ÉCOLE POLYTECHNIQUE
UNIVERSITÉ PARIS-SACLAY

Journal de l'École polytechnique
Mathématiques

Pierre-Emmanuel Chaput & Julien Maubon

Maximal representations of cocompact complex hyperbolic lattices, a uniform approach

Tome 6 (2019), p. 231-281.

<http://jep.centre-mersenne.org/item/JEP_2019__6__231_0>

Publié avec le soutien
du Centre National de la Recherche Scientifique

CENTRE MERSENNE

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org

Journal de l'École polytechnique
Mathématiques
Tome 6, 2019, p. 231–281 DOI: 10.5802/jep.93

MAXIMAL REPRESENTATIONS OF COCOMPACT COMPLEX HYPERBOLIC LATTICES, A UNIFORM APPROACH

by Pierre-Emmanuel Chaput & Julien Maubon

Abstract. — We complete the classification of maximal representations of cocompact complex hyperbolic lattices in Hermitian Lie groups by dealing with the exceptional groups $\mathrm{E}_{6(-14)}$ and $\mathrm{E}_{7(-25)}$. We prove that if ρ is a maximal representation of a cocompact complex hyperbolic lattice $\Gamma \subset \mathrm{SU}(1,n)$, $n > 1$, in an exceptional Hermitian group $G_{\mathbb{R}}$, then $n = 2$ and $G_{\mathbb{R}} = \mathrm{E}_{6(-14)}$, and we describe completely the representation ρ. The case of classical Hermitian target groups was treated by Vincent Koziarz and the second named author [KM17]. However we do not focus immediately on the exceptional cases and instead we provide a more unified perspective, as independent as possible of the classification of the simple Hermitian Lie groups. This relies on the study of the cominuscule representation of the complexification G of the target group $G_{\mathbb{R}}$. As a by-product of our methods, when the target Hermitian group $G_{\mathbb{R}}$ has tube type, we obtain an inequality on the Toledo invariant of the representation $\rho : \Gamma \to G_{\mathbb{R}}$ which is stronger than the Milnor-Wood inequality (thereby excluding maximal representations in such groups).

Résumé (Représentations maximales des réseaux hyperboliques complexes cocompacts : une approche unifiée)

Nous complétons la classification des représentations maximales des réseaux hyperboliques complexes dans les groupes de Lie hermitiens en traitant le cas des groupes exceptionnels $\mathrm{E}_{6(-14)}$ et $\mathrm{E}_{7(-25)}$. Nous montrons que si ρ est une représentation maximale d'un réseau hyperbolique complexe cocompact $\Gamma \subset \mathrm{SU}(1,n)$, avec $n > 1$, dans un groupe hermitien $G_{\mathbb{R}}$ de type exceptionnel, alors $n = 2$ et $G_{\mathbb{R}} = \mathrm{E}_{6(-14)}$, et nous décrivons complètement la représentation ρ. Le cas des groupes hermitiens classiques avait été traité par Vincent Koziarz et le deuxième auteur cité [KM17]. Cependant, nous ne nous restreignons pas immédiatement aux groupes exceptionnels : nous proposons au contraire une approche unifiée, aussi indépendante que possible de la classification des groupes de Lie hermitiens simples. Cette approche repose sur une étude de la représentation cominuscule de la complexification du groupe d'arrivée $G_{\mathbb{R}}$. Dans le cas où $G_{\mathbb{R}}$ est de type tube, nos méthodes permettent en particulier d'établir une inégalité sur l'invariant de Toledo de la représentation $\rho : \Gamma \to G_{\mathbb{R}}$ qui est plus forte que l'inégalité de Milnor-Wood et qui exclut donc la possibilité d'une représentation maximale pour de tels groupes.

2010 Mathematics Subject Classification. — 53C35, 22E40, 32L05, 32Q15, 17B10, 20G05.
Keywords. — Complex hyperbolic lattices, Milnor-Wood inequality, maximal representations, cominuscule representations, exceptional Lie groups, harmonic Higgs bundles, holomorphic foliations.

Contents

1. Introduction

This paper deals with maximal representations of complex hyperbolic lattices in semisimple Hermitian Lie groups with no compact factors.

A *complex hyperbolic lattice* Γ is a lattice in the Lie group $\mathrm{SU}(1,n)$, a finite cover of the group of biholomorphisms of the n-dimensional complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}} = \mathrm{SU}(1,n)/\mathrm{U}(n)$. We shall always assume that our lattice Γ is *cocompact*, or *uniform*, meaning that the quotient $X := \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ is compact. To simplify matters, we also assume in this introduction (except in the statements of our main results) that Γ is torsion free, so that X is also a manifold. The $\mathrm{SU}(1,n)$-invariant Kähler form on $\mathbb{H}^n_{\mathbb{C}}$ with constant holomorphic sectional curvature -1 and the Kähler form it induces on X will both be denoted by ω.

A Lie group $G_{\mathbb{R}}$ is said to be a *real algebraic Hermitian Lie group* (or a *Hermitian group* for short) if it is the connected component $\boldsymbol{G}(\mathbb{R})^\circ$ of the group of real point of an algebraic group $\boldsymbol{G}$ defined over $\mathbb{R}$, if it is semisimple with no compact factors, and if its associated symmetric space $M = G_{\mathbb{R}}/K_{\mathbb{R}}$ is a Hermitian symmetric space (of the noncompact type). This means that the noncompact symmetric space M admits a $G_{\mathbb{R}}$-invariant complex structure, with respect to which the $G_{\mathbb{R}}$-invariant Riemannian metrics are (necessarily) Kähler. The real rank $\mathrm{rk}_{\mathbb{R}}\, G_{\mathbb{R}}$ of $G_{\mathbb{R}}$ coincides with the rank r_M of M as a symmetric space, namely the maximal dimension of a flat subspace in M. Simple Hermitian groups are classified: there are four infinite families of *classical* groups, which up to isogeny are $\mathrm{SU}(p,q)$ with $1 \leqslant p \leqslant q$, $\mathrm{SO}_0(2,p)$ with $p \geqslant 3$, $\mathrm{Sp}(2m,\mathbb{R})$ with $m \geqslant 2$ and $\mathrm{SO}^\star(2m)$ with $m \geqslant 4$; and two *exceptional* groups $\mathrm{E}_{6(-14)}$ and $\mathrm{E}_{7(-25)}$. The real ranks of these groups are respectively p, 2, m, $\lfloor m/2 \rfloor$, 2 and 3.

Let ω_M denote the $G_{\mathbb{R}}$-invariant Kähler form of M, uniquely normalized so that its holomorphic sectional curvatures lie between -1 and $-1/r_M$.

If ρ is a representation (a group homomorphism) from Γ to $G_{\mathbb{R}}$, we define its *Toledo invariant* $\tau(\rho)$ as follows:

$$\tau(\rho) = \frac{1}{n!}\int_X f^\star\omega_M \wedge \omega^{n-1},$$

where $f : \mathbb{H}^n_{\mathbb{C}} \to M$ is a $\mathscr{C}^\infty$ and ρ-equivariant map and $f^\star\omega_M$ is seen as a 2-form on X by Γ-invariance. The Toledo invariant does not depend on the choice of the map f, it depends only on ρ, and in fact, only on the connected component of ρ in $\mathrm{Hom}(\Gamma, G_{\mathbb{R}})$. Moreover, it satisfies the following Milnor-Wood type inequality:

$$|\tau(\rho)| \leqslant r_M \operatorname{vol}(X),$$

a fundamental property established in full generality in [BI07].

The maximal representations $\rho : \Gamma \to G_{\mathbb{R}}$ are those representations for which the Milnor-Wood inequality is an equality.

In [KM17], to the introduction of which we refer for a history of the subject, Koziarz and the second named author classified maximal representations when $n \geqslant 2$, i.e., Γ is not a surface group, and when $G_{\mathbb{R}}$ is a classical group. Their proof uses the standard representations of the classical groups $G_{\mathbb{R}}$ and their complexifications $G := \boldsymbol{G}(\mathbb{C})$ and is therefore quite dependent on the classification of simple Hermitian groups. In the present work, using in a uniform way the *cominuscule* representation of G we give a new proof of the Milnor-Wood inequality relying on algebraic properties of this cominuscule representation shared by all Hermitian groups, see the proof of Theorem 4.6. This proof provides information which, in the case of maximal representations, allows us to extend the classification to all target Hermitian groups.

Theorem A. — *Let Γ be a uniform lattice in $\mathrm{SU}(1,n)$, $n \geqslant 2$. Let $G_{\mathbb{R}}$ be a real algebraic Hermitian Lie group and let M be the Hermitian symmetric space of the noncompact type associated with $G_{\mathbb{R}}$. If ρ is a maximal representation of Γ in $G_{\mathbb{R}}$, then:*

– *each simple factor of $G_{\mathbb{R}}$ is isogenous either to $\mathrm{SU}(p,q)$ for some (p,q) with $q \geqslant n\,p$, or to the exceptional group $\mathrm{E}_{6(-14)}$, the latter being possible if and only if $n = 2$;*

– *if $\tau(\rho) > 0$, there exists a ρ-equivariant holomorphic map $f : \mathbb{H}^n_{\mathbb{C}} \to M$. Moreover it satisfies $f^\star \omega_M = r_M\, \omega$;*

– *if $\tau(\rho) < 0$, there exists a ρ-equivariant antiholomorphic map $f : \mathbb{H}^n_{\mathbb{C}} \to M$. Moreover it satisfies $f^\star \omega_M = -r_M\, \omega$;*

– *in both cases, the map f is unique as it is the unique ρ-equivariant harmonic map $\mathbb{H}^n_{\mathbb{C}} \to M$. It is a totally geodesic embedding. It is also equivariant w.r.t. a uniquely defined homomorphism of Lie groups $\varphi : \mathrm{SU}(1,n) \to G_{\mathbb{R}}$.*

We can also deduce a complete structure result for the maximal representation ρ, namely that ρ is essentially the restriction to Γ of the morphism of Lie groups $\varphi : \mathrm{SU}(1,n) \to G_{\mathbb{R}}$ given by Theorem A.

Corollary B. — *Under the assumptions of Theorem* A*, the representation ρ is reductive, discrete, faithful, and acts cocompactly on the image of f in M. The centralizer $Z_{\mathbb{R}}$ of the image of φ in $G_{\mathbb{R}}$ is compact and there exists a group morphism $\rho_{\mathrm{cpt}} : \Gamma \to Z_{\mathbb{R}}$ such that*

$$\forall\, \gamma \in \Gamma, \quad \rho(\gamma) = \varphi(\gamma)\rho_{\mathrm{cpt}}(\gamma) = \rho_{\mathrm{cpt}}(\gamma)\varphi(\gamma).$$

The group $Z_{\mathbb{R}}$ is described in Lemma 5.10. Moreover, Lemma 5.12 says that $Z_{\mathbb{R}}$ is exactly the subgroup of $G_{\mathbb{R}}$ fixing $f(\mathbb{H}^n_{\mathbb{C}})$ pointwise. Just before Lemma 5.10, we also describe the morphism $\varphi : \mathrm{SU}(1,n) \to G_{\mathbb{R}}$. If $G_{\mathbb{R}}$ is simple, and up to conjugating ρ by an element in $G_{\mathbb{R}}$, it is always given by a diagonal embedding $\mathrm{SU}(1,n) \hookrightarrow \mathrm{SU}(r_M, nr_M) \hookrightarrow G_{\mathbb{R}}$, where $\mathrm{SU}(r_M, nr_M)$ is a subgroup of $G_{\mathbb{R}}$ stable under conjugation by elements of the maximal torus.

The global strategy we adopt here is the same as in [KM17]. First, we use known results (see 3.1) to get a harmonic ρ-equivariant map $f : \mathbb{H}^n_{\mathbb{C}} \to M$ and a Higgs bundle $(\overline{E}, \overline{\theta})$ on the quotient $X = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ associated to a (reductive) representation $\rho : \Gamma \to G_{\mathbb{R}}$. Then, we translate the Milnor-Wood inequality into an inequality involving degrees of subbundles of $\overline{E}$ (see 4.2). This inequality is then proved (in 4.4) using the Higgs-stability properties of $(\overline{E}, \overline{\theta})$, or rather the *leafwise* Higgs-stability properties of the pull-back (E, θ) of $(\overline{E}, \overline{\theta})$ to the projectivized tangent bundle $\mathbb{P}T_X$ of X with respect to the *tautological foliation* on $\mathbb{P}T_X$ (see Sections 3.2 and 3.3).

Although classical target groups were already treated in [KM17], we decided not to focus immediately on the exceptional cases and instead to provide a more unified perspective, as independent as possible of the classification of the simple Hermitian Lie groups, in the spirit of [BGPR17]. To achieve this, instead of considering the Higgs vector bundle associated with the standard representation of the complexification G of $G_{\mathbb{R}}$ (which is only defined in the classical cases), we work with the Higgs vector bundle (E, θ) associated with the *cominuscule* representation $\mathbb{E}$ of G defined by M (and a choice of invariant complex structure), see Section 2.2. This representation is such that M is holomorphically embedded as a locally closed subset of the projectivization $\mathbb{P}\mathbb{E}$ of $\mathbb{E}$: this is sometimes called the *first canonical embedding* of the compact dual $\check{M}$, see [NT76, p. 651].

On the algebraic side, we present in Section 2.4 a general construction of a self-dual graded subspace $\mathbb{V}^r$ of $\mathbb{E}$ associated with an element of $\mathfrak{m}^{\pm}$ of rank r (here, $\mathfrak{g} = \mathfrak{k} \oplus (\mathfrak{m}^+ \oplus \mathfrak{m}^-)$ is the Cartan decomposition of the Lie algebra of G). The subspace $\mathbb{V}^r$ is in fact a module under the action of a complex subgroup $\overline{L}_r$ of G, and the self-duality of $\mathbb{V}^r$ corresponds to the fact that $\overline{L}_r$ admits a noncompact real form $\overline{L}_{r,\mathbb{R}}$ which is a simple (if $G_{\mathbb{R}}$ is simple) Hermitian Lie group whose associated symmetric space M_r is of *tube type* and has rank r, see Section 2.4.4. On the geometric side, this construction produces a leafwise Higgs subsheaf $\mathscr{V}$ of $(E, \theta) \to \mathbb{P}T_X$ associated with the holomorphic component of the Higgs field θ (see Section 4.3), whose existence is then used to prove the Milnor-Wood inequality. To be a bit more precise, on a *generic* fiber of $(E, \theta) \to \mathbb{P}T_X$, the leafwise Higgs subsheaf we define admits a purely representation theoretic description. The algebraic counterparts of the generic objects are first introduced and studied in Section 2. This is then used in Section 4.3 to define the subsheaf and prove that it has the desired properties.

One interesting by-product of this unified approach is that it allows to exclude a priori the possibility of maximal representations in any tube type real algebraic Hermitian Lie group, and in particular in $\mathrm{E}_{7(-25)}$. Recall that up to isogeny the simple tube type Hermitian groups are $\mathrm{SU}(p,p)$, $\mathrm{SO}_0(2,p)$, $\mathrm{Sp}(2m,\mathbb{R})$, $\mathrm{SO}^\star(2m)$ with m even, and $\mathrm{E}_{7(-25)}$. See also Section 2.4.4. Indeed, we prove in Section 5.1 that for tube type targets the representation ρ satisfies a stronger inequality than the Milnor-Wood inequality.

Proposition C. — *Let Γ be a uniform lattice in $\mathrm{SU}(1,n)$, with $n \geqslant 2$, and let $X = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$. Assume that the real algebraic Hermitian Lie group $G_\mathbb{R}$ has tube type and let r_M be the real rank of $G_\mathbb{R}$. Let ρ be a representation $\Gamma \to G_\mathbb{R}$. Then*

$$|\tau(\rho)| \leqslant \max\Big\{r_M - 1, \frac{r_M}{2}\cdot\frac{n+1}{n}\Big\} \operatorname{vol}(X) < r_M \operatorname{vol}(X).$$

When the representation ρ is maximal, the results of [KM17] imply that the restriction of the leafwise Higgs subsheaf $\mathscr{V}$ corresponding to the module $\mathbb{V}^{r_M}$ to almost all leaves $\mathcal{L}$ of the tautological foliation on $\mathbb{P}T_X$ is a Higgs subbundle of $(E,\theta)_{|\mathcal{L}}$. Moreover, the self-duality of $\mathbb{V}^{r_M}$ means that the holomorphic subbundles $\mathscr{V}_i$ composing $\mathscr{V} = \bigoplus_{0\leqslant i\leqslant r_M} \mathscr{V}_i$ have the symmetries of a weight r_M real variation of Hodge structure: $\overline{\mathscr{V}}_{r_M-i} \simeq \mathscr{V}_i$. This is the Higgs bundle analogue, in the higher dimensional setting we are in, of the fact that maximal representations of surface groups stabilize maximal tube type subdomains in M (see the remark below). We also refer to Section 2.6 for a discussion of the cominuscule representation $\mathbb{E}$ and the submodule $\mathbb{V}^r$ from the Hodge theoretic point of view developed in [Gro94, SZ10].

