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Lower bounds for generalized unit regulators

par James SUNDSTROM

Résumé. En 1999 Friedman et Skoruppa ont introduit une méthode de minoration du régulateur relatif $\mathrm{Reg}(L/K)$ d'une extension L/K de corps de nombres. Ce régulateur est défini en utilisant le sous-groupe $E_{L/K}$ des unités relatives de L/K. Puisque $\mathrm{Reg}(L/K)$ apparaît dans la série $\Theta_{E_{L/K}}$ associé à $E_{L/K}$, toute inégalité entre $\Theta_{E_{L/K}}$ et $\Theta'_{E_{L/K}}$ induit une minoration de ce régulateur. On peut appliquer la même méthode à d'autres sous-groupes E du groupe des unités d'un corps de nombres L. Dans cet article nous considérons le cas où $E = E_{L/K_1} \cap E_{L/K_2}$, où K_1 et K_2 sont des corps quadratiques réels; Le régulateur associé croît alors exponentiellement en fonction du degré de L sur $\mathbb{Q}$.

Abstract. In 1999, Friedman and Skoruppa published a method to derive lower bounds for the relative regulator of an extension L/K of number fields. The relative regulator is defined using the subgroup $E_{L/K}$ of relative units of L/K. It appears in the theta series $\Theta_{E_{L/K}}$ associated to $E_{L/K}$, so an inequality relating $\Theta_{E_{L/K}}$ and $\Theta'_{E_{L/K}}$ provides an inequality for $\mathrm{Reg}(L/K)$. This same technique can be applied to other subgroups E of the units of a number field L. In this paper, we consider the case $E = E_{L/K_1} \cap E_{L/K_2}$, where K_1 and K_2 are real quadratic fields; the corresponding regulator grows exponentially in $[L : \mathbb{Q}]$.

1. Introduction

This paper demonstrates how a technique of Friedman and Skoruppa [8] can be generalized. Before proceeding to the generalization, we first review their paper. Friedman and Skoruppa proved lower bounds for the relative regulator $\mathrm{Reg}(L/K)$ associated to an extension L/K of number fields. A relative regulator was defined by Bergé and Martinet [3, 4, 5]. Friedman and Skoruppa considered a slightly different version, defined as follows.

Given a number field K, let $\mathcal{O}_K$ denote the algebraic integers of K, with unit group $\mathcal{O}_K^*$ and roots of unity $\mu_K \subseteq \mathcal{O}_K^*$. Let $\mathcal{A}_K$ be the set of

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archimedean places of K; for each $v \in \mathcal{A}_K$, let

$$e_v = \begin{cases} 1 & \text{if } v \text{ is real} \\ 2 & \text{if } v \text{ is complex.} \end{cases}$$

Let $r_1(K)$ and $r_2(K)$ be, respectively, the number of real and complex places of K.

Given an extension of number fields L/K, let $E_{L/K}$ denote the group of relative units of L/K, i.e.,

$$E_{L/K} = \{\epsilon \in \mathcal{O}_L^* \mid \mathrm{N}_{L/K}(\epsilon) \in \mu_K\}.$$

Note that $E_{L/K}$ has rank $r = r_{L/K} := |\mathcal{A}_L| - |\mathcal{A}_K|$. Let $\epsilon_1, \ldots, \epsilon_r$ be fundamental relative units (free generators for $E_{L/K}$ modulo torsion). For each $w \in \mathcal{A}_K$, fix some $\tilde{w} \in \mathcal{A}_L$ lying above w. Let $\mathcal{A}'_L$ denote the remaining places of L after each $\tilde{w}$ is removed from $\mathcal{A}_L$. Then the relative regulator of L/K is defined by

$$\mathrm{Reg}(L/K) = \left| \det(e_v \log|\epsilon_j|_v)_{\substack{v \in \mathcal{A}'_L \\ 1 \le j \le r}} \right|.$$

Costa and Friedman [7] proved that

$$\mathrm{Reg}(L/K) = \frac{1}{[\mathcal{O}_K^* : \mu_K \, \mathrm{N}_{L/K}(\mathcal{O}_L^*)]} \frac{\mathrm{Reg}(L)}{\mathrm{Reg}(K)} \le \frac{\mathrm{Reg}(L)}{\mathrm{Reg}(K)}.$$

Hence a lower bound for $\mathrm{Reg}(L/K)$ is a lower bound for $\mathrm{Reg}(L)/\mathrm{Reg}(K)$ as well. Furthermore, $\mathrm{Reg}(L/\mathbb{Q}) = \mathrm{Reg}(L)$; of course, this was already clear from the definition of $\mathrm{Reg}(L/\mathbb{Q})$. Thus a lower bound for relative regulators includes a lower bound for the classical regulator as a special case.

To any subgroup E of $\mathcal{O}_L^*$, we can associate a theta series Θ_E. Let $E_{\mathrm{tor}} = E \cap \mu_L$ denote the torsion subgroup of E. Let $E_{\mathbb{R}} = E \otimes \mathbb{R}$, and fix a Haar measure μ on $E_{\mathbb{R}}$, so that $\mu(E_{\mathbb{R}}/E)$ is the volume of any fundamental domain for the action of E on $E_{\mathbb{R}}$.

There is an embedding of $E_{\mathbb{R}}$ into $\mathbb{R}_+^{\mathcal{A}_L}$ given by

$$x = \sum_j \epsilon_j \otimes \xi_j \mapsto (x_v)_{v \in \mathcal{A}_L}, \qquad x_v = \prod_j |\epsilon_j|_v^{\xi_j}. \tag{1.1}$$

For $a \in L$ and $x \in E_{\mathbb{R}}$, set

$$\|ax\|^2 = \sum_{v \in \mathcal{A}_L} e_v |a|_v^2 x_v^2.$$

For any fractional ideal $\mathfrak{a}$ of L and any $t > 0$, define

$$\Theta_E(t; \mathfrak{a}) = \frac{\mu(E_{\mathbb{R}}/E)}{|E_{\mathrm{tor}}|} + \sum_{\substack{a \in \mathfrak{a}/E \\ a \ne 0}} \int_{E_{\mathbb{R}}} \exp\left(-c_{\mathfrak{a}} t \|ax\|^2\right) \mathrm{d}\mu(x), \tag{1.2}$$

where the sum is taken over a complete set of representatives for the nonzero E-orbits in $\mathfrak{a}$, and

$$c_{\mathfrak{a}} = \pi \left(\sqrt{|\mathrm{disc}(L)|}\, \mathrm{N}_{L/\mathbb{Q}}(\mathfrak{a}) \right)^{-2/[L:\mathbb{Q}]}.$$

Friedman and Skoruppa give a proof that Θ_E is well-defined: it is independent of the choice of representatives a, and the sum is absolutely convergent. They also observe that $t^{[L:\mathbb{Q}]/2}\Theta_E(t;\mathfrak{a})$ is an increasing function [8, Proposition 2.1]. Differentiating, it follows that

$$\Theta_E(t;\mathfrak{a}) + \frac{2}{[L:\mathbb{Q}]} t\Theta_E'(t;\mathfrak{a}) \geq 0\,. \tag{1.3}$$

Since the definition of Θ_E involves $\mu(E_{\mathbb{R}}/E)$ as a constant term, this inequality can be understood as a lower bound for $\mu(E_{\mathbb{R}}/E)$. In particular, if we take $E = E_{L/K}$, then it is fairly natural to normalize μ by $\mu(E_{\mathbb{R}}/E) = \mathrm{Reg}(L/K)$. Thus, by estimating the integrals in the definition of Θ_E, we will obtain the desired lower bound for $\mathrm{Reg}(L/K)$.

As a first step to understanding these integrals, Friedman and Skoruppa use the Mellin transform to prove the following (see [8, Proposition 3.1]).

Proposition 1.1. *With notation as above (including $E = E_{L/K}$),*

$$\int_{E_{\mathbb{R}}} \exp(-t\|ax\|^2)\, \mathrm{d}\mu(x) = A \prod_{w\in\mathcal{A}_K} f_w(a_w + \log t)\,,$$

where $A = 2^{-r_{L/K}}\pi^{-r_2(L)/2}$,

$$f_w(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-se_w[L:K]y}\Gamma(s)^{p_w+q_w}\Gamma\left(s+\frac{1}{2}\right)^{q_w} \mathrm{d}s\,,$$

$$a_w = \frac{2}{[L:K]}\log|\mathrm{N}_{L/K}(a)|_w\,.$$

Here c is any positive number, and p_w and q_w are respectively the number of real and complex places of L extending $w \in \mathcal{A}_K$.

This proposition puts the integrals into a more tractable form; instead of estimating the original integrals, it suffices to understand f_w. More precisely, setting $y = \log t$, inequality (1.3) becomes

$$\begin{aligned} &\frac{\mathrm{Reg}(L/K)}{\#\mu_L} \\ &\quad \geq A \sum_{\substack{a\in\mathfrak{a}/E \\ a\neq 0}} \left\{ -1 - \frac{2}{[L:\mathbb{Q}]}\sum_{w\in\mathcal{A}_K}\frac{f_w'}{f_w}(a_w+y)\right\} \prod_{w\in\mathcal{A}_K} f_w(a_w+y)\,, \end{aligned} \tag{1.4}$$

so we need to estimate f_w and f_w'/f_w. This is accomplished by the saddle-point method, as described in de Bruijn [6].

Once we have estimates for f_w and f'_w/f_w, we can substitute them into inequality (1.4) and we are essentially done. Since inequality (1.4) holds for any $y \in \mathbb{R}$, it remains only to choose a y which gives a good bound. However, our estimates for f_w and f'_w/f_w depend on p_w and q_w. We would prefer to have lower bounds for $\mathrm{Reg}(L/K)$ that do not require such detailed information about the places of L. Hence we make some effort to transform the bounds in terms of the p_w and q_w into bounds depending only on $[L : K]$ and $r_1(L)$.

In short, Friedman and Skoruppa's method consists of four main steps:

(1) Use the Mellin transform to replace the Θ_E integrals with complex integrals.
(2) Use the saddle-point method to estimate the complex integrals.
(3) Replace these estimates with estimates that do not depend on the p_w and q_w.
(4) Substitute these estimates into inequality (1.3) to get lower bounds for the regulator.

In this paper, we apply these methods to a generalized regulator for a number field L containing two real quadratic fields K_1 and K_2. Specifically, we consider the regulator associated to $E = E_{L/K_1} \cap E_{L/K_2}$. Section 2 defines this regulator. Section 3 computes the necessary inverse Mellin transform; we find that we need to study a triple integral. (This is the main difficulty in generalizing Friedman and Skoruppa's technique. In their paper, the corresponding multiple integral splits into a product of single integrals; here, the saddle point method must be applied directly to the three-dimensional integral. That estimate is the technical heart of this paper.) Sections 4–5 carry out step 2. Section 4 summarizes some results in single-variable calculus which will be needed; many of these results are quite similar (or identical) to results from Friedman and Skoruppa's original paper. Then Section 5 applies these results to study the relevant triple integrals. Step 3 is done in Sections 6 and 7. Once again, Section 6 provides some simple results, which are applied in Section 7. Finally, Section 8 completes the argument, proving that the generalized regulator $\mathrm{Reg}_{K_1,K_2}(L)$ grows exponentially in $[L : \mathbb{Q}]$. For $[L : \mathbb{Q}] \geq 4000$,

$$\frac{\mathrm{Reg}_{K_1,K_2}(L)}{\#\mu_L} > (4.28 \times 10^{-9}) \cdot 1.199^{[L:\mathbb{Q}]}.$$

2. The Generalized Regulator

Let K_1 and K_2 be distinct real quadratic fields, and L a number field containing the compositum $K := K_1K_2$. Let $m = [L : K] = [L : \mathbb{Q}]/4$. Let $\mathcal{A}_{K_1} = \{w_1, w_2\}$ and $\mathcal{A}_{K_2} = \{w_3, w_4\}$ be the sets of archimedean places of K_1 and K_2. Let $\mathcal{A}_K = \{w_{13}, w_{14}, w_{23}, w_{24}\}$ denote the set of archimedean places of K, labeled so that w_{ij} extends $w_i \in \mathcal{A}_{K_1}$ and $w_j \in \mathcal{A}_{K_2}$. Note that for any $i \in \{1,2\}$ and $j \in \{3,4\}$,

$$\sum_{\substack{v\in\mathcal{A}_L\\ v|w_i}} e_v = \sum_{\substack{v\in\mathcal{A}_L\\ v|w_j}} e_v = 2m, \qquad \sum_{\substack{v\in\mathcal{A}_L\\ v|w_{ij}}} e_v = m. \tag{2.1}$$

For any $w \in \mathcal{A}_{K_1K_2}$, let p_w and q_w denote respectively the number of real and complex places of L extending w, so that $p_w + 2q_w = m$. Let E_i denote the relative units of L/K_i, and define $E = E_1 \cap E_2$. Let E_{tor} denote the torsion subgroup of E.

We define a generalized regulator $\mathrm{Reg}_{K_1,K_2}(L)$ as follows. Let $\epsilon_1, \ldots, \epsilon_r$ ($r = |\mathcal{A}_L| - 3$) be free generators of E/E_{tor}. Let $\tilde{\mathcal{A}}_K$ be a set containing any three places of K, and select one place of L above each place in $\tilde{\mathcal{A}}_K$. Let $\mathcal{A}'_L$ denote $\mathcal{A}_L$ with these three places removed. Define

$$\mathrm{Reg}_{K_1,K_2}(L) = \left|\det(e_v \log|\epsilon_j|_v)_{\substack{v\in\mathcal{A}'_L\\ 1\le j\le r}}\right|.$$

Lemma 2.1. $\mathrm{Reg}_{K_1,K_2}(L)$ *is well-defined, i.e., it is independent of the choice of the* ϵ_j *and of* $\mathcal{A}'_L$.

Proof. Define $\lambda\colon \mathcal{O}_L^* \to \mathbb{R}^{\mathcal{A}_L}$ by

$$\lambda(\epsilon) = (e_v \log|\epsilon|_v)_{v\in\mathcal{A}_L}.$$

Define $\boldsymbol{x}_1 \in \mathbb{R}^{\mathcal{A}_L}$ by

$$(\boldsymbol{x}_1)_v = \begin{cases} e_v & \text{if } v \mid w_1,\\ 0 & \text{otherwise.}\end{cases}$$

Define similarly $\boldsymbol{x}_2$ with respect to w_2 and $\boldsymbol{x}_3$ with respect to w_3. Let M denote the matrix with columns $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \lambda(\epsilon_1), \ldots, \lambda(\epsilon_r)$. Note that $|\det(M)|$ is the covolume of the lattice generated by $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, and $\lambda(\mathcal{O}_L^*)$, so it is independent of our choice of the ϵ_j. Row operations show that

$$|\det(M)| = |\det(M')|\,\mathrm{Reg}_{K_1,K_2}(L),$$

where M' is the 3×3 matrix defined by

$$M' = \Bigg(\sum_{v|w_i \text{ and } v|w_j} e_v\Bigg)_{\substack{1\le i\le 3\\ 1\le j\le 3}}.$$

(The precise row operations to be used depend on $\tilde{\mathcal{A}}_K$, but the result is the same.) Since $|\det(M')| = 4m^3$, this proves that $\mathrm{Reg}_{K_1,K_2}(L) = \frac{1}{4m^3}|\det(M)|$, so $\mathrm{Reg}_{K_1,K_2}(L)$ is well-defined. □

3. Theta series

Let $E \subset \mathcal{O}_L^*$ be as in the previous section. Let $G = \mathbb{R}_+^{\mathcal{A}_L}$ and let $H = \mathbb{R}_+^{\mathcal{A}_{K_1} \cup \{w_3\}}$. Let μ_G denote the natural Haar measure on G, namely

$$\mathrm{d}\mu_G(g) = \prod_{v \in \mathcal{A}_L} \frac{\mathrm{d}g_v}{g_v}.$$

Define μ_H similarly. Let μ be the Haar measure on $E_{\mathbb{R}}$, normalized so that

$$\mu(E_{\mathbb{R}}/E) = \mathrm{Reg}_{K_1,K_2}(L).$$

Define $\delta\colon G \to H$ by $\delta(g) = (h_w)_{w \in \mathcal{A}_{K_1} \cup \{w_3\}}$, where

$$h_w = \prod_{v|w} g_v^{e_v}.$$

Using the embedding $E_{\mathbb{R}} \to G$ from (1.1), we get an exact sequence $1 \to E_{\mathbb{R}} \to G \to H \to 1$.

Let $\sigma\colon H \to G$ be a section of δ. (We will choose a particular section σ below.) Define an isomorphism $\phi\colon E_{\mathbb{R}} \times H \to G$ by $\phi(x,h) = x\sigma(h)$.

Lemma 3.1. $2^{r_2(L)}\mu_G \circ \phi = \mu \times \mu_H$.

Proof. Since $\mu_G \circ \phi$ is a Haar measure on $E_{\mathbb{R}} \times H$, we know that $c\mu_G \circ \phi = \mu \times \mu_H$ for some constant c. Consider $E_{\mathbb{R}}$, G, H, and $\mathbb{R}_+$ as real vector spaces. Choose any $v_{13}, v_{23}, v_{14} \in \mathcal{A}_L$ such that v_{ij} extends w_i and w_j. Define $g_{13}, g_{23}, g_{14} \in G$ by

$$(g_{ij})_v = \begin{cases} \exp(1/e_v) & \text{if } v = v_{ij}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\delta(g_{13}), \delta(g_{23}), \delta(g_{14})$ is a basis for H, so we can define the section $\sigma\colon H \to G$ by $\delta(g_{ij}) \mapsto g_{ij}$.

