \tilde{\pi}(\tau_1, \ldots, \tau_n)) \tag{4.12}$$

for all $(\tau_1, \ldots, \tau_n) \in G$ and all Y.

Now we proceed exactly as we did in Lemma 4.12. We can certainly pick a point $a = (a_1, \ldots, a_n) \in S$ such that for all $(\tau_1, \ldots, \tau_n)$ in some G-open neighbourhood W of a, the only root of (4.12), in some complex neighbourhood U of 0, is 0 itself. However, we know that

$$X = \phi(Y, \tilde{\pi}(\tau_1, \ldots, \tau_n))$$

is a root of (4.12) identically. Fixing $Y = \operatorname{Im} a_1$, we see that

$$\phi(\operatorname{Im} a_1, \tilde{\pi}(\tau_1, \ldots, \tau_n))$$

gets arbitrarily close to 0 within W (it vanishes at a). So as $(\tau_1, \ldots, \tau_n)$ varies within W, ϕ is a root of (4.12), and lies inside of U. The only such root is 0, so we must have

$$\phi(\operatorname{Im} a_1, \tilde{\pi}(\tau_1, \ldots \tau_n)) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in W$. By analytic continuation, this holds for all

$$(\tau_1, \ldots, \tau_n) \in G.$$

Recalling the definition of ϕ, we get

$$F\left(j(\tau_1), \chi(\tau_1) - \frac{3}{\pi \operatorname{Im} a_1} f(\tau_1), \ldots, j(\tau_n), \chi(\tau_n) - \frac{3}{\pi v_n (\operatorname{Im} a_1)} f(\tau_n)\right) = 0$$

for all $(\tau_1, \ldots, \tau_n) \in G$. Hence we are in the situation of Lemma 4.10, so we get

$$F\left(j(\tau_1), \chi^*(\tau_1), \ldots, j(\tau_n), \chi^*(\tau_n)\right) = 0,$$

for all $(\tau_1, \ldots, \tau_n) \in G$, as required.

We have now dealt with the case where $\tilde{P}$ is the zero polynomial. So we suppose that $\tilde{P} \neq 0$ and look for a contradiction. Since

$$\tilde{P}(y, \tilde{\pi}(\tau_1, \ldots, \tau_n)) = 0$$

for $(\tau_1, \ldots, \tau_n) \in S$, there is an irreducible factor Q of $\tilde{P}$ with this same property.

Suppose some coefficient of y^k in Q vanishes on S. Then we repeat the entire process, removing redundant coefficients to get a polynomial $\tilde{Q}$. Again, an irreducible component of $\tilde{Q}$ must vanish on S. Then we can remove redundant coefficients from this irreducible component, and so on.

We continue repeating this process until it terminates with an irreducible polynomial

$$R(y, \tilde{\pi}(\tau_1, \ldots, \tau_n)),$$

which vanishes on S, with the property that none of the coefficients of y^k in R vanish on S. If R were the zero polynomial, then working backwards we see that $\tilde{P}$ should have been the zero polynomial, which we have assumed is not the case. So $R \neq 0$. In particular, R is nonconstant as a polynomial in y.

Hence, since none of the coefficients of y^k in R vanish on S, we can extract an algebraic function ψ such that

$$y = \psi(\tilde{\pi}(\tau_1, \ldots, \tau_n))$$

for all $(\tau_1, \ldots, \tau_n) \in S$. By our earlier comment, we know that the smallest $\mathcal{H}$-special variety containing S is G, a GUT variety. So we are in the situation of Lemma 4.12, hence y is constant on S, which is a contradiction. $\square$

We can reformulate 4.9 into the following slightly cleaner statement.

Corollary 4.13. *Let V be an irreducible subvariety of $\mathbb{C}^{2n}$ and let $\mathcal{Z} = \pi^{-1}(V)$. Then $\mathcal{Z}^{alg}$ is just the union of the weakly $\mathcal{H}$-special subvarieties of $\mathcal{Z}$.*

In the next section we use this to prove the central result of the document.