To prove Theorem A, one then needs to deduce that the ρ-equivariant harmonic map f is (anti-)holomorphic from the facts we just mentioned. To this end, thanks to Proposition C and [KM17], we only need to deal with the case of $\mathrm{E}_{6(-14)}$. Maximal representations in this exceptional group are treated in Sections 5.2 and 5.3 where we prove that they exist if and only if $n = 2$, in which case they are essentially induced by a homomorphism $\varphi : \mathrm{SU}(1,2) \to \mathrm{SU}(2,4) \to \mathrm{E}_{6(-14)}$.

Remark. — The assumption $n \geqslant 2$ is of course essential in all these results. Maximal representations of lattices of $\mathrm{SU}(1,1)$, i.e., surface groups, form a very interesting and intensely studied subclass of representations, see e.g. [Her91, Xia00, MX02, BGPG03, BGPG06, BIW10, GW12]. They are also reductive, faithful and discrete, and they stabilize a tube type subdomain of rank r_M in M, but they exist for all target Hermitian groups and they are not in general "induced" by a homomorphism $\mathrm{SU}(1,1) \to G_\mathbb{R}$. On the contrary they define rich moduli spaces (in the case $G_\mathbb{R} = \mathrm{SU}(1,1)$, this is the Teichmüller space).

Accordingly, if $n = 1$, the first inequality of Proposition C is nothing but the Milnor-Wood inequality. One might ask whether this inequality is sharp when $n \geqslant 2$ and what could be said about representations achieving the optimal bound (whatever it may be). Note that this question is not restricted to tube type targets. For instance if $G_\mathbb{R} = \mathrm{SU}(p,q)$, Theorem A says that the Milnor-Wood inequality is sharp when $q \geqslant pn$, but says nothing about the other cases (when $p = 2$ and $q < 2n$, a better bound is known [KM08]).

The interesting (and probably difficult) question of finding the sharp bound for a given pair $(n, G_{\mathbb{R}})$ for which no maximal representation exists, and of characterizing the representations achieving this bound remains open.

Remark. — As we mentioned, our main results are true for lattices with torsion, see Remark 5.16. The assumption that Γ is uniform should not be necessary, and we believe that our method can be adapted to cover the case of non uniform lattices, but we decided to leave that aside for a future work.

Acknowledgements. — We thank the anonymous referees for their detailed and helpful comments, which led to substantial improvements in the presentation of the article. We thank them in particular for bringing the references [Gro94, SZ10] to our attention.

2. Submodule of a cominuscule representation associated with a nilpotent element

Here we develop the algebraic material that we will need in Section 4 to give a new and unified proof of the Milnor-Wood inequality.

2.1. Hermitian symmetric spaces. — We begin by recalling useful facts and definitions concerning Hermitian symmetric spaces and setting up our notation. Good references for what follows are [Wol72, AMRT10, Hel01, Kna02].

Let $G_{\mathbb{R}}$ be a simple real algebraic Hermitian Lie group and $K_{\mathbb{R}}$ a maximal compact subgroup of $G_{\mathbb{R}}$. By our convention, this means that $G_{\mathbb{R}}$ is a simple noncompact Lie group whose associated symmetric space $M = G_{\mathbb{R}}/K_{\mathbb{R}}$ is a Hermitian symmetric space of the noncompact type (M is irreducible since $G_{\mathbb{R}}$ is simple), and that moreover $G_{\mathbb{R}}$ is the connected component of the group of real points $\boldsymbol{G}(\mathbb{R})$ of an algebraic group $\boldsymbol{G}$ defined over $\mathbb{R}$. Observe that for example the connected component $\mathrm{Isom}^\circ(M)$ of the group of isometries of a Hermitian symmetric space of the noncompact type M is a real algebraic Hermitian Lie group, see e.g. [AMRT10, p. 106]. The group of complex points $\boldsymbol{G}(\mathbb{C})$ of $\boldsymbol{G}$ will be denoted by G.

We shall first assume that the complex Lie group G is simply connected. This assumption simplifies the exposition of the paper and the arguments of our proofs. We shall see in Remarks 5.14 and 5.15 how to deal with the general case.

The *rank* of the symmetric space M, or equivalently the *real rank* of $G_{\mathbb{R}}$, will be denoted by r_M. Recall that a k-dimensional *flat* in M is the image of a totally geodesic isometric embedding of $\mathbb{R}^k$ in M. By definition the rank r_M of M is the maximal dimension of a flat in M. Maximal flats are conjugated under $G_{\mathbb{R}}$ and maximal flats through the origin $eK_{\mathbb{R}}$ are conjugated under $K_{\mathbb{R}}$. Equivalently, in the Hermitian setting, one can consider *polydiscs* in M, that is, images of totally geodesic homomorphic embeddings of a polydisc Δ^k in M. Then r_M is also the maximal (complex) dimension of a polydisc in M (maximal polydiscs are complexifications of maximal flats).

The fact that M is an irreducible Hermitian symmetric space is equivalent to the fact that the Lie algebra $\mathfrak{k}_{\mathbb{R}}$ of $K_{\mathbb{R}}$ has a 1-dimensional center $\mathfrak{z}_{\mathbb{R}}$ and that the centralizer of $\mathfrak{z}_{\mathbb{R}}$ in the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $G_{\mathbb{R}}$ is $\mathfrak{k}_{\mathbb{R}}$, see e.g. [Kna02]. If $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{m}_{\mathbb{R}}$ is a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$, the complex structure at the origin is given by the adjoint action on $\mathfrak{m}_{\mathbb{R}}$ of a suitably normalized element of $\mathfrak{z}_{\mathbb{R}}$. It follows that any maximal Abelian subspace $\mathfrak{t}_{\mathbb{R}}$ of $\mathfrak{k}_{\mathbb{R}}$ contains $\mathfrak{z}_{\mathbb{R}}$ and is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. The corresponding subgroup $T_{\mathbb{R}} \subset K_{\mathbb{R}} \subset G_{\mathbb{R}}$ is a torus.

The Lie algebra $\mathfrak{g}$ of G is the complexification of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $G_{\mathbb{R}}$. We denote by $\mathfrak{z} \subset \mathfrak{t} \subset \mathfrak{k} \subset \mathfrak{g}$ the complexifications of the subalgebras $\mathfrak{z}_{\mathbb{R}} \subset \mathfrak{t}_{\mathbb{R}} \subset \mathfrak{k}_{\mathbb{R}}$ and by $Z \subset T \subset K$ the corresponding complex algebraic subgroups of G. We also let $\mathfrak{m}$ be the complexification of $\mathfrak{m}_{\mathbb{R}}$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a Cartan decomposition of $\mathfrak{g}$. In particular, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$.

Let $z \neq 0$ be an element in the center $\mathfrak{z}$ of $\mathfrak{k}$. Since the adjoint action of $\mathfrak{z}_{\mathbb{R}}$ gives the complex structure of $\mathfrak{m}_{\mathbb{R}}$, $\mathrm{ad}(z)_{|\mathfrak{m}}$ has exactly two opposite eigenvalues. We choose z so that these eigenvalues are 2 and -2 and we let

$$\mathfrak{m}^+ = \{x \in \mathfrak{m} \mid \mathrm{ad}(z)x = 2\,x\}, \quad \mathfrak{m}^- = \{x \in \mathfrak{m} \mid \mathrm{ad}(z)x = -2\,x\}$$

be the corresponding eigenspaces. These are abelian subspaces of $\mathfrak{m}$.

This in turn implies that $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{m}^+$ is a maximal parabolic subalgebra of $\mathfrak{g}$. Let P be the corresponding parabolic subgroup of G and $\check{M}$ the projective variety G/P. Then $\check{M}$ is a symmetric space of the compact type called the *compact dual* of M. Indeed if we let $\check{\mathfrak{g}} = \mathfrak{k}_{\mathbb{R}} \oplus i\mathfrak{m}_{\mathbb{R}}$, $\check{\mathfrak{g}}$ is the Lie algebra of a real compact form $\check{G}_{\mathbb{R}}$ of G, and $\check{M} = \check{G}_{\mathbb{R}}/K_{\mathbb{R}}$. The symmetric space M is then an open $G_{\mathbb{R}}$-orbit in $\check{M} = G/P$.

Example 2.1. — For $p \leqslant q$ let $G_{\mathbb{R}} = \mathrm{SU}(p,q)$ be the special unitary group of the Hermitian form of signature (p,q) on $\mathbb{C}^{p+q}$ whose matrix in the canonical basis is $I_{p,q} = \mathrm{diag}(-1, \dots, -1, 1, \dots, 1)$. Then one can choose $K_{\mathbb{R}} = \mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))$ and the symmetric space M identifies with the set of p-planes in $\mathbb{C}^{p+q}$ on which the Hermitian form is negative definite. It has rank p. As a bounded symmetric domain, it identifies with $\{Y \in \mathscr{M}_{p,q}(\mathbb{C}) \mid I_p - Y^{\star}Y \text{ is positive definite}\}$. We have $G = \mathrm{SL}(p+q, \mathbb{C})$, $K = \mathrm{S}(\mathrm{GL}(p) \times \mathrm{GL}(q))$, T is the torus of diagonal matrices in G, Z is the 1-dimensional subgroup of diagonal matrices of the form $\mathrm{diag}(\lambda^q, \dots, \lambda^q, \lambda^{-p}, \dots, \lambda^{-p})$ for $\lambda \in \mathbb{C}^{\star}$, $\mathfrak{m} \simeq \mathbb{C}^{pq} \times \mathbb{C}^{pq}$ is the subspace of block anti-diagonal matrices in $\mathfrak{sl}(p+q, \mathbb{C})$. The compact real form of G is $\mathrm{SU}(p+q)$ and the compact dual $\check{M}$ identifies with the Grassmannian of p-planes in $\mathbb{C}^{p+q}$, of which M is an open subset.

Let R be the set of roots of G. For $\alpha \in R$, $\mathfrak{g}_{\alpha} \subset \mathfrak{g}$ is the root space of α, $\mathfrak{sl}(\alpha)$ is the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{\alpha}$ and $\mathfrak{g}_{-\alpha}$ and $\mathrm{SL}(\alpha)$ the corresponding subgroup of G.

A root space $\mathfrak{g}_{\alpha}$, $\alpha \in R$, is either a subspace of $\mathfrak{k}$ or a subspace of $\mathfrak{m}$, and the root α is said to be *compact* or *noncompact* accordingly. Equivalently, a root α is compact if and only if $\langle \alpha, z \rangle = 0$. We have:

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha : \langle \alpha, z \rangle = 0} \mathfrak{g}_{\alpha}, \quad \mathfrak{m} = \bigoplus_{\alpha : \langle \alpha, z \rangle \neq 0} \mathfrak{g}_{\alpha}, \quad \mathfrak{m}^+ = \bigoplus_{\alpha : \langle \alpha, z \rangle = 2} \mathfrak{g}_{\alpha}, \quad \mathfrak{m}^- = \bigoplus_{\alpha : \langle \alpha, z \rangle = -2} \mathfrak{g}_{\alpha}.$$

We choose a basis Π of R so that the roots in $R(\mathfrak{m}^+)$ are positive roots. If $\alpha \in R$, we write $\alpha = \sum_{\beta\in\Pi} n_\beta(\alpha)\beta$ the expression of α in terms of the simple roots. The support $\operatorname{supp}(\alpha)$ of $\alpha \in R$ is the set $\{\beta \in \Pi \mid n_\beta(\alpha) \neq 0\}$.

The Weyl group of R is denoted by W. If $\alpha \in R$, $\alpha^\vee \in \mathfrak{t}$ denotes the corresponding coroot: it is defined by the relation $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee\rangle\alpha$ for all $\beta \in \mathfrak{t}^*$, where $s_\alpha \in W$ is the reflexion corresponding to α (in particular, $s_\alpha(\alpha) = -\alpha$ and hence $\langle \alpha, \alpha^\vee\rangle = 2$). If $\alpha = \sum_{\beta\in\Pi} n_\beta(\alpha)\beta$ is the expression of a root α in the simple roots, then $\alpha^\vee = \sum_{\beta\in\Pi} n_\beta(\alpha)\,\frac{\langle\beta,\alpha^\vee\rangle}{\langle\alpha,\beta^\vee\rangle}\,\beta^\vee$. If $\beta \in \Pi$ is a simple root, ϖ_β denotes the fundamental weight corresponding to β. The set of simple roots Π is a basis of $\mathfrak{t}^\star$, $\Pi^\vee = \{\beta^\vee \mid \beta \in \Pi\}$ is a basis of $\mathfrak{t}$ and $\{\varpi_\beta \mid \beta \in \Pi\}$ is by definition the basis of $\mathfrak{t}^\star$ dual to $\Pi^\vee$.

In this paper, we use the convention that if the root system R of $\mathfrak{g}$ is simply laced (equivalently, of type A, D or E), then all the roots are long. Therefore short roots exist only if R is not simply laced. There can be at most two root lengths in R. Moreover if there are two root lengths the ratio $\frac{\text{long root length}}{\text{short root length}}$ equals $\sqrt{2}$ because Hermitian symmetric spaces exist only when the type of G is A, B, C, D, E_6 or E_7 (in particular, G_2 does not appear). We recall that for $\alpha, \beta \in R$, the ratio $\langle\alpha,\beta^\vee\rangle/\langle\beta,\alpha^\vee\rangle$ equals (root length of $\alpha)^2$/(root length of $\beta)^2$.

Linearly independent positive roots $\delta_1, \ldots, \delta_s$ are said to be *strongly orthogonal* if for all $i \neq j$, $\delta_i \pm \delta_j$ is not a root. By [HC56], or [Hel01, Ch. VIII, §7], we have

Fact 2.2. — All maximal sets of noncompact strongly orthogonal roots have cardinality r_M.

If $(\delta_1, \ldots, \delta_{r_M})$ is such a set then $\mathfrak{g}_{-\delta_1} \oplus \cdots \oplus \mathfrak{g}_{-\delta_{r_M}}$ is the tangent space to a maximal polydisc through the origin in M.

2.2. The cominuscule representation of G associated with M

2.2.1. *The representation*

Definition 2.3. — A simple root $\beta \in \Pi$, is *cominuscule* w.r.t. the root system R if $n_\beta(\alpha) \in \{-1, 0, 1\}$ for all $\alpha \in R$, or equivalently if $n_\beta(\Theta) = 1$, where Θ is the highest root of R.

An irreducible representation of G whose highest weight is equal to ϖ_β for β a cominuscule root of R is called a *cominuscule representation.*

The following is well-known, but we include a proof.

Proposition 2.4. — *There is a unique simple noncompact root. This root is long and cominuscule.*

Notation 2.5. — This unique simple noncompact root will be denoted by ζ.

Proof. — Since R is irreducible, the highest root Θ of R is unique and Θ has a positive coefficient on every simple root. Now, if there are more than one simple noncompact root, or if $n_\zeta(\Theta) > 1$ for some noncompact simple root ζ, then $\langle\Theta, z\rangle > 2$.

Contradiction. Hence ζ is unique and cominuscule. Assume that ζ is short (so that R is not simply laced). Then, since Θ is long,

$$\langle \Theta, \zeta^\vee \rangle = \langle \zeta, \Theta^\vee \rangle \frac{(\text{root length of } \Theta)^2}{(\text{root length of } \zeta)^2} \geqslant 2.$$

Since $s_\zeta(\Theta)$ is a positive root, this implies that $n_\zeta(\Theta) \geqslant 2$, contradicting the fact that ζ is cominuscule. □

Note that the maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{k} \oplus \mathfrak{m}^+$ of $\mathfrak{g}$ is the standard parabolic defined by the root ζ:

$$\mathfrak{p} = \mathfrak{t} \oplus \bigoplus_{\alpha : n_\zeta(\alpha) \geqslant 0} \mathfrak{g}_\alpha.$$

Recall that P is the corresponding parabolic subgroup of G and that the compact dual of M is $\check{M} = G/P$. By [Mur59], since G is simply connected, the Picard group $\mathrm{Pic}(G/P)$ is isomorphic to the group of characters $\mathscr{X}(P)$ of P. Since $\mathfrak{p} = \mathfrak{z} \oplus [\mathfrak{p}, \mathfrak{p}]$, $\mathscr{X}(P)$ is isomorphic to $\mathbb{Z}$ and thus it is generated by the smallest positive character of P, namely $\varpi := \varpi_\zeta$. Here we embed $\mathscr{X}(P)$ in $\mathfrak{t}^*$ via $\lambda \mapsto d\lambda$, the differential of λ at the unit element of T. Moreover, the isomorphism $\iota : \mathscr{X}(P) \simeq \mathrm{Pic}(G/P)$ is given by $\lambda \mapsto (G \times \mathbb{C}_\lambda)/P$, where $\mathbb{C}_\lambda$ denotes the 1-dimensional P-module defined by λ. In particular, $\mathscr{L} := \iota(\varpi)$ is a generator of $\mathrm{Pic}(\check{M})$.