As before, let $\epsilon_1, \ldots, \epsilon_r$ be a $\mathbb{Z}$-basis for E/E_{tor}. Then the $x_j = \epsilon_j \otimes 1$ form an $\mathbb{R}$-basis of $E_{\mathbb{R}}$. Extend this to a basis for G by adjoining the three vectors g_{ij}. It follows that

$$c\mu_G([x_1, \ldots, x_r, g_{13}, g_{23}, g_{14}]) = \mu([x_1, \ldots, x_r]) \cdot \mu_H([\delta(g_{13}), \delta(g_{23}), \delta(g_{14})]),$$

where $[\ldots]$ denotes convex hull. We have $\mu([x_1, \ldots, x_r]) = \mathrm{Reg}_{K_1,K_2}(L)$ by the normalization of μ. The convex hull of the $\delta(g_{ij})$ is the "unit cube", so it has volume 1. It is easily seen that $\mu_G([x_1, \ldots, x_r, g_{13}, g_{23}, g_{14}]) = 2^{-r_2(L)}\,\mathrm{Reg}_{K_1,K_2}(L)$. Hence $c = 2^{r_2(L)}$. □

For a fractional ideal $\mathfrak{a}$ of L and $t > 0$, recall the theta series $\Theta_E(t;\mathfrak{a})$ defined in (1.2). We will use the Mellin transform to study this function.

First, we define some notation. For any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and any $\kappa \in [0,1]$, set

$$\alpha_\kappa(z) = \kappa \log \Gamma(z) + (1-\kappa) \log \Gamma\left(z + \frac{1}{2}\right).$$

For $i \in \{1,2\}$ and $j \in \{3,4\}$, let

$$k_{ij} = \frac{1}{m}(p_{w_{ij}} + q_{w_{ij}})\,. \tag{3.1}$$

Given $s = (s_1, s_2, s_3) \in \mathbb{C}^3$, define

$$s_{13} = s_1 + s_3\,, \quad s_{23} = s_2 + s_3\,, \quad s_{14} = s_1\,, \quad s_{24} = s_2\,. \tag{3.2}$$

Let $\mathcal{R}$ denote the region

$$\mathcal{R} = \{s \in \mathbb{C}^3 \mid \text{all } \mathrm{Re}(s_{ij}) > 0\}. \tag{3.3}$$

For a given $\vec{\kappa} = (\kappa_{13}, \kappa_{14}, \kappa_{23}, \kappa_{24})$, we define a function $\alpha_{\vec{\kappa}} \colon \mathcal{R} \to \mathbb{C}$ by

$$\alpha_{\vec{\kappa}}(s) = \sum_{\substack{i \in \{1,2\} \\ j \in \{3,4\}}} \alpha_{\kappa_{ij}}(s_{ij})\,. \tag{3.4}$$

Let α denote $\alpha_{\vec{k}}$, where $\vec{k} = (k_{13}, k_{14}, k_{23}, k_{24})$ as in (3.1).

For $g \in G$, set

$$\|g\|^2 = \sum_{v \in \mathcal{A}_L} e_v g_v^2$$

and

$$\Psi(g) = \int_{E_\mathbb{R}} \exp(-\|gx\|^2)\, \mathrm{d}\mu(x)\,.$$

We want to evaluate $\Psi(g)$; since $\Psi(g)$ depends only on g modulo $E_\mathbb{R}$, it suffices to consider $\psi = \Psi \circ \sigma$. Now we compute the Mellin transform of ψ:

$$\begin{aligned}
(M\psi)(s) &= (M\psi)(s_1, s_2, s_3) \\
&= \int_H \Psi(\sigma(h)) h^s \, \mathrm{d}\mu_H(h) \\
&= \int_H \left(\int_{E_\mathbb{R}} \exp(-\|x\sigma(h)\|^2)\, \mathrm{d}\mu(x) \right) h^s \mathrm{d}\mu_H(h) \\
&= \int_{E_\mathbb{R} \times H} \exp(-\|\phi(x,h)\|^2) \delta(\phi(x,h))^s \, (\mathrm{d}\mu \times \mathrm{d}\mu_H)(x,h) \\
&= 2^{r_2(L)} \int_G \exp(-\|g\|^2) \delta(g)^s \, \mathrm{d}\mu_G(g)\,.
\end{aligned}$$

Next observe that

$$\delta(g)^s = \left(\prod_{v \mid w_1} g_v^{e_v} \right)^{s_1} \left(\prod_{v \mid w_2} g_v^{e_v} \right)^{s_2} \left(\prod_{v \mid w_3} g_v^{e_v} \right)^{s_3} = \prod_{\substack{i \in \{1,2\} \\ j \in \{3,4\}}} \prod_{v \mid w_{ij}} g_v^{e_v s_{ij}}\,.$$

It follows that

$$\begin{aligned}(M\psi)(s) &= 2^{r_2(L)} \int_G \exp(-\|g\|^2) \prod_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} \prod_{v|w_{ij}} g_v^{e_v s_{ij}} \,\mathrm{d}\mu_G(g) \\ &= 2^{r_2(L)} \prod_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} \prod_{v|w_{ij}} \int_G e^{-e_v g_v^2} g_v^{e_v s_{ij}} \frac{\mathrm{d}g_v}{g_v} \\ &= 2^{r_2(L)-|\mathcal{A}_L|} \prod_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} \Gamma\left(\frac{s_{ij}}{2}\right)^{p_{w_{ij}}} \left(2^{-s_{ij}}\Gamma(s_{ij})\right)^{q_{w_{ij}}} \\ &= 2^{-|\mathcal{A}_L|}\pi^{-r_2(L)/2} \prod_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} \Gamma\left(\frac{s_{ij}}{2}\right)^{p_{w_{ij}}+q_{w_{ij}}} \Gamma\left(\frac{s_{ij}+1}{2}\right)^{q_{w_{ij}}} \\ &= 2^{-|\mathcal{A}_L|}\pi^{-r_2(L)/2} \exp\left(m\alpha\left(\frac{1}{2}s\right)\right),\end{aligned}$$

where we have used the identity $2^{-s}\Gamma(s) = (2\sqrt{\pi})^{-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})$.

Setting $h = \delta(g)$ and taking an inverse Mellin transform,

$$\begin{aligned}\Psi(g) &= \psi(h) \\ &= \frac{2^{-|\mathcal{A}_L|}\pi^{-r_2(L)/2}}{(2\pi i)^3} \iiint h_{w_1}^{-s_1} h_{w_2}^{-s_2} h_{w_3}^{-s_3} \exp\left(m\alpha\left(\frac{1}{2}s\right)\right) \mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3 \\ &= \frac{2^{3-|\mathcal{A}_L|}\pi^{-r_2(L)/2}}{(2\pi i)^3} \iiint h_{w_1}^{-2s_1} h_{w_2}^{-2s_2} h_{w_3}^{-2s_3} \exp(m\alpha(s))\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3\,,\end{aligned}$$

where the integral with respect to s_j is taken from $c_j - i\infty$ to $c_j + i\infty$. Given $a \in L^*$ and $t > 0$, define $g \in G$ by $g_v = \sqrt{t}|a|_v$. Then for any $x \in E_{\mathbb{R}}$, $t\|ax\|^2 = \|gx\|^2$, so

$$\Psi(g) = \int_{E_{\mathbb{R}}} \exp(-t\|ax\|^2)\,\mathrm{d}\mu(x)$$

is the integral that appears in Θ_E. For $w \in \mathcal{A}_{K_i}$,

$$h_w = \prod_{v|w} g_v^{e_v} = \prod_{v|w} t^{e_v/2}|a|_v^{e_v} = t^m |\mathrm{N}_{L/K_i}(a)|_w\,.$$

Let $A = 2^{3-|\mathcal{A}_L|}\pi^{-r_2(L)/2}$ and let $a_w = \frac{1}{m}\log|\mathrm{N}_{L/K_i}(a)|_w$. For any $y \in \mathbb{R}$, define

$$g_{y,a}(s) = -2(a_{w_1}+y)s_1 - 2(a_{w_2}+y)s_2 - 2(a_{w_3}+y)s_3 + \alpha(s)$$

and

$$f(y,a) = \frac{1}{(2\pi i)^3} \int_{c_3-i\infty}^{c_3+i\infty} \int_{c_2-i\infty}^{c_2+i\infty} \int_{c_1-i\infty}^{c_1+i\infty} \exp(m g_{y,a}(s))\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3\,.$$

The preceding work shows that $\Psi(g) = Af(\log t, a)$. Hence

$$\Theta_E(t;\mathfrak{a}) = \frac{\mu(E_{\mathbb{R}}/E)}{|E_{\mathrm{tor}}|} + A \sum_{\substack{a\in\mathfrak{a}/E\\ a\neq 0}} f(\log(c_{\mathfrak{a}}t), a)\,.$$

Inequality (1.3) says that $\Theta_E(t;\mathfrak{a}) + \frac{1}{2m}t\Theta'_E(t;\mathfrak{a}) \geq 0$. Using the above formula, and choosing t so that $y = \log(c_{\mathfrak{a}}t)$, this proves that for any $y \in \mathbb{R}$,

$$\frac{\mu(E_{\mathbb{R}}/E)}{|E_{\mathrm{tor}}|} \geq A \sum_{\substack{a\in\mathfrak{a}/E\\ a\neq 0}} \left(-1 - \frac{1}{2m}\frac{f'}{f}(y,a)\right) f(y,a)\,. \tag{3.5}$$

Next we want to choose y such that $-\frac{f'}{f}(y,a) \geq 2m$ for all a. Then we can drop terms for $a \neq 1$ to conclude that

$$\frac{\mu(E_{\mathbb{R}}/E)}{|E_{\mathrm{tor}}|} \geq A\left(-1 - \frac{1}{2m}\frac{f'}{f}(y,1)\right) f(y,1)\,. \tag{3.6}$$

This is done by the saddle-point method. In order to apply the saddle-point method, we first need to know that there is a saddle point. That is, we would like to find a point (s_1, s_2, s_3) where

$$\frac{\partial g_{y,a}}{\partial s_1} = \frac{\partial g_{y,a}}{\partial s_2} = \frac{\partial g_{y,a}}{\partial s_3} = 0\,.$$

This means we need to solve

$$\begin{aligned} a_{w_1} + y &= \frac{1}{2}\alpha'_{k_{13}}(s_1+s_3) + \frac{1}{2}\alpha'_{k_{14}}(s_1)\,,\\ a_{w_2} + y &= \frac{1}{2}\alpha'_{k_{23}}(s_2+s_3) + \frac{1}{2}\alpha'_{k_{24}}(s_2)\,,\\ a_{w_3} + y &= \frac{1}{2}\alpha'_{k_{13}}(s_1+s_3) + \frac{1}{2}\alpha'_{k_{23}}(s_2+s_3)\,. \end{aligned}$$

Note that for any $k \in (0,1]$, $\alpha'_k\colon (0,\infty) \to \mathbb{R}$ is strictly increasing and surjective.

Lemma 3.2. *This system has a unique solution in $\mathcal{R}\cap\mathbb{R}^3$. (See* (3.3) *for the definition of $\mathcal{R}$.)*

Proof. The given system of equations is equivalent to

$$\frac{1}{2}\alpha'_{k_{24}}(s_2) = a_{w_1} + a_{w_2} - a_{w_3} + y - \frac{1}{2}\alpha'_{k_{14}}(s_1)\,, \tag{3.7}$$

$$\frac{1}{2}\alpha'_{k_{23}}(s_2+s_3) = a_{w_3} - a_{w_1} + \frac{1}{2}\alpha'_{k_{14}}(s_1)\,, \tag{3.8}$$

$$\frac{1}{2}\alpha'_{k_{13}}(s_1+s_3) = a_{w_1} + y - \frac{1}{2}\alpha'_{k_{14}}(s_1)\,. \tag{3.9}$$

Note that equation (3.7) determines $s_2 > 0$ as a strictly decreasing function of $s_1 > 0$, and equation (3.8) determines $s_2 + s_3 > 0$ as a strictly increasing function of $s_1 > 0$. Therefore these two equations determine s_3 as a strictly increasing function of s_1. Under this correspondence, $s_3 \to \infty$ as $s_1 \to \infty$. On the other hand, equation (3.9) determines s_3 as a strictly decreasing function of s_1. Under this correspondence, $s_3 \to \infty$ as $s_1 \to 0$. Now we have two functions $s_1 \mapsto s_3$, and the solutions of the system correspond to choices of s_1 at which these functions are equal. It follows from what we have said that there is exactly one solution. □

4. Single-variable calculus

Before proceeding to the triple-integral estimates we need, we record some single-variable lemmas which will be useful. Throughout this section, we assume $\frac{1}{2} \le \kappa \le 1$, $m > 0$, and $\sigma > 0$ (sometimes adding an additional assumption on m where helpful). Recall the formula [1, 6.4.10]: for $n \ge 1$ and $\mathrm{Re}(s) > 0$,

$$(4.1) \qquad \frac{\Psi^{(n)}(s)}{n!} = (-1)^{n+1} \sum_{k=0}^{\infty} \frac{1}{(s+k)^{n+1}}, \qquad \Psi = \frac{\Gamma'}{\Gamma}.$$

Of course, this Ψ is not the same as the Ψ in the previous section. This abuse of notation is committed for consistency with Friedman and Skoruppa's paper, and should cause no confusion.

Lemma 4.1. *If $m\kappa \ge 4$, then*

$$\frac{\sqrt{\sigma^2 \alpha_\kappa''(\sigma)}}{1.25^{m\kappa[\sigma]/2}} < \sqrt{2} \qquad \text{and} \qquad \frac{\sigma^2 \alpha_\kappa''(\sigma)}{1.25^{m\kappa[\sigma]/2}} < 2.$$

If $m\kappa \ge 30$, then

$$\frac{\sqrt{\sigma^2 \alpha_\kappa''(\sigma)}}{(\frac{73}{72})^{m\kappa[\sigma]/2}} < \sqrt{2}.$$

Proof. The first inequality is given in the proof of Friedman and Skoruppa's Lemma 5.5; as the other inequalities are proven in the same way, the proof is repeated here. Note that

$$\sigma^2 \alpha_\kappa''(\sigma) \le \sigma^2 \Psi'(\sigma) < 1 + \sigma,$$

where the last inequality follows from estimating the sum (4.1) by an integral. Thus we need to show that $\sqrt{1+\sigma}/1.25^{m\kappa[\sigma]/2} < \sqrt{2}$. We see that $\sqrt{1+\sigma}/1.25^{m\kappa[\sigma]/2}$ is maximized as $\sigma \to 1^-$, because $1.25^2 > \sqrt{3/2}$. □

Lemma 4.2. *Let ϵ be 0 or 1. Suppose $u > 0$ and $m\kappa > 2$. Then*

$$\int_{u\sigma}^{\infty} |t^{\epsilon} e^{m\alpha(\sigma+it)}| \,\mathrm{d}t \le \frac{e^{m\alpha(\sigma)} \sigma^{1+\epsilon} u^{\epsilon-1}(1+u^2)}{(m\kappa-2)(1+\frac{u^2}{4})^{m\kappa[\sigma]/2}(1+u^2)^{m\kappa/2}},$$

where $[\sigma]$ denotes the greatest integer less than or equal to σ.

Proof. This is Friedman and Skoruppa's Lemma 5.3. □

Lemma 4.3. *Let D be given with $0 < D \leq m^{1/3}\sqrt{\kappa}$ and assume that $m\kappa > 2$. Define $\delta = D/(m^{1/3}\sqrt{\alpha_\kappa''(\sigma)})$. Then*

$$\int_{|t|>\delta} \left|e^{m\alpha_\kappa(\sigma+it)}\right| \mathrm{d}t \leq \frac{\sqrt{2\pi}e^{m\alpha_\kappa(\sigma)}}{\sqrt{m\alpha_\kappa''(\sigma)}} \left(\frac{2^{3/2}\sigma\sqrt{m\alpha_\kappa''(\sigma)}}{\sqrt{\pi}(m\kappa-2)1.25^{m\kappa[\sigma]/2}2^{m\kappa/2}} + \frac{2^{3/2}e^{-m^{1/3}D^2/4}}{\sqrt{\pi}m^{1/6}D} \right),$$

and

$$\int_{|t|>\delta} e^{-mt^2\alpha_\kappa''(\sigma)/2}\,\mathrm{d}t \leq \frac{2e^{-m^{1/3}D^2/2}}{m^{2/3}D\sqrt{\alpha_\kappa''(\sigma)}}.$$

where $[\sigma]$ denotes the greatest integer less than or equal to σ.