5. André–Oort for χ^*

Since there are no obvious χ- or (j, χ)-special varieties in $\mathbb{C}^n$, it is not clear what an André–Oort statement should look like. We can, however, formulate meaningful André–Oort statements for χ^*. In this section we state and prove 5.8, which is the main theorem of the document, an André–Oort theorem for j and χ^*. The map π will throughout be defined as before, namely

$$\pi(\tau_1, \ldots, \tau_n) = (j(\tau_1), \chi^*(\tau_1), \ldots, j(\tau_n), \chi^*(\tau_n)).$$

The proof follows the standard strategy explicated in [12] very closely, and we will borrow ideas freely from there. Readers familiar with the strategy will be aware of the piece that is currently missing. We need some number-theoretic lower bound in order to force $\pi^{-1}(V)$ to contain many points of a given height. This will force a real algebraic arc to exist in $\pi^{-1}(V)$, so that we can apply the results of the previous section. For the case of j, the lower bound comes from the size of certain Galois orbits, which are known by a result of Siegel to be sufficiently large. Our approach essentially comes down to that same lower bound of Siegel, but first we have to do some work to ensure that the bound still applies to χ^*-special points.

Proposition 5.1 (Masser). *For a quadratic point $\tau \in \mathcal{H}$, we have*

$$\mathbb{Q}(\chi^*(\tau)) \subseteq \mathbb{Q}(j(\tau)).$$

Proof. Masser proves this in the Appendix of [9] for a function he calls ψ, which is $E_2^* E_4/E_6$. Since χ^* lies in $\mathbb{Q}(\psi, j)$, the result follows for χ^*. □

A careful look at Masser's proof of the above yields the following stronger result.

Proposition 5.2. *Let $\tau \in \mathcal{H}$ be a quadratic point and consider the algebraic numbers $j(\tau)$ and $\chi^*(\tau)$. Let σ be a Galois conjugation acting on $\mathbb{Q}(j(\tau)) \supseteq \mathbb{Q}(\chi^*(\tau))$. Let τ' be a quadratic point such that $j(\tau') = \sigma(j(\tau))$. Then $\chi^*(\tau') = \sigma(\chi^*(\tau))$.*

Proof. This comes entirely from close inspection of Masser's work (the appendix in [9]). Let d be the discriminant of the quadratic number τ, and suppose that d is not equal to $3k^2$ for some odd k. Define some rational functions $\beta^\tau_{i,k}$ such that $\beta^\tau_{i,k}(j(\tau))$ are the coefficients of the Taylor expansion of Φ_d about the point $(j(\tau), j(\tau))$. This we can certainly do, and we get

$$\Phi_d(X,Y) = \sum_{(i,k)\neq(0,0)} \beta^\tau_{i,k}(j(\tau))(X - j(\tau))^i (Y - j(\tau))^k.$$

It appears that the rational functions $\beta^\tau_{i,k}$ will differ with τ. However, we will show that, for the τ and τ' defined in the hypotheses of the theorem, we do have $\beta^\tau_{i,k} = \beta^{\tau'}_{i,k}$.

Since Φ_d has rational coefficients, any Galois conjugation preserves the left hand side of the above. So we get

$$\begin{aligned}\Phi_d(X,Y) &= \sum \sigma(\beta^\tau_{i,k}(j(\tau)))(X - \sigma(j(\tau)))^i (Y - \sigma(j(\tau)))^k \\ &= \sum \beta^\tau_{i,k}(j(\tau'))(X - j(\tau'))^i (Y - j(\tau'))^k.\end{aligned}$$

We also have

$$\Phi_d(X,Y) = \sum_{(i,k)\neq(0,0)} \beta^{\tau'}_{i,k}(j(\tau'))(X - j(\tau'))^i (Y - j(\tau'))^j,$$

so by uniqueness of Taylor coefficients, the rational functions β^τ_i and $\beta^{\tau'}_j$ are equal. On pages 118 and 119 of [9], $\psi(\tau)$ is expressed as a fixed $\mathbb{Q}$-rational function p in the $\beta^\tau_{i,k}(j(\tau))$ and $j(\tau)$. The equality

$$\psi(\tau) = p(j(\tau), \beta^\tau_{i,k}(j(\tau)))$$

holds whenever τ has discriminant d and $\beta^\tau_{i,j}$ are the Taylor coefficients of Φ_d about $(j(\tau), j(\tau))$. Since τ' and τ have the same discriminant (both satisfy $\Phi_d(j(\rho), j(\rho)) = 0$), this equation holds for both τ and τ'. Since $\beta_i^\tau = \beta_i^{\tau'}$ we get

$$\begin{aligned}\sigma\psi(\tau) &= p(\sigma(j(\tau)), \beta_{i,k}^\tau(\sigma(j(\tau)))) \\ &= p(j(\tau'), \beta_{i,k}^\tau(j(\tau'))) \\ &= p(j(\tau'), \beta_{i,k}^{\tau'}(j(\tau'))) \qquad \text{since } \beta_{i,k}^\tau = \beta_{i,k}^{\tau'} \\ &= \psi(\tau').\end{aligned}$$