Let now $\mathbb{E}$ be the cominuscule irreducible representation of G whose highest weight is $\varpi = \varpi_\zeta$. Let $\mathbb{E}_\varpi$ be the ϖ-eigenspace of $\mathbb{E}$. Then $\mathbb{E}_\varpi$ is 1-dimensional and its stabilizer in G is P. This gives a G-equivariant holomorphic and isometric embedding of $\check{M} = G/P$ in the projective space $\mathbb{P}\mathbb{E}$. It is called the *first canonical embedding* of $\check{M}$, and since $\mathbb{E}_\varpi \simeq \mathbb{C}_\varpi$ as P-modules, we have $\mathscr{L} \simeq \mathscr{O}_{\mathbb{P}\mathbb{E}}(-1)$. See e.g. [NT76] for more details.

Remark 2.6. — The complex group G may have several cominuscule simple roots or representations. In fact, cominuscule simple roots are in one to one correspondence with Hermitian real forms $G_\mathbb{R}$ (or $\check{G}_\mathbb{R}$) of G. When we will need a case by case argument, we will rather use the classification of the cominuscule roots than the classification of the real groups $G_\mathbb{R}$. This correspondence is shown in the first two columns of Table 1.

We now begin our study of the cominuscule representation $\mathbb{E}$ of G.

Notation 2.7. — We denote by μ_0 the lowest weight of $\mathbb{E}$ and by $X(\mathbb{E})$ the set of weights of $\mathbb{E}$. For $\chi \in X(\mathbb{E})$, we write $\mathbb{E}_\chi$ for the corresponding weight space. Recall that ϖ is the highest weight of $\mathbb{E}$.

The fact that $\mathbb{E}$ is cominuscule has the following consequence on the weights of $\mathbb{E}$.

Lemma 2.8. — *For any weight χ of $\mathbb{E}$, and any root $\alpha \in R$, $\langle \chi, \alpha^\vee \rangle \in \{-2, -1, 0, 1, 2\}$, and the equality $\langle \chi, \alpha^\vee \rangle = \pm 2$ implies that α is short (hence that R is not simply laced).*

Proof. — For the highest weight ϖ of $\mathbb{E}$, the results follows from the facts that $\langle\varpi,\alpha^\vee\rangle = n_\zeta(\alpha)\,\langle\zeta,\alpha^\vee\rangle/\langle\alpha,\zeta^\vee\rangle$ and ζ is a long root. The result still holds if ϖ is replaced by $w\cdot\varpi$, where $w\in W$ is arbitrary, and since any weight of $\mathbb{E}$ is in the convex hull of $W\cdot\varpi$, it holds for any weight. $\square$

We deduce that the structure of $\mathbb{E}$ with respect to the action of $\mathfrak{g}_\alpha$ for α a long root is particularly simple.

Lemma 2.9. — *Let α be a long root and let χ be a weight of $\mathbb{E}$. We have*

$$\mathfrak{g}_{-\alpha}\cdot\mathbb{E}_\chi = \begin{cases}\mathbb{E}_{\chi-\alpha} & \textit{if } \langle\chi,\alpha^\vee\rangle = 1,\\ \{0\} & \textit{otherwise.}\end{cases}$$

Proof. — Let α be long and let $\mathfrak{sl}(\alpha)$ be the Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2$ corresponding to α. Let $S\subset\mathbb{E}$ be the $\mathfrak{sl}(\alpha)$-submodule generated by $\mathbb{E}_\chi$. By Lemma 2.8 and the representation theory of $\mathfrak{sl}_2$, any irreducible component V of S is an $\mathfrak{sl}(\alpha)$-module of dimension 1 or 2.

We therefore have only three possibilities. The first case is when $V=V_\chi$, $\mathfrak{g}_\alpha\cdot V_\chi=\{0\}$ and $\mathfrak{g}_{-\alpha}\cdot V_\chi=\{0\}$. In this case, $\langle\chi,\alpha^\vee\rangle=0$. The second case is when $V=V_\chi\oplus V_{\chi-\alpha}$, $\mathfrak{g}_\alpha\cdot V_\chi=\{0\}$ and $\mathfrak{g}_{-\alpha}\cdot V_\chi=V_{\chi-\alpha}$. In this case, $\langle\chi,\alpha^\vee\rangle=1$ (and $\langle\chi-\alpha,\alpha^\vee\rangle=-1$). The third (symmetric) case is when $V=V_\chi\oplus V_{\chi+\alpha}$, $\mathfrak{g}_\alpha\cdot V_\chi=V_{\chi+\alpha}$ and $\mathfrak{g}_{-\alpha}\cdot V_\chi=\{0\}$. In this case, $\langle\chi,\alpha^\vee\rangle=-1$ (and $\langle\chi+\alpha,\alpha^\vee\rangle=1$).

If $\langle\chi,\alpha^\vee\rangle=1$, we deduce that $S=S_\chi\oplus S_{\chi-\alpha}$ and that $\mathfrak{g}_{-\alpha}\cdot S_\chi=S_{\chi-\alpha}$. We have $s_\alpha(\chi)=\chi-\alpha$ so $\dim(\mathbb{E}_\chi)=\dim(\mathbb{E}_{\chi-\alpha})$. The lemma is proved in this case. If $\langle\chi,\alpha^\vee\rangle\leqslant 0$, we see that $\mathfrak{g}_{-\alpha}\cdot\mathbb{E}_\chi=\{0\}$. $\square$

2.2.2. *The grading*

Notation 2.10. — Let $z_{\mathtt{max}}$ denote the number $\langle\varpi,z\rangle$.

Remark 2.11. — The precise value of $z_{\mathtt{max}}$ will not play any role in the following, however it has already been computed in [KM10, p. 214-216]): with our choice of $z\in\mathfrak{z}$ we have $z_{\mathtt{max}}=2\dim\check{M}/c_1(\check{M})$, where $c_1(\check{M})$ is the first Chern number of $\check{M}$. For example if $G_\mathbb{R}=\mathrm{SU}(p,q)$, $z_{\mathtt{max}}=2pq/(p+q)$.

Proposition 2.12. — *The set $\{\langle\chi,z\rangle\mid\chi\in X(\mathbb{E})\}$ is the set*

$$\{z_{\mathtt{max}}, z_{\mathtt{max}}-2,\dots,z_{\mathtt{max}}-2r_M\}.$$

Proof. — It follows from [RRS92, Th. 2.1] that the W-orbit of the weight ϖ is exactly the set of weights of the form $\varpi-\sum_{i=1}^k\delta_i$, where $(\delta_i)_{1\leqslant i\leqslant k}$ is a family of noncompact long strongly orthogonal roots. In fact, the orbit $W\cdot\varpi$ is in bijection with W/W_P, so the equivalence between items (c) and (e) in the cited theorem proves the claim. For any i, we have $\langle\delta_i,z\rangle=2$, thus we have the equality of sets

$$\{\langle\mu,z\rangle\mid\mu\in W\cdot\varpi\}=\{z_{\mathtt{max}},z_{\mathtt{max}}-2,\dots,z_{\mathtt{max}}-2r_M\}.$$

In particular, $\langle\mu_0, z\rangle = z_{\mathtt{max}} - 2r_M$ and for $\chi\in X(\mathbb{E})$, we have $z_{\mathtt{max}} - 2r_M \leqslant \langle\chi, z\rangle \leqslant z_{\mathtt{max}}$. The result of the proposition now follows from the fact that 2 is a divisor of $\langle\alpha, z\rangle$ for any root α. □

Now we can introduce the grading of $\mathbb{E}$.

Definition 2.13. — For a relative integer i, let

$$\mathbb{E}_i := \bigoplus_{\chi:\langle\chi,z\rangle = z_{\mathtt{max}} - 2i} \mathbb{E}_\chi.$$

This grading corresponds to the decomposition of $\mathbb{E}$ into irreducible K-modules.

Proposition 2.14. — *The K-modules $\mathbb{E}_i$ are irreducible.*

Proof. — This might be well-known to experts, but we include a proof for completeness. We give a case by case argument and use Table 1.

In type A_{n-1}, we have $\mathbb{E} = \wedge^{r_M}(\mathbb{C}^{r_M} \oplus \mathbb{C}^{n-r_M})$ and thus $\mathbb{E}_i = \wedge^{r_M - i}\mathbb{C}^{r_M} \otimes \wedge^i\mathbb{C}^{n-r_M}$: this is an irreducible $\mathrm{S}(\mathrm{GL}_{r_M} \times \mathrm{GL}_{n-r_M})$-module.

In type B_n, we have $\mathbb{E} = \mathbb{C}^{2n+1} = \mathbb{C} \oplus \mathbb{C}^{2n-1} \oplus \mathbb{C}$, and each summand is an irreducible Spin_{2n-1}-module, hence an irreducible K-module. In type (D_n, ϖ_1) the situation is similar.

In type (D_n, ϖ_n), $\mathbb{E}$ is the spinor representation of the spin group and, according to [Che97], we have $\mathbb{E} = \wedge^0\mathbb{C}^n \oplus \wedge^2\mathbb{C}^n \oplus \cdots \oplus \wedge^{2p}\mathbb{C}^n$ (where $p = \lfloor n/2\rfloor$). Thus $\mathbb{E}_i = \wedge^{2i}\mathbb{C}^n$, and this is an irreducible GL_n-module.

For the types E_6 and E_7, we use models of these exceptional Lie algebras and their minuscule representations, as given for example in [Man06]. In type E_6, we have $\mathbb{E} = \mathbb{C} \oplus V_{16} \oplus V_{10}$, where V_{16} is a spinor representation and V_{10} the vector representation of the spin group Spin_{10}. In type E_7, we have $\mathbb{E} = \mathbb{C} \oplus V_{27} \oplus V'_{27} \oplus \mathbb{C}$, where V_{27} and V'_{27} are the two minuscule representations of a group of type E_6. In both cases, the K-modules $\mathbb{E}_i$ are irreducible.

We now deal with the case of type C_n. The following representation-theoretic argument has been suggested to us by an anonymous referee that we would like to thank. We denote by $\mathbb{C}^{2n} = \mathbb{C}^n_a \oplus \mathbb{C}^n_b$ a symplectic $2n$-dimensional space, with $\mathbb{C}^n_a$ and $\mathbb{C}^n_b$ supplementary isotropic subspaces. We then have $\mathbb{E} = \left(\wedge^n\mathbb{C}^{2n}\right)_\omega$, where the symbol ω means that we take in $\wedge^n\mathbb{C}^{2n}$ the irreducible Sp_{2n}-submodule containing the highest weight line $\wedge^n\mathbb{C}^n_a$. More precisely, by [FH91, Th. 17.5], we have a decomposition of $\wedge^n\mathbb{C}^{2n}$ as an Sp_{2n}-module as follows: $\wedge^n\mathbb{C}^{2n} = \mathbb{E} \oplus \omega \wedge (\wedge^{n-2}\mathbb{C}^{2n})$.

Thus, $\wedge^{n-i}\mathbb{C}^n_a \otimes \wedge^i\mathbb{C}^n_b = \mathbb{E}_i \oplus \omega \wedge (\wedge^{n-i-1}\mathbb{C}^n_a) \otimes \wedge^{i-1}\mathbb{C}^n_b$. We now consider this as a representation of $K = \mathrm{GL}_n$. As K-modules, we have $\mathbb{C}^n_b \simeq (\mathbb{C}^n_a)^\star$. We thus get $\wedge^{n-i}\mathbb{C}^n_a \otimes \wedge^{n-i}\mathbb{C}^n_a \simeq \mathbb{E}_i \oplus \wedge^{n-i-1}\mathbb{C}^n_a \otimes \wedge^{n-i+1}\mathbb{C}^n_a$. From this last equation, it follows that $\mathbb{E}_i$ is the Cartan square of $\wedge^{n-i}\mathbb{C}^n_a$ and therefore it is an irreducible GL_n-module. □

Proposition 2.15. — *We have the following properties:*

(a) $\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_{r_M}$.

(b) $\mathbb{E}_0 = \mathbb{E}_\varpi$.

(c) $\mathbb{E}_{i+1} = \mathfrak{m}^- \cdot \mathbb{E}_i$.

(d) *The map* $\mathbb{E}_0 \otimes \mathfrak{m}^- \to \mathbb{E}_1$ *is an isomorphism.*

Proof. — Only the last two points need a proof. Let $U(\mathfrak{m}^-)$ denote the enveloping algebra of $\mathfrak{m}^-$. The third point follows from the fact that $\mathbb{E} = U(\mathfrak{m}^-) \cdot \mathbb{E}_\varpi$ and the fact that for α a root of $\mathfrak{m}^-$, we have $\langle \alpha, z\rangle = -2$. The last point follows by Schur's lemma since $\mathfrak{m}^-$ and $\mathbb{E}_1$ are irreducible $\mathfrak{k}$-modules and $\mathbb{E}_0$ is 1-dimensional. □

Remark 2.16. — Those statements are well-known. In the case where $\mathbb{E}$ is of tube type, they are proved in [Gro94, Prop. 5.2].

2.3. Dominant orthogonal sequences. — In [Kos12], Kostant introduced his so-called "chain cascade" of orthogonal roots. Here we will need a version of his algorithm where we impose that all the roots of the chain cascade have a positive coefficient on ζ. Note that a similar algorithm is used in [BM15].

We define an integer q and, for any integer i such that $1 \leqslant i \leqslant q$, a root α_i together with the subset $\Pi_i \subset \Pi$, by the following inductive process:

– We let $\Pi_1 = \Pi$.

– Assuming that $\alpha_1, \ldots, \alpha_{i-1}$ and $\Pi_1, \ldots, \Pi_i$ have been defined, we let α_i be the highest root of the root system $R(\Pi_i)$ generated by Π_i.

– We let $\Sigma_i \subset \Pi_i$ be the set of simple roots β such that $\langle \alpha_i^\vee, \beta\rangle \neq 0$.

– If $\zeta \in \Sigma_i$, then $q = i$ and the algorithm terminates. Otherwise, Π_{i+1} is the connected component of $\Pi_i \smallsetminus \Sigma_i$ containing ζ.

Definition 2.17. — If $(\alpha_i)_{1\leqslant i\leqslant q}$ is the sequence defined by this process, we say that it is the *maximal dominant orthogonal sequence* for ϖ. More generally, the sequences $(\alpha_i)_{1\leqslant i\leqslant r}$ for $1 \leqslant r \leqslant q$ are called the *dominant orthogonal sequences*.

Example 2.18. — In type A_{n-1} with the standard base $(\beta_1, \ldots, \beta_{n-1})$, and for $\zeta = \beta_p$ with $2p \leqslant n$, we get $\Pi_i = \{\beta_i, \ldots, \beta_{n-i}\}$ and the maximal dominant orthogonal sequence is $(\sum_{i=1}^{n-1} \beta_i, \sum_{i=2}^{n-2} \beta_i, \ldots, \sum_{i=p}^{n-p} \beta_i)$, so $q = p$.

For later use, we record in Table 1 below what are the dominant orthogonal sequences $(\alpha_1, \ldots, \alpha_r)$ in all cases. The number r varies from 1 to r_M, see Proposition 2.19 (2).

The following proposition essentially adapts the results of [Kos12] to our context and explains our terminology.

Proposition 2.19. — *Let* $(\alpha_i)_{1\leqslant i\leqslant q}$ *be the maximal dominant orthogonal sequence for* ϖ. *Then*

(1) *The roots* α_i *are long and strongly orthogonal.*

(2) $q = r_M$.

(3) *For any integer* $i \leqslant r_M$, $\alpha_1^\vee + \cdots + \alpha_i^\vee$ *is a dominant coweight: for all* β *in* Π, *we have* $\langle \beta, \alpha_1^\vee + \cdots + \alpha_i^\vee\rangle \geqslant 0$.

(4) $\varpi - \alpha_1 - \cdots - \alpha_{r_M}$ *is the lowest weight* μ_0 *of the irreducible* G*-module* $\mathbb{E}$.