Proof. These inequalities can be found in Friedman and Skoruppa (see their proof of Lemma 5.4). □

Lemma 4.4. *Suppose $m\kappa > 2$. Then for any $0 < D < m^{1/3}\sqrt{\kappa}$,*

$$\int_{-\infty}^{\infty} \left|e^{m\alpha_\kappa(\sigma+it)}\right| \mathrm{d}t < \frac{\sqrt{2\pi}e^{m\alpha_\kappa(\sigma)}}{\sqrt{m\alpha_\kappa''(\sigma)}} \left(1 + \frac{2^{3/2}\sigma\sqrt{m\alpha_\kappa''(\sigma)}}{\sqrt{\pi}(m\kappa-2)1.25^{m\kappa[\sigma]/2}2^{m\kappa/2}} + \frac{2^{3/2}e^{-m^{1/3}D^2/4}}{\sqrt{\pi}m^{1/6}D} + \frac{e^{D^4/(4m^{1/3}\kappa)}-1}{D^4/(4m^{1/3}\kappa)}\frac{3}{4m\kappa} \right),$$

where $[\sigma]$ again denotes the greatest integer less than or equal to σ. If $m \geq 1000$, then

$$\int_{-\infty}^{\infty} \left|e^{m\alpha_\kappa(\sigma+it)}\right| \mathrm{d}t < \frac{\sqrt{2\pi}e^{m\alpha_\kappa(\sigma)}}{\sqrt{m\alpha_\kappa''(\sigma)}} \cdot 1.00205\,.$$

Proof. The first claim essentially comes from Lemma 5.4 of Friedman and Skoruppa, which estimates

$$\int_{-\infty}^{\infty} e^{m(\alpha_\kappa(\sigma+it)-iyt)}\,\mathrm{d}t$$

for $y = \alpha_\kappa'(\sigma)$. However, that lemma has two extra terms which are not needed here. Friedman and Skoruppa bound the integral over $|t| \geq \delta := D/(m^{1/3}\sqrt{\alpha_\kappa''(\sigma)})$ by replacing $e^{m(\alpha_\kappa(\sigma+it)-iyt)}$ with $|e^{m\alpha_\kappa(\sigma+it)}|$, so that part of the argument works in this case without any change. That is,

$$\int_{|t|>\delta} \left|e^{m\alpha_\kappa(\sigma+it)}\right| \mathrm{d}t \leq \frac{\sqrt{2\pi}e^{m\alpha_\kappa(\sigma)}}{\sqrt{m\alpha_\kappa''(\sigma)}} \left(\frac{2^{3/2}\sigma\sqrt{m\alpha_\kappa''(\sigma)}}{\sqrt{\pi}(m\kappa-2)1.25^{m\kappa[\sigma]/2}2^{m\kappa/2}} + \frac{2^{3/2}e^{-m^{1/3}D^2/4}}{\sqrt{\pi}m^{1/6}D} \right),$$

Next, we have

$$\begin{aligned}\int_{-\delta}^{\delta} &\left|e^{m\alpha_\kappa(\sigma+it)-m\alpha_\kappa(\sigma)}\right| \,\mathrm{d}t \\ &= \int_{-\delta}^{\delta} e^{-\frac{1}{2}m\alpha_\kappa''(\sigma)t^2}\,\mathrm{d}t + \int_{-\delta}^{\delta}\left(\left|e^{m\alpha_\kappa(\sigma+it)-m\alpha_\kappa(\sigma)}\right| - e^{-\frac{1}{2}m\alpha_\kappa''(\sigma)t^2}\right)\mathrm{d}t \\ &< \frac{\sqrt{2\pi}}{\sqrt{m\alpha_\kappa''(\sigma)}} + \int_{-\delta}^{\delta}\left(\left|e^{m\alpha_\kappa(\sigma+it)-m\alpha_\kappa(\sigma)}\right| - e^{-\frac{1}{2}m\alpha_\kappa''(\sigma)t^2}\right)\mathrm{d}t\,.\end{aligned}$$

Note that

$$\left|e^{m\alpha_\kappa(\sigma+it)-e^{m\alpha_\kappa(\sigma)}}\right| - e^{-\frac{1}{2}m\alpha_\kappa''(\sigma)t^2} = \left(e^{m\,\mathrm{Re}(\rho(t))}-1\right)e^{-\frac{1}{2}m\alpha_\kappa''(\sigma)t^2},$$

where $\rho(t) = \alpha_\kappa(\sigma+it) - iyt + \frac{1}{2}\alpha_\kappa''(\sigma)t^2$. Therefore we can bound the last integral in the same way that Friedman and Skoruppa bounded the integral of $(e^{m\rho(t)}-1)e^{-\frac{1}{2}m\alpha_\kappa''(\sigma)t^2}$, except that we do not get a term coming from $\mathrm{Im}(\rho)$. We conclude that

$$\int_{-\delta}^{\delta}\left(\left|e^{m\alpha_\kappa(\sigma+it)-m\alpha_\kappa(\sigma)}\right| - e^{-\frac{1}{2}m\alpha_\kappa''(\sigma)t^2}\right)\mathrm{d}t < \frac{\sqrt{2\pi}}{\sqrt{m\alpha_\kappa''(\sigma)}}\,\frac{e^{D^4/(4m^{1/3}\kappa)}-1}{D^4/(4m^{1/3}\kappa)}\,\frac{3}{4m\kappa}\,.$$

Now suppose $m \geq 1000$. Then

$$\begin{aligned}\frac{2^{3/2}\sigma\sqrt{m\alpha_\kappa''(\sigma)}}{\sqrt{\pi}(m\kappa-2)1.25^{m\kappa[\sigma]/2}2^{m\kappa/2}} &= \frac{4}{m\kappa^{3/2}\sqrt{\pi}}\,\frac{\sqrt{m\kappa/2}}{2^{m\kappa/2}}\,\frac{m\kappa}{m\kappa-2}\,\frac{\sqrt{\sigma^2\alpha_\kappa''(\sigma)}}{1.25^{m\kappa[\sigma]/2}} \\ &< \frac{10^{-76}}{m}\,;\end{aligned}\tag{4.2}$$

bounds for the first three terms are obvious, and the last term is addressed by Lemma 4.1. Thus

$$\begin{aligned}\int_{-\infty}^{\infty}&\left|e^{m\alpha_\kappa(\sigma+it)}\right|\,\mathrm{d}t \\ &< \frac{\sqrt{2\pi}e^{m\alpha_\kappa(\sigma)}}{\sqrt{m\alpha_\kappa''(\sigma)}}\left(1 + \frac{10^{-76}}{m} + \frac{2^{3/2}e^{-m^{1/3}D^2/4}}{\sqrt{\pi}m^{1/6}D} + \frac{e^{D^4/(2m^{1/3})}-1}{D^4/(2m^{1/3})}\,\frac{3}{2m}\right).\end{aligned}$$

Set $D = 1.76$. The quantity in parentheses is decreasing in m for $m > 0$, so the claimed bound follows by substituting $m = 1000$. □

Lemma 4.5. *Suppose* $m \geq 1000$. *Let* $0 < D \leq m^{1/3}\sqrt{\kappa}$ *be given, and again set* $\delta = D/(m^{1/3}\sqrt{\alpha_\kappa''(\sigma)})$. *Then*

$$\int_{|t|>\delta}\left|e^{m\alpha_\kappa(\sigma+it)}\right|\,\mathrm{d}t < \frac{\sqrt{2\pi}e^{m\alpha_\kappa(\sigma)}}{\sqrt{m\alpha_\kappa''(\sigma)}}\,\frac{1}{m}\cdot\left(10^{-76} + \frac{2^{3/2}m^{5/6}e^{-m^{1/3}D^2/4}}{\sqrt{\pi}D}\right).$$

Proof. This is immediate from Lemma 4.3 and inequality (4.2). □

Lemma 4.6. *Let $C > 0$ be given. Then*

$$\int_{-\infty}^{\infty} e^{-Ct^2}\,\mathrm{d}t = \sqrt{\pi}C^{-1/2}, \qquad \int_{0}^{\infty} e^{-Ct^2}t\,\mathrm{d}t = \frac{1}{2}C^{-1},$$
$$\int_{0}^{\infty} e^{-Ct^2}t^3\,\mathrm{d}t = \frac{1}{2}C^{-2}.$$

Proof. Make the substitution $u = Ct^2$. For any $n \geq 0$, we have

$$\int_{0}^{\infty} e^{-Ct^2}t^n\,\mathrm{d}t = \int_{0}^{\infty} e^{-u}(u/C)^{(n+1)/2}\,\frac{\mathrm{d}u}{2u} = \frac{1}{2}\Gamma\Big(\frac{n+1}{2}\Big)C^{-(n+1)/2}. \quad \square$$

Lemma 4.7. *Let any $\sigma > 0$ and $0 \leq \kappa \leq 1$ be given. Define $\rho = \rho_{\kappa,\sigma} : \mathbb{R} \to \mathbb{C}$ by*

$$\rho(t) = \alpha_\kappa(\sigma + it) - \alpha_\kappa(\sigma) - i\alpha'_\kappa(\sigma)t + \frac{1}{2}\alpha''_\kappa(\sigma)t^2\,;$$

i.e., ρ is the error in the degree-2 Taylor approximation to $\alpha_\kappa(\sigma+it)$. Then for any $t \in \mathbb{R}$,

$$|\mathrm{Im}(\rho(t))| \leq -\frac{\alpha_\kappa^{(3)}(\sigma)}{3!}|t|^3, \qquad |\mathrm{Re}(\rho(t))| \leq \frac{\alpha_\kappa^{(4)}(\sigma)}{4!}t^4\,. \tag{4.3}$$

If $|t| \leq \sigma$, then

$$0 \leq \mathrm{Re}(\rho(t)) \leq \frac{\alpha''_\kappa(\sigma)}{4}t^2\,; \tag{4.4}$$

if $|t| \leq \frac{\sigma}{3\sqrt{2}}$, then

$$0 \leq \mathrm{Re}(\rho(t)) \leq \frac{\alpha''_\kappa(\sigma)}{72}t^2\,.$$

Furthermore, $\mathrm{Im}(\alpha_\kappa(\sigma+it))$ *is odd and* $\mathrm{Re}(\alpha_\kappa(\sigma+it))$ *is even as a function of t. Thus* $\mathrm{Im}(\rho(t))$ *is an odd function and* $\mathrm{Re}(\rho(t))$ *is an even function.*

Proof. The odd/even statement is proven by Friedman and Skoruppa, as well as the fact that $\mathrm{Re}(\rho(t)) \geq 0$ for $|t| \leq \sigma$. See the proof of their Lemma 5.1.

From (4.1), we know that $|\alpha_\kappa^{(3)}(\sigma + it)|$ and $|\alpha_\kappa^{(4)}(\sigma + it)|$ (considered as functions of t) are both maximized at $t = 0$, with $\alpha_\kappa^{(3)}(\sigma) < 0$ and $\alpha_\kappa^{(4)}(\sigma) > 0$. Now apply the Taylor remainder theorem to $\mathrm{Im}(\rho(t))$: since

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\,\mathrm{Im}(\alpha_\kappa(\sigma + it)) = -\,\mathrm{Im}(\alpha_\kappa^{(3)}(\sigma + it))\,,$$

we see that for any $t \in \mathbb{R}$, there exists θ_t between 0 and t such that

$$|\mathrm{Im}(\rho(t))| = \left|\frac{-\,\mathrm{Im}(\alpha_\kappa^{(3)}(\sigma + i\theta_t))}{3!}t^3\right| \leq -\frac{\alpha_\kappa^{(3)}(\sigma)}{3!}|t|^3\,.$$

This proves (4.3) for the imaginary part; the proof for the real part is identical.

For any $\sigma > 0$, (4.1) shows that

$$\frac{\sigma^2\Psi^{(3)}(\sigma)}{4!} = \frac{1}{4}\sum_{k=0}^{\infty}\frac{\sigma^2}{(\sigma+k)^4} < \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{(\sigma+k)^2} = \frac{\Psi'(\sigma)}{4}.$$

It follows that $\sigma^2\alpha_\kappa^{(4)}(\sigma)/4! < \alpha_\kappa''(\sigma)/4$. Thus for $|t| \le \sigma$,

$$|\mathrm{Re}(\rho(t))| \le \frac{\alpha_\kappa^{(4)}(\sigma)}{4!}t^4 \le \frac{\sigma^2\alpha_\kappa^{(4)}(\sigma)}{4!}t^2 \le \frac{\alpha_\kappa''(\sigma)}{4}t^2.$$

The same argument works for $|t| \le \frac{\sigma}{3\sqrt{2}}$. □

Lemma 4.8. *Let $R > 0$ be given. Then for any $0 \le u \le R$ and any $v \in \mathbb{R}$, we have*

$$\left|\mathrm{Re}(e^{u+iv} - 1)\right| \le u\frac{e^R - 1}{R} + \frac{v^2}{2}.$$

Proof. This is inequality (5.11) from Friedman and Skoruppa. □

Lemma 4.9. *For $m \ge 1000$,*

$$\int_{|t|>\frac{\sigma}{3\sqrt{2}}} \left|te^{m\alpha_\kappa(\sigma+it)}\right| \,\mathrm{d}t < \frac{0.0002557\sigma e^{m\alpha_\kappa(\sigma)}}{m^{3/2}\sqrt{\alpha_\kappa''(\sigma)}}.$$

Proof. By Lemma 4.2, since $\kappa \ge 1/2$,

$$\begin{aligned}\int_{|t|>\frac{\sigma}{3\sqrt{2}}} \left|te^{m\alpha_\kappa(\sigma+it)}\right| \,\mathrm{d}t &\le 2\cdot\frac{\frac{19}{18}\sigma^2 e^{m\alpha_\kappa(\sigma)}}{(m\kappa-2)(\frac{73}{72})^{m\kappa[\sigma]/2}(\frac{19}{18})^{m\kappa/2}}\\ &= \frac{19}{9(\kappa-\frac{2}{m})}\cdot\frac{\sqrt{m}}{(\frac{19}{18})^{m\kappa/2}}\cdot\frac{\sqrt{\sigma^2\alpha_\kappa''(\sigma)}}{(\frac{73}{72})^{m\kappa[\sigma]/2}}\cdot\frac{\sigma e^{m\alpha_\kappa(\sigma)}}{m^{3/2}\sqrt{\alpha_\kappa''(\sigma)}}.\end{aligned}$$

We have $\sqrt{m}/(19/18)^{m\kappa/2} \le \sqrt{m}/(19/18)^{m/4} \le \sqrt{1000}/(19/18)^{1000/4} < 0.0000853$. Combining this with Lemma 4.1, we conclude that

$$\begin{aligned}\int_{|t|>\frac{\sigma}{3\sqrt{2}}} \left|te^{m\alpha_\kappa(\sigma+it)}\right| \,\mathrm{d}t &< \frac{19}{9(\frac{1}{2}-\frac{2}{1000})}\cdot 0.0000853\cdot\sqrt{2}\cdot\frac{\sigma e^{m\alpha_\kappa(\sigma)}}{m^{3/2}\sqrt{\alpha_\kappa''(\sigma)}}\\ &< \frac{0.0002557\sigma e^{m\alpha_\kappa(\sigma)}}{m^{3/2}\sqrt{\alpha_\kappa''(\sigma)}}.\end{aligned}$$ □

Lemma 4.10. *For $m \ge 1000$,*

$$\int_{|t|>\frac{\sigma}{3\sqrt{2}}} \left|e^{m\alpha_\kappa(\sigma+it)}\right| \,\mathrm{d}t < \frac{0.00003429 e^{m\alpha_\kappa(\sigma)}}{m\sqrt{\alpha_\kappa''(\sigma)}}.$$

Proof. The argument is the same as the proof of Lemma 4.9; by Lemma 4.2,

$$\begin{aligned}\int_{|t|>\frac{\sigma}{3\sqrt{2}}} \left|e^{m\alpha_\kappa(\sigma+it)}\right| \, \mathrm{d}t &\le 2\cdot \frac{\frac{19\sqrt{2}}{6}\sigma}{(m\kappa-2)(\frac{73}{72})^{m\kappa[\sigma]/2}(\frac{19}{18})^{m\kappa/2}} \\ &= \frac{19\sqrt{2}}{3(\kappa-\frac{2}{m})}\cdot\frac{1}{(\frac{19}{18})^{m\kappa/2}}\cdot\frac{\sqrt{\sigma^2\alpha''_\kappa(\sigma)}}{(\frac{73}{72})^{m\kappa[\sigma]/2}}\cdot\frac{1}{m\sqrt{\alpha''_\kappa(\sigma)}} \\ &< \frac{19\sqrt{2}}{3(\frac{1}{2}-\frac{2}{1000})}\cdot\frac{1}{(\frac{19}{18})^{1000/4}}\cdot\sqrt{2}\cdot\frac{1}{m\sqrt{\alpha''_\kappa(\sigma)}} \\ &< \frac{0.00003429}{m\sqrt{\alpha''_\kappa(\sigma)}}. \qquad \square\end{aligned}$$

Lemma 4.11. *For any $m>0$,*

$$\int_{-\frac{\sigma}{3\sqrt{2}}}^{\frac{\sigma}{3\sqrt{2}}} \left|te^{m\alpha_\kappa(\sigma+it)}\right| \, \mathrm{d}t < \frac{\frac{72}{35}e^{m\alpha_\kappa(\sigma)}}{m\alpha''_\kappa(\sigma)}.$$

Proof. Lemma 4.7 shows that, for $|t|\le\sigma/(3\sqrt{2})$,

$$\left|e^{m\alpha_\kappa(\sigma+it)}\right| \le e^{m\alpha_\kappa(\sigma)-m\alpha''_\kappa(\sigma)t^2/2+m\alpha''_\kappa(\sigma)t^2/72} = e^{m\alpha_\kappa(\sigma)-(35/72)m\alpha''_\kappa(\sigma)t^2}.$$

Hence

$$\int_{-\frac{\sigma}{3\sqrt{2}}}^{\frac{\sigma}{3\sqrt{2}}} \left|te^{m\alpha_\kappa(\sigma+it)}\right| \, \mathrm{d}t < 2\int_0^\infty te^{m\alpha_\kappa(\sigma)-(35/72)m\alpha''_\kappa(\sigma)t^2}\,\mathrm{d}t = \frac{\frac{72}{35}e^{m\alpha_\kappa(\sigma)}}{m\alpha''_\kappa(\sigma)}. \quad \square$$

Lemma 4.12. *For any $\sigma>0$, $\alpha''_{1/2}(\sigma/\sqrt{2})>\alpha''_1(\sigma)=\Psi'(\sigma)$.*

Proof. The duplication formula for Ψ [1, 6.3.8] says that

$$\alpha'_{1/2}(\sigma) = \frac{1}{2}\Psi(\sigma)+\frac{1}{2}\Psi\left(\sigma+\frac{1}{2}\right) = \Psi(2\sigma)-\log 2.$$

Differentiating, $\alpha''_{1/2}(\sigma)=2\Psi'(2\sigma)$. Thus we must prove that $2\Psi'(\sqrt{2}\sigma)>\Psi'(\sigma)$. Estimating the sum (4.1) by integrals, we find that

$$\frac{1}{\sigma^2}+\frac{1}{(\sigma+1)} < \Psi'(\sigma) < \frac{1}{\sigma^2}+\frac{1}{(\sigma+1)^2}+\frac{1}{\sigma+1}.$$