When τ is $3k^2$ for some odd k, the exact same argument still goes through, except the rational function p is replaced by q, which is some other (still fixed and explicit) rational function. Both p and q are written out on pages 118 and 119 of [9], but we will write them here for completeness[3].

$$p(j, \beta_{i,k}) = \frac{9j(\beta_{2,0} - \beta_{1,1} + \beta_{0,2})}{\beta_{0,1}} + \frac{3(7j - 6912)}{2(j - 1728)}.$$
$$q(j, \beta_{i,k}) = \frac{9j(\beta_{4,0} - \beta_{3,1} + \beta_{2,2} - \beta_{1,3} + \beta_{0,4})}{\beta_{0,1}} + \frac{3(7j - 6912)}{2(j - 1728)}.$$

In either case we get $\sigma\psi(\tau) = \psi(\tau')$. Since $\chi^* = r(j, \psi)$ for a $\mathbb{Q}$-rational function r, we get $\sigma(\chi^*(\tau)) = \chi^*(\tau')$ as required. ☐

Corollary 5.3. *Let K be a number field. There are positive constants $c, \delta > 0$ with the following property. Let $\tau \in \mathbb{D}$ be a quadratic point of discriminant D. Then there are $\gg D^\delta$ distinct quadratic points $\tau' \in \mathbb{D}$, of height at most cD, such that $(j(\tau'), \chi^*(\tau'))$ is a Galois conjugate, over K, of the point $(j(\tau), \chi^*(\tau))$.*

Proof. For quadratic points $\sigma \in \mathbb{D}$, let $H(\sigma)$ be the height of σ and $D(\sigma)$ the discriminant. It is known that the number of distinct Galois conjugates of $j(\sigma)$ over $\mathbb{Q}$ is bounded from below by a positive power of $D(\sigma)$. This follows from the Siegel lower bound [17] for class numbers of quadratic fields. See Pila [12] for more details.

Since $[K : \mathbb{Q}]$ is a fixed constant, the number of Galois conjugates of $j(\tau)$ over K is therefore $\gg D^\delta$. Each Galois conjugate θ_i of $j(\tau)$ over K yields a distinct $\tau_i \in \mathbb{D}$, such that $\theta_i(j(\tau)) = j(\tau_i)$. Moreover, $D(\tau_i) = D(\tau) = D$.

By work of Pila [12], there is a constant c such that, for any $\sigma \in \mathbb{D}$,

$$H(\sigma) \le cD(\sigma).$$

Hence each τ_i has $H(\tau_i) \le cD$.

Finally, by 5.2, we have $(j(\tau_i), \chi^*(\tau_i)) = (\theta_i(j(\tau)), \theta_i(\chi^*(\tau)))$. ☐

[3]The reader may note a strange-looking asymmetry in p and q, namely the $\beta_{0,1}$ in the denominator. Why not $\beta_{1,0}$? Masser in fact proves in his work that $\beta_{0,1} = \beta_{1,0}$, so really there is no asymmetry.

Corollary 5.3 gives us exactly the lower bound we need to work with the Pila–Wilkie theorem. Shortly we will use this bound and the Pila–Wilkie theorem 4.1 to prove our main theorem, 5.8. First we have a proposition demonstrating the ideas in the simplest case; it also serves as the base case for an inductive argument we use in 5.8.

Proposition 5.4 (André–Oort for (j, χ^*), in 2 dimensions)**.** *Let $C \subseteq \mathbb{C}^2$ be an irreducible algebraic curve. Then C contains only finitely many (j, χ^*)-special points.*

Proof. Suppose that C contained infinitely many special points. Since special points are algebraic, this tells us that C can in fact be defined over $\overline{\mathbb{Q}}$ and thus over a number field K.

