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Ernst-Ulrich GEKELER

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On Drinfeld modular forms of higher rank

par Ernst-Ulrich GEKELER

to the memory of David Goss

Résumé. Nous étudions les formes modulaires pour le groupe $\Gamma = \mathrm{GL}(r, \mathbb{F}_q[T])$ sur l'espace symétrique Ω^r de Drinfeld, où $r \geq 2$. Parmi nos résultats, on a l'existence d'une racine $(q-1)$-ième (à une constante près) h de la fonction discriminant Δ, la description de la (dé-)croissance des formes élémentaires $g_1, g_2, \ldots, g_{r-1}, \Delta$ dans le domaine fondamental $\mathcal{F}$ de Γ, et la réduction de ces formes sur la partie centrale $\mathcal{F}_{\boldsymbol{o}}$ de $\mathcal{F}$. Nous étudions avec plus de détail le cas de $r = 3$.

Abstract. We study Drinfeld modular forms for the modular group $\Gamma = \mathrm{GL}(r, \mathbb{F}_q[T])$ on the Drinfeld symmetric space Ω^r, where $r \geq 2$. Results include the existence of a $(q-1)$-th root (up to constants) h of the discriminant function Δ, the description of the growth/decay of the standard forms $g_1, g_2, \ldots g_{r-1}$, Δ on the fundamental domain $\mathcal{F}$ of Γ, and the reduction of these forms on the central part $\mathcal{F}_{\boldsymbol{o}}$ of $\mathcal{F}$. The results are exemplified in detail for $r = 3$.

Introduction

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field and $A = \mathbb{F}_q[T]$ be the polynomial ring in an indeterminate T, with field of fractions $K = \mathbb{F}_q(T)$. Furthermore, $K_\infty = \mathbb{F}_q((1/T))$ is the completion of K at infinity, with completed algebraic closure $\mathbb{C}_\infty$. The Drinfeld symmetric space $\Omega^r \subset \mathbb{P}^{r-1}(\mathbb{C}_\infty)$, where $r \geq 2$, is acted upon by $\Gamma := \mathrm{GL}(r, A)$, and the quotient $\Gamma \backslash \Omega^r$ parametrizes classes of A-lattices Λ of rank r in $\mathbb{C}_\infty$, that is, of Drinfeld modules of rank r. Such a Drinfeld module ϕ, corresponding to $\boldsymbol{\omega} \in \Omega^r$, is given by an operator polynomial

$$\phi_T(X) = TX + g_1 X^q + \cdots + g_{r-1} X^{q^{r-1}} + g_r X^{q^r},$$

where the coefficients $g_i = g_i(\boldsymbol{\omega})$ depend on $\boldsymbol{\omega}$, and the discriminant $\Delta := g_r$ is nowhere zero. The dependence is such that the g_i are modular forms for Γ, i.e., holomorphic, with a functional equation of the usual type under

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$\boldsymbol{\omega} \longmapsto \gamma\boldsymbol{\omega}$ ($\gamma \in \Gamma$), and regular at infinity. For $r = 2$, such Drinfeld modular forms (and their generalizations to congruence subgroups of $\Gamma = \mathrm{GL}(2, A)$) were introduced by David Goss in his 1977 Harvard thesis and his papers [10, 11, 12], and further studied by the present author in the 1980's. The aim of this work is to generalize results known for $r = 2$ (notably about the growth/decay of such forms, and the location of their zeroes) to larger ranks r.

The plan of the paper is as follows.

In the first section, we sketch the background on Drinfeld modules/modular forms and introduce notation. It doesn't contain any new material. In the second section, the relationship between Ω^r and the Bruhat–Tits building $\mathcal{BT}$ of $\mathrm{PGL}(r, K_\infty)$ is explained. This enables us to visualize the fundamental domain $\mathcal{F} \subset \Omega^r$ for Γ via a standard Weyl chamber W in the realization $\mathcal{BT}(\mathbb{R})$ of $\mathcal{BT}$.

We introduce the basic division functions μ_i ($1 \leq i \leq r$) in Section 3. The μ_i form an $\mathbb{F}$-basis of the T-torsion of the generic Drinfeld module $\phi^{\boldsymbol{\omega}}$, where $\boldsymbol{\omega}$ runs through Ω^r. They are modular forms of negative weight -1 for the congruence subgroup $\Gamma(T)$ of Γ, and are the key objects to get control over the g_i and Δ. As a first consequence, we construct the form h, which satisfies $h^{q-1} = \frac{(-1)^r}{T}\Delta$ and is modular of weight $(q^r - 1)/(q - 1)$ and type 1, see Theorem 3.8.

The systematic study of the μ_i is given in Section 4. We give the increments of $\log_q |\mu_i(\boldsymbol{\omega})|$, regarded as functions on the Weyl chamber W, when $\boldsymbol{k} \in W(\mathbb{Z})$ is replaced by a neighboring vertex $\boldsymbol{k'}$ (Proposition 4.10). From this we deduce similar results for Δ and the g_i (Theorem 4.13 and its Corollaries 4.15 and 4.16). These results contain certain combinatorial numbers $v_{\boldsymbol{k},i}^{(\ell)}$, which are investigated in the fifth section. We find an explicit and easy-to-evaluate expression in (5.3), which gives the final version Theorem 5.5 of Theorem 4.13 on the increments of $\log_q |\Delta(\boldsymbol{\omega})|$. We also find the direction of largest descent of $|\Delta|$; surprisingly, it strongly depends on the starting point $\boldsymbol{k}$ (Theorem 5.9).

In Section 6 we study the behavior of $g_1, \ldots, g_{r-1}, g_r = \Delta$ on

$$\mathcal{F}_{\boldsymbol{o}} = \{(\omega_1, \ldots, \omega_{r-1}, 1) \in \Omega^r \mid |w_1| = \cdots = |w_{r-1}| = 1\}$$

and the canonical reductions of the vanishing loci $V(g_i) \cap \mathcal{F}_{\boldsymbol{o}}$ in $\Omega^r(\overline{\mathbb{F}})$ (Theorem 6.2). In particular, $V(g_i) \cap \mathcal{F}_{\boldsymbol{o}}$ is non-empty.

In the final section, the case of $r = 3$ is considered in more detail. Besides tables with values of some of the functions treated, we give a brief study of g_1 at the wall $\mathcal{F}_2$ of $\mathcal{F}$ (where the zeroes of g_1 are located), and of g_2 at $\mathcal{F}_1$ (which encompasses the zeroes of g_2).

Notation.

- $\mathbb{F}$ denotes throughout the finite field $\mathbb{F}_q$ with q elements, with algebraic closure $\overline{\mathbb{F}}$, and $\mathbb{F}^{(m)}$ is the unique field extension of degree m of $\mathbb{F}$ in $\overline{\mathbb{F}}$.
- $A = \mathbb{F}[T]$ is the polynomial ring in an indeterminate T, with field of fractions $K = \mathbb{F}(T)$. The completion at infinity of K is $K_\infty = \mathbb{F}((\pi))$, with ring of integers $O_\infty = \mathbb{F}[\![\pi]\!]$, where $\pi := T^{-1}$. We write $\mathbb{C}_\infty$ for the completed algebraic closure of K_∞, $O_{\mathbb{C}_\infty}$ for its ring of integers, and fix an identification of $\overline{\mathbb{F}}$ with the residue class field of $O_{\mathbb{C}_\infty}$. Then $x \longmapsto \overline{x}$ is the canonical map from $O_{\mathbb{C}_\infty}$ to $\overline{\mathbb{F}}$, with congruence relation $x \equiv y \Longleftrightarrow \overline{x} = \overline{y}$. We normalize the absolute value $|\cdot|$ on K_∞ by $|T| = q$ and also write $|\cdot|$ for its unique extension to $\mathbb{C}_\infty$.
- $\log : \mathbb{C}_\infty^* \longrightarrow \mathbb{Q}$ is the map $x \longmapsto \log_q |x|$, and $\deg : A \longrightarrow \{-\infty\} \cup \mathbb{N}_0$ is the degree map, with $\deg(0) = -\infty$, with the usual conventions. For some fixed natural number $r \geq 2$, G denotes the group scheme $\mathrm{GL}(r)$, with its center Z of scalar matrices, and $\Gamma = G(A) = \mathrm{GL}(r, A)$.
- $\#(X)$ is the cardinality of the set X,
- $G \setminus X$ the space of G-orbits of the group G that acts on X.
- $\sum'_I$ (resp. $\prod'_I$) is the sum (or product) over the non-zero elements of the index set I.
- $(x_1 : \cdots : x_r)$ are projective coordinates in $\mathbb{P}^{r-1}$; mostly we normalize $x_r = 1$; in this case we write $(a_1, \ldots, a_{r-1}, a_r) = (a_1, \ldots, a_{r-1}, 1)$ for the corresponding point.

1. The basic set-up

(see e.g. [2], [5], [13, §4], [16]).

A *lattice* in $\mathbb{C}_\infty$ is a discrete $\mathbb{F}$-subspace Λ of $\mathbb{C}_\infty$, i.e., Λ intersects each ball in finitely many points. With such a Λ, we associate its *lattice function* $e_\Lambda : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$,

$$e_\Lambda(z) = z \sideset{}{'}\prod_{\lambda \in \Lambda} (1 - z/\lambda)\,, \tag{1.1}$$

where the prime $(\ )'$ indicates the product (or sum in other contexts) over the non-zero elements λ of Λ. Then e_Λ is an entire, surjective, $\mathbb{F}$-linear function with kernel Λ, and may be written

$$e_\Lambda(z) = z + \sum_{n \geq 1} \alpha_n(\Lambda) z^{q^n}.$$

The α_i are modular forms of weight $q^n - 1$, i.e.,

$$\alpha_n(c\Lambda) = c^{1-q^n} \alpha_n(\Lambda) \text{ if } c \in \mathbb{C}_\infty^*\,.$$

The *Eisenstein series* $E_k(\Lambda)$ is

$$E_k(\Lambda) = \sum_{\lambda\in\Lambda}{}' \lambda^{-k}, \tag{1.2}$$

which accordingly has weight k. Suppose that Λ is an A-lattice, that is, a free A-module of some rank $r \in \mathbb{N}$. With Λ we associate the Drinfeld A-module ϕ^Λ, which is characterized by the polynomial

$$\phi_T^\Lambda = TX + g_1(\Lambda)X^q + \cdots + g_{r-1}(\Lambda)X^{q^{r-1}} + g_r(\Lambda)X^{q^r}, \tag{1.3}$$

where the coefficients $g_1, \ldots, g_r$ are elements of $\mathbb{C}_\infty$ and the *discriminant* $\Delta(\Lambda) = g_r(\Lambda)$ is non-zero. The relation with Λ is through the functional equation

$$e_\Lambda(Tz) = \phi_T(e_\Lambda(z)), \tag{1.4}$$

which allows to determine the $\alpha_n(\Lambda)$ from the $g_i(\Lambda)$ and vice versa. In particular, one finds

$$g_i(c\Lambda) = c^{1-q^i} g_i(\Lambda). \tag{1.5}$$

Through $\Lambda \rightsquigarrow \phi^\Lambda$, isomorphism classes of Drinfeld A-modules of rank r correspond $1-1$ to classes of A-lattices of rank r up to scaling.

From now on we assume $r \geq 2$. Choosing an A-basis $\{\omega_1, \ldots, \omega_r\}$, the discreteness condition on Λ says that $\{\omega_1, \ldots, \omega_r\}$ is K_∞-linearly independent. Therefore we define the *Drinfeld symmetric space*

$$\begin{aligned} \Omega^r &:= \left\{ (\omega_1 : \ldots : \omega_r) \in \mathbb{P}^{r-1}(\mathbb{C}_\infty) \,\middle|\, \omega_1, \ldots, \omega_r \ K_\infty\text{-linearly independent} \right\} \\ &= \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \cup H, \end{aligned} \tag{1.6}$$

where H runs through the hyperplanes of $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$ defined over K_∞. The point set Ω^r has a natural structure as rigid analytic space [3, 8] over $\mathbb{C}_\infty$, namely as an open admissible subspace of $\mathbb{P}^{r-1}/\mathbb{C}_\infty$. Let Γ be the group $\mathrm{GL}(r, A)$, which acts as a matrix group from the left on $\mathbb{P}(\mathbb{C}_\infty)$, stabilizing Ω^r. By the above we find that the map

$$\left\{ \begin{matrix} \text{classes up to scaling of} \\ A\text{-lattices } \Lambda \text{ of rank } r \end{matrix} \right\} = \left\{ \begin{matrix} \text{isomorphism classes of} \\ \text{Drinfeld } A\text{-modules of rank } r \end{matrix} \right\} \xrightarrow{\cong} \Gamma \setminus \Omega^r, \tag{1.7}$$

which associates with the class of Λ the point $(\omega_1 : \cdots : \omega_r)$ determined by a basis $\{\omega_1, \ldots, \omega_r\}$ of Λ, is well-defined and bijective.