(G,ϖ)	$\mathfrak{g}_{\mathbb{R}}$, r_M		$\alpha_1^\vee+\cdots+\alpha_r^\vee$	α_r
(A_{n-1},ϖ_p), $2p\leqslant n$ $\overset{1}{\circ}\cdots\circ\overset{p}{\bullet}\circ\cdots\overset{n-1}{\circ}$	$\mathfrak{su}(p,n-p)$ $r_M=p$	if $2r<n$:	$0\cdots0\,\overset{r}{1}\,0\cdots0\,\overset{n-r}{1}\,0\cdots0$	$0\cdots0\,\overset{r}{1}\cdots\overset{n-r}{1}\,0\cdots0$
		if $2r=n$:	$0\cdots0\,\overset{p}{2}\,0\cdots0$	$0\cdots0\,\overset{p}{1}\,0\cdots0$
(B_n,ϖ_1) $\overset{1}{\bullet}\circ\cdots\circ\circ\Rightarrow\overset{n}{\circ}$	$\mathfrak{so}(2,2n-1)$ $r_M=2$	if $r=1$:	$0\,1\,0\cdots0\,0\Rightarrow0$	$1\,2\cdots2\,2\Rightarrow2$
		if $r=2$:	$2\,0\cdots0\,0\Rightarrow0$	$1\,0\cdots0\,0\Rightarrow0$
(C_n,ϖ_n) $\overset{1}{\circ}\cdots\circ\circ\Leftarrow\overset{n}{\bullet}$	$\mathfrak{sp}(2n,\mathbb{R})$ $r_M=n$	if $r<n$:	$0\cdots0\,\overset{r}{2}\,0\cdots0\,0\Leftarrow0$	$0\cdots0\,\overset{r}{2}\cdots2\,2\Leftarrow1$
		if $r=n$:	$0\cdots0\,0\Leftarrow2$	$0\cdots0\,0\Leftarrow1$
(D_n,ϖ_1) $\overset{1}{\bullet}\circ\cdots\circ\begin{smallmatrix}\circ\ n-1\\ \circ\ n\end{smallmatrix}$	$\mathfrak{so}(2,2n-2)$ $r_M=2$	if $r=1$:	$0\,1\,0\cdots0\,0\begin{smallmatrix}0\\0\end{smallmatrix}$	$1\,2\cdots2\,2\begin{smallmatrix}1\\1\end{smallmatrix}$
		if $r=2$:	$2\,0\cdots0\,0\begin{smallmatrix}0\\0\end{smallmatrix}$	$1\,0\cdots0\,0\begin{smallmatrix}0\\0\end{smallmatrix}$
(D_n,ϖ_n) $\overset{1}{\circ}\circ\cdots\circ\begin{smallmatrix}\circ\ n-1\\ \bullet\ n\end{smallmatrix}$	$\mathfrak{so}^\star(2n)$ $r_M=\lfloor n/2\rfloor$	if $2r\leqslant n-2$:	$0\cdots0\,\overset{2r}{1}\,0\cdots0\,0\begin{smallmatrix}0\\0\end{smallmatrix}$	$0\cdots0\,1\,\overset{2r}{2}\cdots2\,2\begin{smallmatrix}1\\1\end{smallmatrix}$
		if $2r=n-1$:	$0\cdots0\,0\begin{smallmatrix}1\\1\end{smallmatrix}$	$0\cdots0\,1\begin{smallmatrix}1\\1\end{smallmatrix}$
		if $2r=n$:	$0\cdots0\,0\begin{smallmatrix}0\\2\end{smallmatrix}$	$0\cdots0\,0\begin{smallmatrix}0\\1\end{smallmatrix}$
(E_6,ϖ_1) $\overset{1}{\bullet}\circ\overset{\circ}{\circ}\circ\circ$	$\mathfrak{e}_{6(-14)}$ $r_M=2$	if $r=1$:	$0\,0\,\overset{1}{0}\,0\,0$	$1\,2\,\overset{2}{3}\,2\,1$
		if $r=2$:	$1\,0\,\overset{0}{0}\,0\,1$	$1\,1\,\overset{0}{1}\,1\,1$
(E_7,ϖ_7) $\circ\circ\overset{\circ}{\circ}\circ\circ\overset{7}{\bullet}$	$\mathfrak{e}_{7(-25)}$ $r_M=3$	if $r=1$:	$1\,0\,\overset{0}{0}\,0\,0\,0$	$2\,3\,\overset{2}{4}\,3\,2\,1$
		if $r=2$:	$0\,0\,\overset{0}{0}\,0\,1\,0$	$0\,1\,\overset{1}{2}\,2\,2\,1$
		if $r=3$:	$0\,0\,\overset{0}{0}\,0\,0\,2$	$0\,0\,\overset{0}{0}\,0\,0\,1$

Table 1. Dominant orthogonal sequences $(\alpha_1,\ldots,\alpha_r)$ for $1\leqslant r\leqslant r_M$ (We do not indicate all the roots α_i, but rather the sum of the corresponding coroots $\alpha_1^\vee+\cdots+\alpha_r^\vee$, by indicating on a Dynkin diagram the values $\langle\alpha_1^\vee+\cdots+\alpha_r^\vee,\beta\rangle$ for all simple roots β. In the last column, we express the root α_r which is also the smallest root α such that $\langle\alpha_1^\vee+\cdots+\alpha_r^\vee,\alpha\rangle=2$.)

Proof. — Kostant [Kos12, Lem. 1.6] proved the strong orthogonality. For the convenience of the reader, we recall his arguments in our context. The root α_i is the highest root of $R(\Pi_i)$, and Π_i contains the long root ζ, so α_i is long. By construction, $\langle\alpha_i^\vee,\alpha_j\rangle=0$ if $j>i$, so α_i and α_j are orthogonal. Since they are both long, they are strongly orthogonal, otherwise $\alpha_i\pm\alpha_j$ would be longer than α_i. This proves (1).

For (2), by Fact 2.2, it is enough to prove that $(\alpha_1,\dots,\alpha_q)$ is a maximal sequence of orthogonal roots (see also [Kos12, Th. 1.8]). Let $\alpha\in R$ be such that $\langle\alpha_i^\vee,\alpha\rangle=0$ holds for all i. We can and will assume that $\alpha>0$. Let i be the greatest integer such that $\alpha\in R(\Pi_i)$. By maximality of i there exists a simple root β in $\operatorname{supp}(\alpha)\cap\Sigma_i$. Since α_i is dominant on $R(\Pi_i)$, we have $\langle\alpha_i^\vee,\alpha\rangle\geqslant\langle\alpha_i^\vee,\beta\rangle>0$, a contradiction to the assumption made on α.

For the third point, let $i\leqslant r_M$ and let $\beta\in\Pi$.

– If $\beta\in\Pi_i$, by construction, $\langle\alpha_j^\vee,\beta\rangle=0$ for $j<i$. Since α_i is the highest root of Π_i, $\langle\alpha_i^\vee,\beta\rangle\geqslant0$. Thus $\langle\alpha_1^\vee+\cdots+\alpha_i^\vee,\beta\rangle=\langle\alpha_i^\vee,\beta\rangle\geqslant0$.

– If $\beta\in\Sigma_{i-1}$, then $\langle\alpha_{i-1}^\vee,\beta\rangle\geqslant1$ and $\langle\alpha_j^\vee,\beta\rangle=0$ for $j<i-1$. Since α_i is long, $\langle\alpha_i^\vee,\beta\rangle\geqslant-1$. Thus, $\langle\alpha_1^\vee+\cdots+\alpha_i^\vee,\beta\rangle\geqslant0$.

– If $\beta\in\Sigma_j$ for $j<i-1$, then, by construction of Π_i, $\langle\alpha_i^\vee,\beta\rangle=0$. By induction on i, $\langle\alpha_1^\vee+\cdots+\alpha_{i-1}^\vee,\beta\rangle\geqslant0$, so $\langle\alpha_1^\vee+\cdots+\alpha_i^\vee,\beta\rangle\geqslant0$.

For (4), by Lemma 2.8 and since α_i is long, we have $\langle\varpi,\alpha_i^\vee\rangle=1$. Thus $s_{\alpha_i}(\varpi)=\varpi-\langle\varpi,\alpha_i^\vee\rangle\alpha_i=\varpi-\alpha_i$. Therefore $s_{\alpha_1}\cdots s_{\alpha_{r_M}}(\varpi)=\varpi-\alpha_1-\cdots-\alpha_{r_M}$ is a weight of $\mathbb{E}$. We prove that it is a lowest weight. Let $\beta\in\Pi$. If $\beta\neq\zeta$, then

$$\langle\varpi-\alpha_1-\cdots-\alpha_{r_M},\beta^\vee\rangle=\langle-\alpha_1-\cdots-\alpha_{r_M},\beta^\vee\rangle,$$

and this is non-positive by (3) and because all the roots α_i have the same length. If $\beta=\zeta$, then we compute that $\langle\varpi-\alpha_1-\cdots-\alpha_{r_M},\zeta^\vee\rangle=1-\langle\alpha_{r_M},\zeta^\vee\rangle\leqslant0$ since $\zeta\in\Sigma_{r_M}$. □

We make the following observations.

Proposition 2.20. — *Let $1\leqslant r\leqslant r_M$, $(\alpha_1,\dots,\alpha_r)$ be the corresponding dominant orthogonal sequence, and $h_r=\alpha_1^\vee+\cdots+\alpha_r^\vee$. Then:*

(1) *h_r is dominant: for any positive root α, we have $\langle\alpha,h_r\rangle\geqslant0$.*
(2) *For any root α, we have $\langle\alpha,h_r\rangle\leqslant2$.*
(3) *α_r is the smallest root α such that $\langle\alpha,h_r\rangle=2$.*
(4) *If a root α satisfies $\langle\alpha,h_r\rangle=2$, then $\langle\alpha,z\rangle=2$.*
(5) *We have $\langle\varpi,h_r\rangle=r$.*

Proof. — Recall Table 1. The first point has been proved in Proposition 2.19 (3). Let Θ be the highest root of the root system of G, which can be found for example in [Bou68]. Since h_r is dominant, the second item follows from the fact that $\langle\Theta,h_r\rangle=2$ in all cases.

For the third item, we have indicated the root α_r in the last column of the table. It readily follows that it is the smallest root α such that $\langle\alpha,h_r\rangle=2$. Since it has coefficient 1 on ζ, we have $\langle\alpha_r, z\rangle = 2$. Any root α as in the fourth item will be bigger than α_r, so the claim follows.

To prove that $\langle\varpi, h_r\rangle = r$, recall that if $(\alpha_1, \dots, \alpha_{r_M})$ is the maximal dominant orthogonal sequence, the weight $\varpi - \sum_1^{r_M} \alpha_i$ is the lowest weight μ_0 by Proposition 2.19 (4). Thus, $\langle\varpi - \mu_0, h_r\rangle = \langle\alpha_1 + \dots + \alpha_{r_M}, \alpha_1^\vee + \dots + \alpha_r^\vee\rangle = 2r$, since $\langle\alpha_i, \alpha_j^\vee\rangle = 2\delta_{i,j}$ by orthogonality of the roots α_i. Moreover, h_r is part of some $\mathfrak{sl}_2$-triple (x, h_r, y), so $\langle\mu_0, h_r\rangle = -\langle\varpi, h_r\rangle$, by the representation theory of $\mathfrak{sl}_2$. This proves that $\langle\varpi, h_r\rangle = r$. □

2.4. The self-dual Higgs submodule associated with a nilpotent element

We explain in this section that an element $y \in \mathfrak{m}^-$ defines a special graded subspace in $\mathbb{E}$. This algebraic construction will be used in Section 4.3 to produce the leafwise Higgs subsheaves needed to prove the Milnor-Wood inequality.

2.4.1. *K-orbits in $\mathfrak{m}^-$: ranks of nilpotent elements.* — We begin by describing the K-orbits in $\mathfrak{m}^-$, introducing the notion of *rank*, and then we choose nice representatives in the orbits to simplify our work. Let us recall the following result, which is proved e.g. in [HC56], [Hel01, Ch. VIII, §7], or [Wol72].

Proposition 2.21. — *The orbits of the complex group K in $\mathfrak{m}^-$ are parametrized by integers $r \in \{0, \dots, r_M\}$. A representative of each orbit is $y_{\alpha_1} + \dots + y_{\alpha_r}$, where $(\alpha_1, \dots, \alpha_{r_M})$ is the maximal dominant orthogonal sequence.*

Definition 2.22. — Let $y \in \mathfrak{m}^-$. The *rank* $r(y)$ of y is the integer r such that y is in the K-orbit of $y_{\alpha_1} + \dots + y_{\alpha_r}$.

Example 2.23. — In case $\mathfrak{g}$ has type A, $\mathfrak{m}^-$ identifies with a space of matrices, and the rank as defined above of an element in $\mathfrak{m}^-$ coincides with its rank as a matrix.

The following proposition is a characterization of the rank $r(y)$ of $y \in \mathfrak{m}^-$ in terms of its action on $\mathbb{E}$. If $y \in \mathfrak{g}$, we will denote by $\boldsymbol{y} \in \operatorname{End}(\mathbb{E})$ the image of y by the representation of $\mathfrak{g}$ on $\mathbb{E}$.

Proposition 2.24. — *Let $y = y_{\alpha_1} + \dots + y_{\alpha_{r(y)}}$ with $(\alpha_1, \dots, \alpha_{r(y)})$ a dominant orthogonal sequence. We have $\boldsymbol{y}^{r(y)+1} = 0$ and $\boldsymbol{y}^{r(y)}(\mathbb{E}_0) = \mathbb{E}_{\varpi - \alpha_1 - \dots - \alpha_{r(y)}}$, so in particular $\boldsymbol{y}^{r(y)}(\mathbb{E}_0) \neq \{0\}$. In other words, the rank of y is the order of nilpotency of $\boldsymbol{y}$.*

Proof. — We have $y_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$. For any pair (χ, δ) with $\chi \in X(\mathbb{E})$ and δ a long root, $\chi - 2\delta$ cannot be a weight of $\mathbb{E}$ by Lemma 2.9. We deduce from Proposition 2.19 (1) that $\boldsymbol{y}_{\alpha_i}^2 = 0$. On the other hand, for any i, j with $i \neq j$, the maps $\boldsymbol{y}_{\alpha_i}$ and $\boldsymbol{y}_{\alpha_j}$ commute since $\alpha_i + \alpha_j$ is not a root by Proposition 2.19 (1) also. Thus we get

$$\boldsymbol{y}^k = k! \sum \boldsymbol{y}_{\alpha_{j_1}} \circ \dots \circ \boldsymbol{y}_{\alpha_{j_k}},$$

where the sum is over the increasing sequences $1 \leqslant j_1 < \dots < j_k \leqslant r(y)$.

In particular $\boldsymbol{y}^{r(y)+1} = 0$, and $\boldsymbol{y}^{r(y)} = r(y)!\, \boldsymbol{y}_{\alpha_1} \circ \dots \circ \boldsymbol{y}_{\alpha_{r(y)}}$. By Lemma 2.9, we have $\boldsymbol{y}_{\alpha_1} \circ \dots \circ \boldsymbol{y}_{\alpha_{r(y)}} \cdot \mathbb{E}_\varpi = \mathbb{E}_{\varpi - \alpha_1 - \dots - \alpha_{r(y)}}$, so the proposition is proved. □

For the remaining of this section, we let $1 \leqslant r \leqslant r_M$ be an integer and $y = y_r := y_{\alpha_1} + \cdots + y_{\alpha_r} \in \mathfrak{m}^-$, where $(\alpha_1, \ldots, \alpha_r)$ is the dominant orthogonal sequence of length r obtained by the algorithm of Definition 2.17. By Proposition 2.21, there is no loss of generality in considering this particular element y. We denote by h_r the element $\alpha_1^\vee + \cdots + \alpha_r^\vee$ spelled out in Table 1. Recall that $\boldsymbol{y}$ is y acting on $\mathbb{E}$ via the cominuscule representation of $\mathfrak{g}$, and that r is the order of nilpotency of $\boldsymbol{y}$.

2.4.2. *Definition of* $\mathbb{V}^r$ *and the Higgs property.* — It is a very well-known idea that the nilpotent element $\boldsymbol{y}$ of $\mathrm{End}(\mathbb{E})$ uniquely defines an increasing filtration $\mathbb{W}^r_\bullet$ of $\mathbb{E}$, called the *Deligne weight filtration of* $\boldsymbol{y}$, or *Jacobson-Morozov filtration.* This idea appears in [Del80, §1.6.13] in the context of Hodge theory and in [McG02, §§3.2 & 3.4] in the context of nilpotent orbits in Lie algebras.