Define a set $Z \subseteq \mathbb{D}$ by

$$Z = \{\tau \in \mathbb{D} : (j(\tau), \chi^*(\tau)) \in C\}.$$

Then Z is definable. If it contains an arc of a real algebraic curve, then by 4.9 it must be all of $\mathcal{H}$, which is impossible since j and χ^* are algebraically independent. Hence Z^{alg} is empty. We will show that $Z \setminus Z^{\mathrm{alg}} = Z$ contains "many" (ie. a positive power of T) quadratic points of a given height T, contradicting the Pila–Wilkie theorem.

Since C contains infinitely many special points, we have infinitely many distinct quadratic points $\tau \in \mathbb{D}$ with $(j(\tau), \chi^*(\tau)) \in C$. In particular, we can find such a τ with arbitrarily large discriminant D. Hence by 5.3, there are $\gg D^\delta$ quadratic points $\tau' \in \mathbb{D}$, of height at most cD, such that $(j(\tau'), \chi^*(\tau'))$ is a Galois conjugate of $(j(\tau), \chi^*(\tau))$ over K.

Since it is a Galois conjugate of $(j(\tau), \chi^*(\tau))$, we know that

$$(j(\tau'), \chi^*(\tau')) \in C,$$

hence all of the τ' lie in Z. So there are $\gg D^\delta$ quadratic points (of height at most cD) in $Z = Z \setminus Z^{\mathrm{alg}}$, which contradicts the Pila–Wilkie Theorem for any $\epsilon < \delta$. □

In more dimensions, the fundamental ideas for dealing with special points by counting Galois conjugates are exactly the same; we have the following. (Compare with Theorem 11.2 of [12].)

Proposition 5.5. *Suppose $V \subseteq \mathbb{C}^{2n}$ is a variety defined over a number field K. Write V^{sp} for the union of all positive-dimensional (j, χ^*)-special subvarieties of V. Suppose that V^{sp} is a variety. Then $V \setminus V^{sp}$ contains only finitely many (j, χ^*)-special points.*

Proof. Let $\mathcal{Z} = \pi^{-1}(V)$ and $Z = \mathcal{Z} \cap \mathbb{D}^n$. Then Z is definable.

The set $\mathcal{Z}^{\mathrm{alg}}$ consists of $\mathcal{Z}^{\mathrm{sp}} = \pi^{-1}(V^{\mathrm{sp}})$ as well as possibly some weakly $\mathcal{H}$-special varieties; but the weakly $\mathcal{H}$-special varieties can contain no quadratic points. Hence, if we denote by $N(X, T)$ the number of quadratic points in X up to height T, we have

$$N(Z \setminus Z^{\mathrm{sp}}, T) \leq N(Z \setminus Z^{\mathrm{alg}}, T) \ll_{\epsilon} T^{\epsilon}$$

for any $\epsilon > 0$; the last bound coming from the Pila–Wilkie Counting Theorem. Here $Z^{\mathrm{sp}} = \mathcal{Z}^{\mathrm{sp}} \cap \mathbb{D}^n$.

Suppose for a contradiction that $V \setminus V^{\mathrm{sp}}$ contains infinitely many (j, χ^*)-special points. Then we can find quadratic points

$$u = (\tau_1, \ldots, \tau_n) \in Z \setminus Z^{\mathrm{sp}}$$

of arbitrarily large discriminant D. By 5.3, there are $\gg D^{\delta}$ quadratic points $u' \in \mathbb{D}^n$, with height at most cD, such that $\pi(u')$ is a Galois conjugate of $\pi(u)$ over K. This gives us $\gg D^{\delta}$ quadratic points (of height at most cD) in $Z \setminus Z^{\mathrm{sp}}$. Choosing any $\epsilon < \delta$, we get a contradiction to the Pila–Wilkie theorem for sufficiently large D. $\square$

So we have some control over the special points that can arise in a given variety. The next step is to deal with the positive-dimensional special subvarieties.

Definition 5.6. A $\mathcal{H}$-special (or (j, χ^*)-special, or j-special, etc.) variety is called *basic* if it has no constant factors. That is, if the set S_0, from the definition of a special variety, is empty.