From now on we normalize projective coordinates of $\boldsymbol{\omega} := (\omega_1 : \cdots : \omega_r)$ on Ω^r by assuming $\omega_r = 1$, and write $(\omega_1, \ldots, \omega_r) = (\omega_1, \ldots, \omega_{r-1}, 1)$ for the corresponding point. Then $\gamma = (\gamma_{i,j}) \in \Gamma$ acts as

$$\gamma\boldsymbol{\omega} = \mathrm{aut}(\gamma, \boldsymbol{\omega})^{-1} \left(\ldots, \sum_i \gamma_{i,j}\omega_j, \ldots \right) \tag{1.8}$$

with $\mathrm{aut}(\gamma, \boldsymbol{\omega}) = \sum_{1 \le j \le n} \gamma_{n,j}\omega_j$. If $\Lambda_{\boldsymbol{\omega}}$ denotes the lattice $\sum_{1 \le i \le r} A\omega_i$, the function

$$\begin{aligned} g_i : \Omega^r &\longrightarrow \mathbb{C}_\infty \qquad (1 \le i \le r) \\ \boldsymbol{\omega} &\longmapsto g_i(\boldsymbol{\omega}) := g_i(\Lambda_{\boldsymbol{\omega}}) \end{aligned}$$

satisfies

$$g_i(\gamma\boldsymbol{\omega}) = \mathrm{aut}(\gamma, \boldsymbol{\omega})^{q^i-1}(\boldsymbol{\omega}). \tag{1.9}$$

Furthermore, g_i is holomorphic on Ω^r in the rigid analytic sense.

Regarding $g_1, \ldots, g_r = \Delta$ as indeterminates of respective weights $q^i - 1$, the open subscheme M^r given by $\Delta \neq 0$ of

$$\overline{M}^r := \mathrm{Proj}\ \mathbb{C}_\infty[g_1, \ldots, g_r]$$

is a moduli scheme for Drinfeld A-modules of rank r over $\mathbb{C}_\infty$, that is

$$\begin{aligned} \Gamma \setminus \Omega^r &\xrightarrow{\cong} M^r(\mathbb{C}_\infty) \\ \text{class of } \boldsymbol{\omega} &\longmapsto (g_1(\boldsymbol{\omega}) : \cdots : g_r(\boldsymbol{\omega})) \end{aligned} \tag{1.10}$$

is a bijection compatible with the analytic structures on both sides. Now $\overline{M}^r$ is a natural compactification of M^r ($\overline{M}^r$ is a projective $\mathbb{C}_\infty$-scheme containing M^r as an everywhere dense open subscheme), so we can give the following ad hoc definition.

Definition 1.11. A modular form of weight $k \in \mathbb{N}_0$ and type m (where m is a class in $\mathbb{Z}/(q-1)$) for $\Gamma = \mathrm{GL}(r, A)$ is a function $f : \Omega^r \longrightarrow \mathbb{C}_\infty$ that

(i) satisfies $f(\gamma\boldsymbol{\omega}) = \frac{\mathrm{aut}(\gamma,\boldsymbol{\omega})^k}{\det(\gamma)^m} f(\boldsymbol{\omega})$, $\gamma \in \Gamma$, $\boldsymbol{\omega} \in \Omega^r$;
(ii) is holomorphic and
(iii) is analytic along the divisor $(\Delta = 0)$ of $\overline{M}^r(\mathbb{C}_\infty)$.

Condition (iii) needs some explanation, which in the case $r = 2$ can be found e.g. in [5]. It is best understood in the following examples.

Examples 1.12.

(i) $g_i : \boldsymbol{\omega} \longmapsto g_i(\boldsymbol{\omega}) = g_i(\Lambda_{\boldsymbol{\omega}})$ is a modular form of weight $q^i - 1$ and type 0;
(ii) ditto for $\alpha_i : \boldsymbol{\omega} \longmapsto \alpha_i(\boldsymbol{\omega}) := \alpha_i(\Lambda_{\boldsymbol{\omega}})$;
(iii) For $k > 0$, $E_k : \boldsymbol{\omega} \longmapsto E_k(\boldsymbol{\omega}) := E_k(\Lambda_{\boldsymbol{\omega}})$ is modular of weight k and type 0. It doesn't vanish identically if and only if $k \equiv 0 \,(\bmod\ q-1)$.
(iv) In Theorem 3.8 we will present a $(q-1)$-th root h of $\Delta = g_n$ (more precisely, $h^{q-1} = \frac{(-1)^r}{T}\Delta$) which is modular of weight $(q^r-1)/(q-1)$ and type 1.

It can be shown that the $\mathbb{C}_\infty$-algebra of all modular forms of type 0 is a polynomial ring

$$\mathbb{C}_\infty[g_1, \ldots, g_r] = \mathbb{C}_\infty[\alpha_1, \ldots, \alpha_r] = \mathbb{C}_\infty[E_{q-1}, E_{q^2-1}, \ldots, E_{q^r-1}],$$

and the $\mathbb{C}_\infty$-algebra of all modular forms of arbitrary types is $\mathbb{C}_\infty[g_1, \dots, g_{r-1}, h]$, but we will not use this fact in the present work.

1.13. We define the set (recall that $\omega_r = 1$)

$$\mathcal{F} := \{\boldsymbol{\omega} \in \Omega^r \mid \boldsymbol{\omega} \text{ satisfies (i) and (ii) below}\},$$

where

(i) $|\omega_1| \geq |\omega_2| \geq \cdots \geq |\omega_r|$;
(ii) for $1 \leq i < r$, $|\omega_i| = \min_{a_{i+1},\dots,a_r \in A^{r-i}} |\omega_i - \sum_{j>i} a_j\omega_j|$.

As is shown in [4], $\mathcal{F}$ is an open admissible subspace of the analytic space Ω^r and a fundamental domain for Γ on Ω^r, in the sense that

1.14. Each $\boldsymbol{\omega} \in \Omega^r$ is Γ-equivalent with at least one and at most finitely many points of $\mathcal{F}$.

As uniqueness of the representative fails, this is much weaker than the classical notion of fundamental domain, but is the best we can achieve in our non-archimedean environment. Moreover,

1.15. If $\boldsymbol{\omega} \in \mathcal{F}$ and $x = \sum_{1 \leq i \leq r} a_i\omega_i$ ($a_i \in K_\infty$) belongs to the K_∞-space generated by $\{\omega_i \mid 1 \leq i \leq r\}$, then $|x| = \max_i |a_i\omega_i|$.

Since modular forms are determined by their restrictions to $\mathcal{F}$, natural questions arise.

Questions 1.16.

- *Describe the behavior of the g_i on $\mathcal{F}$, i.e., their absolute values $|g_i(\boldsymbol{\omega})|$;*
- *Describe $|g_i(\boldsymbol{\omega})|$ if $\boldsymbol{\omega}$ "tends to infinity";*
- *What are the zero loci $V(g_i) \cap \mathcal{F}$ of the g_i?*

and similar questions for other natural modular forms like α_n, E_k. We will find satisfactory answers to some of these as far as the g_i (and the E_k) are concerned, and leave the case e.g. of the α_n for further study.

2. Geometry of Ω^r and the Bruhat–Tits building $\mathcal{BT}$

(see [1, 2, 16]).

2.1. We let G be the reductive group scheme $\mathrm{GL}(r)$, where $r \geq 2$, with center Z of scalar matrices, B the standard Borel subgroup of upper triangular matrices and $T \subset B$ the standard torus of diagonal matrices.

The Bruhat–Tits building $\mathcal{BT}$ of $G(K_\infty)/Z(K_\infty)$ is a contractible simplicial complex endowed with an effective simplicial action of $G(K_\infty)/Z(K_\infty)$. Its set of vertices is

$$V(\mathcal{BT}) = \text{set of homothety classes } [L] \text{ of } O_\infty\text{-lattices}$$
$$(= \text{free } O_\infty\text{-submodules } L \text{ up to scaling) of rank } r \text{ of } K_\infty^r.$$

As $G(K_\infty)$ acts transitively on $V(\mathcal{BT})$, it may be identified with $G(K_\infty)/Z(K_\infty)\cdot\mathcal{K}$, where $\mathcal{K} = G(O_\infty)$ is the stabilizer of the standard lattice $L_0 = O_\infty^r$. The vertices $[L_0],\dots,[L_m]$ form a simplex if and only if they are represented by lattices $L_0,\dots,L_m$ such that $L_0 \supsetneq L_1 \supsetneq L_2 \supsetneq \cdots \supsetneq L_m \supsetneq \pi L_0$. Thus

- simplices have dimensions less or equal to $r-1$;
- each simplex is contained in a simplex of maximal dimension $r-1$;
- simplices are naturally ordered up to cyclic permutations of their vertices.

2.2. As usual, we write $\mathcal{BT}(\mathbb{R})$ for the realization of $\mathcal{BT}$, $\mathcal{BT}(\mathbb{Q})$ for the subset of $\mathcal{BT}(\mathbb{R})$ of points with rational barycentric coordinates, and $\mathcal{BT}(\mathbb{Z})$ for the set $V(\mathcal{BT})$ of vertices.

Let $\mathfrak{A}$ be the apartment of $\mathcal{BT}$ defined by the torus T, i.e., the full subcomplex with set of vertices

$$\mathfrak{A}(\mathbb{Z}) = V(\mathfrak{A}) = T(K_\infty)[L_0] = \{[L_{\boldsymbol{k}}] \mid \boldsymbol{k} = (k_1,\dots,k_r) \in \mathbb{Z}^r\},$$

where

$$L_{\boldsymbol{k}} = (\pi^{-k_1}O_\infty,\dots,\pi^{-k_r}O_\infty) \subset K_\infty^r.$$

Clearly, $L_0 = L_{\boldsymbol{o}}$, where $\boldsymbol{o} = (0,\dots,0)$ and $[L_{\boldsymbol{k}}] = [L_{\boldsymbol{k}'}]$ if and only if $\boldsymbol{k}'-\boldsymbol{k} = (k,k,\dots,k)$ for some $k \in \mathbb{Z}$. $\mathfrak{A}(\mathbb{R})$ is an euclidean affine space with translation group $(T(K_\infty)/Z(K_\infty)T(O_\infty))\otimes\mathbb{R} \cong \mathbb{R}^{r-1}$. As we dispose of the natural origin $O = [L_0]$, we identify $\mathfrak{A}(\mathbb{R})$ with $(T(K_\infty)/Z(K_\infty)T(O_\infty))\otimes\mathbb{R}$.

We let $\{\alpha_i \mid 1 \le i \le r-1\}$ be the simple roots of T with respect to the Borel subgroup B. That is, $\alpha_i \in \mathrm{Hom}(T,\mathbb{G}_m)$ is the homomorphism

$$\begin{pmatrix} t_1 & \dots & 0 \\ & \ddots & \\ 0 & \dots & t_r \end{pmatrix} \to t_i/t_{i+1}$$

from T to the multiplicative group $\mathbb{G}_m$. It induces the linear form, also denoted by $\alpha_i : \mathfrak{A}(\mathbb{R}) \longrightarrow \mathbb{R}$ given on integral points by $[L_{\boldsymbol{k}}] \longmapsto k_i - k_{i+1}$.

The choice of B determines the Weyl chamber $W = \{x \in \mathfrak{A}(\mathbb{R}) \mid \alpha_i(x) \ge 0 \text{ for } i = 1,2,\dots,r-1\}$. We let $W_i := \{x \in W \mid \alpha_i(x) = 0\}$ be the i-th wall of W. As a matter of fact, W is a fundamental domain (in the classical sense) for the action of $\Gamma = G(A)$ on $\mathcal{BT}(\mathbb{R})$. That is, each point $x \in \mathcal{BT}(\mathbb{R})$ ist Γ-equivalent with a unique $y \in W$ (although $\gamma \in \Gamma$ with $\gamma x = y$ need not be uniquely determined). We write $W(\mathbb{Z})$ for $W \cap \mathfrak{A}(\mathbb{Z})$, $W(\mathbb{Q})$ for $W \cap \mathfrak{A}(\mathbb{Q})$, etc.

2.3. There is a natural map that relates the symmetric space Ω^r with $\mathcal{BT}$. We first note that, by the theorem of Goldman–Iwahori [9], $\mathcal{BT}(\mathbb{R})$ may be naturally identified with the space of homothety classes of real-valued non-archimedean norms on the K_∞-vector space K_∞^r. Here the vertex $[L]$

corresponds to the class $[\nu]$ of norms whose unit ball is the O_∞-lattice L in K_∞^r. (For the description of $\lambda(x)$ for non-integral points of $\mathcal{BT}(\mathbb{R})$, see [2, Chapitre II]) Observing that each $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r = 1) \in \Omega^r$ determines a norm $\nu_{\boldsymbol{\omega}}$ with values in $q^{\mathbb{Q}} \cup \{0\}$ through

$$\nu_{\boldsymbol{\omega}}(x_1, \ldots, x_r) := \Big| \sum_{1 \leq i \leq r} x_i \boldsymbol{\omega}_i \Big|,$$

we let

$$\lambda : \Omega^r \longrightarrow \mathcal{BT}(\mathbb{Q})$$

be the map induced by $\boldsymbol{\omega} \longmapsto \nu_{\boldsymbol{\omega}}$. This *building map* has the following properties:

- λ regarded as a map to $\mathcal{BT}(\mathbb{Q})$ is surjective;
- λ is $G(K_\infty)$-equivariant.

2.4. The description of λ is at the base of describing the geometry of Ω^r. Viz, the pre-images $\lambda^{-1}(\sigma)$ of simplices σ of $\mathcal{BT}$ are affinoid spaces (even rational subdomains of $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$), which are glued together according to the incidence relations in $\mathcal{BT}$. In what follows, we describe the pre-images of vertices v. Since $G(K_\infty)$ acts transitively, it suffices to restrict to the case $v = [L_o]$.