Hence

$$\begin{aligned}2\Psi'(\sqrt{2}\sigma)-\Psi'(\sigma) &> 2\left(\frac{1}{2\sigma^2}+\frac{1}{(\sqrt{2}\sigma+1)}\right)-\left(\frac{1}{\sigma^2}+\frac{1}{(\sigma+1)^2}+\frac{1}{\sigma+1}\right) \\ &= \frac{(2-\sqrt{2})\sigma^2+(3-2\sqrt{2})\sigma}{(\sigma+1)^2(\sqrt{2}\sigma+1)} \\ &> 0. \qquad \square\end{aligned}$$

5. Estimation of f and f'/f

Let $m > 0$ be given. (We will primarily be interested in $m \geq 1000$, but we will note which lemmas hold for all $m > 0$ and which require $m \geq 1000$.) Let $\frac{1}{2} \leq k_{ij} \leq 1$ be given for $i \in \{1, 2\}$ and $j \in \{3, 4\}$, and let $\alpha = \alpha_{\vec{k}}$ as in (3.4). Also let $\vec{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ be given. For $s = (s_1, s_2, s_3) \in \mathcal{R}$, where $\mathcal{R}$ was defined in (3.3), define

$$g(s) = -\vec{y} \cdot s + \alpha(s)$$

and

$$G(s) = \exp(mg(s)).$$

For a given $a \in L^*$ and $y \in \mathbb{R}$, if we take $\vec{y} = (2(a_{w_1} + y), 2(a_{w_2} + y), 2(a_{w_3} + y))$, then $g = g_{y,a}$. Let $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ be the unique point in $\mathcal{R} \cap \mathbb{R}^3$ at which

$$\frac{\partial g}{\partial s_1} = \frac{\partial g}{\partial s_2} = \frac{\partial g}{\partial s_3} = 0;$$

$\vec{\sigma}$ exists by Lemma 3.2. We are interested in

$$\begin{aligned} f &= \frac{1}{(2\pi i)^3} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \int_{\sigma_3 - i\infty}^{\sigma_3 + i\infty} G(s)\, \mathrm{d}s_3\, \mathrm{d}s_2\, \mathrm{d}s_1 \\ &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\vec{\sigma} + i\vec{t})\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 . \end{aligned}$$

To simplify notation, define

$$A_{ij} = \alpha''_{k_{ij}}(\sigma_{ij}) \tag{5.1}$$

and $\Pi_{ij} = A_{13}A_{23}A_{14}A_{24}/A_{ij}$. (Recall from (3.2) the definition of σ_{ij}.) Let $A_{\min} = \min(A_{ij})$, and let $\Pi_{\max} = \max(\Pi_{ij}) = A_{13}A_{23}A_{14}A_{24}/A_{\min}$. (Without loss of generality, we will assume that $A_{23} = A_{\min}$ and $A_{24} \leq A_{13}$ wherever this helps.[1]) Define

$$P = \Pi_{13} + \Pi_{23} + \Pi_{14} + \Pi_{24};$$

i.e., $P = P_3(A_{13}, A_{14}, A_{23}, A_{24})$, where P_3 is the degree-3 elementary symmetric polynomial.

[1] Note that the only relation among t_{13}, t_{14}, t_{23}, and t_{24} is $t_{13} + t_{24} = t_{14} + t_{23}$. Since this is symmetric, we are free to choose any A_{ij} as $A_{\min}$. After choosing $A_{23} = A_{\min}$, we are still free to swap A_{13} and A_{24} if necessary.

Define

$$H(\vec{t}) := \exp\left(m\left(g(\vec{\sigma}) - \sum_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} A_{ij}t_{ij}^2/2\right)\right)$$
$$= G(\vec{\sigma})\exp\left(-m\sum_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} A_{ij}t_{ij}^2/2\right).$$

The idea is that $H(\vec{t})$ is a good approximation to $G(\vec{\sigma}+i\vec{t})$; we obtain $H(\vec{t})$ from $G(s) = \exp(mg(s))$ by replacing each $\alpha_{k_{ij}}(s_{ij})$ in $g(s) = -\vec{y}\cdot s + \sum \alpha_{k_{ij}}(s_{ij})$ with its degree-two Taylor approximation (as a function of t_{ij}). The fact that $\vec{\sigma}$ is a critical point ensures that the linear terms cancel. The main term in our estimate for f comes from integrating H:

Lemma 5.1. *We have*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{P}}.$$

Proof. It is well-known (see, e.g., [6], page 71) that if $A = (a_{ij})$ is an $n\times n$ positive-definite symmetric matrix, then

$$\int_{T\in\mathbb{R}^n} \exp\Big(-\frac{1}{2}\sum a_{ij}T_iT_j\Big)\,\mathrm{d}T = (2\pi)^{n/2}\det(A)^{-1/2}.$$

In this case,

$$\det(A) = m^3\det\begin{pmatrix} A_{13}+A_{14} & 0 & A_{13}\\ 0 & A_{23}+A_{24} & A_{23}\\ A_{13} & A_{23} & A_{13}+A_{23}\end{pmatrix} = m^3P. \qquad \square$$

As a simple consequence of this lemma, we can evaluate some other integrals which will be useful later:

Lemma 5.2. *Let any $i\in\{1,2\}$ and $j\in\{3,4\}$ be given. Then*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\vec{t})t_{ij}^4\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{P}}\frac{3}{m^2A_{ij}^2}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\vec{t})t_{ij}^6\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{P}}\frac{15}{m^3A_{ij}^3}.$$

Proof. First observe that

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\vec{t})t_{ij}^2\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\ &= -\frac{2}{m}\frac{\partial}{\partial A_{ij}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\ &= -\frac{2}{m}\frac{\partial}{\partial A_{ij}}\frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{P}} \\ &= \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{5/2}P^{3/2}}\frac{\partial P}{\partial A_{ij}}.\end{aligned}$$

Note that $\partial P/\partial A_{ij}$ does not depend on A_{ij}; for example, $\partial P/\partial A_{13} = A_{14}A_{23} + A_{14}A_{24} + A_{23}A_{24}$. Hence

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\vec{t})t_{ij}^4\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\ &= -\frac{2}{m}\frac{\partial}{\partial A_{ij}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\vec{t})t_{ij}^2\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\ &= -\frac{2}{m}\frac{\partial}{\partial A_{ij}}\left(\frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{5/2}P^{3/2}}\frac{\partial P}{\partial A_{ij}}\right) \\ &= 3\frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{7/2}P^{5/2}}\left(\frac{\partial P}{\partial A_{ij}}\right)^2,\end{aligned}$$

so the first claim will follow once we check that

$$\frac{\partial P/\partial A_{ij}}{P} < \frac{1}{A_{ij}}. \tag{5.2}$$

This is obvious; for example,

$$A_{13}\frac{\partial P}{\partial A_{13}} = A_{13}A_{14}A_{23} + A_{13}A_{14}A_{24} + A_{13}A_{23}A_{24} < P.$$

The second claim is proven identically. □

As in Lemma 4.5, choose D such that $0 < D \leq 1000^{1/3}/\sqrt{2}$. (This ensures $D \leq m^{1/3}\sqrt{k_{ij}}$ for all i, j when $m \geq 1000$.) Define

$$\delta_{ij} = \frac{D}{m^{1/3}\sqrt{A_{ij}}}.$$

Let $\Delta \subseteq \mathbb{R}^3$ denote the set

$$\Big\{(t_1, t_2, t_3) \in \mathbb{R}^3 \;\Big|\; |t_{14}| \leq \delta_{14}, |t_{24}| \leq \delta_{24}, |t_{13}| \leq \delta_{13}\Big\}.$$

Recall that we want to prove that the integrals of G and H have the same asymptotic behavior. We will do so by showing that H is a good approximation to G inside of Δ, and that the contributions to the integrals outside of Δ are (asymptotically) negligible.

Lemma 5.3. *Suppose $m \geq 1000$. Then*

$$\iiint_{\mathbb{R}^3\setminus\Delta} |G(\vec{\sigma}+i\vec{t})| \,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{\Pi_{23}}}\frac{1}{m}\cdot 3.013\left(10^{-76}+\frac{2^{3/2}m^{5/6}e^{-m^{1/3}D^2/4}}{\sqrt{\pi}D}\right).$$

Proof. Recall that

$$|\Gamma(\sigma+it)| \leq \Gamma(\sigma) \qquad \text{for any } \sigma>0 \text{ and } t\in\mathbb{R}. \tag{5.3}$$

Hence for any $s \in \mathcal{R}$ with $\mathrm{Re}(s_{23}) = \sigma_{23}$, we have

$$|G(s)| \leq e^{m\alpha_{k_{23}}(\sigma_{23})}|\exp(m(-\vec{y}\cdot s+\alpha_{k_{13}}(s_{13})+\alpha_{k_{14}}(s_{14})+\alpha_{k_{24}}(s_{24})))|. \tag{5.4}$$

We can bound the triple integral of the right-hand side by splitting it into three single integrals. (We will use this strategy several more times in this section.) In order to do so, we will need to change variables from (t_1, t_2, t_3) to (t_{14}, t_{24}, t_{13}), so that the right-hand side of (5.4) becomes a product of three single-variable functions. The change-of-variable matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

has determinant 1, so the substitution does not introduce a Jacobian factor. Using Lemmas 4.4 and 4.5 to bound the resulting single integrals, we find that

$$\int_{|t_{14}|>\delta_{14}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(\vec{\sigma}+i\vec{t})| \,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{\Pi_{23}}}\frac{1}{m}\cdot 1.00205^2\left(10^{-76}+\frac{2^{3/2}m^{5/6}\exp(-m^{1/3}D^2/4)}{\sqrt{\pi}D}\right).$$

We get the same bound for $\int_{-\infty}^{\infty}\int_{|t_{24}|>\delta_{24}}\int_{-\infty}^{\infty}$ and for $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{|t_{13}|>\delta_{13}}$. Note that $3\cdot 1.00205^2 < 3.013$. ☐

Lemma 5.4. *We have*

$$\iiint_{\mathbb{R}^3\setminus\Delta} H(\vec{t}) \,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{\Pi_{23}}}\frac{1}{m}\cdot 3\sqrt{\frac{2}{\pi}}\frac{m^{5/6}e^{-m^{1/3}D^2/2}}{D}.$$

Proof. Arguing as in the proof of Lemma 5.3, Lemmas 4.3 and 4.6 show that

$$\int_{|t_{14}|>\delta_{14}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(\vec{t})\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1$$
$$\le G(\vec{\sigma}) \left(\frac{2e^{-m^{1/3}D^2/2}}{m^{2/3} D \sqrt{A_{14}}} \right) \sqrt{\frac{2\pi}{mA_{24}}} \sqrt{\frac{2\pi}{mA_{13}}}$$
$$= \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{\Pi_{23}}} \frac{1}{m} \cdot \sqrt{\frac{2}{\pi}} \frac{m^{5/6} e^{-m^{1/3}D^2/2}}{D}.$$

We get the same bound for $\int_{-\infty}^{\infty} \int_{|t_{24}|>\delta_{24}} \int_{-\infty}^{\infty}$ and for $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{|t_{13}|>\delta_{13}}$. The result follows. □

Define $\rho_{13}(t)$ as in Lemma 4.7, i.e.,

$$\rho_{13}(t) = \alpha_{k_{13}}(\sigma_{13} + it) - \alpha_{k_{13}}(\sigma_{13}) - i\alpha'_{k_{13}}(\sigma_{13})t + \frac{1}{2}\alpha''_{k_{13}}(\sigma_{13})t^2;$$

define ρ_{23}, ρ_{14}, and ρ_{24} similarly. Then define

$$\rho(\vec{t}) = \sum_{\substack{i\in\{1,2\} \\ j\in\{3,4\}}} \rho_{ij}(t_{ij}),$$

so that

$$G(\vec{\sigma} + i\vec{t}) - H(\vec{t}) = H(\vec{t})(e^{m\rho(\vec{t})} - 1).$$

Lemma 5.5. *For any* $\vec{t} \in \mathbb{R}^3$,

$$|\mathrm{Im}(\rho(\vec{t}))| \le \frac{\sqrt{2}}{3} \sum_{\substack{i\in\{1,2\} \\ j\in\{3,4\}}} A_{ij}^{3/2} |t_{ij}|^3, \tag{5.5}$$

$$|\mathrm{Re}(\rho(\vec{t}))| \le \frac{1}{2} \sum_{\substack{i\in\{1,2\} \\ j\in\{3,4\}}} A_{ij}^2 t_{ij}^4. \tag{5.6}$$

Furthermore, if $\vec{t} \in \Delta$ *and* $A_{23} = A_{min}$, *then* $|\mathrm{Re}(\rho(\vec{t}))| \le 42 D^4 m^{-4/3}$.

Proof. By Lemma 4.7,

$$|\mathrm{Im}(\rho(\vec{t}))| \le -\frac{1}{6} \sum_{\substack{i\in\{1,2\} \\ j\in\{3,4\}}} \alpha_{k_{ij}}^{(3)}(\sigma_{ij}) |t_{ij}|^3,$$

$$|\mathrm{Re}(\rho(\vec{t}))| \le \frac{1}{24} \sum_{\substack{i\in\{1,2\} \\ j\in\{3,4\}}} \alpha_{k_{ij}}^{(4)}(\sigma_{ij}) t_{ij}^4.$$

Friedman and Skoruppa proved [8, Lemma 5.2] that for any integer $n \geq 2$, any $0 < \kappa \leq 1$, and any $\sigma > 0$,

$$\frac{|\alpha_\kappa^{(n)}(\sigma)|}{(\alpha_\kappa''(\sigma))^{n/2}} \leq \frac{(n-1)!}{\kappa^{\frac{n}{2}-1}};$$

thus the previous inequalities and $k_{ij} \geq \frac{1}{2}$ imply inequalities (5.5) and (5.6).

When $\vec{t} \in \Delta$,

$$A_{13}^2 t_{13}^4 \leq A_{13}^2 \delta_{13}^4 = A_{13}^2 \left(\frac{D}{m^{1/3}\sqrt{A_{13}}} \right)^4 = \frac{D^4}{m^{4/3}},$$

and similarly for $A_{14}^2 t_{14}^4$ and $A_{24}^2 t_{24}^4$. By Jensen's inequality, we know that for any $n \geq 1$ and any $x, y, z \in \mathbb{R}$,

$$|x+y+z|^n \leq 3^{n-1}(|x|^n + |y|^n + |z|^n).$$

In particular,

$$t_{23}^4 = (t_{13} - t_{14} + t_{24})^4 \leq 27(t_{13}^4 + t_{14}^4 + t_{24}^4).$$

For $A_{23} = A_{\min}$ and $\vec{t} \in \Delta$, it follows that

$$A_{23}^2 t_{23}^4 \leq 27[A_{13}^2 t_{13}^4 + A_{14}^2 t_{14}^4 + A_{24}^2 t_{24}^4] \leq 81 D^4 m^{-4/3}.$$

Then inequality (5.6) says that $|\mathrm{Re}(\rho(\vec{t}))| \leq 42 D^4 m^{-4/3}$. □

It follows from Lemma 4.7 that $\mathrm{Re}(\rho)$ is an even function and $\mathrm{Im}(\rho)$ is an odd function, in the sense that

$$\mathrm{Re}(\rho(t_1, t_2, t_3)) = \mathrm{Re}(\rho(-t_1, -t_2, -t_3)).$$

Thus $H(\vec{t})\,\mathrm{Im}(e^{m\rho(\vec{t})} - 1) = H(\vec{t}) e^{m\,\mathrm{Re}(\rho(\vec{t}))} \sin(m\,\mathrm{Im}(\rho(\vec{t})))$ is odd, so

$$\begin{aligned}\iiint_\Delta (G(\vec{\sigma} + i\vec{t}) - H(\vec{t}))\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 &= \iiint_\Delta H(\vec{t})(e^{m\rho(\vec{t})} - 1)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\ &= \iiint_\Delta H(\vec{t})\,\mathrm{Re}(e^{m\rho(\vec{t})} - 1)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1.\end{aligned}$$

Now we use this fact to bound the integral.

Lemma 5.6. *Assume $A_{23} = A_{min}$. Then*

$$\left| \iiint_\Delta (G(\vec{\sigma} + i\vec{t}) - H(\vec{t}))\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \right| < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} \frac{1}{m} \left(6 \frac{e^R - 1}{R} + \frac{80}{3} \right),$$

where $R = 42 D^4 m^{-1/3}$.