Every weakly $\mathcal{H}$-special variety S arises as the product of a basic $\mathcal{H}$-special variety B with some number of constant factors q_i (if all the q_i are quadratic points then S is special). When this happens, we say that S is the *translate* of B by the factors q_i. The following lemma tells us which basic special varieties have translates lying in the preimage of a given variety V.

Lemma 5.7. *Let $V \subseteq \mathbb{C}^{2n}$ be a variety and define $\mathcal{Z} = \pi^{-1}(V)$. There is a finite collection $\mathcal{B}$ of basic $\mathcal{H}$-special varieties with the property that every maximal, positive-dimensional, weakly $\mathcal{H}$-special subvariety of $\mathcal{Z}$ is a translate of γB, for some $B \in \mathcal{B}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})^k$.*

Proof. This is identical to Proposition 10.2 of [12]. In the presence of 4.9, the proof carries over exactly. $\square$

Finally, we combine 5.5 and 5.7 in an inductive argument to prove our main theorem.

Theorem 5.8 (André–Oort for (j, χ^*))**.** *Let $V \subseteq \mathbb{C}^{2n}$ be a variety. Then V contains only finitely many maximal (j, χ^*)-special subvarieties.*

Proof. There is a subvariety $\tilde{V} \subseteq V$, defined over $\overline{\mathbb{Q}}$, containing all the algebraic points of V. So we may assume that V is defined over $\overline{\mathbb{Q}}$ (and thus over a number field K).

We will proceed by induction on n. The base case is 5.4. The conclusion holds by 5.5 if V^{sp} is variety. So it is sufficient to prove that V^{sp} is a variety, under the assumption that 5.8 holds for $m < n$.

By 5.7, there are finitely many basic $\mathcal{H}$-special varieties, $B \in \mathcal{B}$, such that every maximal $\mathcal{H}$-special subvariety of $\pi^{-1}(V)$ is a translate of some γB. A maximal (j, χ^*)-special subvariety of V is the Zariski closure of $\pi(S)$, for some maximal $\mathcal{H}$-special subvariety $S \subseteq \pi^{-1}(V)$. Therefore any maximal (j, χ^*)-special subvariety of V is the translate (by some special points $(j(\tau_i), \chi^*(\tau_i))$) of one of a finite collection $\mathcal{C}$ of basic (j, χ^*)-special varieties. (The twists by elements of $\mathrm{SL}_2(\mathbb{Z})$ have no effect since j and χ^* are modular.)

So it is enough to show that, given some basic special $C \in \mathcal{C}$, there are only finitely many translates of C which are maximal (j, χ^*)-special subvarieties of V. Such a C will be a subvariety of $\mathbb{C}^{2k}$ for some k.

The possible translates of C are elements of $\mathbb{C}^{2(n-k)}$, namely the set of points[4]

$$\begin{aligned} V' = \{(j_1, \chi_1, \ldots, j_{n-k}, \chi_{n-k}) &: \text{the translate of } C \text{ by} \\ &(j_1, \chi_1, \ldots, j_{n-k}, \chi_{n-k}) \text{ is contained in } V\}. \end{aligned}$$

This is an algebraic subvariety of $\mathbb{C}^{2(n-k)}$. The translates of C which yield special subvarieties of V are the (j, χ^*)-special points of V'. The translates which yield *maximal* special subvarieties are the (j, χ^*)-special points of $V' \setminus (V')^{\mathrm{sp}}$. By our inductive assumption, there are only finitely many such points. Thus V^{sp}, which consists of finitely many translates of the finitely many basic special varieties in $\mathcal{C}$, is a variety. So we can conclude by 5.5. □

Corollary 5.9 (André–Oort for χ^*)**.** *Let $V \subseteq \mathbb{C}^n$ be a variety. Then V contains only finitely many maximal χ^*-special subvarieties.*

Proof. Consider a variety $V' \subseteq \mathbb{C}^{2n}$, defined as

$$V' = \{(J_1, X_1, \ldots, J_n, X_n) : (X_1, \ldots, X_n) \in V\}.$$

Given a maximal χ^*-special subvariety S of V, there is a corresponding (j, χ^*)-special subvariety $S' \subseteq V'$, such that the projection of S' onto the X_i coordinates (which correspond to χ^*) is S. By 5.8, it is enough to show that S' is a maximal (j, χ^*)-special subvariety of V'.