2.5. As is immediate from the definition of λ, each $(\omega_1, \ldots, \omega_{r-1}, 1) \in \lambda^{-1}([L_o])$ satisfies $|\omega_1| = \cdots = |\omega_r| = 1$. We let $x \longmapsto \overline{x}$ be the reduction map from the valution ring $O_{\mathbb{C}_\infty}$ to its residue class field $\overline{\mathbb{F}}$. For $\omega_1, \ldots, \omega_r \in O_{\mathbb{C}_\infty}$ with $|\omega_i| = 1$, we have: $\{\omega_1, \ldots, \omega_r\}$ is K_∞-linearly independent $\Longleftrightarrow$ $\{\omega_1, \ldots, \omega_r\}$ is O_∞-linearly independent $\Longleftrightarrow$ $\{\overline{\omega}_1, \ldots, \overline{\omega}_r\}$ is $\mathbb{F}$-linearly independent, by Nakayama's lemma. Hence $\lambda^{-1}([L_0])$ is the inverse image under the reduction map $\mathbb{P}^{r-1}(\mathbb{C}_\infty) = \mathbb{P}^{r-1}(O_{\mathbb{C}_\infty}) \xrightarrow{\text{red}} \mathbb{P}^{r-1}(\overline{\mathbb{F}})$ of the complement of the union of the finitely many hyperplanes $H \subset \mathbb{P}^{r-1}(\overline{\mathbb{F}})$ which are defined over $\mathbb{F}$.

In contrast with the normalization $\omega_r = 1$ of (1.7), we assume until the end of §2.5 that points $\boldsymbol{\omega} = (\omega_1 : \cdots : \omega_r)$ of $\mathbb{P}^{r-1}(\mathbb{C}_\infty)$ are given in coordinates with $\max |\omega_i| = 1$. Let H be defined by the vanishing of the linear form $\ell_H : \mathbb{F}^r \longrightarrow \mathbb{F}$. Using the inclusions $\mathbb{F} \hookrightarrow \overline{\mathbb{F}} \hookrightarrow O_{\mathbb{C}_\infty} \hookrightarrow \mathbb{C}_\infty$, we extend it uniquely to an $O_{\mathbb{C}_\infty}$-linear form also labelled $\ell_H : O_{\mathbb{C}_\infty}^r \longrightarrow O_{\mathbb{C}_\infty}$.

Put $S_H := \{\boldsymbol{\omega} = (\omega_1 : \cdots : \boldsymbol{\omega}_r) \in \mathbb{P}^{r-1}(O_{\mathbb{C}_\infty}) \,|\, |\ell_H(\omega_1, \ldots, \omega_r)| < 1\}$, which is well-defined independently of choices made. Then

$$\lambda^{-1}([L_0]) = \mathbb{P}^{r-1}(O_{\mathbb{C}_\infty}) \setminus \cup S_H\,,$$

where H runs through the hyperplanes of $\mathbb{P}^{r-1}(\mathbb{F})$, i.e., the finitely many points of the dual space $\check{\mathbb{P}}(\mathbb{F})$. It is well-known that such a space is an admissible open affinoid subspace of the analytic space $\mathbb{P}^{r-1}/\mathbb{C}_\infty$, and in fact a rational subdomain [3, 8]. Its canonical reduction is the scheme $\mathbb{P}^{r-1}/\mathbb{F} \setminus \cup H$,

H as above. We put $\Omega^r(\overline{\mathbb{F}}) : \mathbb{P}^{r-1}(\overline{\mathbb{F}}) \setminus \cup H(\overline{\mathbb{F}})$ for its underlying set of geometric points.

2.6. The relationship between the fundamental domains $\mathcal{F} \subset \Omega^r$ and $W \subset \mathfrak{A}(\mathbb{R}) \subset \mathcal{BT}(\mathbb{R})$ is simply

$$\lambda(\mathcal{F}) = W(\mathbb{Q}),\ \lambda^{-1}(W) = \mathcal{F},$$

as a direct consequence of the definitions. For later use, we fix some notation. For $1 \leq i \leq r-1$ we let $\mathcal{F}_i = \lambda^{-1}(W_i) = \{\boldsymbol{\omega} \in \mathcal{F} \mid |\omega_i| = |\omega_{i+1}|\}$ be the i-th wall of $\mathcal{F}$. Recall that we have normalized $\omega_r = 1$. Therefore, for $\boldsymbol{k} = (k_1, k_2, \dots, k_r) \in \mathbb{N}_0^r$ with $k_1 \geq k_2 \geq \cdots \geq k_r = 0$, the pre-image $\mathcal{F}_{\boldsymbol{k}} := \lambda^{-1}([L_{\boldsymbol{k}}])$ of the vertex $[L_{\boldsymbol{k}}]$ of $\mathcal{BT}$ equals $\{\boldsymbol{\omega} \in \mathcal{F} \mid |\omega| = q^{k_i}, 1 \leq i \leq r\}$.

2.7. Next we consider holomorphic functions on Ω^r. For an admissible open $U \subset \Omega^r$, let $\mathcal{O}(U)$ be the ring of holomorphic functions on U, with unit group $\mathcal{O}(U)^*$. For U affinoid, we let $\|f\|_U$ be the spectral norm $\sup_{x\in U} |f(x)|$ of $f \in \mathcal{O}(U)$. It follows from §2.5 that for each vertex v and each $f \in \mathcal{O}(\lambda^{-1}(v))^*$, f has constant absolute value $|f(x)| = \|f\|_{\lambda^{-1}(v)}$. (Upon scaling, we may assume $\|f\|_{\lambda^{-1}(v)} = 1$. Then the reduction $\overline{f}$ of f is a rational function on $\mathbb{P}^{r-1}(\overline{\mathbb{F}})$ with zeroes or poles at most along the $\mathbb{F}$-rational hyperplanes, so f itself has constant absolute value 1.)

Suppose now that $f \in \mathcal{O}(\Omega^r)^*$ is a global unit. Then its absolute value $|f|$ is constant on fibers of λ, that is, $|f|$ may be considered as a function on $\mathcal{BT}(\mathbb{Q})$. Instead of $|f|$, we mostly consider

$$\log f := \log_q |f|.$$

That function interpolates linearly, i.e., if $x = \sum t_i v_i$ belongs to the simplex $\{v_i\}$ with barycentric coordinates t_i, then $\log f(x) = \sum t_i \log f(v_i)$.

2.8. We can say more. Let $e = (v, w)$ be an oriented 1-simplex of $\mathcal{BT}$, an *arrow* for short. We define the *van der Put value* of f on e through

$$P(f)(e) := \log_q \frac{|f(w)|}{|f(v)|} = \log f(w) - \log f(v).$$

It is an integer, which can be determined as follows. Apparently,

(i) $P(f)(\overline{e}) + P(f)(e) = 0$, if $\overline{e}$ is the arrow e with reverse orientation, and

(ii) $\sum_e P(f)(e) = 0$, if the e run through the arrows of a closed path in $\mathcal{BT}$.

Now suppose that $e = (v, w)$ with $v = [L]$, $w = [L']$, where $\pi L \subset L' \subset L$ and $\dim_{\mathbb{F}}(L/L') = 1$. Call such an arrow *special*. By §2.5, the special arrows with origin $o(e) = v$ correspond one-to-one to the points of the dual projective space $\check{\mathbb{P}}(L/\pi L)$ over $\mathbb{F}$.

If f is normalized such that $|f| = 1$ on $\lambda^{-1}(v)$ then its reduction $\overline{f}$ has vanishing order $m \in \mathbb{Z}$ along the hyperplane H of $\mathbb{P}(L/\pi L) = \mathbb{P}^{r-1}(\mathbb{F})$ that corresponds to L' (see §2.5). Then $P(f)(e) = -m$ (positive if $\overline{f}$ has a pole along H). As each e is homotopic with a path composed of special arrows, (i) and (ii) suffice to determine $P(f)(e)$.

We note another property of $P(f)$. As $\overline{f}$ is a rational function on $\mathbb{P}(L/\pi L) \times \overline{\mathbb{F}} \cong \mathbb{P}^{r-1}/\overline{F}$ with zeroes and poles at most at the $\mathbb{F}$-rational hyperplanes, it may be written as

$$\overline{f} = \text{const} \prod \ell_H^{m(H)}$$

with $m(H) \in \mathbb{Z}$, $\sum m(H) = 0$, where H runs through the $\mathbb{F}$-rational hyperplanes and ℓ_H is a linear form corresponding to H. This shows that

(iii) $\sum_{\substack{e \text{ special} \\ o(e)=v}} P(f)(e) = 0$ for each vertex v,

where the sum is extended over the special arrows e with origin $o(e) = v$. We let $\boldsymbol{H}(\mathcal{BT}, \mathbb{Z})$ be the group of $\mathbb{Z}$-valued functions on the set of arrows (=oriented 1-simplices) of $\mathcal{BT}$ that satisfy conditions (i), (ii) and (iii).

Proposition 2.9. *The van der Put map*

$$\begin{aligned} P:\ \mathcal{O}(\Omega^r)^* &\longrightarrow \boldsymbol{H}(\mathcal{BT}, \mathbb{Z}) \\ f &\longmapsto P(f), \end{aligned}$$

where $P(f)$ evaluates on the arrow $e = (v, w)$ as

$$P(f)(e) = \log f(w) - \log f(v) = \log_q \left| \frac{f(w)}{f(v)} \right|$$

is a well-defined group homomorphism and equivariant with respect to the natural actions of $G(K_\infty)$. Its kernel is the subgroup $\mathbb{C}_\infty^*$ of non-zero constant functions on Ω^r.

Proof. The well-definedness comes from the preceding considerations; homomorphy and $G(K_\infty)$-equivariance are then obvious. Further, $\ker(P) = \mathbb{C}_\infty^*$ is a formal consequence of the fact ([16] Proposition 4) that Ω^r is a Stein space [14]. □

Remark 2.10. Marius von der Put defined the above map P and derived its main properties in [15] in the case $r = 2$. This was the starting point for the study of the action of arithmetic groups on $\boldsymbol{H}(\mathcal{BT}, \mathbb{Z})$ in [7]. Our present aim is to calculate the invertible function Δ on Ω^r (and the companion functions $g_1, \ldots, g_{r-1}$)) by determining $P(\Delta)$. In view of §2.6, it suffices to find $P(\Delta)(e)$ for arrows e that belong to the Weyl chamber W.

3. The division functions

For $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{r-1}, 1) \in \Omega^r$, we let $\Lambda_{\boldsymbol{\omega}}$ be the A-lattice $\Lambda_{\boldsymbol{\omega}} = \sum_{1 \leq i \leq r} A\omega_i$, with lattice function $e_{\boldsymbol{\omega}} := e_{\Lambda_{\boldsymbol{\omega}}}$ and Drinfeld module $\phi^{\boldsymbol{\omega}} = \phi^{\Lambda_{\boldsymbol{\omega}}}$. Its T-division polynomial (1.3) may be factored as

$$\phi_T^{\boldsymbol{\omega}} = \Delta(\boldsymbol{\omega}) \prod (X - \mu)\,, \tag{3.1}$$

where μ runs through the set of its zeroes, which form an r-dimensional vector space ${}_T\phi^{\boldsymbol{\omega}}$ over $A/(T) = \mathbb{F}$. If $\{u\}$ is a system of representatives for $\Lambda_{\boldsymbol{\omega}}/T\Lambda_{\boldsymbol{\omega}}$ then ${}_T\phi^{\boldsymbol{\omega}} = \{e_{\boldsymbol{\omega}}(\frac{u}{T})\}$. In particular, the

$$\mu_i(\boldsymbol{\omega}) := e_{\boldsymbol{\omega}}\left(\frac{\omega_i}{T}\right) \quad (1 \leq i \leq r) \tag{3.2}$$

constitute an $\mathbb{F}$-basis of ${}_T\phi^{\boldsymbol{\omega}}$. Given $\boldsymbol{u} = (u_1, \ldots, u_r) \in \mathbb{F}^r$, we let

$$\mu_{\boldsymbol{u}} := \sum_{1 \leq i \leq r} u_i \mu_i\,.$$

As functions of $\boldsymbol{\omega}$ the $\mu_{\boldsymbol{u}}$ are holomorphic (this follows e.g. from Proposition 3.4 below) and vanish nowhere on Ω^r. Furthermore, for $\gamma \in \Gamma = \mathrm{GL}(r, A)$, the functional equation

$$\mu_{\boldsymbol{u}}(\gamma\boldsymbol{\omega}) = \mathrm{aut}(\gamma, \boldsymbol{\omega})^{-1} \mu_{\boldsymbol{u}\gamma}(\boldsymbol{\omega}) \tag{3.3}$$

holds, where $\boldsymbol{u}\gamma$ is right matrix multiplication by γ on the row vector $\boldsymbol{u} \in \mathbb{F}^r = (A/(T))^r$. (The proof is by straightforward calculation and thus omitted.) Hence $\mu_{\boldsymbol{u}}(\gamma\boldsymbol{\omega}) = \mathrm{aut}(\gamma, \boldsymbol{\omega})^{-1}\mu_{\boldsymbol{u}}(\omega)$ if $\gamma \in \Gamma(T) = \{\gamma \in \Gamma \,|\, \gamma \equiv 1 \pmod{T}\}$. That is, $\mu_{\boldsymbol{u}}$ is modular of weight -1 for the congruence subgroup $\Gamma(T)$. It is useful to dispose of the following well-known interpretation as reciprocal of an Eisenstein series.