Proof. By Lemmas 5.5 and 4.8,

$$\begin{aligned}\Big|\iiint_\Delta (G(\vec\sigma+i\vec t)-H(\vec t))\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1\Big| \\ &= \Big|\iiint_\Delta H(\vec t)\,\mathrm{Re}(e^{m\rho(\vec t)}-1)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1\Big| \\ &\le \iiint_\Delta H(\vec t)\left(u(\vec t)\frac{e^R-1}{R}+\frac{v(\vec t)^2}{2}\right)\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1\,,\end{aligned}$$

where

$$u(\vec t) = \frac{1}{2}m(A_{13}^2t_{13}^4 + A_{23}^2t_{23}^4 + A_{14}^2t_{14}^4 + A_{24}^2t_{24}^4)\,,$$
$$v(\vec t) = \frac{\sqrt{2}}{3}m(A_{13}^{3/2}|t_{13}|^3 + A_{23}^{3/2}|t_{23}|^3 + A_{14}^{3/2}|t_{14}|^3 + A_{24}^{3/2}|t_{24}|^3)\,.$$

It follows from Lemma 5.2 that

$$\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty H(\vec t)u(\vec t)\frac{e^R-1}{R}\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2}G(\vec\sigma)}{m^{3/2}\sqrt{P}}\cdot\frac{6}{m}\frac{e^R-1}{R}\,.$$

By Jensen's inequality,

$$\frac{v(\vec t)^2}{2} \le \frac{4}{9}m^2(A_{13}^3t_{13}^6 + A_{14}^3t_{14}^6 + A_{23}^3t_{23}^6 + A_{24}^3t_{24}^6)\,.$$

Hence Lemma 5.2 shows that

$$\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty H(\vec t)\frac{v(\vec t)^2}{2}\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2}G(\vec\sigma)}{m^{3/2}\sqrt{P}}\cdot\frac{80/3}{m}\,. \qquad \square$$

Lemma 5.7. *Suppose* $m \ge 1000$. *Then*

$$f = \frac{1}{(2\pi)^3}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty G(\vec\sigma+i\vec t)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{G(\vec\sigma)}{(2\pi)^{3/2}m^{3/2}\sqrt{P}}(1+\varphi)\,,$$

where

$$|\varphi| < \frac{378.1}{m}\,.$$

Proof. Without loss of generality, assume $A_{23} = A_{\min}$. We split up the integral as

$$\iiint G = \iiint H + \iiint_{\mathbb{R}^3\setminus\Delta} G - \iiint_{\mathbb{R}^3\setminus\Delta} H + \iiint_\Delta (G-H)\,.$$

Then Lemma 5.1 gives the main term in the estimate, and Lemmas 5.3, 5.4, and 5.6 provide the error terms. We get

$$m|\varphi| < \sqrt{\frac{P}{\Pi_{23}}}\left\{3.013\left(10^{-76} + \frac{2^{3/2} m^{5/6} \exp(-m^{1/3}D^2/4)}{\sqrt{\pi} D}\right)\right.$$
$$\left. + 3\sqrt{\frac{2}{\pi}}\frac{m^{5/6} e^{-m^{1/3}D^2/2}}{D}\right\} + 6\frac{e^{42D^4 m^{-1/3}} - 1}{42 D^4 m^{-1/3}} + \frac{80}{3}.$$

Since $P/\Pi_{23} = P/\Pi_{\max} \leq 4$, we have

$$m|\varphi| < 2\left\{3.013\left(10^{-76} + \frac{2^{3/2} m^{5/6} \exp(-m^{1/3}D^2/4)}{\sqrt{\pi} D}\right)\right. \tag{5.7}$$
$$\left. + 3\sqrt{\frac{2}{\pi}}\frac{m^{5/6} e^{-m^{1/3}D^2/2}}{D}\right\} + 6\frac{e^{42D^4 m^{-1/3}} - 1}{42 D^4 m^{-1/3}} + \frac{80}{3},$$

Set $D = 1.01$; note that $D < 1000^{1/3}/\sqrt{2}$, so this choice is valid. A simple derivative check shows that the right-hand side of (5.7) is decreasing for $m \geq 1000$. Setting $m = 1000$ yields $|\varphi| < 378.1/m$. □

Next we need to estimate f'/f, where $f' = \partial f/\partial y$ is given by

(5.8)

$$f' = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -2m(s_1 + s_2 + s_3)G(s)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1$$
$$= -2m(\sigma_1 + \sigma_2 + \sigma_3)f - \frac{2im}{(2\pi)^3}\iiint_{\mathbb{R}^3}(t_1 + t_2 + t_3)G(\vec{\sigma} + i\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1.$$

We want to show that

$$-\frac{1}{2m}\frac{f'}{f} \to (\sigma_1 + \sigma_2 + \sigma_3) \qquad \text{as } m \to \infty,$$

so we need to find an upper bound for the error term

$$-\frac{1}{2m}\frac{f'}{f} - (\sigma_1 + \sigma_2 + \sigma_3) \tag{5.9}$$
$$= \frac{i}{(2\pi)^3 f}\iiint_{\mathbb{R}^3}(t_1 + t_2 + t_3)G(\vec{\sigma} + i\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1.$$

Before attempting to bound the integral, we will need a few more lemmas. Recall that A_{ij} was defined in (5.1), and σ_{ij} was defined in (3.2).

Lemma 5.8. *Let $i_0, i_1 \in \{1, 2\}$ and $j_0, j_1 \in \{3, 4\}$ be given such that $A_{i_0 j_0} \leq A_{i_1 j_1}$. Then $\sigma_{i_0 j_0} > \sigma_{i_1 j_1}/\sqrt{2}$.*

Proof. Recall that $\alpha''_\kappa(\sigma)$ is increasing in κ and decreasing in σ. We have

$$\alpha''_{1/2}(\sigma_{i_0j_0}) \le A_{i_0j_0} \le A_{i_1j_1} \le \alpha''_1(\sigma_{i_1j_1}) < \alpha''_{1/2}(\sigma_{i_1j_1}/\sqrt{2}),$$

where we have used Lemma 4.12 for the last inequality. It follows that $\sigma_{i_0j_0} > \sigma_{i_1j_1}/\sqrt{2}$, as claimed. □

Corollary 5.9. *For any $i \in \{1,2\}$ and $j \in \{3,4\}$,*

$$\frac{\sigma_{ij}}{\sigma_1+\sigma_2+\sigma_3} < 1.$$

If $A_{23} \le A_{14}$, then

$$\frac{\sigma_{14}}{\sigma_1+\sigma_2+\sigma_3} < 2-\sqrt{2}.$$

Similarly, if $A_{24} \le A_{13}$, then

$$\frac{\sigma_{13}}{\sigma_1+\sigma_2+\sigma_3} < 2-\sqrt{2}.$$

Proof. The first claim follows from $\sigma_1+\sigma_2+\sigma_3 = \sigma_{13}+\sigma_{24} = \sigma_{14}+\sigma_{23}$. The last two claims are proven identically; we prove the first using Lemma 5.8:

$$\frac{\sigma_{14}}{\sigma_1+\sigma_2+\sigma_3} = \frac{\sigma_{14}}{\sigma_{23}+\sigma_{14}} = \frac{1}{\frac{\sigma_{23}}{\sigma_{14}}+1} < \frac{1}{\frac{1}{\sqrt{2}}+1} = 2-\sqrt{2}. \qquad \square$$

Corollary 5.10. *Assume $A_{23} \le A_{14}$ and $A_{24} \le A_{13}$. Then for any $i \in \{1,2\}$ and $j \in \{3,4\}$,*

$$\frac{1}{\sqrt{A_{ij}}} < \begin{cases} (2\sqrt{2}-2)(\sigma_1+\sigma_2+\sigma_3) & \text{if } i=1, \\ \sqrt{2}(\sigma_1+\sigma_2+\sigma_3) & \text{if } i=2. \end{cases}$$

Proof. It follows from (4.1) that for any $\sigma > 0$ and any $\frac{1}{2} \le \kappa \le 1$, $\sigma^2\alpha''_\kappa(\sigma) \ge \kappa\sigma^2\Psi'(\sigma) > \kappa \ge \frac{1}{2}$. Thus

$$\frac{1}{\sqrt{A_{ij}}}\frac{1}{\sigma_1+\sigma_2+\sigma_3} = \frac{1}{\sqrt{\sigma_{ij}^2\alpha''_{k_{ij}}(\sigma_{ij})}}\frac{\sigma_{ij}}{\sigma_1+\sigma_2+\sigma_3} < \sqrt{2}\frac{\sigma_{ij}}{\sigma_1+\sigma_2+\sigma_3}.$$

Now use the previous corollary. □

Define

$$\Sigma = \left\{\vec{t} \in \mathbb{R}^3 \,\middle|\, |t_{13}| \le \frac{\sigma_{13}}{3\sqrt{2}},\ |t_{14}| \le \frac{\sigma_{14}}{3\sqrt{2}},\ |t_{24}| \le \frac{\sigma_{24}}{3\sqrt{2}}\right\}.$$

We will split the integral (5.9) into an integral over Σ and an integral over $\mathbb{R}^3 \setminus \Sigma$.

Lemma 5.11. *Assume that* $A_{23} = A_{min}$ *and* $A_{24} \le A_{13}$. *Then, for* $m \ge 1000$,

$$\left| \iiint_{\mathbb{R}^3 \setminus \Sigma} (t_1 + t_2 + t_3) G(\vec{\sigma} + i\vec{t}) \, \mathrm{d}t_3 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \right| < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1 + \sigma_2 + \sigma_3) \cdot \frac{0.0008}{m} .$$

Proof. Note that $t_1 + t_2 + t_3 = t_{13} + t_{24}$, so we consider separately the integrals

$$\iiint_{\mathbb{R}^3 \setminus \Sigma} t_{13} G(\vec{\sigma} + i\vec{t}) \, \mathrm{d}t_3 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \quad \text{and} \quad \iiint_{\mathbb{R}^3 \setminus \Sigma} t_{24} G(\vec{\sigma} + i\vec{t}) \, \mathrm{d}t_3 \, \mathrm{d}t_2 \, \mathrm{d}t_1 ,$$

starting with the t_{13} integral. As in the proof of Lemma 5.3, we can bound this integral by splitting it into a product of three single integrals. By Lemmas 4.9 and 4.4,

$$\begin{aligned}
\int_{|t_{13}| > \frac{\sigma_{13}}{3\sqrt{2}}} \left| t_{13} e^{m\alpha_{k_{13}}(\sigma_{13} + it_{13})} \right| \mathrm{d}t_{13} &\le \frac{0.0002557 \sigma_{13} e^{m\alpha_{k_{13}}(\sigma_{13})}}{m^{3/2}\sqrt{A_{13}}} , \\
\int_{-\infty}^{\infty} \left| e^{m\alpha_{k_{24}}(\sigma_{24} + it_{24})} \right| \mathrm{d}t_{24} &\le \frac{1.00205\sqrt{2\pi} e^{m\alpha_{k_{24}}(\sigma_{24})}}{\sqrt{mA_{24}}} , \\
\int_{-\infty}^{\infty} \left| e^{m\alpha_{k_{14}}(\sigma_{14} + it_{14})} \right| \mathrm{d}t_{14} &\le \frac{1.00205\sqrt{2\pi} e^{m\alpha_{k_{14}}(\sigma_{14})}}{\sqrt{mA_{14}}} .
\end{aligned}$$

Combining these inequalities, we see that

$$\iiint_{|t_{13}| > \frac{\sigma_{13}}{3\sqrt{2}}} t_{13} G(\vec{\sigma} + i\vec{t}) \, \mathrm{d}t_3 \, \mathrm{d}t_2 \, \mathrm{d}t_1 < \frac{0.002 \sigma_{13} G(\vec{\sigma})}{m^{5/2}\sqrt{A_{13}A_{24}A_{14}}} .$$

Similarly, Lemmas 4.11, 4.10, and 4.4 show that

$$\begin{aligned}
\iiint_{|t_{13}| \le \frac{\sigma_{13}}{3\sqrt{2}},\ |t_{24}| > \frac{\sigma_{24}}{3\sqrt{2}}} & t_{13} G(\vec{\sigma} + i\vec{t}) \, \mathrm{d}t_3 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \\
&\le \left(\frac{72}{35} \cdot 0.00003429 \cdot 1.00205 \right) \frac{G(\vec{\sigma})}{m^{5/2}\sqrt{A_{13}^2 A_{24} A_{14}}} \\
&< \frac{0.0002 G(\vec{\sigma})}{m^{5/2}\sqrt{A_{13}^2 A_{24} A_{14}}}
\end{aligned}$$

and that the same bound applies for the integral over $|t_{13}| \le \frac{\sigma_{13}}{3\sqrt{2}}$, $|t_{24}| \le \frac{\sigma_{24}}{3\sqrt{2}}$, $|t_{14}| > \frac{\sigma_{14}}{3\sqrt{2}}$. Therefore

$$\begin{aligned}(5.10)\quad \left|\iiint_{\mathbb{R}^3\setminus\Sigma} t_{13} G(\vec{\sigma}+i\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1\right| \\ < \frac{G(\vec{\sigma})}{m^{5/2}\sqrt{A_{13}A_{24}A_{14}}}\left(0.002\sigma_{13} + \frac{0.0004}{\sqrt{A_{13}}}\right).\end{aligned}$$

Corollary 5.9 says that

$$0.002\sigma_{13} < 0.002(2-\sqrt{2})(\sigma_1+\sigma_2+\sigma_3) < 0.002(\sigma_1+\sigma_2+\sigma_3)\,.$$

Corollary 5.10 shows that

$$\frac{0.0004}{\sqrt{A_{13}}} < 0.0004(2\sqrt{2}-2)(\sigma_1+\sigma_2+\sigma_3) < 0.0004(\sigma_1+\sigma_2+\sigma_3)\,.$$

Since $A_{23} = A_{\min}$, we have $A_{13}A_{24}A_{14} = \Pi_{\max} \ge P/4$. Thus

$$\begin{aligned}\left|\iiint_{\mathbb{R}^3\setminus\Sigma} t_{13} G(\vec{\sigma}+i\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1\right| &< (0.002+0.0004)\frac{G(\vec{\sigma})(\sigma_1+\sigma_2+\sigma_3)}{m^{5/2}\sqrt{A_{13}A_{24}A_{14}}} \\ &\le \frac{G(\vec{\sigma})(\sigma_1+\sigma_2+\sigma_3)}{m^{3/2}\sqrt{P}}\cdot\frac{0.0048}{m} \\ &< \frac{(2\pi)^{3/2}G(\vec{\sigma})(\sigma_1+\sigma_2+\sigma_3)}{m^{3/2}\sqrt{P}}\cdot\frac{0.0004}{m}\,.\end{aligned}$$

Now we use the same method to bound the t_{24} integral. The argument used to prove (5.10) works equally well in this case; that is, the t_{24} integral is bounded above by

$$\frac{G(\vec{\sigma})}{m^{5/2}\sqrt{A_{13}A_{24}A_{14}}}\left(0.002\sigma_{24} + \frac{0.0004}{\sqrt{A_{24}}}\right).$$

Corollaries 5.9 and 5.10 show that

$$0.002\sigma_{24} < 0.002(\sigma_1+\sigma_2+\sigma_3)\,,$$

$$\frac{0.0004}{\sqrt{A_{24}}} < 0.0004\sqrt{2}(\sigma_1+\sigma_2+\sigma_3) < 0.0006(\sigma_1+\sigma_2+\sigma_3)\,.$$

Since $(2\pi)^{-3/2}\cdot 2(0.002+0.0006) < 0.0004$, this proves that

$$\left|\iiint_{\mathbb{R}^3\setminus\Sigma} t_{24} G(\vec{\sigma}+i\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1\right| < \frac{(2\pi)^{3/2}G(\vec{\sigma})(\sigma_1+\sigma_2+\sigma_3)}{m^{3/2}\sqrt{P}}\cdot\frac{0.0004}{m}\,.$$

□

It remains to consider the integral over Σ. The following lemma describes the behavior of the integrand for $\vec{t} \in \Sigma$.

Lemma 5.12. *Assume* $A_{23} = A_{min}$*. Let* $\vec{t} \in \Sigma$ *be given. Then*

$$|\mathrm{Re}(\rho_{23}(t_{23}))| \leq \frac{\alpha''_{k_{23}}(\sigma_{23})}{4} t_{23}^2 ,$$

and for $(i,j) \neq (2,3)$*, we have*

$$|\mathrm{Re}(\rho_{ij}(t_{ij}))| \leq \frac{\alpha''_{k_{ij}}(\sigma_{ij})}{72} t_{ij}^2 .$$

Proof. It follows from Lemma 5.8 that $|t_{23}| < \sigma_{23}$:

$$|t_{23}| = |t_{13} + t_{24} - t_{14}| \leq |t_{13}| + |t_{24}| + |t_{14}| \leq \frac{1}{3}\left(\frac{\sigma_{13}}{\sqrt{2}} + \frac{\sigma_{24}}{\sqrt{2}} + \frac{\sigma_{14}}{\sqrt{2}}\right) < \sigma_{23} .$$

Now use Lemma 4.7. □

Recall that $\mathrm{Re}(\rho)$ is an even function and $\mathrm{Im}(\rho)$ is an odd function, so

$$\begin{aligned}
&\iiint_\Sigma (t_1 + t_2 + t_3) G(\vec{\sigma} + i\vec{t})\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \\
&\qquad = \iiint_\Sigma (t_1 + t_2 + t_3) H(\vec{t}) e^{m\rho(\vec{t})}\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \\
&\qquad = \iiint_\Sigma (t_1 + t_2 + t_3) H(\vec{t}) e^{m\,\mathrm{Re}(\rho)} \cdot \\
&\qquad\qquad\qquad [\cos(m(\mathrm{Im}(\rho))) + i \sin(m\,\mathrm{Im}(\rho))]\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 \\
&\qquad = i \iiint_\Sigma (t_1 + t_2 + t_3) H(\vec{t}) e^{m\,\mathrm{Re}(\rho)} \sin(m\,\mathrm{Im}(\rho))\, \mathrm{d}t_3\, \mathrm{d}t_2\, \mathrm{d}t_1 . \qquad (5.11)
\end{aligned}$$

By Lemma 5.5,

$$|\sin(m\,\mathrm{Im}(\rho(\vec{t})))| \leq |m\,\mathrm{Im}(\rho(\vec{t}))| \leq \frac{\sqrt{2}}{3} m \sum_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} A_{ij}^{3/2} |t_{ij}|^3 .$$

Lemma 5.12 shows that for $\vec{t} \in \Sigma$,

$$|\mathrm{Re}(\rho(\vec{t}))| \leq \frac{1}{72}(A_{13}t_{13}^2 + A_{14}t_{14}^2 + A_{24}t_{24}^2) + \frac{1}{4} A_{14} t_{14}^2 .$$

Hence $H(\vec{t}) e^{m\,\mathrm{Re}(\rho)} \leq \tilde{H}(\vec{t})$, where we define

$$\tilde{H}(\vec{t}) = G(\vec{\sigma}) \exp\left(-m\left(\frac{35}{72} A_{13} t_{13}^2 + \frac{35}{72} A_{14} t_{14}^2 + \frac{35}{72} A_{24} t_{24}^2 + \frac{1}{4} A_{23} t_{23}^2\right)\right).$$

Substituting these inequalities into (5.11), we find that

$$(5.12)\quad \left|\iiint_{\Sigma}(t_1+t_2+t_3)G(\vec{\sigma}+i\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1\right|$$
$$\leq \frac{\sqrt{2}}{3}m\iiint_{\Sigma}|t_1+t_2+t_3|\left(\sum_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} A_{ij}^{3/2}|t_{ij}|^3\right)\tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1\,.$$

Note that $\tilde{H}(\vec{t})$ is obtained from $H(\vec{t})$ by replacing A_{23} with $\frac{1}{2}A_{23}$ and the other A_{ij}'s with $\frac{35}{36}A_{ij}$, so our previous lemmas about H also apply to $\tilde{H}$. In particular, P is replaced by

$$\tilde{P} := P_3\Big(\frac{35}{36}A_{13}, \frac{35}{36}A_{14}, \frac{35}{36}A_{24}, \frac{1}{2}A_{23}\Big).$$

We will need to know how $\tilde{P}$ compares to P.