The defining properties of the filtration $(\mathbb{W}^r_k)_{k\in\mathbb{Z}}$ are that for all $k \in \mathbb{Z}$,

$$\boldsymbol{y}(\mathbb{W}^r_k) \subset \mathbb{W}^r_{k-2}$$

and for all $k \geqslant 1$,

$$\boldsymbol{y}^k : \mathbb{W}^r_k/\mathbb{W}^r_{k-1} \longrightarrow \mathbb{W}^r_{-k}/\mathbb{W}^r_{-k-1} \text{ is an isomorphism.} \tag{2.1}$$

The filtration $(\mathbb{W}^r_k)_{k\in\mathbb{Z}}$ can be defined as follows (see e.g. [SZ85]):

$$\mathbb{W}^r_k = \sum_{\substack{\ell\geqslant 0\\ k+\ell+1\geqslant 0}} \mathrm{Ker}\,\boldsymbol{y}^{k+\ell+1} \cap \mathrm{Im}\,\boldsymbol{y}^\ell = \sum_{\substack{i-j=k\\ i\geqslant 0,\, j\geqslant 0}} \mathrm{Ker}\,\boldsymbol{y}^{i+1} \cap \mathrm{Im}\,\boldsymbol{y}^j. \tag{2.2}$$

Since r is the rank of nilpotency of $\boldsymbol{y}$ acting on $\mathbb{E}$, we have $\mathbb{W}^r_k = \{0\}$ for $k < -r$ and $\mathbb{W}^r_k = \mathbb{E}$ for $k \geqslant r$.

The elements y and h_r fit in an $\mathfrak{sl}_2$-triple (x, h_r, y) for some $x \in \mathfrak{m}^+$, and the subspaces $\mathbb{W}^r_k$ can be alternatively defined by means of this triple:

$$\mathbb{W}^r_k = \bigoplus_{\chi\in X(\mathbb{E}):\langle\chi,h_r\rangle\leqslant k} \mathbb{E}_\chi. \tag{2.3}$$

We combine the gradation $\mathbb{E} = \bigoplus_{i=0}^{r_M} \mathbb{E}_i$ and the weight filtration $(\mathbb{W}^r_k)_{k\in\mathbb{Z}}$ to define the subspaces of $\mathbb{E}$ that will be the center of our attention for the remaining part of Section 2.

Notation 2.25. — Let $\mathbb{V}^r_i = \mathbb{E}_i \cap \mathbb{W}^r_{r-2i}$ for $0 \leqslant i \leqslant r_M$, and let $\mathbb{V}^r = \bigoplus_i \mathbb{V}^r_i \subset \mathbb{E}$.

Observe that $\mathbb{V}^r_0 = \mathbb{E}_0$ and that $\mathbb{V}^r_i \neq \{0\}$ only if $i \leqslant r$. The subspaces $\mathbb{V}^r_i$ can also be described as sums of weight spaces $\mathbb{E}_\chi$.

Proposition 2.26. — *For $\chi \in X(\mathbb{E}_i)$, we have $\langle\chi, h_r\rangle \geqslant r - 2i$. Hence, for all i such that $0 \leqslant i \leqslant r$,*

$$\mathbb{V}^r_i = \bigoplus_{\substack{\chi\in X(\mathbb{E}_i):\\ \langle\chi,h_r\rangle=r-2i}} \mathbb{E}_\chi = \bigoplus_{\substack{\chi\in X(\mathbb{E}):\\ \langle\chi,z\rangle=z_{\max}-2i,\\ \langle\chi,h_r\rangle=r-2i}} \mathbb{E}_\chi.$$

Proof. — Proposition 2.15 and the fact that, for any root α, we have $\langle\alpha, h_r\rangle \leqslant 2$ (see Proposition 2.20) imply the first assertion. It is plain from the description of the subspaces $\mathbb{W}^r_k$ given in (2.3) that the second assertion follows from the first. ☐

The following lemma is reminiscent of Equation (2.1). We will prove a stronger statement in Proposition 2.40. More Hodge-theoretic considerations will be made in Section 2.6.

LEMMA 2.27. — *The linear map* $\boldsymbol{y}^{r-2i}$ *is an isomorphism between* $\mathbb{V}_i^r$ *and* $\mathbb{V}_{r-i}^r$. *In particular,* $\dim \mathbb{V}_{r-i}^r = \dim \mathbb{V}_i^r$.

Proof. — We know that

$$\mathbb{V}_i^r = \mathbb{E}_i \cap \mathbb{W}_{r-2i}^r, \quad \mathbb{V}_{r-i}^r = \mathbb{E}_{r-i} \cap \mathbb{W}_{2i-r}^r,$$

and that for i satisfying $0 \leqslant i \leqslant r/2$, $\boldsymbol{y}^{r-2i}$ is an isomorphism between $\mathbb{W}_{r-2i}^r/\mathbb{W}_{r-2i-1}^r$ and $\mathbb{W}_{2i-r}^r/\mathbb{W}_{2i-r-1}^r$. By Proposition 2.15, $\boldsymbol{y}^{r-2i}$ maps $\mathbb{E}_i$ to $\mathbb{E}_{r-i}$. Since we have $\mathbb{E}_k \cap \mathbb{W}_{r-2k-1}^r = \{0\}$ for all k by Proposition 2.26 and Equation (2.3), we get that $\boldsymbol{y}^{r-2i}$ is an isomorphism between $\mathbb{V}_i^r$ and $\mathbb{V}_{r-i}^r$. □

DEFINITION 2.28. — A subspace $\mathbb{S}$ of $\mathbb{E}$ is a *y-Higgs subspace* if $y \cdot \mathbb{S} \subset \mathbb{S}$ and $\mathfrak{m}_+ \cdot \mathbb{S} \subset \mathbb{S}$.

The following algebraic fact is at the heart of the construction of leafwise Higgs subsheaves in Section 4.

PROPOSITION 2.29. — *The subspace* $\mathbb{V}^r$ *is a y-Higgs subspace of* $\mathbb{E}$. *More precisely, the following inclusions hold:*

- $y \cdot \mathbb{V}_i^r \subset \mathbb{V}_{i+1}^r$.
- $\mathfrak{m}^+ \cdot \mathbb{V}_i^r \subset \mathbb{V}_{i-1}^r$.

Proof. — Using the description of $\mathbb{V}_i^r$ given in Proposition 2.26, the first inclusion holds because $y \in \bigoplus_i \mathfrak{g}_{-\alpha_i}$ and each α_i satisfies $\langle -\alpha_i, z\rangle = \langle -\alpha_i, h_r\rangle = -2$. The second one also holds since for α a root of $\mathfrak{m}^+$, we have $\langle \alpha, z\rangle = 2$ (and $\langle \alpha, h_r\rangle \leqslant 2$ by Proposition 2.20). □

2.4.3. *The module structure of* $\mathbb{V}^r$. — We now show that the subspace $\mathbb{V}^r$ of $\mathbb{E}$ has the structure of an irreducible L_r-module, for L_r a complex reductive group determined by y, whereas the $\mathbb{V}_i^r$ are irreducible H_r-modules, where $H_r = L_r \cap K$.

We begin by defining the relevant subalgebras of $\mathfrak{g}$.

NOTATION 2.30. — Let $\mathfrak{q}_r \subset \mathfrak{g}$ be the parabolic subalgebra defined by the coweight $z - h_r$:

$$\mathfrak{q}_r = \mathfrak{t} \oplus \bigoplus_{\alpha : \langle \alpha, h_r\rangle \leqslant \langle \alpha, z\rangle} \mathfrak{g}_\alpha.$$

A Levi factor of $\mathfrak{q}_r$ is

$$\mathfrak{l}_r = \mathfrak{t} \oplus \bigoplus_{\alpha : \langle \alpha, h_r\rangle = \langle \alpha, z\rangle} \mathfrak{g}_\alpha.$$

Let $\mathfrak{l}_r^{\pm} \subset \mathfrak{l}_r$ be the nilpotent subalgebras of $\mathfrak{l}_r$ defined by

$$\mathfrak{l}_r^{\pm} = \bigoplus_{\alpha : \langle \alpha, h_r\rangle = \langle \alpha, z\rangle = \pm 2} \mathfrak{g}_\alpha.$$

The intersection $\mathfrak{k}\cap\mathfrak{q}_r$ is a parabolic subalgebra of $\mathfrak{k}$ and a Levi factor of it is

$$\mathfrak{h}_r=\mathfrak{t}\oplus\bigoplus_{\alpha:\langle\alpha,h_r\rangle=\langle\alpha,z\rangle=0}\mathfrak{g}_\alpha.$$

We denote by Q_r, L_r and H_r, the connected subgroups of G with Lie algebra $\mathfrak{q}_r$, $\mathfrak{l}_r$ and $\mathfrak{h}_r$ respectively. Such groups exist because $\mathfrak{q}_r$ is a parabolic subalgebra, and $\mathfrak{h}_r$ and $\mathfrak{l}_r$ are Levi subalgebras. We observe that H_r is a Levi subgroup of $K\cap Q_r$: in fact, we have $K\cap Q_r=H_r\cdot R(K\cap Q_r)$, where $R(K\cap Q_r)$ denotes the unipotent radical of $K\cap Q_r$. Since Levi subgroups correspond to Levi subalgebras, L_r is a Levi factor of Q_r.

For the convenience of the reader, the conditions that define the different Lie subalgebras of $\mathfrak{g}$ we are considering are displayed in Table 2.

Lie algebra	$\mathfrak{k}$	$\mathfrak{m}^\pm$	$\mathfrak{q}_r$	$\mathfrak{l}_r$	$\mathfrak{l}_r^\pm$	$\mathfrak{h}_r$
Condition	$z=0$	$z=\pm2$	$h_r\leqslant z$	$h_r=z$	$h_r=z=\pm2$	$h_r=z=0$

Table 2. Lie subalgebras under consideration (We abbreviate the condition $\langle\alpha,z\rangle=0$, resp. $\langle\alpha,h_r\rangle=0$, on a root α as $z=0$, resp. $h_r=0$.)

We have the following lemmas concerning the $\mathbb{V}_i^r$'s.

Lemma 2.31. — *We have $\mathbb{V}_{i+1}^r=\mathfrak{l}_r^-\cdot\mathbb{V}_i^r$ and $\mathbb{V}_{i-1}^r=\mathfrak{l}_r^+\cdot\mathbb{V}_i^r$.*

Proof. — Let us denote by $X(\mathbb{E}_i)$ the set of weights of $\mathbb{E}_i$. We know by Proposition 2.15 that $\mathbb{E}_{i+1}=\mathfrak{m}^-\cdot\mathbb{E}_i$. Thus,

$$\mathbb{E}_{i+1}=\Big(\bigoplus_{\alpha:\langle\alpha,z\rangle=-2}\mathfrak{g}_\alpha\Big)\cdot\Big(\bigoplus_{\chi\in X(\mathbb{E}_i)}\mathbb{E}_\chi\Big)=\bigoplus_{\substack{\chi\in X(\mathbb{E}_i)\\ \alpha\in R(\mathfrak{m}^-)}}\mathbb{E}_{\chi+\alpha}.$$

Given $\chi\in X(\mathbb{E}_i)$ and $\alpha\in R(\mathfrak{m}^-)$, we can make two observations:

– If $\langle\chi,h_r\rangle>r-2i$ then we have $\langle\chi+\alpha,h_r\rangle>r-2(i+1)$ and thus $\mathbb{E}_{\chi+\alpha}\not\subset\mathbb{V}_{i+1}^r$.
– If $\langle\alpha,h_r\rangle>-2$ then $\langle\chi+\alpha,h_r\rangle>r-2(i+1)$ and thus $\mathbb{E}_{\chi+\alpha}\not\subset\mathbb{V}_{i+1}^r$.

The first equality of the lemma follows from these observations. The proof of the second equality is similar. □

Note that $\mathfrak{l}_r^-$ is an $\mathfrak{h}_r$-module. More precisely, we have

Lemma 2.32. — *The modules $\mathbb{V}_i^r$ are irreducible $\mathfrak{h}_r$-modules. The Lie algebra $\mathfrak{l}_r^-$ is an irreducible $\mathfrak{h}_r$-module.*

Proof. — Denote by $\mathfrak{k}^+$, resp. $\mathfrak{h}_r^+$, the sum of the root spaces for positive roots in $\mathfrak{k}$, resp. $\mathfrak{h}_r$. Let i be fixed and such that $\mathbb{V}_i^r\neq\{0\}$. By Proposition 2.15, $\mathbb{E}_i$ is an irreducible $\mathfrak{k}$-module. Let μ_i be the lowest weight of $\mathbb{E}_i$. We have $\mathbb{E}_{\mu_i}\subset\mathbb{V}_i^r$. Since $\mathbb{E}_i$ is irreducible, we have $\mathbb{E}_i=U(\mathfrak{k}^+)\cdot\mathbb{E}_{\mu_i}$ (here U denotes the universal enveloping algebra). Now, arguing as in the proof of the Lemma 2.31, we see that this implies that

$\mathbb{V}_i^r = U(\mathfrak{h}_r^+) \cdot \mathbb{E}_{\mu_i}$. Now, $\mathbb{E}_{\mu_i}$ has dimension 1, hence $\mathbb{V}_i^r$ is indecomposable. Since $\mathfrak{h}_r$ is reductive, this proves that $\mathbb{V}_i^r$ is irreducible. Now, by Lemma 2.31 again, we have $\mathbb{V}_1^r = \mathfrak{l}_r^- \cdot \mathbb{V}_0^r \simeq \mathfrak{l}_r^- \cdot \mathbb{E}_0$: thus $\mathfrak{l}_r^-$ is also irreducible. □

Concerning the submodule $\mathbb{V}^r$, we have

Proposition 2.33

(1) *The subspace $\mathbb{V}^r \subset \mathbb{E}$ is an irreducible Q_r-module.*
(2) *As a consequence, the nilpotent radical of Q_r acts trivially on $\mathbb{V}^r$.*
(3) *$\mathbb{V}_i^r$ is a $K \cap Q_r$-module.*
(4) *The subgroup of elements in G which preserve $\mathbb{V}^r$ is exactly Q_r.*

Proof. — In characteristic 0, given a representation of a connected group, a subspace of this representation is stable under this group if and only if it is stable for the induced action of the Lie algebra. Thus, to prove the proposition, it is enough to deal with the corresponding Lie algebras.

Let us prove that $\mathbb{V}^r$ is stable under $\mathfrak{q}_r$. It is clearly stable under $\mathfrak{t}$. Let α be such that $\mathfrak{g}_\alpha \subset \mathfrak{q}_r$, i.e., $\langle \alpha, h_r\rangle \leqslant \langle \alpha, z\rangle$. Let $v \in \mathbb{E}_\chi \subset \mathbb{V}_i^r$, so that we have $\langle \chi, h_r\rangle = r-2i$. For $x \in \mathfrak{g}_\alpha$, we have $x\cdot v \in \mathbb{E}_{\chi+\alpha} \subset \mathbb{E}_{i-\langle\alpha,z\rangle/2}$. Since $\langle \alpha, h_r\rangle \leqslant \langle \alpha, z\rangle$, we have $\langle \chi+\alpha, h_r\rangle \leqslant r-2i+\langle\alpha, z\rangle$, thus either $x\cdot v = 0$ or $\langle \chi+\alpha, h_r\rangle = r-2i+\langle \alpha, z\rangle$, by Proposition 2.26. In the second case, we get $x \cdot v \in \mathbb{V}^r_{i-\langle\alpha,z\rangle/2}$. Combining Lemmas 2.31 and 2.32, we see that $\mathbb{V}^r$ is an irreducible $\mathfrak{l}_r$-module, hence also an irreducible $\mathfrak{q}_r$-module.

It follows from Engel's theorem that there exists a non-zero vector in $\mathbb{V}^r$ which is annihilated by the nilpotent radical of $\mathfrak{q}_r$. Since this radical is an ideal and $\mathbb{V}^r$ is an irreducible $\mathfrak{q}_r$-module, this radical acts trivially on $\mathbb{V}^r$.

The fact that $\mathbb{V}_i^r$ is a $K\cap Q_r$-module follows because $\mathbb{V}_i^r = \mathbb{E}_i \cap \mathbb{V}^r$, $\mathbb{E}_i$ is K-stable, and $\mathbb{V}^r$ is Q_r-stable.