Proposition 3.4.

$$\mu_{\boldsymbol{u}}(\boldsymbol{\omega})^{-1} = \sum_{\substack{\boldsymbol{a} \in K^r \\ \boldsymbol{a} \equiv T^{-1}\boldsymbol{u} \pmod{A^r}}} \frac{1}{a_1\omega_1 + \cdots + a_r\omega_r}$$

Proof. Let $E_{\boldsymbol{u}}(\boldsymbol{\omega})$ be the right hand side. It is equal to the lattice sum $\sum_{\lambda \in \Lambda_\omega} \frac{1}{T^{-1}\boldsymbol{u\omega} + \lambda}$, where $\boldsymbol{u\omega} = \sum u_i\omega_i$. Next we note that the derivative e'_Λ of a lattice function is the constant 1. Therefore, taking logarithmic derivatives,

$$\frac{1}{e_\Lambda(z)} = \frac{e'_\Lambda(z)}{e_\Lambda(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda}$$

as meromorphic functions on $\mathbb{C}_\infty$. We get

$$E_{\boldsymbol{u}}(\boldsymbol{\omega}) = \sum_{\lambda \in \Lambda_{\boldsymbol{\omega}}} \frac{1}{T^{-1}\boldsymbol{u\omega} + \lambda} = e_{\boldsymbol{\omega}}\left(\frac{\boldsymbol{u\omega}}{T}\right)^{-1} = \mu_{\boldsymbol{u}}(\boldsymbol{\omega})^{-1}. \qquad \square$$

From (3.1) and (1.3) we find

$$\Delta(\boldsymbol{\omega}) = T \prod_{\boldsymbol{u}\in\mathbb{F}^r}{}' \mu_{\boldsymbol{u}}(\boldsymbol{\omega})^{-1} = T \prod_{\boldsymbol{u}\in\mathbb{F}^r}{}' E_{\boldsymbol{u}}(\boldsymbol{\omega}). \tag{3.5}$$

More generally, we may express all the coefficients $g_i(\boldsymbol{\omega})$of $\phi_T^{\boldsymbol{\omega}}$ through the $\mu_{\boldsymbol{u}}$, viz: The polynomial

$$X^{q^r}\phi_T^{\boldsymbol{\omega}}(X^{-1}) = \Delta + g_{r-1}X^{q^r-q^{r-1}} + \cdots + g_1 X^{q^r-q} + TX^{q^r-1}$$

has the $\mu_{\boldsymbol{u}}^{-1}$ ($\boldsymbol{u} \neq \boldsymbol{o}$) as its zeroes; therefore by Vieta

$$g_i(\boldsymbol{\omega}) = T \cdot s_{q^i-1}\{\mu_{\boldsymbol{u}}^{-1} \mid \boldsymbol{o} \neq \boldsymbol{u} \in \mathbb{F}^r\}, \tag{3.6}$$

T times the (q^i-1)-th elementary symmetric function of the $\mu_{\boldsymbol{u}}^{-1} = E_{\boldsymbol{u}}$. Our strategy will be to study the behavior and notably the absolute values of the $\mu_{\boldsymbol{u}}$ on the fundamental domain $\mathcal{F}$ in order to get information about Δ and the g_i.

3.7. We call $\boldsymbol{o} \neq \boldsymbol{u} = (u_1, \ldots, u_r) \in \mathbb{F}^r$ *monic* if $u_i = 1$ for the largest subscript i with $u_i \neq 0$. The monic elements are representatives for the action of $\mathbb{F}^*$ on $\mathbb{F}^r \setminus \{0\}$. Accordingly, $\mu_{\boldsymbol{u}}$ is monic if $\boldsymbol{u}$ is monic.

Theorem 3.8. *We define the function h on Ω^r by*

$$h(\boldsymbol{\omega}) := \prod_{\substack{\boldsymbol{u}\in\mathbb{F}^r \\ \text{monic}}} \mu_{\boldsymbol{u}}(\boldsymbol{\omega})^{-1}.$$

Then $h^{q-1}(\boldsymbol{\omega}) = \frac{(-1)^r}{T}\Delta(\boldsymbol{\omega})$, and h is modular of weight $(q^r-1)/(q-1)$ and type 1 for Γ.

Proof. For $c \in \mathbb{F}^*$ we have $\mu_{c\boldsymbol{u}} = c\mu_{\boldsymbol{u}}$, so

$$T^{-1}\Delta = \prod_{\boldsymbol{u}}{}' \mu_{\boldsymbol{u}}^{-1} = \prod_{\substack{\boldsymbol{u} \text{ monic} \\ c\in\mathbb{F}^*}} \mu_{c\boldsymbol{u}}^{-1} = \prod_{\boldsymbol{u} \text{ monic}} (-\mu_{\boldsymbol{u}}^{1-q}) = (-1)^r h^{q-1},$$

where we have used $\prod_{c\in\mathbb{F}^*} c = -1$ and $(-1)^{(q^r-1)/(q-1)} = (-1)^r$. We must show that for $\gamma \in \Gamma = G(A) = \mathrm{GL}(r, A)$ the relation

$$h(\gamma\boldsymbol{\omega}) = \frac{\mathrm{aut}(\gamma, \boldsymbol{\omega})^{(q^r-1)/(q-1)}}{\det \gamma} h(\boldsymbol{\omega}) \tag{$*$}$$

holds. If $\gamma \in \Gamma(T)$, this follows immediately from (3.3), as in this case $\det(\gamma) = 1$ and $\boldsymbol{u}\gamma = \boldsymbol{u}$ for each $\boldsymbol{u} \in \mathbb{F}^r$. Now Γ is a semi-direct product $G(\mathbb{F})$ and $\Gamma(T)$, and it suffices to verify $(*)$ for $\gamma \in G(\mathbb{F})$.

Let M be the set of monics $\boldsymbol{u} \in \mathbb{F}^r$. For each $\gamma \in G(\mathbb{F})$, the set $M\gamma$ is still a set of representatives of $(\mathbb{F}^r \setminus \{0\})/\mathbb{F}^*$, that is $M\gamma = \{c_{\boldsymbol{u}}(\gamma)\boldsymbol{u} \mid \boldsymbol{u} \in M\}$ with scalars $c_{\boldsymbol{u}}(\gamma) \in \mathbb{F}^*$. Taking the product of (3.3) over the $\boldsymbol{u} \in M$, we find

$$h(\gamma\boldsymbol{\omega}) = \mathrm{aut}(\gamma, \boldsymbol{\omega})^{(q^r-1)/(q-1)} h(\boldsymbol{\omega}) \cdot c^{-1}(\gamma)$$

with $c(\gamma) = \prod_{\boldsymbol{u}\in M} c_{\boldsymbol{u}}(\gamma) \in \mathbb{F}^*$. As $\mathrm{aut}(\gamma, \boldsymbol{u})$ is a factor of automorphy, we find that $c : G(\mathbb{F}) \longrightarrow \mathbb{F}^*$ is a homomorphism, which necessarily is a power of the determinant. To find the exponent, it suffices to test on the matrix $\tau = \mathrm{diag}(t, 1, \ldots, 1)$. Then $\mathrm{aut}(\tau, \boldsymbol{\omega}) = 1$ and

$$c_{\boldsymbol{u}}(\tau) = \begin{cases} 1, & \text{if } \boldsymbol{u} \neq (1, 0, \ldots, 0) \\ t, & \text{if } \boldsymbol{u} = (1, 0, \ldots, 0). \end{cases}$$

This yields $c(\tau) = t = \det(\tau)$ and thus $c(\gamma) = \det(\gamma)$ for each $\gamma \in G(\mathbb{F})$. □

Remark 3.9. We leave aside the question of the "right" normalization of h and Δ, i.e., scalings such that $h^{q-1} = \pm\Delta$. For the case of $r = 2$, the rationality of expansion coefficients yields natural arithmetic normalizations such that $h^{q-1} = -\Delta$ [5].

4. Absolute values of modular forms

In this section we determine $|\mu_i(\boldsymbol{\omega})|$ for $\boldsymbol{\omega} \in \mathcal{F}$ and draw conclusions.

4.1. We assume that $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r)$ with $\omega_r = 1$, $|\omega_i| = q^{k_i}$ with $k_i \in \mathbb{Q}$, $k_1 \geq k_2 \geq \cdots \geq k_r = 0$. Now

$$\mu_i = \mu_i(\boldsymbol{\omega}) = e_{\boldsymbol{\omega}}\left(\frac{\omega_i}{T}\right) = \frac{\omega_i}{T} \prod_{\lambda\in\Lambda_{\boldsymbol{\omega}}}{}' \left(1 - \frac{\omega_i}{T\lambda}\right)$$

and

$$\left|1 - \frac{\omega_i}{T\lambda}\right| = \begin{cases} 1, & \text{if } |T\lambda| > |\omega_i| \\ |\frac{\omega_i}{T\lambda}|, & \text{if } |T\lambda| \leq |\omega_i|. \end{cases}$$

The latter results from §1.15 if $|T\lambda| = |\omega_i|$. Therefore, $|\mu_i|$ is the finite product

$$|\mu_i(\boldsymbol{\omega})| = \left|\frac{\omega_i}{T}\right| \prod_{\substack{\lambda \\ |T\lambda|\leq|\omega_i|}}{}' \left|\frac{\omega_i}{T\lambda}\right|. \tag{4.2}$$

A closer look to this formula reveals (for details, see [4, Proposition 3.4]):

Proposition 4.3.

(i) *For the $\mu_i = \mu_i(\boldsymbol{\omega})$ the following inequalities hold:*

$$|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_r|.$$

For some i with $1 \leq i < r$ we have equality $|\mu_i| = |\mu_{i+1}|$ if and only if $|\omega_i| = |\omega_{i+1}|$.

(ii) *Let $\mu_{\boldsymbol{u}} = \sum_{1\leq i\leq r} u_i\mu_i$ be as in (3.2). The absolute value $|\mu_{\boldsymbol{u}}(\boldsymbol{\omega})|$ equals $\mu_i(\boldsymbol{\omega})$, where i is minimal with $u_i \neq 0$.*

Moreover, under the same assumptions ([4, Corollary 3.6]):

Proposition 4.4. *If $g_i(\boldsymbol{\omega}) = 0$ for some $1 \leq i < r$ then $|\omega_{r-i}| = |\omega_{r-i+1}|$.*

Remarks 4.5.

(1) The reverse numbering in Proposition 4.4 comes from the fact that $\omega_r, \omega_{r-1}, \ldots, \omega_1$ in this order forms a successive minimum basis for $\Lambda_{\boldsymbol{\omega}}$.

(2) Let $V(g_i)$ be the vanishing locus of the function g_i on Ω^r. Proposition 4.4 asserts that $V(g_i) \cap \mathcal{F}$ is contained in $\lambda^{-1}(W_{r-i}) = \mathcal{F}_{r-i}$, see §2.6.

To evaluate (4.2), we may in view of §2.7 assume that $\lambda(\boldsymbol{\omega})$ is a vertex $[L_{\boldsymbol{k}}] \in W(\mathbb{Z})$, i.e., $\boldsymbol{\omega} \in \mathcal{F}_{\boldsymbol{k}}$. Thus, in addition to the assumptions in §4.1, from now on

$$\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbb{N}_0^r\,.$$

4.6. The case $\boldsymbol{k} = \boldsymbol{o} = (0, \ldots, 0)$ is simple. Here (4.2) and Proposition 4.3 give $|\mu_i(\boldsymbol{\omega})| = |T|^{-1} = |\mu_{\boldsymbol{u}}(\boldsymbol{\omega})|$ for each $\boldsymbol{o} \neq \boldsymbol{u} \in \mathbb{F}^r$. With (3.5) we find

$$|\Delta(\boldsymbol{\omega})| = |T|^{q^r} \text{ and } \log \Delta(\boldsymbol{\omega}) = q^r,$$

valid for $\boldsymbol{\omega} \in \mathcal{F}_{\boldsymbol{o}}$.

4.7. For $1 \leq \ell < r$ we let $\boldsymbol{k}_\ell$ be the vector $(1, 1, \ldots, 1, 0, \ldots, 0)$ with ℓ ones. Inside the euclidean space $\mathfrak{A}(\mathbb{R})$, $\{\boldsymbol{k}_\ell\}$ is the set of co-roots of the simple roots $\{\alpha_1, \ldots, \alpha_{r-1}\}$, i.e., $\alpha_i(\boldsymbol{k}_\ell) = \delta_{i,\ell}$ (Kronecker symbol), and $W(\mathbb{Z}) = W \cap \mathfrak{A}(\mathbb{Z})$ is the set of non-negative integral combinations of the $\boldsymbol{k}_\ell$.