Lemma 5.13. *Assume $A_{23}=A_{min}$. Then $\tilde{P} > 0.5841P$.*

Proof. The assumption that $A_{23}=A_{\min}$ ensures that $\Pi_{23}=A_{13}A_{14}A_{24}\geq P/4$. Thus

$$\begin{aligned}\tilde{P} &= \frac{1}{2}\left(\frac{35}{36}\right)^2(\Pi_{13}+\Pi_{14}+\Pi_{24}) + \left(\frac{35}{36}\right)^3\Pi_{23}\\ &= \frac{1}{2}\left(\frac{35}{36}\right)^2 P + \left(\left(\frac{35}{36}\right)^3 - \frac{1}{2}\left(\frac{35}{36}\right)^2\right)\Pi_{23}\\ &\geq \left(\frac{1}{2}\left(\frac{35}{36}\right)^2 + \frac{1}{4}\left(\left(\frac{35}{36}\right)^3 - \frac{1}{2}\left(\frac{35}{36}\right)^2\right)\right)P > 0.5841P\,. \qquad \square\end{aligned}$$

Now we can bound the relevant integrals.

Lemma 5.14. *Assume $A_{23}=A_{min}$ and $A_{24}\leq A_{13}$. Then*

$$\begin{aligned} A_{23}^{3/2}\iiint_{\mathbb{R}^3} t_{23}^4\tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 &< \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{P}}(\sigma_1+\sigma_2+\sigma_3)\cdot\frac{22.206}{m^2}\,,\\ A_{14}^{-1/2}A_{23}^2\iiint_{\mathbb{R}^3} t_{23}^4\tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 &< \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{P}}(\sigma_1+\sigma_2+\sigma_3)\cdot\frac{13.008}{m^2}\,,\\ A_{13}^{3/2}\iiint_{\mathbb{R}^3} t_{13}^4\tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 &< \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{P}}(\sigma_1+\sigma_2+\sigma_3)\cdot\frac{3.441}{m^2}\,,\\ A_{24}^{3/2}\iiint_{\mathbb{R}^3} t_{24}^4\tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 &< \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{P}}(\sigma_1+\sigma_2+\sigma_3)\cdot\frac{5.874}{m^2}\,,\\ A_{13}^{-1/2}A_{24}^2\iiint_{\mathbb{R}^3} t_{24}^4\tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 &< \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{P}}(\sigma_1+\sigma_2+\sigma_3)\cdot\frac{3.441}{m^2}\,. \end{aligned}$$

Proof. We prove the first claim; the rest are proven similarly. Lemma 5.2 shows that

$$\begin{aligned} A_{23}^{3/2} \iiint_{\mathbb{R}^3} t_{23}^4 \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 &< A_{23}^{3/2} \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{\tilde{P}}} \cdot \frac{3}{m^2(\frac{1}{2}A_{23})^2} \\ &< A_{23}^{3/2} \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} \cdot \frac{3 \cdot 2^2/\sqrt{0.5841}}{m^2 A_{23}^2} \\ &< \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} \cdot \frac{15.7014}{m^2\sqrt{A_{23}}}. \end{aligned}$$

Now Corollary 5.10 shows that

$$A_{23}^{3/2} \iiint_{\mathbb{R}^3} t_{23}^4 \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1+\sigma_2+\sigma_3) \cdot \frac{15.7014\sqrt{2}}{m^2}.$$

Note that $15.7014\sqrt{2} < 22.206$. □

Lemma 5.15. *We have*

$$A_{14}^{1/2} A_{23} \iiint_{\mathbb{R}^3} t_{14}^2 t_{23}^2 \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1+\sigma_2+\sigma_3) \cdot \frac{6.690}{m^2},$$

$$A_{13}^{1/2} A_{24} \iiint_{\mathbb{R}^3} t_{13}^2 t_{24}^2 \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1+\sigma_2+\sigma_3) \cdot \frac{3.441}{m^2}.$$

Proof. Recall from the proof of Lemma 5.2 that

$$\iiint_{\mathbb{R}^3} t_{23}^2 H(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 = \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{5/2} P^{3/2}} \frac{\partial P}{\partial A_{23}}.$$

It follows that

$$\begin{aligned} &\iiint_{\mathbb{R}^3} t_{14}^2 t_{23}^2 H(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\ &\qquad = -\frac{2}{m}\frac{\partial}{\partial A_{14}} \iiint_{\mathbb{R}^3} t_{23}^2 H(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\ &\qquad = -\frac{2}{m}\frac{\partial}{\partial A_{14}} \left(\frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{5/2} P^{3/2}} \frac{\partial P}{\partial A_{23}} \right) \\ &\qquad = 3\frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{7/2} P^{5/2}} \frac{\partial P}{\partial A_{14}} \frac{\partial P}{\partial A_{23}} - \frac{2}{m} \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{5/2} P^{3/2}} (A_{13}+A_{24}) \\ &\qquad < 3\frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{7/2} P^{5/2}} \frac{\partial P}{\partial A_{14}} \frac{\partial P}{\partial A_{23}}. \end{aligned}$$

Now inequality (5.2) shows that

$$\iiint_{\mathbb{R}^3} t_{14}^2 t_{23}^2 H(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} \cdot \frac{3}{m^2 A_{14} A_{23}}.$$

We conclude that

$$A_{14}^{1/2} A_{23} \iiint_{\mathbb{R}^3} t_{14}^2 t_{23}^2 \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1$$

$$< A_{14}^{1/2} A_{23} \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{\tilde{P}}} \cdot \frac{3}{m^2(\frac{35}{36}A_{14})(\frac{1}{2}A_{23})}$$

$$< \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} \cdot \frac{8.075}{m^2\sqrt{A_{23}}},$$

because $3 \cdot \frac{36}{35} \cdot 2/\sqrt{0.5841} < 8.075$. Corollary 5.10 completes the proof of the first claim, because $8.075(2\sqrt{2}-2) < 6.690$. The second claim is proven identically. □

Lemma 5.16. *If $A_{23} = A_{min}$, then*

$$A_{23}^{3/2} \iiint_{\mathbb{R}^3} |t_{14} t_{23}^3| \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1 + \sigma_2 + \sigma_3) \cdot \frac{9.849}{m^2}.$$

If furthermore $A_{24} \le A_{13}$, then

$$A_{24}^{3/2} \iiint_{\mathbb{R}^3} |t_{13} t_{24}^3| \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1 + \sigma_2 + \sigma_3) \cdot \frac{3.441}{m^2}.$$

Proof. Using "arithmetic mean $\ge$ geometric mean", the claims follow from the previous two lemmas; for the first claim,

$$A_{23}^{3/2} |t_{14} t_{23}^3| \le \frac{1}{2}\left(A_{14}^{1/2} A_{23} t_{14}^2 t_{23}^2 + A_{14}^{-1/2} A_{23}^2 t_{23}^4\right).$$ □

Lemma 5.17. *Assume $A_{23} = A_{min}$. Then*

$$A_{14}^{3/2} \iiint_{\mathbb{R}^3} |t_{24} t_{14}^3| \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1 + \sigma_2 + \sigma_3) \cdot \frac{3.975}{m^2},$$

$$A_{13}^{3/2} \iiint_{\mathbb{R}^3} |t_{24} t_{13}^3| \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1 + \sigma_2 + \sigma_3) \cdot \frac{3.975}{m^2},$$

$$A_{14}^{3/2} \iiint_{\mathbb{R}^3} |t_{13} t_{14}^3| \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1 + \sigma_2 + \sigma_3) \cdot \frac{2.329}{m^2}.$$

Proof. We prove the first claim; the rest are proven similarly. Note that

$$\iiint_{\mathbb{R}^3} |t_{24} t_{14}^3| \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1$$

$$\le G(\vec{\sigma}) \iiint_{\mathbb{R}^3} |t_{24} t_{14}^3| \exp\left(-\frac{35m}{72}(A_{13}t_{13}^2 + A_{14}t_{14}^2 + A_{24}t_{24}^2)\right)\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1$$

$$= G(\vec{\sigma}) \int_{-\infty}^{\infty} |t_{24}| e^{-(35m/72)A_{24}t_{24}^2}\,\mathrm{d}t_{24} \cdot \int_{-\infty}^{\infty} |t_{14}^3| e^{-(35m/72)A_{14}t_{14}^2}\,\mathrm{d}t_{14}$$

$$\cdot \int_{-\infty}^{\infty} e^{-(35m/72)A_{13}t_{13}^2}\,\mathrm{d}t_{13}.$$

We can use Lemma 4.6 to evaluate these integrals:

$$\begin{aligned}\int_{-\infty}^{\infty} |t_{24}| \exp\Big(-\frac{35m}{72} A_{24} t_{24}^2\Big)\,\mathrm{d}t_{24} &= 2\int_0^{\infty} t_{24} \exp\Big(-\frac{35m}{72} A_{24} t_{24}^2\Big)\,\mathrm{d}t_{24}\\ &= \frac{72/35}{mA_{24}},\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} |t_{14}^3| \exp\Big(-\frac{35m}{72} A_{14} t_{14}^2\Big)\,\mathrm{d}t_{14} &= 2\int_0^{\infty} t_{14}^3 \exp\Big(-\frac{35m}{72} A_{14} t_{14}^2\Big)\,\mathrm{d}t_{14}\\ &= \frac{(72/35)^2}{m^2A_{14}^2},\end{aligned}$$

$$\int_{-\infty}^{\infty} \exp\Big(-\frac{35m}{72} A_{13} t_{13}^2\Big)\,\mathrm{d}t_{13} = \frac{\sqrt{72\pi/35}}{\sqrt{mA_{13}}}.$$

Thus

$$A_{14}^{3/2} \iiint_{\mathbb{R}^3} |t_{24}t_{14}^3| \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \le \frac{22.131G(\vec{\sigma})}{m^{3.5}\sqrt{A_{24}A_{14}A_{13}}} \frac{1}{\sqrt{A_{24}}}.$$

Using $A_{24}A_{14}A_{13} \ge P/4$ and Corollary 5.10,

$$\begin{aligned}A_{14}^{3/2} \iiint_{\mathbb{R}^3} |t_{24}t_{14}^3| \tilde{H}(\vec{t})\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 &< \frac{G(\vec{\sigma})}{m^{3/2}\sqrt{P}}(\sigma_1+\sigma_2+\sigma_3)\cdot\frac{62.596}{m^2}\\ &< \frac{(2\pi)^{3/2}G(\vec{\sigma})}{m^{3/2}\sqrt{P}}(\sigma_1+\sigma_2+\sigma_3)\cdot\frac{3.975}{m^2}.\end{aligned}$$

The second claim is proven identically. The third claim is proven similarly, except that Corollary 5.10 contributes a factor of $(2\sqrt{2}-2)$ instead of $\sqrt{2}$. Note that $3.975(2\sqrt{2}-2)/\sqrt{2} < 2.329$. □

Now we can use inequality (5.12) to estimate f'/f; recall the formula for $f' := \partial f/\partial y$ in (5.8). Since $t_1+t_2+t_3 = t_{13}+t_{24} = t_{14}+t_{23}$, we have

$$\begin{aligned}|t_1+t_2+t_3|&(A_{13}^{3/2}|t_{13}|^3 + A_{23}^{3/2}|t_{23}|^3 + A_{14}^{3/2}|t_{14}|^3 + A_{24}^{3/2}|t_{24}|^3)\\ &\le A_{13}^{3/2}(t_{13}^4 + |t_{24}t_{13}^3|) + A_{23}^{3/2}(|t_{14}t_{23}^3| + t_{23}^4)\\ &\qquad + A_{14}^{3/2}(|t_{13}t_{14}^3| + |t_{24}t_{14}^3|) + A_{24}^{3/2}(|t_{13}t_{24}^3| + t_{24}^4).\end{aligned}$$

Now we can split the integral in (5.12) into eight separate integrals, which we bound using Lemmas 5.14, 5.16, and 5.17 (still assuming that $A_{23} =$

$A_{\min}$ and $A_{24} \le A_{13}$):

$$\left| \iiint_{\Sigma} (t_1 + t_2 + t_3) G(\vec{\sigma} + i\vec{t}) \, \mathrm{d}t_3 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \right| \Big/ \left(\frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1 + \sigma_2 + \sigma_3) \right)$$
$$< \frac{1}{m} \frac{\sqrt{2}}{3} (3.441 + 3.975 + 9.849 + 22.206 + 2.329 + 3.975 + 3.441 + 5.874)$$
$$< \frac{25.9697}{m} .$$

Combining this with Lemma 5.11, we conclude that, for $m \ge 1000$,

$$\left| \iiint_{\mathbb{R}^3} (t_1 + t_2 + t_3) G(\vec{\sigma} + i\vec{t}) \, \mathrm{d}t_3 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \right| < \frac{(2\pi)^{3/2} G(\vec{\sigma})}{m^{3/2}\sqrt{P}} (\sigma_1 + \sigma_2 + \sigma_3) \cdot \frac{25.971}{m} .$$

Now equation (5.9) and Lemma 5.7 show that, for $m \ge 1000$,

$$\begin{aligned} &\frac{1}{\sigma_1 + \sigma_2 + \sigma_3} \left| -\frac{1}{2m} \frac{f'}{f} - (\sigma_1 + \sigma_2 + \sigma_3) \right| \\ &\qquad = \frac{1}{\sigma_1 + \sigma_2 + \sigma_3} \left| \frac{1}{(2\pi)^3 f} \iiint_{\mathbb{R}^3} (t_1 + t_2 + t_3) G(\vec{\sigma} + i\vec{t}) \, \mathrm{d}t_3 \, \mathrm{d}t_2 \, \mathrm{d}t_1 \right| \\ &\qquad \le \frac{1}{|f|} \frac{G(\vec{\sigma})}{(2\pi)^{3/2} m^{3/2} \sqrt{P}} \cdot \frac{25.971}{m} \\ &\qquad \le \frac{1}{1 - \frac{378.1}{m}} \cdot \frac{25.971}{m} . \end{aligned}$$

This proves the following lemma:

Lemma 5.18. *For $m \ge 1000$,*

$$-\frac{1}{2m} \frac{\partial f / \partial y}{f} = (\sigma_1 + \sigma_2 + \sigma_3)(1 + \beta) ,$$

where $|\beta| < \frac{25.971}{m - 378.1}$.

6. Properties of Ψ

In this section, we prove some properties of the digamma function Ψ which will be needed when we study the saddle point $\vec{\sigma}$ in the next section.

Recall the Maclaurin series [1, 23.1.1–2]:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = 1 - \frac{t}{2} + \sum_{n=1}^{\infty} B_{2n} \frac{t^{2n}}{(2n)!} ,$$

where B_n denotes the n^{th} Bernoulli number.

Lemma 6.1. *For any real $t \neq 0$, the function $\frac{t}{e^t-1} - 1 + \frac{t}{2}$ is strictly enveloped by its Maclaurin series. That is,*

$$\frac{t}{e^t - 1} - 1 + \frac{t}{2} - \sum_{n=1}^{N} B_{2n} \frac{t^{2n}}{(2n)!}$$

is positive when N is even and negative when N is odd.

Proof. This follows from the fact that $\frac{t}{e^t-1} = \frac{t}{2}\coth(\frac{t}{2}) - \frac{t}{2}$, and $z \coth z$ is known to be enveloped by its Maclaurin series; see [9, Part I, #154]. □

Recall the asymptotic series [1, 6.3.18, 6.4.11]:

$$\Psi(x) \sim \log x - \frac{1}{2x} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nx^{2n}},$$

$$\Psi^{(k)}(x) \sim (-1)^{k-1} \left(\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} + \sum_{n=1}^{\infty} B_{2n} \frac{(2n+k-1)!}{(2n)! x^{2n+k}} \right).$$

As these series are directly related to the Maclaurin series for $t/(e^t - 1)$, the previous lemma lets us turn these series into inequalities.

Lemma 6.2. *For any $x > 0$, $\Psi(x) - \log(x) + \frac{1}{2x}$ is strictly enveloped by the above asymptotic series; that is,*

$$\Psi(x) - \log x + \frac{1}{2x} + \sum_{n=1}^{N} \frac{B_{2n}}{2nx^{2n}}$$

is positive when N is odd and negative when N is even. A similar result holds for each $\Psi^{(k)}(x)$.