Finally, to prove that the stabilizer of $\mathbb{V}^r$ is exactly Q_r, let us denote by $\mathfrak{stab}(\mathbb{V}^r)\subset\mathfrak{g}$ the Lie subalgebra preserving the subspace $\mathbb{V}^r$ in $\mathbb{E}$. We know by (1) that $\mathfrak{stab}(\mathbb{V}^r)\supset\mathfrak{q}_r$. On the other hand, let α be a root such that $\mathfrak{g}_\alpha \cdot \mathbb{V}^r \subset \mathbb{V}^r$. Let us assume as a first case that $\langle\alpha, z\rangle = -2$, i.e., $\mathfrak{g}_\alpha \subset \mathfrak{m}^-$. Since, by Proposition 2.15(d), the action of $\mathfrak{g}$ on $\mathbb{E}$ induces an isomorphism $\mathbb{E}_1 \simeq \mathbb{E}_\varpi \otimes \mathfrak{m}^-$, we have $\mathfrak{g}_\alpha \cdot \mathbb{V}_0^r = \mathfrak{g}_\alpha \cdot \mathbb{E}_\varpi = \mathbb{E}_{\varpi+\alpha}$. Then $\mathbb{E}_{\varpi+\alpha} \subset \mathbb{V}^r \cap \mathbb{E}_1 = \mathbb{V}_1^r$, so $\langle \alpha, h_r\rangle = -2$, and $\mathfrak{g}_a \subset \mathfrak{q}_r$. Assume now that $\langle \alpha, z\rangle = 0$, i.e., $\mathfrak{g}_\alpha \subset \mathfrak{k}$, and by contradiction that $\langle\alpha, h_r\rangle > 0$. Proposition 2.20 then implies that $\langle \alpha, h_r\rangle = 1$. For any integer $1 \leqslant i \leqslant r$, we cannot have $\langle \alpha, \alpha_i^\vee\rangle = 2$ because this would imply $\alpha = \alpha_i$ and $\langle\alpha, h_r\rangle = 2$. Let i be such that $\langle \alpha, \alpha_i^\vee\rangle > 0$: then $\langle\alpha, \alpha_i^\vee\rangle = 1$. Therefore, $\alpha - \alpha_i$ is a root. Since $\mathfrak{g}_\alpha \subset \mathfrak{k}$, $\mathfrak{g}_\alpha \cdot \mathbb{E}_i \subset \mathbb{E}_i$ and hence $\mathfrak{g}_\alpha \cdot \mathbb{V}_i^r \subset \mathbb{V}_i^r$. Furthermore $\mathfrak{g}_\alpha(\mathbb{V}_0^r) = 0$ since $\mathbb{V}_0^r = \mathbb{E}_\varpi$. It follows that

$$\mathfrak{g}_{\alpha-\alpha_i} \cdot \mathbb{E}_\varpi = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha_i}] \cdot \mathbb{E}_\varpi = \mathfrak{g}_\alpha \cdot (\mathfrak{g}_{-\alpha_i} \cdot \mathbb{E}_\varpi).$$

Again by Proposition 2.15(d), $\mathfrak{g}_{\alpha-\alpha_i} \cdot \mathbb{E}_\varpi = \mathbb{E}_{\varpi+\alpha-\alpha_i}$ and $\mathfrak{g}_{-\alpha_i} \cdot \mathbb{E}_\varpi = \mathbb{E}_{\varpi-\alpha_i}$. But $\mathbb{E}_{\varpi-\alpha_i} \subset \mathbb{V}_1^r$ whereas $\mathbb{E}_{\varpi+\alpha-\alpha_i} \not\subset \mathbb{V}_1^r$ since we assumed $\langle\alpha, h_r\rangle > 0$. This contradicts $\mathfrak{g}_\alpha \cdot \mathbb{V}^r \subset \mathbb{V}^r$.

Let now $\mathrm{Stab}(\mathbb{V}^r) \subset G$ be the subgroup stabilizing $\mathbb{V}^r$. We have $Q_r \subset \mathrm{Stab}(\mathbb{V}^r)$, thus $\mathrm{Stab}(\mathbb{V}^r)$ is parabolic and therefore connected [Hum75, Cor. 23.1.B]. It follows that $Q = \mathrm{Stab}(\mathbb{V}^r)$. ☐

2.4.4. *The self-duality property. Tube type Hermitian symmetric spaces.* — By definition, a Hermitian symmetric space $M = G_{\mathbb{R}}/K_{\mathbb{R}}$ is of *tube type* if it is biholomorphically equivalent to a domain $F \oplus iC$, where $C \subset F$ is a proper open cone in the real vector space F. The simplest example is the upper half-plane, where $F = \mathbb{R}$ and $C = \mathbb{R}_+^*$. For irreducible symmetric spaces, this happens when $G_{\mathbb{R}}$ is isogenous to $\mathrm{SU}(p,p)$, $\mathrm{Spin}(2,m)$, $\mathrm{Sp}(2m,\mathbb{R})$, $\mathrm{SO}^\star(2m)$ for m even, and $\mathrm{E}_{7(-25)}$.

We introduce the following definition.

Definition 2.34. — A sequence $(\gamma_1, \dots, \gamma_r)$ of roots will be called *admissible* if all the roots γ_i are long and bigger than ζ, and if they are pairwise strongly orthogonal.

Note that the dominant orthogonal sequences $(\alpha_1, \dots, \alpha_r)$ are admissible. We can now state:

Proposition 2.35. — *The following assertions are equivalent:*

(1) *M has tube type;*
(2) *For all admissible sequences $\gamma_1, \dots, \gamma_{r_M}$, we have $\gamma_1 + \cdots + \gamma_{r_M} = 2\varpi$;*
(3) $\alpha_1 + \cdots + \alpha_{r_M} = 2\varpi$*;*
(4) *There exists an admissible sequence $\gamma_1, \dots, \gamma_{r_M}$ such that $\gamma_1 + \cdots + \gamma_{r_M} = 2\varpi$;*
(5) $\alpha_1^\vee + \cdots + \alpha_{r_M}^\vee = z$*;*
(6) *$\mathbb{E}$ is self-dual: $\mathbb{E} \simeq \mathbb{E}^\star$ as G-modules;*
(7) *For all i, $\mathbb{E}_i \simeq \mathbb{E}^\star_{r_M - i}$ as K-modules;*
(8) $\dim \mathbb{E}_{r_M} = 1$*;*
(9) $\alpha_{r_M} = \zeta$.

Proof. — Glancing at Table 1, one can readily check that $G_{\mathbb{R}}/K_{\mathbb{R}}$ is of tube type if and only if $z = \alpha_1^\vee + \cdots + \alpha_{r_M}^\vee$, and that this occurs also exactly when $\alpha_{r_M} = \zeta$. Hence (1), (5) and (9) are equivalent.

The equality $z = a_1^\vee + \cdots + \alpha_{r_M}^\vee$ is equivalent to $\langle \alpha_1^\vee + \cdots + \alpha_{r_M}^\vee, \beta\rangle = 2\delta_{\beta,\zeta}$ for all $\beta \in \Pi$ because ζ is the only noncompact simple root and it has the same length as each α_i (they are all long). Since all the roots α_i have the same length, for any root β, the equation $\langle \alpha_1^\vee + \cdots + \alpha_{r_M}^\vee, \beta\rangle = 0$ is equivalent to $\langle \alpha_1 + \cdots + \alpha_{r_M}, \beta^\vee\rangle = 0$. It follows that (5) and (3) are equivalent.

We now prove that items (2), (3), and (4) are equivalent. In fact, we have the trivial implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Moreover, if (4) holds, then let $\gamma_1, \dots, \gamma_{r_M}$ be an admissible sequence such that $\gamma_1 + \cdots + \gamma_{r_M} = 2\varpi$. By [RRS92, Th. 2.1], any admissible sequence can be obtained applying to this sequence an element w of the subgroup of the Weyl group that fixes ϖ. Condition (2) follows from the equality $w(\gamma_1) + \cdots + w(\gamma_{r_M}) = w(2\varpi) = 2\varpi$.

Since the weights of $\mathbb{E}^\star$ are the opposite of the weights of $\mathbb{E}$, the highest weight of $\mathbb{E}^\star$ is $-\mu_0$. Therefore, (6) holds if and only if $\mu_0 = -\varpi$, which is equivalent to (3) by Proposition 2.19 (4).

Assuming (6) and using the decomposition into K-modules $\mathbb{E} = \bigoplus_{i=0}^{r_M} \mathbb{E}_i$, we deduce that $\mathbb{E}_{r_M - i} \simeq \mathbb{E}_i^\star$ as K-modules. Conversely, given (7), we obviously have (8). Assuming now (8), we deduce that $\mathbb{E}_{r_M} = \mathbb{E}_{\mu_0}$. Since $\mathbb{E}_{r_M}$ is a K-module of dimension 1 in this case, it follows that μ_0 vanishes on all the roots of $\mathfrak{k}$, so μ_0 is an integral multiple of ϖ. It follows that $\mu_0 = -\varpi$ and we get (3) as above. □

As we just said, the G-module $\mathbb{E}$ is not always self-dual. However, the subspace $\mathbb{V}^r$ is always a self-dual module under a subgroup $\overline{L}_r$ of G that we now define.

Recall that the Lie algebra $\mathfrak{l}_r$ is by definition generated by the root spaces of the roots α such that $\langle h_r, \alpha\rangle = \langle z, \alpha\rangle$, and that $\mathfrak{l}_r^\pm$ is the subalgebra generated by the root spaces of the roots α such that $\langle h_r, \alpha\rangle = \langle z, \alpha\rangle = \pm 2$. We define $\bar{\mathfrak{l}}_r$ as the Lie subalgebra of $\mathfrak{g}$ generated by the spaces $\mathfrak{l}_r^+$ and $\mathfrak{l}_r^-$. Two particular cases are easy:

Fact 2.36. — If $r = 1$, then $\mathfrak{sl}_2 \simeq \bar{\mathfrak{l}}_r \subset \mathfrak{g}$ is the Lie subalgebra corresponding to the highest root Θ. If M has tube type and $r = r_M$, then $\bar{\mathfrak{l}}_r = \mathfrak{g}$.

Proof. — In case $r = 1$, then $h_r = \Theta^\vee$, so $\langle \alpha, h_r\rangle = 2$ implies $\alpha = \Theta$. Thus $\mathfrak{l}_r^+ = \mathfrak{g}_\Theta$ in this case. When M has tube type and $r = r_M$, we have $h_r = z$ (Proposition 2.35 (5)), so $\mathfrak{l}_r^+ = \mathfrak{m}^+$ in this case. □

We now describe the Lie algebra $\bar{\mathfrak{l}}_r$ in general. To this end, we introduce a definition.

Definition 2.37. — Let R be a (possibly reducible) root system, let Π be a basis of R, and ϖ be a dominant weight. Given a simple root β, we say that β is connected to ϖ in R if there is a simple root γ in the same connected component as β in the Dynkin diagram of R and such that $\langle \varpi, \gamma^\vee\rangle > 0$.

We denote by $\Pi_{h_r=0}$ the set of simple roots $\beta \in \Pi$ such that $\langle \beta, h_r\rangle = 0$.

Lemma 2.38. — *The Lie algebra $\bar{\mathfrak{l}}_r$ is simple. The lowest root $-\Theta$ together with the simple roots which are connected to Θ in $\Pi_{h_r=0}$ form a basis of the root system of $\bar{\mathfrak{l}}_r$.*

Proof. — Recall that ζ denotes the only noncompact simple root (see Notation 2.5). Looking at Table 1, we see that the root ζ cannot be connected to Θ in $\Pi_{h_r=0}$. Thus, if γ is connected to Θ in $\Pi_{h_r=0}$, it satisfies $\langle \gamma, z\rangle = 0$ and, of course, $\langle \gamma, h_r\rangle = 0$. Let us assume that $\langle \Theta, \gamma^\vee\rangle > 0$. Then $\Theta - \gamma$ is a root and it satisfies $\langle h_r, \Theta - \gamma\rangle = \langle z, \Theta - \gamma\rangle = 2$, so $\mathfrak{g}_{\Theta-\gamma} \subset \bar{\mathfrak{l}}_r$, and so $\mathfrak{g}_\gamma \subset \bar{\mathfrak{l}}_r$. By induction, the root spaces of all simple roots connected to Θ and their opposite are in $\bar{\mathfrak{l}}_r$, and, together with $\gamma_{-\Theta}$ and γ_Θ, they generate $\bar{\mathfrak{l}}_r$.

In general, the Dynkin diagram of a Lie algebra has vertices corresponding to the simple roots of the Lie algebra. In our case, given the above description of a basis of $\bar{\mathfrak{l}}_r$, the set of vertices of its Dynkin diagram is the union of some connected components of $\Pi_{h_r=0}$ and the set $\{-\Theta\}$. Moreover, all the connected components of $\Pi_{h_r=0}$ which occur are connected to $-\Theta$. Thus the Dynkin diagram of $\bar{\mathfrak{l}}_r$ is connected and $\bar{\mathfrak{l}}_r$ is simple. □

In Table 3, we indicate in each case which root system for $\bar{\mathfrak{l}}_r$ is obtained (the cases where $r=1$ or M has tube type and $r=r_M$ are not represented, see Fact 2.36).

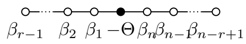
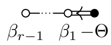
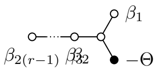
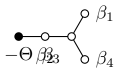
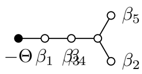
(G,ϖ)	$\bar{\mathfrak{l}}_r$	$R(\bar{\mathfrak{l}}_r)$
(A_{n-1},ϖ_ℓ)		A_{2r-1}
(C_n,ϖ_n)		C_r
(D_n,ϖ_n)		D_{2r-1}
(E_6,ϖ_1), $r=2$		D_5
(E_7,ϖ_7), $r=2$		D_6

Table 3. The Lie algebra $\bar{\mathfrak{l}}_r$ and its root system (The cases where $r=1$ or M has tube type and $r=r_M$ are not represented.)

We consider the root system $R_{\bar{\mathfrak{l}}_r}$ with its basis $(-\Theta,\beta_1,\ldots,\beta_k)$ given by Lemma 2.38. We have

Lemma 2.39. — *The simple root $-\Theta$ in $R(\bar{\mathfrak{l}}_r)$ is cominuscule.*

In other words, the lemma says that any root in $R(\bar{\mathfrak{l}}_r)$ has coefficient at most one on the simple root $-\Theta$. Observe that, as a consequence, $-\Theta$ defines a Hermitian noncompact real form $\overline{L}_{r,\mathbb{R}}$ of $\overline{L}_r$. The element $y=y_r$ can be considered as an element of $\bar{\mathfrak{l}}_r^-$.

Proof. — Since any root in $R(\bar{\mathfrak{l}}_r)$ can be obtained from Θ applying a suitable element of the Weyl group of $\bar{\mathfrak{l}}_r$, it is enough to show the implication

$$|n_{-\Theta}(\alpha)|\leqslant 1 \Longrightarrow |n_{-\Theta}(s_\gamma(\alpha))|\leqslant 1,$$

for any simple root γ in $R(\bar{\mathfrak{l}}_r)$. To prove this, we assume without loss of generality that $n_{-\Theta}(\alpha) \geqslant 0$. If γ is one of the simple roots β_j, then $n_{-\Theta}(\alpha) = n_{-\Theta}(s_\gamma(\alpha))$, so the implication clearly holds. If $\gamma = -\Theta$, we consider two cases. If $n_{-\Theta}(\alpha) = 1$, then α is a negative root. We have $0 \leqslant \langle \alpha, -\Theta^\vee \rangle \leqslant 2$, thus $-1 \leqslant n_{-\Theta}(s_\gamma(\alpha)) \leqslant 1$. If $n_{-\Theta}(\alpha) = 0$, then $\alpha \neq -\Theta$. Since Θ is long, it follows that $|\langle \alpha, -\Theta^\vee \rangle| \leqslant 1$. Thus $|n_{-\Theta}(s_\gamma(\alpha))| \leqslant 1$, as expected. $\square$

As announced, we have the following, which strengthens Lemma 2.27.

Proposition 2.40. — *The $\overline{L}_r$-module generated by $\mathbb{E}_\varpi$ is $\mathbb{V}^r$. It is self-dual.*

Proof. — First, we have $\bar{\mathfrak{l}}_r \subset \mathfrak{l}_r$. This follows from the definition of $\bar{\mathfrak{l}}_r$ and the inclusions $\mathfrak{l}_r^+ \subset \mathfrak{l}_r$, $\mathfrak{l}_r^- \subset \mathfrak{l}_r$.

From the inclusion $\bar{\mathfrak{l}}_r \subset \mathfrak{l}_r$ and Proposition 2.33(1), it follows that the $\overline{L}_r$-submodule generated by $\mathbb{E}_\varpi$ is included in $\mathbb{V}^r$. By Lemma 2.31, $\mathbb{V}^r$ is included in the $\mathfrak{l}_r^-$-submodule generated by $\mathbb{E}_\varpi$. Since $\mathfrak{l}_r^- \subset \bar{\mathfrak{l}}_r$ by definition, we deduce that the $\overline{L}_r$-module generated by $\mathbb{E}_\varpi$ is exactly $\mathbb{V}^r$.