Indeed, if S' were contained in a (j, χ^*)-special subvariety $T \subseteq V'$, with $\dim T > \dim S'$, then by the definition of (j, χ^*)-special varieties, there must be a condition on a χ^*-coordinate which is relaxed in going from S' to T.

[4]We are being slightly lax with our labelling of coordinates here. The constant factors by which we translate our basic varieties can be in any of the pairs of coordinates in $(\mathbb{C}^2)^n$. Since there are only finitely many ways to reorder the coordinates, no issues will arise from allowing the translations to take place in any of the coordinates.

Hence the projection of T onto the X_i coordinates would be a χ^*-special subvariety of V strictly containing S. Contradiction. $\square$

References

[1] F. Diamond & J. Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer, 2005, xv+436 pages.

[2] G. Diaz, "Transcendance et indépendance algébrique: liens entre les points de vue elliptique et modulaire", *Ramanujan J.* **4** (2000), no. 2, p. 157-199.

[3] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, vol. 248, Cambridge University Press, 1998, x+180 pages.

[4] D. S. Dummit & R. M. Foote, *Abstract algebra*, John Wiley & Sons, 2004.

[5] B. Edixhoven, "On the André–Oort conjecture for Hilbert modular surfaces", in *Moduli of abelian varieties (Texel Island, 1999)*, Progress in Mathematics, vol. 195, Birkhäuser, 2001, p. 133-155.

[6] B. Edixhoven & A. Yafaev, "Subvarieties of Shimura varieties", *Ann. Math.* **157** (2003), no. 2, p. 621-645.

[7] B. Klingler & A. Yafaev, "The André–Oort conjecture", *Ann. Math.* **180** (2014), no. 3, p. 867-925.

[8] S. Lang, *Elliptic functions*, Graduate Texts in Mathematics, vol. 112, Springer, 1987, xi+326 pages.

[9] D. Masser, *Elliptic functions and transcendence*, Lecture Notes in Mathematics, vol. 437, Springer, 1975, xiv+143 pages.

[10] Y. Peterzil & S. Starchenko, "Uniform definability of the Weierstrass $\wp$ functions and generalized tori of dimension one", *Sel. Math., New Ser.* **10** (2004), no. 4, p. 525-550.

[11] J. Pila, "Rational points of definable sets and results of André–Oort–Manin–Mumford type", *Int. Math. Res. Not.* **2009** (2009), no. 13, p. 2476-2507.

[12] ———, "O-minimality and the André–Oort conjecture for $\mathbb{C}^n$", *Ann. Math.* **173** (2011), no. 3, p. 1779-1840.

[13] ———, "Modular Ax–Lindemann–Weierstrass with derivatives", *Notre Dame J. Formal Logic* **54** (2013), no. 3-4, p. 553-565.

[14] ———, "O-minimality and Diophantine Geometry", in *Proceedings of the ICM 2014*, KM Kyung Moon Sa, 2014, p. 547-572.

[15] J. Pila & J. Tsimerman, "Ax–Lindemann for $\mathcal{A}_g$", *Ann. Math.* **179** (2014), no. 2, p. 659-681.

[16] T. Schneider, *Einführung in die transzendenten Zahlen*, Grundlehren der Mathematischen Wissenschaften, vol. 81, Springer, 1957.

[17] C. L. Siegel, "Über die Klassenzahl quadratischer Zahlkörper", *Acta Arith.* **1** (1935), p. 83-86.

[18] J. Tsimerman, "The André–Oort conjecture for $\mathcal{A}_g$", *Ann. Math.* **187** (2018), no. 2, p. 379-390.

[19] E. Ullmo & A. Yafaev, "Galois orbits of special subvarieties of Shimura varieties", preprint, 2006.

[20] D. Zagier, "Elliptic modular forms and their applications", in *The 1-2-3 of modular forms*, Universitext, Springer, 2008, p. 1-103.

[21] U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry*, Annals of Mathematics Studies, vol. 181, European Mathematical Society, 2012, xi+160 pages.

Haden Spence
2 Dennis Close,
Aston Clinton,
Buckinghamshire,
HP22 5US, UK
E-mail: haden_spence@btopenworld.com
URL: http://www.maths.ox.ac.uk/people/haden.spence