Proof. Recall (from [2, p. 18]) that

$$\Psi(x) = \log x - \frac{1}{2x} - \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-tx} \, dt.$$

It follows that

$$\begin{aligned} \Psi(x) - \log x + \frac{1}{2x} &+ \sum_{n=1}^{N} \frac{B_{2n}}{2nx^{2n}} \\ &= \sum_{n=1}^{N} \frac{B_{2n}}{2nx^{2n}} - \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) e^{-tx} \, dt \end{aligned}$$

$$= - \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} - \sum_{n=1}^{N} B_{2n} \frac{t^{2n-1}}{(2n)!} \right) e^{-tx} \, dt, \tag{6.1}$$

and the result follows from the previous lemma. Differentiating (6.1) proves the claim for $\Psi^{(k)}$. □

Recall (from [1, 6.3.5]) that $\Psi(x) = -\frac{1}{x} + \Psi(x+1)$; taking derivatives, we get recurrence formulas for the derivatives of Ψ as well. Combining these formulas with the previous lemma, we obtain bounds for Ψ and its derivatives: for any $x > 0$,

$$\Psi(x) < -\frac{1}{x} + \log(x+1) - \frac{1}{2(x+1)} - \frac{1}{12(x+1)^2} + \frac{1}{120(x+1)^4}\,, \tag{6.2}$$

$$\Psi\left(x+\frac{1}{2}\right) > -\frac{1}{x+\frac{1}{2}} + \log(x+\frac{3}{2}) - \frac{1}{2(x+\frac{3}{2})} - \frac{1}{12(x+\frac{3}{2})^2}\,, \tag{6.3}$$

$$0 < \Psi'(x) < \frac{1}{x^2} + \frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{6(x+1)^3}\,, \tag{6.4}$$

$$\Psi'\left(x+\frac{1}{2}\right) > \frac{1}{(x+\frac{1}{2})^2} + \frac{1}{x+\frac{3}{2}} + \frac{1}{2(x+\frac{3}{2})^2} + \frac{1}{6(x+\frac{3}{2})^3} - \frac{1}{30(x+\frac{3}{2})^5}\,, \tag{6.5}$$

$$\Psi''(x) < -\frac{2}{x^3} - \frac{1}{(x+1)^2} - \frac{1}{(x+1)^3} - \frac{1}{2(x+1)^4} + \frac{1}{6(x+1)^6} < 0\,, \tag{6.6}$$

$$0 < \Psi^{(3)}(x) < \frac{6}{x^4} + \frac{2}{(x+1)^3} + \frac{3}{(x+1)^4} + \frac{2}{(x+1)^5}\,. \tag{6.7}$$

Lemma 6.3. *For any $x > 0$,*

$$\begin{aligned}\Psi(x+\frac{1}{2}) - \Psi(x) > &-\frac{1}{x+\frac{1}{2}} - \frac{1}{2(x+\frac{3}{2})} - \frac{1}{12(x+\frac{3}{2})^2}\\ &+\frac{1}{x} + \frac{1}{x+1} - \frac{1}{24(x+1)^2} - \frac{1}{120(x+1)^4} > 0\,.\end{aligned}$$

Proof. Note that $\lambda(t) := \log(1+t) - t + \frac{1}{2}t^2 \geq 0$ for all $t \geq 0$; this follows from $\lambda(0) = 0$ and $\lambda'(t) = t^2/(t+1) \geq 0$ for all $t \geq 0$. Therefore

$$\log\left(x+\frac{3}{2}\right) - \log(x+1) = \log\left(1+\frac{1}{2(x+1)}\right) \geq \frac{1}{2(x+1)} - \frac{1}{8(x+1)^2}\,.$$

Now the first inequality follows immediately from inequalities (6.2) and (6.3). To prove the second inequality, combine terms; we get a rational function with positive coefficients. □

Lemma 6.4. *For any $x > 0$,*

$$\frac{3}{2}\Psi''(x)^2 < \Psi'(x)\Psi^{(3)}(x)\,.$$

Proof. Let ζ denote the Hurwitz zeta function. Recall from equation (4.1) that

$$\Psi^{(n)}(x) = (-1)^{n+1} n!\zeta(n+1,x)\,.$$

Thus the inequality is equivalent to $\zeta(3,x)^2 < \zeta(2,x)\zeta(4,x)$, which follows from Cauchy–Schwarz. □

Lemma 6.5. *For a given $x > 0$, the function $\kappa \mapsto |\alpha_\kappa^{(3)}(x)|/\alpha_\kappa''(x)$ is increasing for $\kappa \in [0,1]$.*

Proof. Observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\kappa}\frac{|\alpha_\kappa^{(3)}(x)|}{\alpha_\kappa''(x)} &= \frac{\mathrm{d}}{\mathrm{d}\kappa}\frac{-\alpha_\kappa^{(3)}(x)}{\alpha_\kappa''(x)} \\ &= \frac{\alpha_\kappa''(x)(\Psi''(x+\frac{1}{2}) - \Psi''(x)) + (\Psi'(x) - \Psi'(x+\frac{1}{2}))\alpha_\kappa^{(3)}(x)}{\alpha_\kappa''(x)^2} \\ &= \frac{\Psi'(x)\Psi''(x+\frac{1}{2}) - \Psi'(x+\frac{1}{2})\Psi''(x)}{\alpha_\kappa''(x)^2},\end{aligned}$$

so we want to prove that $\Psi'(x)\Psi''(x+\frac{1}{2}) - \Psi'(x+\frac{1}{2})\Psi''(x) > 0$. We do so by showing that Ψ''/Ψ' is an increasing function: by Lemma 6.4,

$$\frac{d}{dx}\frac{\Psi''(x)}{\Psi'(x)} = \frac{\Psi'(x)\Psi^{(3)}(x) - \Psi''(x)^2}{\Psi'(x)^2} > 0\,. \qquad \square$$

Lemma 6.6. *For any $x > 0$ and any $\kappa \in [\frac{1}{2}, 1]$,*

$$\frac{|\alpha_\kappa^{(3)}(x)|}{\alpha_\kappa''(x)} > \frac{\Psi'(x) - \Psi'(x+\frac{1}{2})}{\Psi(x+\frac{1}{2}) - \Psi(x)}\,.$$

Proof. Thanks to Lemma 6.5, we may assume $\kappa = \frac{1}{2}$. Recall the duplication formula for Ψ, which says that $\alpha_{1/2}'(x) = \Psi(2x) - \log 2$. Thus we want to show that

$$\frac{2|\Psi''(2x)|}{\Psi'(2x)} - \frac{\Psi'(x) - \Psi'(x+\frac{1}{2})}{\Psi(x+\frac{1}{2}) - \Psi(x)} > 0$$

for $x > 0$. By Lemma 6.3 and inequalities (6.4), (6.5), and (6.6), this function is bounded below by a rational function with positive coefficients. $\square$

7. Properties of the Saddle Point

Now we are prepared to consider the saddle point $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. If $i \in \{1,2\}$, let i' denote the other element of $\{1,2\}$; define j' similarly for $j \in \{3,4\}$. Thus

$$\sigma_{ij} + \sigma_{i'j'} = \sigma_1 + \sigma_2 + \sigma_3 \qquad \text{for any } i \in \{1,2\} \text{ and } j \in \{3,4\}.$$

Recall from Lemma 3.2 that, for a nonzero $a \in \mathcal{O}_L$ and $y \in \mathbb{R}$, the corresponding saddle point is the unique $\vec{\sigma} \in \mathbb{R}^3$ with all $\sigma_{ij} > 0$ which satisfies

$$a_{w_1} + y = \frac{1}{2}\alpha'_{k_{13}}(\sigma_{13}) + \frac{1}{2}\alpha'_{k_{14}}(\sigma_{14})\,, \tag{7.1}$$

$$a_{w_2} + y = \frac{1}{2}\alpha'_{k_{23}}(\sigma_{23}) + \frac{1}{2}\alpha'_{k_{24}}(\sigma_{24})\,, \tag{7.2}$$

$$a_{w_3} + y = \frac{1}{2}\alpha'_{k_{13}}(\sigma_{13}) + \frac{1}{2}\alpha'_{k_{23}}(\sigma_{23})\,. \tag{7.3}$$

In order for our estimates to be useful, we need to choose a y which gives a good lower bound on $\sigma_1 + \sigma_2 + \sigma_3$, independent of a and $\vec{k}$.

Lemma 7.1. *Let $y_0 \in \mathbb{R}$ be given. For any $y \geq y_0$, $\vec{k} \in [\frac{1}{2}, 1]^4$, and $a \in L$, the corresponding saddle point $\vec{\sigma}$ satisfies*

$$\sigma_1 + \sigma_2 + \sigma_3 \geq 2(\alpha'_{1/2})^{-1}(y_0)\,.$$

In particular, if $y \geq -1.18$, then $\sigma_1 + \sigma_2 + \sigma_3 \geq 1.0572$.

Proof. Using the fact that a is an algebraic integer, we add equations (7.1) and (7.2) to obtain

$$2y \leq a_{w_1} + a_{w_2} + 2y = \frac{1}{2}\sum_{j=3}^{4}\Big(\alpha'_{k_{1j}}(\sigma_{1j}) + \alpha'_{k_{2j}}(\sigma_{2j'})\Big).$$

Thus we can choose $j \in \{3, 4\}$ such that

$$2y \leq \alpha'_{k_{1j}}(\sigma_{1j}) + \alpha'_{k_{2j}}(\sigma_{2j'}) \leq \alpha'_{1/2}(\sigma_{1j}) + \alpha'_{1/2}(\sigma_{2j'})\,.$$

Since $\alpha'_{1/2}$ is a concave function, it follows that $y \leq \alpha'_{1/2}((\sigma_{1j} + \sigma_{2j'})/2)$. Thus

$$\sigma_1 + \sigma_2 + \sigma_3 = \sigma_{1j} + \sigma_{2j'} \geq 2(\alpha'_{1/2})^{-1}(y) \geq 2(\alpha'_{1/2})^{-1}(y_0)\,. \qquad \square$$

Recall that, once we choose y such that $-\frac{f'}{f}(y, a) \geq 2m$ for all a, we can ignore all terms in inequality (3.5) except for the $a = 1$ term. It remains to understand the saddle point corresponding to $a = 1$. When $a = 1$, equations (7.1)-(7.3) say that the saddle point $\vec{\sigma}$ satisfies

$$2y = \alpha'_{k_{13}}(\sigma_{13}) + \alpha'_{k_{14}}(\sigma_{14}) = \alpha'_{k_{23}}(\sigma_{23}) + \alpha'_{k_{24}}(\sigma_{24}) = \alpha'_{k_{13}}(\sigma_{13}) + \alpha'_{k_{23}}(\sigma_{23})\,,$$

or equivalently,

$$\alpha'_{k_{24}}(\sigma_{24}) + \alpha'_{k_{14}}(\sigma_{14}) = 2y\,, \tag{7.4}$$

$$\alpha'_{k_{13}}(\sigma_{13}) - \alpha'_{k_{24}}(\sigma_{24}) = 0\,, \tag{7.5}$$

$$\alpha'_{k_{14}}(\sigma_{14}) - \alpha'_{k_{23}}(\sigma_{23}) = 0\,. \tag{7.6}$$

Now we regard $a = 1$ and $y \in \mathbb{R}$ as fixed, and consider properties of σ_{ij} as a function of $\vec{k}$, defined by equations (7.4), (7.5), and (7.6). For any $\vec{k} \in [0, 1]^4$ and any $s_{ij} > 0$, define

$$P_{\vec{k}}(s_{13}, s_{14}, s_{23}, s_{24}) = P_3(\alpha''_{k_{13}}(s_{13}), \alpha''_{k_{14}}(s_{14}), \alpha''_{k_{23}}(s_{23}), \alpha''_{k_{24}}(s_{24}))\,,$$

where P_3 is the degree-3 elementary symmetric polynomial in four variables. Then we wish to find an upper bound for

$$P = P(\vec{k}) = P_{\vec{k}}(\sigma_{13}(\vec{k}), \sigma_{14}(\vec{k}), \sigma_{23}(\vec{k}), \sigma_{24}(\vec{k}))\,,$$

for $\vec{k} \in [\frac{1}{2}, 1]^4$. I claim that $P(\vec{k})$ is maximized when $\vec{k} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Essentially, we will prove this by showing that $\partial P / \partial k_{ij} < 0$. To simplify notation, define

$$\begin{aligned} A_{ij} &= \alpha''_{k_{ij}}(\sigma_{ij}) > 0\,, \\ A^{(3)}_{ij} &= \alpha^{(3)}_{k_{ij}}(\sigma_{ij}) < 0\,, \\ \Delta_{ij} &= \Psi(\sigma_{ij} + \frac{1}{2}) - \Psi(\sigma_{ij}) > 0\,, \\ \Delta'_{ij} &= \Psi'(\sigma_{ij}) - \Psi'(\sigma_{ij} + \frac{1}{2}) > 0\,. \end{aligned}$$

Note that $\Delta_{ij} = -\partial \alpha'_{k_{ij}}(\sigma_{ij}) / \partial k_{ij}$ if we take the derivative with σ_{ij} fixed (not a function of k_{ij}).

Finally, let Q denote the degree-2 elementary symmetric polynomial in 3 variables. For $i \in \{1, 2\}$ and $j \in \{3, 4\}$, define

$$Q_{ij} = Q(A_{ij'}, A_{i'j}, A_{i'j'})\,;$$

e.g., $Q_{13} = A_{14}A_{23} + A_{14}A_{24} + A_{23}A_{24}$.

Lemma 7.2. *For any $i \in \{1, 2\}$ and $j \in \{3, 4\}$,*

$$\frac{\partial \sigma_{ij}}{\partial k_{ij}} = \frac{\Delta_{ij}}{P} Q_{ij}\,, \qquad \frac{\partial \sigma_{ij'}}{\partial k_{ij}} = \frac{\Delta_{ij}}{P} A_{i'1} A_{i'2}\,,$$

$$\frac{\partial \sigma_{i'j}}{\partial k_{ij}} = \frac{\Delta_{ij}}{P} A_{1j'} A_{2j'}\,, \qquad \frac{\partial \sigma_{i'j'}}{\partial k_{ij}} = -\frac{\Delta_{ij}}{P} A_{ij'} A_{i'j}\,.$$

Proof. Applying the implicit function theorem to equations (7.4)-(7.6),

$$\begin{aligned} \begin{pmatrix} \partial\sigma_1/\partial k_{13} \\ \partial\sigma_2/\partial k_{13} \\ \partial\sigma_3/\partial k_{13} \end{pmatrix} &= - \begin{pmatrix} A_{14} & A_{24} & 0 \\ A_{13} & -A_{24} & A_{13} \\ A_{14} & -A_{23} & -A_{23} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ -\Delta_{13} \\ 0 \end{pmatrix} \\ &= \frac{\Delta_{13}}{P} \begin{pmatrix} A_{23}A_{24} \\ -A_{14}A_{23} \\ A_{14}(A_{23} + A_{24}) \end{pmatrix}. \end{aligned}$$

This proves the lemma for derivatives with respect to k_{13}; the rest follow by symmetry. □

Lemma 7.3. *For any $i, j \in \{1, 2\}$,*

$$\frac{\partial P}{\partial k_{ij}} = \frac{\Delta_{ij}}{P}\left(\frac{\Delta'_{ij}}{\Delta_{ij}} PQ_{ij} - |A^{(3)}_{ij'}| A_{i'1}A_{i'2}Q_{ij'} - |A^{(3)}_{i'j}| A_{1j'}A_{2j'}Q_{i'j} - |A^{(3)}_{ij}| Q^2_{ij} + |A^{(3)}_{i'j'}| A_{ij'}A_{i'j}Q_{i'j'}\right).$$

Proof. By symmetry, it suffices to prove the case $(i, j) = (1, 4)$. Recall that we defined $\Pi_{ij} = A_{13}A_{14}A_{23}A_{24}/A_{ij}$, so that $P = \Pi_{24} + \Pi_{23} + \Pi_{14} + \Pi_{13}$. Lemma 7.2 shows that

$$\frac{\partial \Pi_{24}}{\partial k_{14}} = \left(\frac{\Delta_{14}}{P} A^{(3)}_{13} A_{23}A_{24}\right) A_{14}A_{23} + A_{13}\left(\frac{\Delta_{14}}{P} A^{(3)}_{14} Q_{14} + \Delta'_{14}\right) A_{23} + A_{13}A_{14}\left(-\frac{\Delta_{14}}{P} A^{(3)}_{23} A_{13}A_{24}\right),$$

$$\frac{\partial \Pi_{23}}{\partial k_{14}} = \left(\frac{\Delta_{14}}{P} A^{(3)}_{13} A_{23}A_{24}\right) A_{14}A_{24} + A_{13}\left(\frac{\Delta_{14}}{P} A^{(3)}_{14} Q_{14} + \Delta'_{14}\right) A_{24} + A_{13}A_{14}\left(\frac{\Delta_{14}}{P} A^{(3)}_{24} A_{13}A_{23}\right),$$

$$\frac{\partial \Pi_{14}}{\partial k_{14}} = \left(\frac{\Delta_{14}}{P} A^{(3)}_{13} A_{23}A_{24}\right) A_{23}A_{24} + A_{13}\left(-\frac{\Delta_{14}}{P} A^{(3)}_{23} A_{13}A_{24}\right) A_{24} + A_{13}A_{23}\left(\frac{\Delta_{14}}{P} A^{(3)}_{24} A_{13}A_{23}\right),$$

$$\frac{\partial \Pi_{13}}{\partial k_{14}} = \left(\frac{\Delta_{14}}{P} A^{(3)}_{14} Q_{14} + \Delta'_{14}\right) A_{23}A_{24} + A_{14}\left(-\frac{\Delta_{14}}{P} A^{(3)}_{23} A_{13}A_{24}\right) A_{24} + A_{14}A_{23}\left(\frac{\Delta_{14}}{P} A^{(3)}_{24} A_{13}A_{23}\right).$$

Summing these equalities and simplifying yields the desired result. □

We now have formulas for $\partial P/\partial k_{ij}$. If we could prove that these derivatives are always negative, that would complete the proof that P is maximized when $\vec{k} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. But we are not able to prove that, so we proceed in two steps. First, the next lemma narrows down the possibilities for where the maximum could occur. Then it is feasible to check by brute force that the necessary derivatives are negative.