Now we prove that $\mathbb{V}^r$, as an $\overline{L}_r$-module, is self-dual. The highest weight of $\mathbb{V}^r$ as an $\overline{L}_r$-module is the restriction of ϖ, which is the fundamental weight $\varpi_{-\Theta}$ corresponding to the simple root $-\Theta$ in the root system of $\bar{\mathfrak{l}}_r$. The sequence $\alpha_1, \dots, \alpha_r$ is a sequence of long strongly orthogonal roots. They are all bigger than $-\Theta$. In fact, if α_i was not bigger than $-\Theta$, then it would be a linear combination of the roots β_j, so it would satisfy $n_\zeta(\alpha_i) = 0$, contradicting the definition of the sequence $\alpha_1, \dots, \alpha_r$.

In view of (4) in Proposition 2.35, it is enough to prove that $\sum \alpha_i$ and $2\varpi_{-\Theta}$ have the same pairing with the simple roots of $R_{\bar{\mathfrak{l}}_r}$. For $-\Theta$, we have $\langle \sum \alpha_i^\vee, -\Theta^\vee \rangle = -2$ and $\langle 2\varpi_{-\Theta}, -\Theta^\vee \rangle = -2$. For β a simple root connected to Θ in $\Pi_{h_r=0}$, we have $\langle \sum \alpha_i, \beta^\vee \rangle = 0$ because $\langle \beta, \sum \alpha_i^\vee \rangle = 0$. We have $\langle \varpi_{-\Theta}, \beta^\vee \rangle = 0$. This proves that $\mathbb{V}^r$ is self-dual. $\square$

From the equivalences in Proposition 2.35 and Proposition 2.40, and recalling that $\overline{L}_{r,\mathbb{R}}$ is a Hermitian noncompact real form of $\overline{L}_r$, it follows that the symmetric space $M_r := \overline{L}_{r,\mathbb{R}}/(\overline{L}_{r,\mathbb{R}} \cap K_\mathbb{R})$ is a tube type Hermitian symmetric space of rank r.

Remark 2.41. — This construction is related, but not identical, to the theory of *boundary components* of bounded symmetric domains (see e.g. [Wol72, PS69, Sat80, AMRT10]). The topological boundary of the symmetric space M inside its compact dual $\check{M}$ is a union of r_M $G_\mathbb{R}$-orbits, and each $G_\mathbb{R}$-orbit is a union of so-called boundary components, cf. [AMRT10, Def. 3.2 p. 127]. The components are isomorphic to Hermitian symmetric subspaces M_Λ of M, where Λ is a subset of the set of noncompact strongly orthogonal roots, see [AMRT10, Th. 3.3 p. 127] for example. If we take our maximal dominant orthogonal sequence $(\alpha_1, \dots, \alpha_{r_M})$ as the set of strongly orthogonal roots, and $\Lambda = \{\alpha_1, \dots, \alpha_r\}$ as a subset, boundary components corresponding to this subset are symmetric spaces of rank r, as is the subspace M_r, but they are not of tube type unless M already is. The Lie algebra of the stabilizer of M_Λ in $G_\mathbb{R}$ is $\mathfrak{q}_\Lambda \cap \mathfrak{g}_\mathbb{R}$, where $\mathfrak{q}_\Lambda$ is the parabolic subalgebra of $\mathfrak{g}$ corresponding to the coweight

$h_{r_M} - h_r$, whereas we recall that the parabolic subalgebra $\mathfrak{q}_r$ we are considering here corresponds to the coweight $z - h_r$. This shows that $\mathfrak{q}_r = \mathfrak{q}_\Lambda$ if and only if M is of tube type.

Example 2.42. — We give an example of this construction. We assume that we are in the first row of Table 1, namely that G has type A_{p+q-1} for some positive integers $p \leqslant q$ and that $\varpi = \varpi_p$. The group $G_{\mathbb{R}}$ is therefore $\mathrm{SU}(p,q)$ acting on $\mathbb{C}^{p+q}$ preserving a Hermitian form h of signature (p,q) given by the diagonal matrix $\mathrm{diag}(-1,\dots,-1,1,\dots,1)$ in the canonical basis $(e_1,\dots,e_p,e_{p+1},\dots,e_{p+q})$.

Let $y \in \mathfrak{m}^-$ be an element of rank r. We have a natural decomposition $\mathbb{C}^{p+q} = A \oplus N \oplus B \oplus I$, with A, resp. N, B, I generated by $(e_1,\dots,e_r)$, resp. $(e_{r+1},\dots,e_p)$, $(e_{p+1},\dots,e_{p+q-r})$, $(e_{p+q-r+1},\dots,e_{p+q})$, so that y is given by the block matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}.$$

The element $h_r \in \mathfrak{t}$ acts on the Lie algebra $\mathfrak{sl}(p+q,\mathbb{C})$ of G, and we represent the weights of its action as the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ -2 & -1 & -1 & 0 \end{pmatrix},$$

where the blocks correspond to the decomposition $A \oplus N \oplus B \oplus I$ of $\mathbb{C}^{p+q}$. Similarly, the weights of the action of the central element z are given as follows

$$\begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ -2 & -2 & 0 & 0 \\ -2 & -2 & 0 & 0 \end{pmatrix}.$$

Thus the Lie algebra $\mathfrak{q}_r$, $\mathfrak{l}_r$ and $\mathfrak{h}_r$ are respectively

$$\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & 0 \\ * & 0 & * & * \end{pmatrix}, \quad \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ * & 0 & 0 & * \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

We see that Q_r is exactly the stabilizer of the flag $(N \subset N \oplus A \oplus I)$, that the subgroup L_r of G corresponding to $\mathfrak{l}_r$ is $\mathrm{S}(\mathrm{GL}(N) \times \mathrm{GL}(A \oplus I) \times \mathrm{GL}(B))$ and that H_r is the block diagonal group $\mathrm{S}(\mathrm{GL}(N) \times \mathrm{GL}(A) \times \mathrm{GL}(I) \times \mathrm{GL}(B))$. Finally the intersections of $G_{\mathbb{R}} = \mathrm{SU}(p,q)$ with these two latter groups are isomorphic to $\mathrm{S}(\mathrm{U}(p-r) \times \mathrm{U}(r,r) \times \mathrm{U}(q-r))$ and $\mathrm{S}(\mathrm{U}(p-r) \times \mathrm{U}(r) \times \mathrm{U}(r) \times \mathrm{U}(q-r))$ respectively, so that $\overline{L}_{r,\mathbb{R}} \simeq \mathrm{SU}(r,r)$.

On the other hand, we have $\mathbb{E} = \wedge^p(N \oplus A \oplus I \oplus B)$ and it is easy to check that $\mathbb{V}^r = \wedge^{p-r} N \otimes \wedge^r (A \oplus I)$, which illustrates that the stabilizer of $\mathbb{V}^r$ is Q_r.

2.5. The slope of the H_r-modules $\mathbb{V}_i^r$. — Our results so far allow to compute the *slope* of the H_r-modules $\mathbb{V}_i^r$. We first need two definitions and a lemma. Given an algebraic group H, we denote by $X(H)$ its character group considered as a $\mathbb{Z}$-module with additive law defined by: $(\chi+\psi)(h)=\chi(h)\psi(h)$.

Definition 2.43. — Let H be a reductive group and $\mathbb{F}$ be an H-module. The *slope* of $\mathbb{F}$ is the element $\mu_H(\mathbb{F})=\det(\mathbb{F})/\dim(\mathbb{F})$ in $X(H)\otimes_{\mathbb{Z}}\mathbb{Q}$. We say that $\mathbb{F}$ is a *polystable* H-module if $\mathbb{F}$ is a direct sum of irreducible H-modules of the same slope.

Lemma 2.44. — *Let H be a reductive group.*

– *Let $\mathbb{F}$ be a polystable H-module. For any H-submodule $\mathbb{S}$ of $\mathbb{F}$, $\mu_H(\mathbb{S})=\mu_H(\mathbb{F})$ and $\mu_H(\mathbb{F}/\mathbb{S})=\mu_H(\mathbb{F})$.*

– *Let $\mathbb{F}$, $\mathbb{F}'$ be polystable H-modules. Then $\mathbb{F}\otimes\mathbb{F}'$ is a polystableH-module, whose slope is $\mu_H(\mathbb{F})+\mu_H(\mathbb{F}')$.*

– *Let $\mathbb{F}$ be a polystable H-module. Then $\mathbb{F}^\star$ is a polystable H-module, whose slope is $-\mu_H(\mathbb{F})$.*

Proof. — The first assertion is clear because H is reductive and so any submodule is a direct sum of irreducible components. We prove the second assertion. Let Z be the center of H. It is known that restriction to Z yields an injection $X(H)\hookrightarrow X(Z)$. In fact, by [Bor69, Prop. 14.2], we have $H=Z\cdot(H,H)$, and any character of H is trivial on (H,H).

Let us first assume that $\mathbb{F}$ and $\mathbb{F}'$ are irreducible H-modules. By Schur's lemma, there are characters χ,ψ of Z such that $\forall\, g\in Z, \forall\, w\in\mathbb{F}, \forall\, w'\in\mathbb{F}', g\cdot w=\chi(g)w$ and $g\cdot w'=\psi(g)w'$. Therefore, in $X(Z)\otimes\mathbb{Q}$, we have $\chi=\mu_H(\mathbb{F})_{|Z}$ and $\psi=\mu_H(\mathbb{F}')_{|Z}$. Let now $\mathbb{U}\subset\mathbb{F}\otimes\mathbb{F}'$ be an irreducible component. We have

$$\forall\, u\in\mathbb{U},\ \forall\, g\in Z,\quad g\cdot u=\chi(g)\psi(g)u.$$

Therefore $\mu_H(\mathbb{U})_{|Z}=\chi+\psi$. Thus, $(\mu_H(\mathbb{F})+\mu_H(\mathbb{F}'))_{|Z}=\mu_H(\mathbb{U})_{|Z}$. Since restriction of characters to Z is injective, we have $\mu_H(\mathbb{U})=\mu_H(\mathbb{F})+\mu_H(\mathbb{F}')$.

Let now $\mathbb{F}$ and $\mathbb{F}'$ be arbitrary H-modules, and write the decomposition into irreducible submodules: $\mathbb{F}=\bigoplus\mathbb{F}_i$ and $\mathbb{F}'=\bigoplus\mathbb{F}'_j$. Let i,j be fixed and let $\mathbb{U}\subset\mathbb{F}_i\otimes\mathbb{F}'_j$ be an irreducible factor. We have proved that $\mu_H(\mathbb{U})=\mu_H(\mathbb{F}_i)+\mu_H(\mathbb{F}'_j)$. Since $\mathbb{F}$ and $\mathbb{F}'$ are polystable, we have $\mu_H(\mathbb{F}_i)=\mu_H(\mathbb{F})$ and $\mu_H(\mathbb{F}'_j)=\mu_H(\mathbb{F}')$. Thus, $\mathbb{U}$ has slope $\mu_H(\mathbb{F})+\mu_H(\mathbb{F}')$ and the lemma is proved.

The third item follows from the identity $\det\mathbb{F}^\star=-\det\mathbb{F}$ in $X(H)$. □

We apply this lemma to the H_r-modules $\mathbb{V}_i^r$:

Proposition 2.45. — *We have $\mu_{H_r}(\mathbb{V}_i^r)=\mu_{H_r}(\mathbb{V}_0^r)+i\mu_{H_r}(\mathbb{V}_0^{r\star}\otimes\mathbb{V}_1^r)$.*

Proof. — The H_r-modules $\mathbb{V}_i^r$ are irreducible by Lemma 2.32, thus polystable. The same holds for the H_r-module $\mathfrak{l}_r^-\simeq\mathbb{V}_0^{r\star}\otimes\mathbb{V}_1^r$, and we have $\mu_{H_r}(\mathfrak{l}_r^-)=\mu_{H_r}(\mathbb{V}_0^{r\star}\otimes\mathbb{V}_1^r)$. Let i be fixed. By Lemma 2.44, $\mathbb{V}_i^r\otimes\mathfrak{l}_r^-$ is polystable. Since, by Lemma 2.31, $\mathbb{V}_{i+1}^r$ is a quotient of $\mathbb{V}_i^r\otimes\mathfrak{l}_r^-$, it follows that $\mu_{H_r}(\mathbb{V}_{i+1}^r)=\mu_{H_r}(\mathbb{V}_i^r)+\mu_{H_r}(\mathfrak{l}_r^-)$. □

2.6. Interpretation in terms of homogeneous variations of Hodge structure

As we learned from one of the referees, the cominuscule representation $\mathbb{E}$ of G has been interpreted in the language of Hodge structures, first by Gross [Gro94] in the tube type case, then by Sheng and Zuo [SZ10] in general. Indeed, the representation $\mathbb{E}$ defines a canonical homogeneous complex variation of Hodge structure ($\mathbb{C}$-VHS) $E_{CY} := \bigoplus_{p+q=r_M} E^{p,q}$ on M or on any quotient $\Lambda\backslash M$, for Λ a discrete subgroup of $G_{\mathbb{R}}$. The Hodge bundle $E^{p,q}$ is just the bundle over $M = G_{\mathbb{R}}/K_{\mathbb{R}}$ associated with the $K_{\mathbb{R}}$-module $\mathbb{E}_{r_M-p}$. Geometrically, the corresponding filtration is given by the successive osculating tangent spaces to the cone over the subvariety $M \subset \mathbb{PE}$. This $\mathbb{C}$-VHS is of particular interest because it is of Calabi-Yau type, meaning that $\dim E^{r_M,0} = 1$ and $E^{r_M-1,1} \simeq E^{r_M,0} \otimes \Omega^1_M$. It is a real variation of Hodge structure ($\mathbb{R}$-VHS) of weight r_M precisely when M is of tube type. This is exactly the same thing as saying, as we did, that $\mathbb{E}$ is self-dual in this case.

From this point of view, what we do here can be interpreted in the following way. Let us work on the projectivized tangent bundle $\mathbb{P}T_M$ of M. The notion of rank of an element of $\mathfrak{m}^-$ naturally extends to elements in the tangent bundle of M, and the bundle $\mathbb{P}T_M$ is the union of r_M $G_{\mathbb{R}}$-orbits $\mathrm{Orb}_1, \dots, \mathrm{Orb}_{r_M}$, where $\mathrm{Orb}_r := \{[\eta] \in \mathbb{P}T_M \mid \mathrm{rk}(\eta) = r\}$. It is also the quotient $G_{\mathbb{R}}/(K_{\mathbb{R}} \cap Z_r)$, where Z_r is the centralizer of a nilpotent element y in $\mathfrak{m}^-$ of rank r. The sets $\cup_{s\leqslant r}\mathrm{Orb}_s$ are the characteristic varieties of Mok [Mok89]. Let us fix $1 \leqslant r \leqslant r_M$ and call π_M the projection $\mathbb{P}T_M \to M$. The definition of the weight filtration $\mathbb{W}^r_\bullet$ of $\mathbb{E}$ associated with a nilpotent element $y \in \mathfrak{m}^-$ or rank r can also be extended to give an increasing holomorphic weight filtration on the restriction of the $\mathbb{C}$-VHS $\pi_M^\star E_{CY}$ to Orb_r, compatible with the decreasing Hodge filtration $(\bigoplus_{p\geqslant k} \pi_M^\star E^{p,q})_k$, and this defines a homogeneous family of "complex" mixed Hodge structures on Orb_r. When M has tube type, $\pi_M^\star E_{CY}$ is an $\mathbb{R}$-VHS and we have a real mixed Hodge structure on each fiber of $\pi_M^\star E_{CY}$ above Orb_r.

Example 2.46. — We keep the notations of Example 2.42, assuming now that we are in the tube type case: the group $G_{\mathbb{R}}$ is $\mathrm{SU}(p,p)$ with maximal compact subgroup $K_{\mathbb{R}} = \mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(p))$. Let C, resp. D, be the subspace of $\mathbb{C}^{2p}$ generated by $(e_1, \dots, e_p)$, resp. $(e_{p+1}, \dots, e_{2p})$. In this case, the cominuscule representation is

$$\mathbb{E} = \wedge^p(C \oplus D) = \bigoplus_{m+n=p} \wedge^m C \otimes \wedge^n D.$$

Whenever V is a complex vector space, we denote by $\overline{V}$ the complex vector space V with the $\mathbb{C}$-action defined by $\lambda\cdot v = \overline{\lambda}v$. Note that, with this convention, a real structure on V is the same as an involutory $\mathbb{C}$-linear isomorphism $\overline{V} \to V$. As representations of $K_{\mathbb{R}}$, $\overline{C} \simeq C^\star$, $\overline{D} \simeq D^\star$, and $\wedge^{2p}(C \oplus D)$ is the trivial representation. We get a $K_{\mathbb{R}}$-equivariant isomorphism

$$\overline{\wedge^m C \otimes \wedge^n D} \simeq \wedge^m \overline{C} \otimes \wedge^n \overline{D} \simeq \wedge^m C^\star \otimes \wedge^n D^\star \simeq (\wedge^m C \otimes \wedge^n D)^\star \simeq \wedge^{p-m} C \otimes \wedge^{p-n} D,$$

where the last isomorphism is given by the perfect pairing

$$(\wedge^m C \otimes \wedge^n D) \otimes (\wedge^{p-m} C \otimes \wedge^{p-n} D) \longrightarrow \wedge^p C \otimes \wedge^p D \simeq \wedge^{2p}(C \oplus D) \simeq \mathbb{C}.$$

Therefore we get a real structure on $\mathbb{E}$, invariant by $K_{\mathbb{R}}$, using the isomorphism

$$\overline{\mathbb{E}} = \bigoplus_{m+n=p} \overline{\wedge^m C \otimes \wedge^n D} \simeq \bigoplus_{m+n=p} \wedge^n C \otimes \wedge^m D = \mathbb{E}.$$

Since we have tautologically, for this real structure, $\overline{\wedge^m C \otimes \wedge^n D} = \wedge^n C \otimes \wedge^m D$, this defines an $\mathbb{R}$-VHS of weight $p = r_M$ on the symmetric space $M = G_{\mathbb{R}}/K_{\mathbb{R}}$. This is the homogeneous VHS defined in [Gro94].