4.8. Recall that "log" is the real-valued function $\log_q |\cdot|$ on $\mathbb{C}_\infty^*$. As $\log \mu_i(\boldsymbol{\omega})$ depends only on the coordinates $\boldsymbol{k} \in \mathbb{N}_0^r$ of $\boldsymbol{\omega}$, we write $\log \mu_i(\boldsymbol{k})$ for that quantity. It is fully determined by the ascending length filtration on the $\mathbb{F}$-vector space $\Lambda_{\boldsymbol{\omega}}$. To make this precise, we need the

Definition 4.9. For $\boldsymbol{k}$ as before and $1 \leq i \leq r$, we put

$$V_{\boldsymbol{k},i} := \{(a_{i+1}, \ldots, a_r) \in A^{r-i} \mid \deg a_j < k_i - k_j,\, i < j \leq r\},$$

an $\mathbb{F}$-vector subspace of A^{r-i} of dimension $(r-i)k_i - (k_{i+1} + \cdots + k_r)$. (Although $k_r = 0$, it is useful to keep it present in the notation.) For $i \leq \ell < r$ we define the subset

$$V_{\boldsymbol{k},i}^{(\ell)} := \left\{ \boldsymbol{a} = (a_{i+1}, \ldots, a_r) \in V_{\boldsymbol{k},i} \;\middle|\; \begin{array}{r} \max\limits_{i<j\leq\ell}(k_j + \deg a_j) \\ < \max\limits_{i<j\leq r}(k_j + \deg a_j) \\ \text{or } \boldsymbol{a} = \boldsymbol{o} \end{array} \right\}.$$

Further, $v_{\boldsymbol{k},i} := \#(V_{\boldsymbol{k},i})$, $v_{\boldsymbol{k},i}^{(\ell)} = \#(V_{\boldsymbol{k},i}^{(\ell)})$. The condition defining $V_{\boldsymbol{k},i}^{(\ell)}$ is empty for $\ell = i$, so $V_{\boldsymbol{k},i}^{(i)} = V_{\boldsymbol{k},i}$, and $V_{k,i}^{(r-1)} \subset V_{\boldsymbol{k},i}^{(r-2)} \subset \cdots \subset V_{\boldsymbol{k},i}^{(i)}$.

We are mainly interested in the growth of $\log \mu_i(\boldsymbol{k})$ under $\boldsymbol{k} \rightsquigarrow \boldsymbol{k}' := \boldsymbol{k} + \boldsymbol{k}_\ell$, which is described by the quantities just introduced.

Proposition 4.10. *Let* $1 \leq i \leq r$, $1 \leq \ell < r$. *Then*

$$\log \mu_i(\boldsymbol{k}+\boldsymbol{k}_\ell) - \log \mu_i(\boldsymbol{k}) = \begin{cases} v_{\boldsymbol{k},i}^{(\ell)}, & i \leq \ell \\ 0, & i > \ell. \end{cases}$$

Proof. Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_r) \in \mathcal{F}_{\boldsymbol{k}}$, $\boldsymbol{\omega}' = (T\omega_1, \ldots, T\omega_\ell, \omega_{\ell+1}, \ldots, \omega_r) \in \mathcal{F}_{\boldsymbol{k}'}$ with $\boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{k}_\ell$. If $i > \ell$ then the product (4.2) for $|\mu_i(\boldsymbol{\omega})|$ doesn't change upon replacing $\boldsymbol{\omega}$ with $\boldsymbol{\omega}'$. So assume $i \leq \ell$. The factors $|\frac{\omega_i}{T\lambda}|$ in (4.2) correspond to

$$\lambda = a_{i+1}\omega_{i+1} + \cdots + a_r\omega_r, \text{ where } \boldsymbol{o} \neq \boldsymbol{a} = (a_{i+1}, \ldots, a_r) \in V_{\boldsymbol{k},i}.$$

Again replacing $\boldsymbol{\omega}$ with $\boldsymbol{\omega}'$, such a factor is multiplied by q if $|a_{i+1}\omega_{i+1} + \cdots + a_\ell\omega_\ell| < |\lambda|$ (i.e., $\boldsymbol{a} \in V_{\boldsymbol{k},i}^{(\ell)}$), and is unchanged if $|a_{i+1}\omega_{i+1} + \cdots + a_\ell\omega_\ell| = |\lambda|$, as follows from §1.15. Ditto, $|\frac{\omega_i'}{T}| = q|\frac{\omega_i}{T}|$. Beyond those factors coming from the product for $|\mu_i(\boldsymbol{\omega})|$, the product (4.2) for $|\mu_i(\boldsymbol{\omega}')|$ contains factors $|\frac{\omega_i'}{T\lambda'}|$ with $|\omega_i| < |T\lambda'| \leq |\omega_i'|$, but for these, due to §4.1 applied to the primed situation, $|T\lambda'| = |\omega_i'|$ holds, and so they don't contribute to the product. □

Recall that $W(\mathbb{Z}) = W \cap \mathfrak{A}(\mathbb{Z})$ is ordered through the product order on the coefficients $a_\ell \in \mathbb{N}_0$ of $\boldsymbol{k} = \sum a_\ell \boldsymbol{k}_\ell$. We extend this order to $W(\mathbb{Q})$, i.e., allow coefficients in $\mathbb{Q}_{\geq 0}$.

Corollary 4.11. *The function* $\log \mu_i$ *on* $W(\mathbb{Q})$ *strictly increases in directions* $\boldsymbol{k}_\ell$ *for* $\ell \geq i$ *and is constant in directions* $\boldsymbol{k}_\ell$, $\ell < i$. *In particular,* $\log \mu_r$ *is constant on* $W(\mathbb{Q})$ *with value* -1, *and for* $i < r$, $\boldsymbol{k}_i$ *is a direction of maximal growth of* $\log \mu_i$.

Proof. This is Proposition 4.10, together with the fact that $\log \mu_i$ interpolates linearly from $W(\mathbb{Z})$ to $W(\mathbb{Q})$, the inequalities $v_{\boldsymbol{k},i}^{(r-1)} \leq v_{\boldsymbol{k},i}^{(r-2)} \leq \cdots \leq v_{\boldsymbol{k},i}^{(i)}$, and §4.6. □

Next, for $\boldsymbol{o} \neq \boldsymbol{u} \in \mathbb{F}^r$ let $\mu_{\boldsymbol{u}} = \sum u_i\mu_i$ be as in the last section. As before, $\log \mu_{\boldsymbol{u}}(\boldsymbol{\omega})$ depends only on $\boldsymbol{k} = \lambda(\boldsymbol{\omega})$, so we write $\log \mu_{\boldsymbol{u}}(\boldsymbol{k})$ for $\log \mu_{\boldsymbol{u}}(\boldsymbol{\omega})$, and similarly $\log \Delta(\boldsymbol{k})$ for $\log \Delta(\boldsymbol{\omega})$. With Proposition 4.3 we find

$$\sum_{\boldsymbol{u}\in\mathbb{F}^r}{}' \log \mu_{\boldsymbol{u}}(\boldsymbol{k}) = (q-1) \sum_{1\leq i\leq r} q^{r-i} \log \mu_i(\boldsymbol{k}), \tag{4.12}$$

which gives a similar equation for the increment under $\boldsymbol{k} \rightsquigarrow \boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{k}_\ell$.

Theorem 4.13.

(i) *Let e be the arrow $e = (\boldsymbol{k}, \boldsymbol{k}') = ([L_{\boldsymbol{k}}], [L_{\boldsymbol{k}'}])$ in $W(\mathbb{Z})$, where $\boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{k}_\ell$, $\boldsymbol{k}_\ell = (1, 1, \ldots, 1, 0, \ldots, 0)$ with ℓ ones. The van der Put function $P(\Delta)$ evaluates on e as*

$$P(\Delta)(e) = -(q-1) \sum_{1 \le i \le \ell} q^{r-i} v_{\boldsymbol{k},i}^{(\ell)}$$

with the numbers $v_{\boldsymbol{k},i}^{(\ell)}$ of Definition 4.9. Ditto,

$$P(h)(e) = -\sum_{1 \le i \le \ell} q^{r-i} v_{\boldsymbol{k},i}^{(\ell)}.$$

(ii) *For $\boldsymbol{\omega} \in \mathcal{F}_{\boldsymbol{k}}$ the formula*

$$\log \Delta(\boldsymbol{\omega}) = q^r + \sum_e P(\Delta)(e)$$

holds, where e runs through the arrows of shape $(\boldsymbol{k}', \boldsymbol{k}' + \boldsymbol{k}_\ell)$ of any path in $W(\mathbb{Z})$ with origin $\boldsymbol{o}$ and endpoint $\boldsymbol{k}$.

Proof. (i) is (4.12) combined with (3.5). For (ii) we also use §4.6. □

Remarks 4.14.

(i) The sum in the formula for $\log \Delta(\boldsymbol{\omega})$ could more suggestively be written as a path integral $\int_{\boldsymbol{o}}^{\boldsymbol{k}} P(\Delta)(e)de$, which depends only on the homotopy class of the path connecting $\boldsymbol{o}$ to $\boldsymbol{k}$ in $W(\mathbb{Z})$.

(ii) The arrows $(\boldsymbol{o}, \boldsymbol{k}_\ell)$ are those emanating from $\boldsymbol{o}$ in the unique $(r-1)$-simplex σ_0 in W that contains $\boldsymbol{o}$. For $\boldsymbol{k}_\ell, \boldsymbol{k}_m$ with $\ell \neq m$ and the arrow $e = (\boldsymbol{k}_\ell, \boldsymbol{k}_m)$, we may calculate $P(\Delta)(e)$ as the difference $P(\Delta)(\boldsymbol{o}, \boldsymbol{k}_m) - P(\Delta)(\boldsymbol{o}, \boldsymbol{k}_\ell)$. As each arrow e in $W(\mathbb{Z})$ belongs to a unique translate $\sigma_{\boldsymbol{k}} = \boldsymbol{k} + \sigma_0$ of σ_0 (i.e., if e is not parallel with some $\boldsymbol{k}_\ell$, it has a unique representation as $e = (\boldsymbol{k} + \boldsymbol{k}_\ell, \boldsymbol{k} + \boldsymbol{k}_m)$ with some $1 \le \ell, m < r$), we find similarly $P(\Delta)(e) = P(\Delta)(\boldsymbol{k}, \boldsymbol{k} + \boldsymbol{k}_m) - P(\Delta)(\boldsymbol{k}, \boldsymbol{k} + \boldsymbol{k}_\ell)$.

Below there are some consequences of the preceding considerations.

Corollary 4.15. *The function Δ is strictly monotonically decreasing on $W(\mathbb{Q})$.*

Proof. All the numbers $v_{\boldsymbol{k},i}^{(\ell)}$ are strictly positive, so this follows from Theorem 4.13 (i) and §2.7. □

Suppose that $\boldsymbol{x} \in W(\mathbb{Q})$ doesn't lie on the wall W_{r-i}, $1 \le i < r$. For $\boldsymbol{\omega} \in \lambda^{-1}(\boldsymbol{x})$ we have $|\omega_{r-i}| > |\omega_{r-i+1}|$, thus by Proposition 4.3 (i) $|\mu_{r-i}(\boldsymbol{\omega})| > |\mu_{r-i+1}(\boldsymbol{\omega})|$. By Proposition 4.3 (ii) each of the $(q^i - 1)$ values $\mu_{\boldsymbol{u}}(\boldsymbol{\omega})$ where $\boldsymbol{o} \neq \boldsymbol{u} = (u_1, \ldots, u_r) \in \mathbb{F}^r$, $u_1 = u_2 = \cdots = u_{r-i} = 0$, is strictly less in absolute value than any $\mu_{\boldsymbol{u}}(\boldsymbol{\omega})$ with some $u_1, \dots, u_{r-i} \neq 0$. Hence the reverse inequality holds for the reciprocals $\mu_{\boldsymbol{u}}(\boldsymbol{\omega})^{-1}$, and the term

$$\prod_{\substack{\boldsymbol{u}\in\mathbb{F}^r \\ u_1=\cdots=u_{r-i}=0}}' \mu_{\boldsymbol{u}}(\boldsymbol{\omega})^{-1}$$

dominates (and hence determines the absolute value) in the sum for the elementary symmetric function $s_{q^i-1}\{\mu_{\boldsymbol{u}}(\boldsymbol{\omega})^{-1}\}$.

By (3.6) and describing the $\mu_{\boldsymbol{u}}$ through the μ_i, we find the following result, which complements Proposition 4.4.

Corollary 4.16. *The coefficient form g_i has no zeroes on $\mathcal{F} \setminus \mathcal{F}_{r-i}$. For $\boldsymbol{\omega} \in \mathcal{F} \setminus \mathcal{F}_{r-i}$, $\log g_i(\boldsymbol{\omega})$ depends only on $\boldsymbol{x} = \lambda(\boldsymbol{\omega})$, and is given by*

$$\log g_i(\boldsymbol{\omega}) = 1 - (q-1) \sum_{0\leq j<i} q^j \log \mu_{r-j}(\boldsymbol{\omega}).$$

If $\boldsymbol{\omega} \in \mathcal{F}_{r-i}$, the right hand side is still an upper bound for $\log g_i(\boldsymbol{\omega})$, which is attained in $\lambda^{-1}(\boldsymbol{x})$. In particular, $\log g_1(\boldsymbol{\omega})$ is constant with value q on $\mathcal{F} \setminus \mathcal{F}_{r-1}$ and $\log g_1(\boldsymbol{\omega}) \leq q$ for $\boldsymbol{\omega} \in \mathcal{F}_{r-1}$.

Proof. The assertion for $\boldsymbol{\omega} \in \mathcal{F} \setminus \mathcal{F}_{r-i}$ has been shown, and it is obvious that the right hand side is an upper bound if $\boldsymbol{\omega} \in \mathcal{F}_{r-i}$. The set of those $\boldsymbol{\omega}' \in X := \lambda^{-1}(\boldsymbol{x})$ where $|g_i(\boldsymbol{\omega}')|$ is less than the upper bound is the inverse image of a closed proper subvariety of the canonical reduction of X, and is therefore strictly contained in X. □

As we have seen, the vanishing locus of g_i satisfies

$$\lambda(V(g_i) \cap \mathcal{F}) \subset W_{r-i}(\mathbb{Q}).$$

This is in stark contrast with the behavior of Eisenstein series, which all have their zeroes in $\mathcal{F}_{r-1}$.