Lemma 7.4. *There exist $\kappa_1, \kappa_2 \in [\frac{1}{2}, 1]$ such that*

$$\max_{\vec{k} \in [\frac{1}{2}, 1]^4} P(\vec{k}) = P\left(\kappa_1, \frac{1}{2}, \frac{1}{2}, \kappa_2\right) = P\left(\frac{1}{2}, \kappa_1, \kappa_2, \frac{1}{2}\right).$$

Proof. First observe that $P(\kappa_1, \frac{1}{2}, \frac{1}{2}, \kappa_2) = P(\frac{1}{2}, \kappa_1, \kappa_2, \frac{1}{2})$ is automatic, by symmetry. Now we prove that if P is maximized at $\vec{k}$, then $\vec{k}$ has the form

$(\kappa_1, \frac{1}{2}, \frac{1}{2}, \kappa_2)$ or $(\frac{1}{2}, \kappa_1, \kappa_2, \frac{1}{2})$. This is immediate from the following claim: for every $\vec{k} \in [\frac{1}{2}, 1]^4$,

$$\frac{\partial P}{\partial k_{13}}, \frac{\partial P}{\partial k_{24}} < 0 \qquad \text{or} \qquad \frac{\partial P}{\partial k_{14}}, \frac{\partial P}{\partial k_{23}} < 0\,. \tag{7.7}$$

Lemma 6.6 says that $\Delta'_{14}/\Delta_{14} < |A_{14}^{(3)}|/A_{14}$. Thus

$$-|A_{14}^{(3)}|Q_{14}^2 + \frac{\Delta'_{14}}{\Delta_{14}} PQ_{14} < |A_{14}^{(3)}| \left(\frac{P}{A_{14}} - Q_{14} \right) Q_{14} = |A_{14}^{(3)}| \frac{A_{13}A_{23}A_{24}}{A_{14}} Q_{14}\,.$$

Substituting this into the previous lemma,

$$\frac{\partial P}{\partial k_{14}} < \frac{\Delta_{14}}{P} \Big(- |A_{13}^{(3)}|A_{23}A_{24}Q_{13} - |A_{24}^{(3)}|A_{13}A_{23}Q_{24} + |A_{14}^{(3)}| \frac{A_{13}A_{23}A_{24}}{A_{14}} Q_{14} + |A_{23}^{(3)}|A_{13}A_{24}Q_{23} \Big).$$

Consequently,

$$\begin{aligned} \frac{P}{\Delta_{14}A_{14}\Pi_{14}^2} \frac{\partial P}{\partial k_{14}} &< -\frac{|A_{13}^{(3)}|}{A_{13}^2} \frac{Q_{13}}{A_{14}A_{23}A_{24}} - \frac{|A_{24}^{(3)}|}{A_{24}^2} \frac{Q_{24}}{A_{13}A_{14}A_{23}} \\ &\qquad + \frac{|A_{14}^{(3)}|}{A_{14}^2} \frac{Q_{14}}{A_{13}A_{23}A_{24}} + \frac{|A_{23}^{(3)}|}{A_{14}^2} \frac{Q_{23}}{A_{13}A_{14}A_{24}} \\ &= -\frac{|A_{13}^{(3)}|}{A_{13}^2} \frac{Q_{13}}{\Pi_{13}} - \frac{|A_{24}^{(3)}|}{A_{24}^2} \frac{Q_{24}}{\Pi_{24}} + \frac{|A_{14}^{(3)}|}{A_{14}^2} \frac{Q_{14}}{\Pi_{14}} + \frac{|A_{23}^{(3)}|}{A_{14}^2} \frac{Q_{23}}{\Pi_{23}}\,. \end{aligned}$$

Hence $\partial P/\partial k_{14}$ is negative if

$$\frac{|A_{13}^{(3)}|}{A_{13}^2} \frac{Q_{13}}{\Pi_{13}} + \frac{|A_{24}^{(3)}|}{A_{24}^2} \frac{Q_{24}}{\Pi_{24}} \ge \frac{|A_{14}^{(3)}|}{A_{14}^2} \frac{Q_{14}}{\Pi_{14}} + \frac{|A_{23}^{(3)}|}{A_{14}^2} \frac{Q_{23}}{\Pi_{23}}\,.$$

Similarly, $\partial P/\partial k_{23}$ is negative if this inequality holds; if this inequality holds in the reverse direction, then $\partial P/\partial k_{13}$ and $\partial P/\partial k_{24}$ are negative. This proves claim (7.7). □

Now we need only check that $\partial P/\partial k_{13}$ and $\partial P/\partial k_{24}$ are negative when $\vec{k}$ lies in the square

$$S = \left[\frac{1}{2}, 1\right] \times \left\{\frac{1}{2}\right\} \times \left\{\frac{1}{2}\right\} \times \left[\frac{1}{2}, 1\right].$$

By symmetry, it suffices to check $\partial P/\partial k_{13} < 0$. If $\vec{k}$ lies in a subset of S of the form

$$S' = [k_{13}^{\min}, k_{13}^{\max}] \times \left\{\frac{1}{2}\right\} \times \left\{\frac{1}{2}\right\} \times [k_{24}^{\min}, k_{24}^{\max}]\,,$$

we need an upper bound for $\partial P/\partial k_{13}$ as a function of $k_{13}^{\min}, k_{13}^{\max}, k_{24}^{\min}$, and $k_{24}^{\max}$. Then we partition the interval $[\frac{1}{2}, 1]$ into N equal subintervals of length $\frac{1}{2N}$, thereby partitioning S into N^2 equal subsquares. By choosing N sufficiently large, we will see that $\partial P/\partial k_{13} < 0$.

In order to bound $\partial P/\partial k_{13}$ on S', we will need upper/lower bounds for the σ_{ij}.

Lemma 7.5. *Let $y \in \mathbb{R}$ be fixed. Let $\frac{1}{2} \leq k_{13}^{min} \leq k_{13}^{max} \leq 1$ and $\frac{1}{2} \leq k_{24}^{min} \leq k_{24}^{max} \leq 1$ be given. Define*

$$\sigma_{13}^{min} = \sigma_{13}\left(k_{13}^{min}, \frac{1}{2}, \frac{1}{2}, k_{24}^{max}\right), \qquad \sigma_{13}^{max} = \sigma_{13}\left(k_{13}^{max}, \frac{1}{2}, \frac{1}{2}, k_{24}^{min}\right),$$
$$\sigma_{24}^{min} = \sigma_{24}\left(k_{13}^{max}, \frac{1}{2}, \frac{1}{2}, k_{24}^{min}\right), \qquad \sigma_{24}^{max} = \sigma_{24}\left(k_{13}^{min}, \frac{1}{2}, \frac{1}{2}, k_{24}^{max}\right),$$

Let $\vec{k} = (k_{13}, \frac{1}{2}, \frac{1}{2}, k_{24})$ for $k_{13} \in [k_{13}^{min}, k_{13}^{max}]$ and $k_{24} \in [k_{24}^{min}, k_{24}^{max}]$. Then

$$\sigma_{13}^{min} \leq \sigma_{13}(\vec{k}) \leq \sigma_{13}^{max},$$
$$\sigma_{24}^{min} \leq \sigma_{24}(\vec{k}) \leq \sigma_{24}^{max},$$
$$\sigma_{14}(\vec{k}) = \sigma_{23}(\vec{k}) = \frac{\sigma_{13}(\vec{k}) + \sigma_{24}(\vec{k})}{2}.$$

Proof. The inequalities are immediate from Lemma 7.2: $\partial\sigma_{13}/\partial k_{13}$ and $\partial\sigma_{24}/\partial k_{24}$ are positive, while $\partial\sigma_{13}/\partial k_{24}$ and $\partial\sigma_{24}/\partial k_{13}$ are negative. Also, $\sigma_{14} = \sigma_{23}$ follows from $\alpha'_{1/2}(\sigma_{14}) = \alpha'_{1/2}(\sigma_{23})$, and then the last claim follows from $\sigma_{13} + \sigma_{24} = \sigma_{14} + \sigma_{23}$. □

The lemma implies that, for $\vec{k} \in S'$ as above,

$$A_{13}(\vec{k}) \leq \alpha''_{k_{13}^{\max}}(\sigma_{13}^{\min}).$$

(Recall that $\alpha''_{\kappa}(x)$ is strictly decreasing as a function of x, and strictly increasing as a function of κ.) Similarly, the A_{ij}, the $|A_{ij}^{(3)}|$, the Δ_{ij}, and the Δ'_{ij} are clearly all monotone in the σ_{ij} and k_{ij}, so this technique gives us upper/lower bounds for every term in Lemma 7.3.

Lemma 7.6. *Suppose $y = -1.18$ and $\vec{k} \in [\frac{1}{2}, 1]^4$. Then*

$$P(\vec{k}) \leq P\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) < 111.78.$$

Proof. Use the upper bound described in the previous paragraph, partitioning S into 30^2 subsquares. □

For $\vec{k} \in [0, 1]^4$, recall that the function $\alpha_{\vec{k}}$ is defined by

$$\alpha_{\vec{k}}(s_1, s_2, s_3) = \alpha_{k_{13}}(s_1 + s_3) + \alpha_{k_{23}}(s_2 + s_3) + \alpha_{k_{14}}(s_1) + \alpha_{k_{24}}(s_2).$$

Define

$$g(\vec{k}) = \alpha_{\vec{k}}(\vec{\sigma}(\vec{k})) - 2y(\sigma_1(\vec{k}) + \sigma_2(\vec{k}) + \sigma_3(\vec{k}))$$
$$= \alpha_{\vec{k}}(\vec{\sigma}(\vec{k})) - \vec{y} \cdot \vec{\sigma}(\vec{k}),$$

where $\vec{y} = (2y, 2y, 2y)$.

Lemma 7.7. *The function g is concave on $[0,1]^4$.*

Proof. First observe that $\partial g/\partial k_{13}$ is

$$\left(\log\Gamma(\sigma_{13}) - \log\Gamma\left(\sigma_{13} + \frac{1}{2}\right)\right) + \left(\sum_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} \alpha'_{k_{ij}}(\sigma_{ij}) \frac{\partial \sigma_{ij}}{\partial k_{13}}\right) - 2y \frac{\partial(\sigma_1+\sigma_2+\sigma_3)}{\partial k_{13}}.$$

Recall that $\sigma_{13}+\sigma_{24} = \sigma_1+\sigma_2+\sigma_3 = \sigma_{23}+\sigma_{14}$. Along with equations (7.4)–(7.6), this yields

$$\begin{aligned}
&\sum_{\substack{i\in\{1,2\}\\ j\in\{3,4\}}} \alpha'_{k_{ij}}(\sigma_{ij}) \frac{\partial \sigma_{ij}}{\partial k_{13}} \\
&\quad = \alpha'_{k_{13}}(\sigma_{13}) \frac{\partial \sigma_{13}}{\partial k_{13}} + \alpha'_{k_{24}}(\sigma_{24}) \frac{\partial \sigma_{24}}{\partial k_{13}} + \alpha'_{k_{14}}(\sigma_{14}) \frac{\partial \sigma_{14}}{\partial k_{13}} + \alpha'_{k_{23}}(\sigma_{23}) \frac{\partial \sigma_{23}}{\partial k_{13}} \\
&\quad = \alpha'_{k_{13}}(\sigma_{13}) \frac{\partial \sigma_{13}}{\partial k_{13}} + \alpha'_{k_{13}}(\sigma_{13}) \frac{\partial \sigma_{24}}{\partial k_{13}} + \alpha'_{k_{14}}(\sigma_{14}) \frac{\partial \sigma_{14}}{\partial k_{13}} + \alpha'_{k_{14}}(\sigma_{14}) \frac{\partial \sigma_{23}}{\partial k_{13}} \\
&\quad = [\alpha'_{k_{13}}(\sigma_{13}) + \alpha'_{k_{14}}(\sigma_{14})] \frac{\partial(\sigma_1+\sigma_2+\sigma_3)}{\partial k_{13}} \\
&\quad = 2y \frac{\partial(\sigma_1+\sigma_2+\sigma_3)}{\partial k_{13}}.
\end{aligned}$$

Thus we have $\partial g/\partial k_{13} = \log\Gamma(\sigma_{13}) - \log\Gamma(\sigma_{13} + \frac{1}{2})$. Identical arguments show that

$$\frac{\partial g}{\partial k_{ij}} = \log\Gamma(\sigma_{ij}) - \log\Gamma\left(\sigma_{ij} + \frac{1}{2}\right)$$

for all $i \in \{1,2\}$ and $j \in \{3,4\}$. It follows that, for any $i_0, i_1 \in \{1,2\}$ and $j_0, j_1 \in \{3,4\}$,

$$\frac{\partial}{\partial k_{i_1 j_1}} \frac{\partial g}{\partial k_{i_0 j_0}} = -\Delta_{i_0 j_0} \frac{\partial \sigma_{i_0 j_0}}{\partial k_{i_1 j_1}}.$$

Recall that $\Delta_{i_0 j_0} > 0$. Hence, to prove that g is convex, we must prove that the matrix

$$M = \begin{pmatrix}
\frac{\partial\sigma_{13}}{\partial k_{13}} & \frac{\partial\sigma_{13}}{\partial k_{14}} & \frac{\partial\sigma_{13}}{\partial k_{23}} & \frac{\partial\sigma_{13}}{\partial k_{24}} \\
\frac{\partial\sigma_{14}}{\partial k_{13}} & \frac{\partial\sigma_{14}}{\partial k_{14}} & \frac{\partial\sigma_{14}}{\partial k_{23}} & \frac{\partial\sigma_{14}}{\partial k_{24}} \\
\frac{\partial\sigma_{23}}{\partial k_{13}} & \frac{\partial\sigma_{23}}{\partial k_{14}} & \frac{\partial\sigma_{23}}{\partial k_{23}} & \frac{\partial\sigma_{23}}{\partial k_{24}} \\
\frac{\partial\sigma_{24}}{\partial k_{13}} & \frac{\partial\sigma_{24}}{\partial k_{14}} & \frac{\partial\sigma_{24}}{\partial k_{23}} & \frac{\partial\sigma_{24}}{\partial k_{24}}
\end{pmatrix}$$

is positive semidefinite. We do so by checking that the principal minors of M are nonnegative. Since $\sigma_{13} + \sigma_{24} = \sigma_{14} + \sigma_{23}$, we have $\det(M) = 0$. Lemma 7.2 shows that the 1×1 principal minors are all positive.

Now we consider the 2×2 principal minors. By symmetry, it suffices to consider

$$\det \begin{pmatrix} \frac{\partial \sigma_{13}}{\partial k_{13}} & \frac{\partial \sigma_{13}}{\partial k_{14}} \\ \frac{\partial \sigma_{14}}{\partial k_{13}} & \frac{\partial \sigma_{14}}{\partial k_{14}} \end{pmatrix} \qquad \text{and} \qquad \det \begin{pmatrix} \frac{\partial \sigma_{13}}{\partial k_{13}} & \frac{\partial \sigma_{13}}{\partial k_{24}} \\ \frac{\partial \sigma_{24}}{\partial k_{13}} & \frac{\partial \sigma_{24}}{\partial k_{24}} \end{pmatrix}.$$

Lemma 7.2 shows that both determinants are positive.

It remains to consider the 3×3 principal minors; by symmetry, we need only consider the leading 3×3 minor. Once again, it is trivial to check that the determinant is positive using Lemma 7.2. □

Lemma 7.8. *Let $y = -1.18$. Then the minimum value of $g(\vec{k})$ for $\vec{k} \in [\frac{1}{2}, 1]^4$ occurs when $\vec{k} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.*

Proof. By concavity of g, it suffices to check the vertices of $[\frac{1}{2}, 1]^4$. By symmetry, we need only consider

$$g\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \approx 3.49963,$$
$$g\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \approx 3.72,$$
$$g\left(1, 1, \frac{1}{2}, \frac{1}{2}\right) \approx 3.92,$$
$$g\left(1, \frac{1}{2}, \frac{1}{2}, 1\right) \approx 3.97,$$
$$g\left(1, 1, 1, \frac{1}{2}\right) \approx 4.15,$$
$$g\left(1, 1, 1, 1\right) \approx 4.35.$$
□

8. Conclusion

Lemmas 5.18 and 7.1 show that, whenever $m \geq 1000$ and $y \geq -1.18$, we have $-\frac{1}{2m}\frac{f'}{f}(y, a) > 1$. Hence inequality (3.6) holds:

$$\frac{\mathrm{Reg}_{K_1,K_2}(L)}{\#\mu_L} \geq 2^{3-|\mathcal{A}_L|}\pi^{-r_2(L)/2}\left(-1 - \frac{1}{2m}\frac{f'}{f}(y, 1)\right) f(y, 1).$$

We also have, in the notation of the previous section,

$$f(y, 1) \geq \frac{\exp(mg(\vec{k}))}{(2\pi)^{3/2} m^{3/2} \sqrt{P(\vec{k})}} \left(1 - \frac{378.1}{m}\right),$$

$$-\frac{1}{2m}\frac{f'}{f}(y,1) \geq (\sigma_1+\sigma_2+\sigma_3)\left(1-\frac{25.971}{m-378.1}\right).$$

Thus Lemmas 7.1, 7.6, and 7.8 show that, for $m \geq 1000$,

$$f(-1.18,1) > \frac{e^{3.49962m}}{(2\pi)^{3/2}m^{3/2}\sqrt{111.78}}\left(1-\frac{378.1}{m}\right) > (1.866\times 10^{-8})e^{3.4995m},$$

$$-1-\frac{1}{2m}\frac{f'}{f}(-1.18,1) \geq -1+1.0572\left(1-\frac{25.971}{m-378.1}\right) > 0.0287.$$

Hence for $m \geq 1000$,

$$\frac{\mathrm{Reg}_{K_1,K_2}(L)}{\#\mu_L} > (4.28\times 10^{-9})e^{3.4995m}2^{-|\mathcal{A}_L|}\pi^{-r_2(L)/2}.$$

Note that $|\mathcal{A}_L| + r_2(L) = [L:\mathbb{Q}] = 4m$. Thus,

$$2^{-|\mathcal{A}_L|}\pi^{-r_2(L)/2} = 2^{-4m+r_2(L)}\pi^{-r_2(L)/2} \geq 16^{-m},$$

as $2^{-4m+r_2}\pi^{-r_2/2}$ is minimized (as a function of r_2) when $r_2 = 0$. We conclude that, for $[L:K] \geq 1000$,

$$\frac{\mathrm{Reg}_{K_1,K_2}(L)}{\#\mu_L} \geq (4.28\times 10^{-9})e^{3.4995m}16^{-m} > (4.28\times 10^{-9})\cdot 2.0686^m.$$

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