Proposition 4.17. *The vanishing locus $V(E_k)$ of the k-th Eisenstein series E_k $(0 < k \equiv 0 \,(\mathrm{mod}\, q-1))$ intersected with $\mathcal{F}$ is contained in $\mathcal{F}_{r-1}$.*

Proof. Suppose that $\boldsymbol{\omega} \in \mathcal{F} \setminus \mathcal{F}_{r-1}$, i.e., $|\omega_{r-1}| > |\omega_r| = 1$. Then the terms of maximal absolute value in

$$E_k(\boldsymbol{\omega}) = \sum_{\boldsymbol{a}\in A^r}' \frac{1}{(a_1\omega_1 + \cdots + a_r\omega_r)^k}$$

are those with $a_1 = \cdots = a_{r-1} = 0$, $a_r \in \mathbb{F}^*$. But $\sum_{a_r\in\mathbb{F}^*} a_r^{-k} = -1$, so $E_{\boldsymbol{k}}(\boldsymbol{\omega}) = -1+$ terms of lower size cannot vanish. □

5. The increments of $\log \Delta$

In this section we perform some more detailed calculations with the numbers $v_{\boldsymbol{k},i}^{(\ell)}$ of Definition 4.9. We keep the set-up of the last section: $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{N}_0^r$, $k_1 \geq k_2 \geq \cdots \geq k_r = 0$, and $1 \leq i \leq \ell < r$. The increment $-P(\Delta)(\boldsymbol{k}, \boldsymbol{k} + \boldsymbol{k}_\ell)$ under $\boldsymbol{k} \rightsquigarrow \boldsymbol{k} + \boldsymbol{k}_\ell$ of the function $\log(\prod'_{\boldsymbol{u} \in \mathbb{F}^r} \mu_{\boldsymbol{u}})$ on $W(\mathbb{Z})$ is expressed in Theorem 4.13 through the $v_{\boldsymbol{k},i}^{(\ell)}$. For brevity, we label it as

$$I_{\boldsymbol{k}}^{(\ell)} := -P(\Delta)(\boldsymbol{k}, \boldsymbol{k} + \boldsymbol{k}_\ell). \tag{5.1}$$

We further define for $\nu \in \mathbb{N}_0$:

$$\begin{aligned} s_\nu^{(\ell)} &= \#\{j \mid \ell < j \leq r \text{ and } k_j = \nu\} \\ t_\nu^{(\ell)} &= \#\{j \mid i < j \leq \ell \text{ and } k_j = \nu\} \\ r_\nu &= \#\{j \mid 1 \leq j \leq r \text{ and } k_j = \nu\}. \end{aligned}$$

Further, for $0 \leq m < k_1$,

$$\begin{aligned} b_\ell(m) &= \sum_{0 \leq \nu \leq m} s_\nu^{(\ell)} \\ c(m) &= \sum_{0 \leq \nu \leq m} (m - \nu) r_\nu, \end{aligned}$$

all of which depend on the fixed data $\boldsymbol{k}, i, \ell$.

Any $\boldsymbol{a} = (a_{i+1}, \dots, a_r) \in V_{\boldsymbol{k},i}$ (cf. Definition 4.9) will be written as $\boldsymbol{a} = (\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)})$, $\boldsymbol{a}^{(1)} = (a_{i+1}, \dots, a_\ell) \in A^{\ell - i}$, $\boldsymbol{a}^{(2)} = (a_{\ell+1}, \dots, a_r) \in A^{r-\ell}$. For $0 \leq m < k_i - k_r = k_i$, put

$$V(m) := \left\{ \boldsymbol{a}^{(2)} \;\middle|\; \max_{\ell < j \leq r} (\deg a_j + k_j) = m \right\}.$$

Further (as $\deg 0 = -\infty$), $V(-\infty) := \{0\}$. Then

$$V := \dot{\bigcup_{m < k_i}} V(m)$$

is an $\mathbb{F}$-vector space of dimension $\sum_{i < j \leq r} (k_i - k_j)$, which exhausts all possibilities for $\boldsymbol{a}^{(2)}$, and

$$V_{\boldsymbol{k},i}^{(\ell)} = \left\{ (\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}) \in V_{\boldsymbol{k},i} \;\middle|\; \begin{array}{l} \max_{i < j \leq \ell} (\deg a_j + k_j) < m \\ \text{if } \boldsymbol{a}^{(2)} \in V(m),\ m \geq 0, \\ \text{and } \boldsymbol{a}^{(1)} = \boldsymbol{o} \text{ if } \boldsymbol{a}^{(2)} = \boldsymbol{o} \end{array} \right\}.$$

Further, for any fixed $0 \leq m < k_i$, the disjoint union

$$W(m) := \dot{\bigcup_{m' \leq m}} V(m')$$

is an $\mathbb{F}$-space of dimension $\sum_{0\le\nu\le m}(m+1-\nu)s_\nu^{(\ell)}$, as we see from counting conditions for $\boldsymbol{a}^{(2)}$ to belong to $W(m)$. Hence, by evaluating $\#W(m) - \#W(m-1)$ and a small calculation, we find

$$\#V(m) = (q^{b_\ell(m)} - 1)q^{\sum_{\nu\le m}(m-\nu)s_\nu^{(\ell)}}. \tag{5.2}$$

For each $\boldsymbol{a}^{(2)} \in V(m)$, where $m \ge 0$, some $\boldsymbol{a}^{(1)}$ yields an element $(\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)})$ of $V_{\boldsymbol{k},i}^{(\ell)}$ if and only if $\deg a_j < m - k_j$ $(i < j \le \ell)$. Such $\boldsymbol{a}^{(1)}$ form an $\mathbb{F}$-vector space of dimension $\sum_{i<j\le\ell}(m-k_j) = \sum_{0\le\nu<m}(m-\nu)t_\nu^{(\ell)}$. So

$$\begin{aligned} v_{\boldsymbol{k},i}^{(\ell)} &= 1 + \sum_{0\le m<k_i} \#V(m)\cdot q^{\sum_{0\le\nu<m}(m-\nu)t_\nu^{(\ell)}} \\ &= 1 + \sum_{0\le m<k_i} (q^{b_\ell(m)} - 1)q^{\sum_{0\le\nu\le m}(m-\nu)(s_\nu^{(\ell)}+t_\nu^{(\ell)})}. \end{aligned}$$

Note that $s_\nu^{(\ell)} + t_\nu^{(\ell)} = \#\{j > i \mid k_j = \nu\}$. If now $j \le i$ with $k_j = \nu$ then $\nu = k_j \ge k_i > m$, so we may replace $s_\nu^{(\ell)} + t_\nu^{(\ell)}$ with $\{j \mid 1 \le j \le r,\, k_j = \nu\} = r_\nu$ in the above sum. Therefore,

$$v_{\boldsymbol{k},i}^{(\ell)} = 1 + \sum_{0\le m<k_i} (q^{b_\ell(m)} - 1)q^{c(m)}. \tag{5.3}$$

Hence the increment under $\boldsymbol{k} \rightsquigarrow \boldsymbol{k} + \boldsymbol{k}_\ell$ of $\log(\prod'_{\boldsymbol{u}\in\mathbb{F}^r}\mu_{\boldsymbol{u}})$ is given by

$$\begin{aligned} I_{\boldsymbol{k}}^{(\ell)} &= (q-1)\sum_{1\le i\le\ell} q^{r-i} v_{\boldsymbol{k},i}^{(\ell)} \\ &= (q-1)\sum_{1\le i\le\ell} q^{r-i}\Big(1 + \sum_{0\le m<k_i}(q^{b_\ell(m)} - 1)q^{c(m)}\Big) \\ &= q^r - q^{r-\ell} + (q-1)\sum_{0\le m<k_1}(q^{b_\ell(m)} - 1)q^{c(m)} \sum_{\substack{1\le i\le\ell \\ k_i>m}} q^{r-i}. \end{aligned} \tag{5.4}$$

Note that the condition $k_i > m$ in the last sum is an upper bound for i; it decreases if m increases. Although complicated, the formula is explicit and easy to evaluate. So our final result for $P(\Delta)$ is

Theorem 5.5. *Let $e = (\boldsymbol{k}, \boldsymbol{k}')$ with $\boldsymbol{k}' = \boldsymbol{k} + \boldsymbol{k}_\ell$ be as in Theorem 4.13. Then*

$$P(\Delta)(e) = -(q^r - q^{r-\ell}) - (q-1)\sum_{0\le m<k_1}(q^{b_\ell(m)} - 1)q^{c(m)} \sum_{\substack{1\le i\le\ell \\ k_i>m}} q^{r-i}.$$

We may read off several qualitative properties. How does $I_{\boldsymbol{k}}^{(\ell)}$ change under $\ell \rightsquigarrow \ell+1$, where $1 \leq \ell < r-1$? We first observe that

$$b_{\ell+1}(m) = \begin{cases} b_\ell(m) - 1, & \text{if } k_{\ell+1} \leq m \\ b_\ell(m), & \text{if } k_{\ell+1} > m \end{cases} \tag{5.6}$$

and $b_\ell(m+1) \geq b_\ell(m)$. Further,

$$c(m+1) = c(m) + \sum_{0 \leq \nu \leq m} r_\nu \,,$$

where $\sum_{0 \leq \nu \leq m} r_\nu \geq r_0 > 0$. By (5.4), comparing termwise,

$$\begin{aligned} I_{\boldsymbol{k}}^{(\ell+1)} - I_{\boldsymbol{k}}^{(\ell)} & \\ &= (q-1)q^{r-\ell-1} + (q-1) \sum_{0 \leq m < k_{\ell+1}} (q^{b_\ell(m)} - 1) q^{c(m)} q^{r-\ell-1} \\ &\qquad - (q-1)^2 \sum_{k_{\ell+1} \leq m < k_1} q^{b_\ell(m)-1+c(m)} \sum_{\substack{1 \leq i \leq \ell \\ k_i > m}} q^{r-i} \\ &=: (q-1)q^{r-\ell-1} + (q-1) \sum_{0 \leq m < k_{\ell+1}} B(m) \\ &\qquad - (q-1)^2 \sum_{k_{\ell+1} \leq m < k_1} B(m) \,, \end{aligned} \tag{5.7}$$

where the last equation defines the $B(m)$ for $m < k_{\ell+1}$, $m \geq k_{\ell+1}$, respectively. (5.7) holds since for $m < k_{\ell+1}$, $b_{\ell+1}(m) = b_\ell(m)$ but

$$\sum_{\substack{1 \leq i \leq \ell+1 \\ k_i > m}} q^{r-i} = \sum_{\substack{1 \leq i \leq \ell \\ k_i > m}} q^{r-i} + q^{r-\ell-1} \,,$$

and for $m \geq k_{\ell+1}$, $b_{\ell+1}(m) = b_\ell(m) - 1$, but the sum $\sum_{\substack{1 \leq i \leq \ell \\ k_i > m}} q^{r-i}$ doesn't change upon $\ell \rightsquigarrow \ell+1$. Note that all the $B(m)$ are positive. We claim

$$q^{r-\ell-1} + \sum_{0 \leq m < k_{\ell+1}} B(m) < (q-1) B(k_{\ell+1}) \,, \tag{5.8}$$

provided that $k_{\ell+1} < k_1$.

Proof.

$$\begin{aligned}
q^{r-\ell-1} + &\sum_{0\leq m<k_{\ell+1}} B(m) \\
&\leq q^{r-\ell-1} \sum_{0\leq m<k_{\ell+1}} q^{b_\ell(m)+c(m)} \\
&\leq q^{r-\ell-1} \sum_{0\leq m<k_{\ell+1}} q^{b_\ell(k_{\ell+1})-1+c(m)} \leq q^{r-\ell-2+b_\ell(k_{\ell+1})+c(k_{\ell+1})} \\
&\leq q^{r-3+b_\ell(k_{\ell+1})+c(k_{\ell+1})} < (q-1)q^{b_\ell(k_{\ell+1})+c(k_{\ell+1})-1}q^{r-1} \\
&\leq (q-1)B(k_{\ell+1})\,. \qquad \square
\end{aligned}$$

As a consequence of (5.7) and (5.8), $I_{\boldsymbol{k}}^{(\ell+1)} - I_{\boldsymbol{k}}^{(\ell)}$ is negative if there is at least one m with $k_{\ell+1} \leq m < k_1$, i.e., if $k_{\ell+1} < k_1$. Otherwise, $I_{\boldsymbol{k}}^{(\ell+1)} - I_{\boldsymbol{k}}^{(\ell)}$ is positive. In view of (5.1) we have shown the following result.

Theorem 5.9. *Let $\boldsymbol{k} = (k_1, k_2, \ldots, k_r) \in \mathbb{N}_0^r$ with $k_1 \geq k_2 \geq \cdots \geq k_r = 0$, $1 \leq \ell < r$ and e_ℓ the arrow $(\boldsymbol{k}, \boldsymbol{k} + \boldsymbol{k}_\ell)$ in $W(\mathbb{Z})$. Suppose that $k_1 = \cdots = k_t > k_{t+1}$. The values of $P(\Delta)$ satisfy*

$$P(\Delta)(e_1) > P(\Delta)(e_2) > \cdots > P(\Delta)(e_t) \\ < P(\Delta)(e_{t+1}) < \cdots < P(\Delta)(e_{r-1})\,.$$

That is, e_t points to the well-defined direction of largest decay of $|\Delta|$ from $\mathcal{F}_{\boldsymbol{k}}$.

6. The vanishing of modular forms on $\mathcal{F}_{\boldsymbol{o}}$

We describe the zero loci of the g_i in $\mathcal{F}_{\boldsymbol{o}}$ and their canonical reductions.

6.1. We let $\|f\| = \|f\|_{\mathcal{F}_{\boldsymbol{o}}}$ be the spectral norm of the holomorphic function f on $\mathcal{F}_{\boldsymbol{o}}$, and denote by "$\equiv$" the congruence of elements of $O_{\mathbb{C}_\infty}$ modulo its maximal ideal, and $\overline{x}$ = reduction of $x \in O_{\mathbb{C}_\infty}$ in its residue class field $\overline{\mathbb{F}}$. Thus from Corollary 4.16 along with (4.2), $\|g_i\| = q^i$ for $1 \leq i \leq r$, including the case $g_r = \Delta$. As $g_i = Ts_{q^i-1}\{\mu_{\boldsymbol{u}}^{-1} \,|\, 0 \neq \boldsymbol{u} \in \mathbb{F}^r\}$, we have for $\boldsymbol{\omega} \in \mathcal{F}_{\boldsymbol{o}}:\ |g_i(\boldsymbol{\omega})| < \|g_i\| \iff |s_{q^i-1}\{T^{-1}\mu_{\boldsymbol{u}}^{-1}\}| < 1$. Now by (4.2),

$$T\mu_{\boldsymbol{u}}(\boldsymbol{\omega}) \equiv \boldsymbol{\omega}_{\boldsymbol{u}} = \sum_{1\leq i\leq r} u_i\omega_i\,.$$

Hence the above is equivalent with $|s_{q^i-1}\{\boldsymbol{\omega}_{\boldsymbol{u}}^{-1}\}| < 1$ and with $\alpha_i(\boldsymbol{\omega}) \equiv 0$, where the α_i are the coefficients of the lattice function

$$e_{L_{\boldsymbol{\omega}}} = z \prod_{\boldsymbol{u}\in\mathbb{F}^r}{}' \left(1 - \frac{z}{\boldsymbol{\omega}_{\boldsymbol{u}}}\right) = \sum_{0\leq i\leq r} \alpha_i(\boldsymbol{\omega}) z^{q^i} \quad (\alpha_0 = 1)\,,$$

$L_{\boldsymbol{\omega}} := \sum_{1\leq i\leq r} \mathbb{F}\omega_i$. (Of course the present α_i, those of (1.1), mustn't be confused with the roots α_i of Sections 3 and 4, which don't appear in this section.)

More conceptually we have

$$\begin{aligned}\phi_T^{\boldsymbol{\omega}}(X) &= TX \prod_{\boldsymbol{u}}{}' \left(1 - \frac{X}{\mu_{\boldsymbol{u}}}\right) = TX + \sum_{1\leq i\leq r} g_i(\boldsymbol{\omega}) X^{q^i} \\ &= Te_{L'}(X) \quad (\text{where } L' = \textstyle\sum_{1\leq i\leq r} \mathbb{F}\mu_i) \\ &= e_{TL'}(TX).\end{aligned}$$

As $TL' \equiv L_{\boldsymbol{\omega}}$ (i.e., the respective basis vectors satisfy $T\mu_i \equiv \omega_i$),

$$e_{TL'}(X) = X + \sum_{1\leq i\leq r} T^{-q^i} g_i(\boldsymbol{\omega}) X^{q^i} \equiv \sum_{0\leq i\leq r} \alpha_i(\boldsymbol{\omega}) X^{q^i} = e_{L_{\boldsymbol{\omega}}}(X),$$

where the congruence is coefficientwise. Together, the condition $\alpha_i(\boldsymbol{\omega}) \equiv 0$ for $|g_i(\boldsymbol{\omega})| < \|g_i\|$ depends only on the reduction $\overline{L} = \sum_{1\leq i\leq r} \mathbb{F}\overline{\omega}_i$ of $L_{\boldsymbol{\omega}}$ in $\overline{\mathbb{F}}$. We let $\overline{\alpha}_i(\overline{\boldsymbol{\omega}})$ be the respective coefficient of $e_{\overline{L}}$ (which of course equals the reduction of $\alpha_i(\boldsymbol{\omega})$), regarded as a function of $\overline{\boldsymbol{\omega}} \in \Omega^r(\mathbb{F})$.

Theorem 6.2. *We let $V(g_i)\cap\mathcal{F}_{\boldsymbol{o}}$ be the vanishing locus of g_i on $\mathcal{F}_{\boldsymbol{o}}$. Its image under the canonical reduction map* red : $\mathcal{F}_{\boldsymbol{o}} \longrightarrow \Omega^r(\overline{\mathbb{F}})$ *is the vanishing locus $V(\overline{\alpha}_i)$. In particular, $V(g_i)\cap\mathcal{F}_{\boldsymbol{o}}$ is non-empty.*

Proof. From the preceding, red : $V(g_i)\cap\mathcal{F}_{\boldsymbol{o}} \longrightarrow \Omega^r(\overline{\mathbb{F}})$ takes its values in $V(\overline{\alpha}_i)$. Once surjectivity onto $V(\overline{\alpha}_i)$ is established, the non-emptiness of $V(g_i)\cap\mathcal{F}_{\boldsymbol{o}}$ results from the non-emptiness of $V(\overline{\alpha}_i)$, which in turn is a consequence of [6, (1.12)]. (For example $\overline{\alpha}_1,\ldots,\overline{\alpha}_{r-1}$ have a common zero at $\overline{\boldsymbol{\omega}}$ if the entries of $\overline{\omega}_1,\ldots,\overline{\omega}_{r-1},\overline{\omega}_r = 1)$ lie in $\mathbb{F}^{(r)}$.)

To show the surjectivity of red : $V(g_i)\cap\mathcal{F}_{\boldsymbol{o}} \longrightarrow V(\overline{\alpha}_i)$, it suffices, by Hensel's lemma, to verify that at least one of the partial derivatives $\frac{\partial}{\partial\omega_j}(T^{-q^i}g_i)(\boldsymbol{\omega})$ at $\boldsymbol{\omega} \in \mathrm{red}^{-1}(V(\overline{\alpha}_i))$ has absolute value 1. Fix such an $\boldsymbol{\omega}$, and let $D_j = \frac{\partial}{\partial\omega_j}$. Then

$$|D_j(T^{-q^i}g_i)(\boldsymbol{\omega}) = 1| \Longleftrightarrow |D_j\alpha_i(\boldsymbol{\omega})| = 1 \Longleftrightarrow D_j\overline{\alpha}_i(\overline{\boldsymbol{\omega}}) \neq 0 \text{ in } \overline{\mathbb{F}}.$$

(By abuse of notation, we also write D_j for the derivative with respect to $\overline{\omega}_j$.) In the proposition below we show that the determinant

$$\det_{1\leq i,j<r}(D_j\overline{\alpha}_i(\overline{\boldsymbol{\omega}}))$$

doesn't vanish (regardless of the (non-) vanishing of $\overline{\alpha}_i(\overline{\boldsymbol{\omega}})$), which gives the result. □

Proposition 6.3. *Let $\omega_1, \ldots, \omega_r \in \overline{\mathbb{F}}$ be $\mathbb{F}$-linearly independent with lattice $\Lambda_{\boldsymbol{\omega}} = \sum \mathbb{F}\omega_i$ and lattice function*

$$e_{\Lambda_{\boldsymbol{\omega}}}(z) = z \prod_{\lambda \in \Lambda_{\boldsymbol{\omega}}}{}' (1 - z/\lambda) = z + \sum_{1 \leq i \leq r} \alpha_i(\boldsymbol{\omega}) z^{q^i}.$$

Write D_j for $\frac{\partial}{\partial \omega_j}$. Then for all $r' \leq r$, the functional determinant

$$\det_{1 \leq i,j \leq r'} (D_j \alpha_i(\boldsymbol{\omega}))$$

doesn't vanish.

Proof. For $i \geq 0$, we let $e_i(\boldsymbol{\omega})$ be the $(q^i - 1)$-th Eisenstein series of $\Lambda_{\boldsymbol{\omega}}$,

$$e_i(\boldsymbol{\omega}) = \sum_{\boldsymbol{a}=(a_1,\ldots,a_r) \in \mathbb{F}^r}{}' (a_1\omega_1 + \cdots + a_r\omega_r)^{1-q^i}$$

(which gives $e_0(\boldsymbol{\omega}) = -1$). It is known ([6, (1.5) and (1.6)]) that for $k > 0$

$$\alpha_k = \sum_{0 \leq i < k} \alpha_i (e_{k-i})^{q^i}$$

holds. Thus for any $D = D_1, \ldots, D_r$,

$$D(\alpha_k) = \sum_{1 \leq i < k} D(\alpha_i) e_{k-i}^{q^i} + D(e_k),$$

which implies that for $r' \leq r$,

$$\det_{1 \leq i,j \leq r'} (D_j(\alpha_i)) = \det_{1 \leq i,j \leq r'} (D_j(e_i)).$$

We will show the non-vanishing of the right hand side. For any $\mathbb{F}$-linear map $\varphi : \Lambda_{\boldsymbol{\omega}} \longrightarrow \mathbb{F}$ we define

$$M(\varphi) := \sum_{\lambda \in \Lambda_{\boldsymbol{\omega}}}{}' \frac{\varphi(\lambda)}{\lambda}.$$

Then $D_j(e_i)(\boldsymbol{\omega}) = \sum'_{\boldsymbol{a} \in \mathbb{F}^r} \frac{a_j}{(a_1\omega_1 + \cdots + a_r\omega_r)^{q^i}} = M(\varphi_j)^{q^i}$, where $\varphi_j : (a_1\omega_1 + \cdots + a_r\omega_r) \longmapsto a_j$.

Hence $\det_{1 \leq i,j \leq r'}(D_j(e_i)(\boldsymbol{\omega})) = \det_{1 \leq i,j \leq r'}(M(\varphi_j)^{q^i})$ is a determinant of Moore type ([13, 1.13]), which doesn't vanish if and only if the $M(\varphi_j)$ are $\mathbb{F}$-linearly independent, where $1 \leq j \leq r'$. Now

$$\begin{aligned} M : \operatorname{Hom}_{\mathbb{F}}(\Lambda_{\boldsymbol{\omega}}, \mathbb{F}) &\longrightarrow \overline{\mathbb{F}} \\ \varphi &\longmapsto M(\varphi) \end{aligned}$$

is linear, and the $M(\varphi_j)$ $(1 \leq j \leq r)$ are linearly independent provided M is injective. This is asserted by the next lemma. ☐

Lemma 6.4. *Let V be a finite-dimensional $\mathbb{F}$-subspace of $\overline{\mathbb{F}}$. For any non-trivial functional $\varphi : V \longrightarrow \mathbb{F}$, the quantity*

$$M(\varphi) = \sum_{v \in V}{}' \frac{\varphi(v)}{v}$$

doesn't vanish.

Proof. Let U be the kernel of φ, $x \in V \setminus U$. Write

$$\begin{aligned} M(\varphi) &= \sum_{c \in \mathbb{F}} \sum_{u \in U}{}' \frac{\varphi(u+cx)}{u+cx} = \varphi(x) \sum_{c \in \mathbb{F}} \sum_{u \in U}{}' \frac{c}{u+cx} \\ &= \varphi(x) \sum_{0 \neq c \in \mathbb{F}} \sum_{u \in U} \frac{1}{c^{-1}u + x} = -\varphi(x) \sum_{u \in U} \frac{1}{u+x}. \end{aligned}$$

Let e_U be the lattice function of U; then

$$\frac{1}{e_U(x)} = \left(\frac{e_U'}{e_U}\right)(x) = \sum_{u \in U} \frac{1}{x-u}$$

by logarithmic derivation; so $M(\varphi) = -\frac{\varphi(x)}{e_U(x)} \neq 0$. □

Now the proof of Theorem 6.2 is complete.

7. The case $r = 3$

As an example for the preceding, we present more details in the case $r = 3$. Again, $\boldsymbol{k} = (k_1, k_2, k_3)$ with $k_1 \geq k_2 \geq k_3 = 0$, $1 \leq i \leq 3$, and $\ell = 1, 2$, and e is the arrow $(\boldsymbol{k}, \boldsymbol{k} + \boldsymbol{k}_\ell)$ in $W(\mathbb{Z})$. Proposition 4.10 yields the following values for $P(\mu_i)(e)$.

	$\ell = 1$	$\ell = 2$
$i = 3$	0	0
$i = 2$	0	q^{k_2}
$i = 1$	$q^{2k_1 - k_2}$	$q^{k_2+1}(q^{2k_1-2k_2-1}+1)/(q+1)$

Table 7.1. Values for $P(\mu_i)(e)$

From specializing (5.4) (or directly from Theorem 4.13 and Table 7.1, which in this case is easier), we find

$$\begin{aligned} P(\Delta)(e) &= -(q-1)q^{2k_1-k_2+2} && (\ell = 1) \\ &= -\frac{(q-1)}{(q+1)}q^{k_2+1}(q^{2k_1-2k_2+1}+q^2+q+1) && (\ell = 2). \end{aligned} \tag{7.2}$$

Below we draw the fundamental domain W and the first few values of $P(\Delta)$ on the arrows of $W(\mathbb{Z})$. The vertex $\boldsymbol{k} = (k_1, k_2, 0)$ is labelled by (k_1, k_2). Arrows $a, b, \ldots, \ell$ are oriented east or northeast.

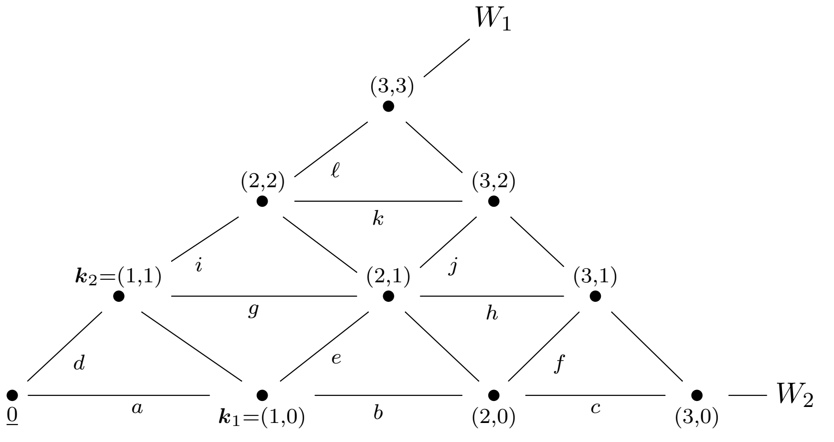

FIGURE 7.3. The Weyl chamber W

For simplicity, we give the values of $-(q-1)^{-1}P(\Delta)$ on the oriented arrows $a, \ldots, \ell$.

(a)	q^2	(g)	q^3
(b)	q^4	(h)	q^5
(c)	q^6	(i)	$q^2(q+1)$
(d)	$q(q+1)$	(j)	$q^2(q^2+1)$
(e)	$q(q^2+1)$	(k)	q^4
(f)	$q(q^4-q^3+q^2+1)$	(l)	$q^3(q+1)$

7.4. The behavior of g_1 and g_2 is easy to describe. First, $g_1(\boldsymbol{\omega})$ is constant with value q^q on $\mathcal{F} \setminus \mathcal{F}_2$, and that value is an upper bound for $|g_1(\boldsymbol{\omega})|$ for $\boldsymbol{\omega} \in \mathcal{F}_2$ (attained in $\lambda^{-1}(\lambda(\boldsymbol{\omega}))$).

Let $\|\cdot\|_{\boldsymbol{k}}$ denote the spectral norm of holomorphic functions on $\mathcal{F}_{\boldsymbol{k}}$. By abuse of notation, we also write $P(f)(e) = P(f)(\boldsymbol{k}, \boldsymbol{k}') := \log_q \|f\|_{\boldsymbol{k}'} - \log_q \|f\|_{\boldsymbol{k}}$ even when $f \neq 0$ possibly has zeroes. Then Corollary 4.16 together with Table 7.1 shows that

$$P(g_2)(\boldsymbol{k}, \boldsymbol{k}+\boldsymbol{k}_\ell) = -(q-1)q^{k_2+1} \quad \text{if } \ell = 2 \text{ and } 0 \text{ if } \ell = 1.$$

Hence the spectral norm of g_2 on $\mathcal{F}_{\boldsymbol{k}}$ (which agrees with its absolute value if $\boldsymbol{k} \notin W_1$) is obtained by integrating $P(g_2)(e)$ along any path in $W(\mathbb{Z})$ from $\boldsymbol{o}$ to $\boldsymbol{k}$, taking into account that $\|g_2\|_{\boldsymbol{o}} = q^{q^2}$.

7.5. At $\mathcal{F}_{\boldsymbol{k}}$ with $\boldsymbol{k} \in W_{3-i}(\mathbb{Z})$, the g_i ($i = 1, 2$) can have smaller absolute values than their spectral norms, or even zeroes. This can be analyzed similar to the case $\boldsymbol{k} = \boldsymbol{o}$ handled in the last section. We restrict to do this in the most simple cases of

- g_1 on $\mathcal{F}_{\boldsymbol{k}}$, $\boldsymbol{k} = (k, 0, 0)$, $k > 0$ and
- g_2 on $\mathcal{F}_{\boldsymbol{k}}$, $\boldsymbol{k} = (1, 1, 0)$.

7.6. We consider $\boldsymbol{k} = (k, 0, 0)$ with $k > 0$. Note that $(\omega_1, \omega_2, 1) \longmapsto (T^k\omega_1, \omega_2, 1)$ is an isomorphism $\mathcal{F}_{\boldsymbol{o}} \xrightarrow{\cong} \mathcal{F}_{\boldsymbol{k}}$ of analytic spaces, which we use to describe the canonical reduction from $\mathcal{F}_{\boldsymbol{k}}$ to $\Omega^3(\overline{\mathbb{F}})$.

As $g_1(\boldsymbol{\omega}) = (T^q - T)E_{q-1}(\boldsymbol{\omega})$ with the Eisenstein series E_{q-1} (see, e.g. [5, 2.10]) and $\|E_{q-1}\|_{\boldsymbol{k}} = 1$ (which follows as in the proof of Proposition 4.17), we only have to study the reduction of E_{q-1}. Now for $\boldsymbol{\omega} \in \mathcal{F}_{\boldsymbol{k}}$,

$$E_{q-1}(\boldsymbol{\omega}) = \sum_{(a,b,c)\in A^3}{}' \frac{1}{(a\omega_1 + b\omega_2 + c)^{q-1}} \equiv \sum_{(b,c)\in\mathbb{F}^2}{}' \frac{1}{(b\omega_2 + c)^{q-1}},$$

where $\equiv$ is congruence modulo the maximal ideal of $O_{\mathbb{C}_\infty}$. Hence

$$|E_{q-1}(\boldsymbol{\omega})| < 1 \Longleftrightarrow \sum_{(b,c)\in\mathbb{F}^2}{}' \frac{1}{(b\overline{\omega}_2 + c)^{q-1}} = 0 \Longleftrightarrow \overline{\omega}_2 \in \mathbb{F}^{(2)} \setminus \mathbb{F},$$

where the last equivalence is well-known (e.g. [6, Corollary 2.9]). As the zeroes of the finite rank-two Eisenstein series $\sum_{(b,c)\in\mathbb{F}^2}(b\overline{\omega}+c)^{1-q}$ are simple (loc. cit.), they may be lifted to zeroes of E_{q-1}. Therefore the reduction map

$$\begin{aligned} \mathrm{red} : \mathcal{F}_{\boldsymbol{k}} &\longrightarrow \Omega^3(\overline{\mathbb{F}}) \\ (T\omega_1, \omega_2, 1) &\longmapsto (\overline{\omega}_1, \overline{\omega}_2, 1) \end{aligned}$$

restricted to $V(g_1) \cap \mathcal{F}_{\boldsymbol{k}} = V(E_{q-1}) \cap \mathcal{F}_{\boldsymbol{k}}$ is onto

$$Y := \left\{(\omega_1, \omega_2, 1) \in \Omega^3(\overline{\mathbb{F}}) \,\middle|\, \omega_2 \in \mathbb{F}^{(2)} \setminus \mathbb{F}\right\} = \coprod_{\omega_2 \in \mathbb{F}^{(2)}\setminus\mathbb{F}} \left\{\omega_1 \in \overline{\mathbb{F}} \setminus \mathbb{F}^{(2)}\right\} \times \{\omega_2\},$$

which is not connected.

7.7. Next we describe the form g_2 on $\mathcal{F}_{\boldsymbol{k}}$, where $\boldsymbol{k} = (1, 1, 0)$. This is more complicated, as g_2 is not an Eisenstein series.

Instead, we have $g_2 = Ts_{q^2-1}\{\mu_{\boldsymbol{u}}^{-1} \mid \boldsymbol{o} \neq \boldsymbol{u} \in \mathbb{F}^3\}$ (see (3.6)). Now for $\boldsymbol{\omega} = (\omega_1, \omega_2, 1) \in \mathcal{F}_{\boldsymbol{k}}$,

$$|\mu_1(\boldsymbol{\omega})| = |\mu_2(\boldsymbol{\omega})| = 1 > |\mu_3(\boldsymbol{\omega})| = q^{-1}.$$

In fact

$$|\mu_i(\boldsymbol{\omega})| \equiv \frac{\omega_i}{T} \prod_{c\in\mathbb{F}}{}' \left(1 - c\frac{\omega_i}{T}\right) = \left(\frac{\omega_i}{T}\right) - \left(\frac{\omega_i}{T}\right)^q \text{ for } i = 1,\, 2,$$

while $\mu_3(\omega) = T^{-1}+$ terms of smaller size. Therefore, for any $\mu_{\boldsymbol{u}} = a\mu_1 + b\mu_2 + c\mu_3$ ($\boldsymbol{o} \neq \boldsymbol{u} = (a,b,c) \in \mathbb{F}^3$),

$$|\mu_{\boldsymbol{u}}(\boldsymbol{\omega})| = q^{-1} \text{ if } (a,b) = (0,0) \text{ and } |\mu_{\boldsymbol{u}}(\boldsymbol{\omega})| = 1 \text{ if } (a,b) \neq (0,0),$$

in which case

$$\mu_{\boldsymbol{u}}(\boldsymbol{\omega}) \equiv \left(\frac{a\omega_1 + b\omega_2}{T}\right) - \left(\frac{a\omega_1 + b\omega_2}{T}\right)^q. \tag{7.8}$$

Consider the polynomial $\Delta(\boldsymbol{\omega})^{-1}\phi_T^{\boldsymbol{\omega}}(X)$:

$$\frac{T}{\Delta}X + \frac{g_1}{\Delta}X^q + \frac{g_2}{\Delta}X^{q^2} + X^{q^3} = \prod_{\boldsymbol{u}\in\mathbb{F}^3}(X - \mu_{\boldsymbol{u}}). \tag{7.9}$$

(All the functions g_1, g_2, Δ, $\mu_{\boldsymbol{u}}$ have to be evaluated at $\boldsymbol{\omega} \in \mathcal{F}_{\boldsymbol{k}}$.) From Figure 7.3 and §7.4, $|\frac{T}{\Delta}| < 1$, $|\frac{g_1}{\Delta}| = 1$ and $|\frac{g_2}{\Delta}| \leq 1$. Therefore the polynomial in (7.9) satisfies

$$\Delta^{-1}\phi_T(X) \equiv \left(\prod{}'(X - \overline{\mu})\right)^q =: (X^{q^2} + sX^q + tX)^q,$$

where $\overline{\mu}$ runs through the rank-two $\mathbb{F}$-lattice L in $\overline{\mathbb{F}}$ generated by the canonical reductions $\overline{\mu}_1 = \overline{(\omega_1/T)} - \overline{(\omega_1/T)}^q$ and $\overline{\mu}_2 = \overline{(\omega_2/T)} - \overline{(\omega_2/T)}^q$. Here $X^{q^2} + sX^q + tX$ is the monic $\mathbb{F}$-linear polynomial associated with $L \subset \overline{\mathbb{F}}$. In the coordinate functions $\overline{\omega}_1$, $\overline{\omega}_2$ on the canonical reduction $\Omega^3(\overline{\mathbb{F}})$ of $\mathcal{F}_{\boldsymbol{k}}$ (i.e., $\overline{\omega}_i = \overline{(\omega_i/T)}$, $i = 1, 2$) we can state:

$$|g_2(\boldsymbol{\omega})| < \|g_2\|_{\boldsymbol{k}} \iff |\frac{g_2(\boldsymbol{\omega})}{\Delta(\boldsymbol{\omega})}| < 1 \iff s = 0 \iff \frac{\overline{\omega}_1 - \overline{\omega}_1^q}{\overline{\omega}_2 - \overline{\omega}_2^q} \in \mathbb{F}^{(2)}$$

(and that quantity is then necessarily in $\mathbb{F}^{(2)} \backslash \mathbb{F}$). That is, red : $\mathcal{F}_{\boldsymbol{k}} \longrightarrow \Omega^3(\overline{\mathbb{F}})$ maps $V(g_2) \cap \mathcal{F}_{\boldsymbol{k}}$ to the set

$$Y = \left\{ (\overline{\omega}_1, \overline{\omega}_2, 1) \in \Omega^3(\overline{\mathbb{F}}) \;\middle|\; \frac{\overline{\omega}_1 - \overline{\omega}_1^q}{\overline{\omega}_2 - \overline{\omega}_2^q} \in \mathbb{F}^{(2)} \right\}.$$

With similar but more complicated considerations not presented here, we find for arbitrary $\mathcal{F}_{\boldsymbol{k}} \subset \mathcal{F}_1$ (i.e., $\boldsymbol{k} = (k,k,0)$ with $k \geq 1$) the same condition: For $\boldsymbol{\omega} \in \mathcal{F}_{\boldsymbol{k}}$ with canonical reduction $(\overline{\omega}_1, \overline{\omega}_2, 1)$, inequality $|g_2(\boldsymbol{\omega})| < \|g_2\|_{\boldsymbol{k}}$ holds if and only if $(\overline{\omega}_1, \overline{\omega}_2, 1) \in Y$.

Unlike the case studied in §7.6, we cannot immediately conclude that red : $V(g_2) \cap \mathcal{F}_{\boldsymbol{k}} \longrightarrow Y$ is surjective, as the trivial case of Hensel's lemma doesn't apply. So these questions and their generalizations to larger r need more investigation.

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Ernst-Ulrich Gekeler
FR Mathematik
Universität des Saarlandes
Campus E2 4
66123 Saarbrücken, Germany
E-mail: gekeler@math.uni-sb.de