If now y is an element in $\mathfrak{m}^-$ of rank r, we moreover have the decompositions $C = A \oplus N$ and $D = B \oplus I$, as in Example 2.42, where the restriction of the action of y to A gives an isomorphism $A \simeq I$. The (p-shifted) increasing weight filtration $\mathbb{W}^r_\bullet$ of $\mathbb{E}$ associated to the action of y is given by

$$\mathbb{W}^r_s = \bigoplus_{\substack{i-\ell \leqslant s-p \\ i+j+k+\ell=p}} \wedge^i A \otimes \wedge^j N \otimes \wedge^k B \otimes \wedge^\ell I,$$

whereas the decreasing Hodge filtration $\mathbb{F}^\bullet$ is given by

$$\mathbb{F}^t = \bigoplus_{\substack{i+j \geqslant t \\ i+j+k+\ell=p}} \wedge^i A \otimes \wedge^j N \otimes \wedge^k B \otimes \wedge^\ell I.$$

Therefore the graded parts are:

$$\operatorname{Gr}^{\mathbb{F}}_t \operatorname{Gr}^{\mathbb{W}^r}_s \mathbb{E} = \bigoplus_{\substack{i-\ell = s-p \\ i+j=t \\ i+j+k+\ell=p}} \wedge^i A \otimes \wedge^j N \otimes \wedge^k B \otimes \wedge^\ell I.$$

The intersection $Z_r \cap K_{\mathbb{R}}$ of $K_{\mathbb{R}}$ with the centralizer Z_r of y is the group

$$\{(a,b,c) \in \mathrm{U}(r) \times \mathrm{U}(p-r) \times \mathrm{U}(p-r) \mid \det(a^2 bc) = 1\}.$$

As $(Z_r \cap K_{\mathbb{R}})$-modules, we have:

$$\begin{aligned}
\overline{\wedge^i A \otimes \wedge^j N \otimes \wedge^k B \otimes \wedge^\ell I} &\simeq \wedge^i A^\star \otimes \wedge^j N^\star \otimes \wedge^k B^\star \otimes \wedge^\ell I^\star \\
&\simeq (\wedge^i A^\star \otimes \wedge^j N^\star \otimes \wedge^k B^\star \otimes \wedge^\ell I^\star) \otimes \wedge^{2p}(A \oplus N \oplus B \oplus I) \\
&\simeq \wedge^{r-i} A \otimes \wedge^{p-r-j} N \otimes \wedge^{p-r-k} B \otimes \wedge^{r-\ell} I \\
&\simeq \wedge^{r-\ell} A \otimes \wedge^{p-r-j} N \otimes \wedge^{p-r-k} B \otimes \wedge^{r-i} I \\
&\simeq \wedge^{i'} A \otimes \wedge^{j'} N \otimes \wedge^{k'} B \otimes \wedge^{\ell'} I
\end{aligned}$$

with $i' = r - \ell$, $j' = p - r - j$, $k' = p - r - k$ and $\ell' = r - i$ (on the fourth line we used the isomorphism $I \simeq A$). Therefore, if $i - \ell = s - p$ and $i + j = t$, then $i' - \ell' = i - \ell = s - p$ and $i' + j' = p - j - \ell = s - (i + j) = s - t$. Hence the real structure on $\mathbb{E}$ defined as above with these isomorphisms is such that

$$\overline{\operatorname{Gr}^{\mathbb{F}}_t \operatorname{Gr}^{\mathbb{W}^r}_s \mathbb{E}} = \operatorname{Gr}^{\mathbb{F}}_{s-t} \operatorname{Gr}^{\mathbb{W}^r}_s \mathbb{E}$$

and the Hodge filtration induces a real Hodge structure of weight s on $\operatorname{Gr}^{\mathbb{W}^r}_s \mathbb{E}$, so that the two filtrations $\mathbb{W}^r_\bullet$ and $\mathbb{F}^\bullet$ indeed define a real mixed Hodge structure on $\mathbb{E}$.

However, again from the Hodge structures point of view, our purpose here is a bit different. We would like to construct a canonical "weak sub-$\mathbb{R}$-VHS" of weight r of the restriction of the $\mathbb{C}$-VHS $\pi_M^\star E_{CY}$ to Orb_r. This is the role played by the submodule $\mathbb{V}^r$. The meaning of "weak $\mathbb{R}$-VHS" is that if V^r denotes the equivariant holomorphic bundle $\mathbb{V}^r\otimes\mathscr{O}_{\mathrm{Orb}_r}$ on Orb_r with its Gauss-Manin connection ∇, and if $F^\bullet$ denotes the Hodge filtration on V^r defined by the filtration $\mathbb{F}^\bullet$ of $\mathbb{E}$, then $(V^r,F^\bullet)$ does have the symmetries of an $\mathbb{R}$-VHS by Proposition 2.40, but does not satisfy Griffiths transversality $\nabla(F^t)\subset F^{t-1}\otimes\Omega^1_{\mathrm{Orb}_r}$. Rather, $(V^r,F^\bullet)$ satisfies a weak transversality condition by Proposition 2.29. Indeed we have $\nabla(F^t\otimes\mathscr{O}_{\mathbb{P}T_M}(-1)_{|\,\mathrm{Orb}_r})\subset F^{t-1}$, where $\mathscr{O}_{\mathbb{P}T_M}(-1)$ is the tautological line bundle on $\mathbb{P}T_M$. This will allow us to construct *leafwise* Higgs subsheaves (but not plain Higgs subsheaves) in Section 4.3.

3. Higgs bundles on foliated Kähler manifolds

3.1. Harmonic Higgs bundles. — We keep the notation of the previous sections. In particular, $G_\mathbb{R}$ is a simple real algebraic Hermitian group, $K_\mathbb{R}$ its maximal compact subgroup, $M=G_\mathbb{R}/K_\mathbb{R}$ the associated irreducible Hermitian symmetric space of the noncompact type, G and K are the complexifications of $G_\mathbb{R}$ and $K_\mathbb{R}$, and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ and $\mathfrak{g}_\mathbb{R}=\mathfrak{k}_\mathbb{R}\oplus\mathfrak{m}_\mathbb{R}$ are the associated Cartan decompositions of the Lie algebras of G and $G_\mathbb{R}$.

Let now (Y,ω_Y) be a compact Kähler manifold, $\Gamma=\pi_1(Y)$ its fundamental group, and $\rho:\Gamma\to G_\mathbb{R}$ be a *reductive* representation (group homomorphism) of Γ into $G_\mathbb{R}$. The assumption that ρ is reductive means that the real Zariski closure of $\rho(\Gamma)$ in $G_\mathbb{R}$ is a reductive group.

In this case, by [Cor88], there exists a ρ-equivariant *harmonic* map f from the universal cover $\widetilde{Y}$ of Y to the symmetric space $M=G_\mathbb{R}/K_\mathbb{R}$ associated with $G_\mathbb{R}$. The fact that Y is Kähler and the nonpositivity of the complexified sectional curvature of M imply by a Bochner formula due to [Sam78, Siu80] that the map f is *pluriharmonic* (i.e., its restriction to 1-dimensional complex submanifolds of Y is still harmonic), and that the image of its (complexified) differential at every point $y\in Y$ is an abelian subalgebra of $T^\mathbb{C}_{f(y)}M$ identified with $\mathfrak{m}$.

By the work of Hitchin and Simpson, this gives a *harmonic $G_\mathbb{R}$-Higgs principal bundle* (P_K,θ) on Y. We will now briefly describe the construction and the properties of such a Higgs bundle. Details and proofs can be found in the original papers [Hit87, Hit92, Sim88, Sim92].

There is a flat principal bundle $P_{G_\mathbb{R}}\to Y$ of group $G_\mathbb{R}$ associated with the representation ρ. The ρ-equivariant map $f:\widetilde{Y}\to G_\mathbb{R}/K_\mathbb{R}$ defines a reduction of its structure group to $K_\mathbb{R}$, i.e., a principal $K_\mathbb{R}$ bundle $P_{K_\mathbb{R}}\subset P_{G_\mathbb{R}}$. The differential of f can be seen as a 1-form with values in $P_{K_\mathbb{R}}(\mathfrak{m}_\mathbb{R}):=(P_{K_\mathbb{R}}\times\mathfrak{m}_\mathbb{R})/K_\mathbb{R}$, the vector bundle associated with the adjoint action of $K_\mathbb{R}$ on $\mathfrak{m}_\mathbb{R}$.

If we enlarge the structure group of $P_{K_\mathbb{R}}$ to K, the pluriharmonicity of f implies that the K-principal bundle $P_K\to Y$ is a holomorphic bundle and that the (1,0)-part $d^{1,0}f:T^{1,0}Y\to T^\mathbb{C}M$ of the complexified differential of f defines a holomorphic section θ of $P_K(\mathfrak{m})\otimes\Omega^1_Y$, where $P_K(\mathfrak{m})$ is the holomorphic vector bundle associated with the principal bundle P_K and the adjoint representation of K on $\mathfrak{m}$. The section θ is called the *Higgs field* and satisfies the integrability condition $[\theta,\theta]=0$ as a section of $P_K(\mathfrak{m})\otimes\Omega^2_Y$. The pair (P_K,θ) is called a *$G_{\mathbb{R}}$-Higgs principal bundle* on Y, see e.g. [BGPG06, §2.2].

If now $\mathbb{E}$ is a (complex) representation of G we can construct the associated holomorphic vector bundle $E:=P_K(\mathbb{E})$ over Y. Since $\mathbb{E}$ is a representation of $\mathfrak{g}$ and not only of $\mathfrak{k}$, we have a morphism $P_K(\mathfrak{m})\to P_K(\mathrm{End}(\mathbb{E}))=\mathrm{End}(E)$. Thus, the Higgs field θ can be seen as a holomorphic 1-form with values in the endomorphisms of E, i.e., a section of $\mathrm{End}(E)\otimes\Omega^1_Y$. The pair (E,θ) is called a *$G_{\mathbb{R}}$-Higgs vector bundle* on Y. The harmonic map f, seen as a reduction of the structure group of P_K to the compact subgroup $K_{\mathbb{R}}$, together with a $K_{\mathbb{R}}$-invariant metric on $\mathbb{E}$, gives a Hermitian metric on (E,θ) called the *harmonic metric*.

The existence of this harmonic metric and the fact that P_K comes from a flat principal $G_{\mathbb{R}}$ bundle imply that for any representation $\mathbb{E}$ of G, the associated Higgs vector bundle $(E,\theta)\to Y$ is Higgs polystable of degree 0, see [Sim88]. To explain what Higgs polystability means, we first define Higgs subsheaves of the Higgs bundle (E,θ). A coherent subsheaf $\mathscr{F}$ of $\mathscr{O}_Y(E)$ is a *Higgs subsheaf* if it is invariant by the Higgs field, i.e., if it satisfies $\theta(\mathscr{F})\subset\mathscr{F}\otimes\Omega^1_Y$. The Higgs vector bundle (E,θ) is said to be *Higgs stable* if for any Higgs subsheaf $\mathscr{F}$ of (E,θ) such that $0<\mathrm{rk}\,\mathscr{F}<\mathrm{rk}\,E$, we have $\mu_{\omega_Y}(\mathscr{F})<\mu_{\omega_Y}(E)$, where $\mu_{\omega_Y}(\mathscr{F})$ is the slope of $\mathscr{F}$, i.e., its degree (computed w.r.t. the Kähler form ω_Y of Y) divided by its rank. The Higgs bundle (E,θ) is *Higgs polystable* if it is a direct sum of Higgs stable Higgs vector bundles of the same slope. Note that here E is flat as a C^∞ bundle, so that its degree is zero.

Remark 3.1. — Since moreover we assumed that $G_{\mathbb{R}}$ is a Hermitian group, then as a K-representation we have $\mathfrak{m}=\mathfrak{m}^+\oplus\mathfrak{m}^-$ and the Higgs field θ on the principal bundle P_K (or on any associated vector bundle E) has two components $\beta\in P_K(\mathfrak{m}^-)\otimes\Omega^1_Y$ and $\gamma\in P_K(\mathfrak{m}^+)\otimes\Omega^1_Y$. The vanishing of γ, resp. β, means that the harmonic map f is holomorphic, resp. antiholomorphic. The component β, resp. γ, will be called the holomorphic, resp. antiholomorphic, component of the Higgs field θ.

3.2. Harmonic Higgs bundles on foliated Kähler manifolds. — Assume now that the base Kähler manifold Y of the harmonic $G_{\mathbb{R}}$-Higgs vector bundle $(E,\theta)\to Y$ of degree 0 admits a smooth holomorphic foliation by complex curves and that this foliation $\mathscr{T}$ admits a transverse volume form Ω. This means that Ω is a nowhere vanishing semipositive closed $(d-1,d-1)$-form (d is the dimension of Y) such that the interior product of Ω with any vector tangent to the foliation $\mathscr{T}$ is zero.

Our goal in this section is to briefly explain the interplay between the Higgs bundle and the foliation (equipped with its transverse volume form). Details can be found in [KM17, §2.2].

We first weaken the notion of Higgs subsheaves of (E,θ) to leafwise Higgs subsheaves by asking only an invariance by the Higgs field along the leaves. More precisely we now consider the Higgs field as a section of $\mathrm{Hom}(E \otimes L, E)$, where L is the holomorphic line subbundle of T_Y tangent of the foliation $\mathscr{T}$. A *leafwise Higgs subsheaf* $\mathscr{F}$ of (E,θ) is a subsheaf of $\mathscr{O}_Y(E)$ such that $\theta(\mathscr{F} \otimes L) \subset \mathscr{F}$.

We define the *foliated degree* $\deg_{\mathscr{T}} \mathscr{F}$ and the *foliated slope* $\mu_{\mathscr{T}}(\mathscr{F})$ of a coherent sheaf $\mathscr{F}$ on Y by wedging the first Chern class of $\mathscr{F}$ with the transverse volume form Ω:

$$\deg_{\mathscr{T}} \mathscr{F} := \int_Y c_1(\mathscr{F}) \wedge \Omega, \quad \text{and} \quad \mu_{\mathscr{T}}(\mathscr{F}) = \frac{\deg_{\mathscr{T}} \mathscr{F}}{\mathrm{rk}\, \mathscr{F}}.$$

The link between the notions of slope (and stability) for modules and for sheaves used in this paper will be given by Fact 4.5 below.

In order to state the needed leafwise polystability property of the Higgs bundle (E,θ) w.r.t. the foliated degree, we first recall a few definitions.

An open subset of Y is called *big* if it is the complement of an analytic set of codimension at least 2 in Y. Two line bundles on Y which have isomorphic restrictions to a big open subset of Y are in fact isomorphic over Y.

Let $\mathscr{F}$ be a subsheaf of $\mathscr{O}_Y(E)$. The complement of the biggest subset of Y where $\mathscr{F}$ is the sheaf of sections of a subbundle F of E is called the *singular locus* $\mathscr{S}(\mathscr{F})$ of $\mathscr{F}$. Equivalently, $\mathscr{S}(\mathscr{F})$ is the analytic subset of Y where the quotient sheaf $\mathscr{O}_Y(E)/\mathscr{F}$ is not locally free. Recall that a subsheaf $\mathscr{F}$ of $\mathscr{O}_Y(E)$ is *saturated* if $\mathscr{O}_Y(E)/\mathscr{F}$ is torsion free. The *saturation* of $\mathscr{F}$ is the kernel of the morphism $\mathscr{O}_Y(E) \to (\mathscr{O}_Y(E)/\mathscr{F})_{\mathrm{tf}}$, where $(\mathscr{O}_Y(E)/\mathscr{F})_{\mathrm{tf}}$ denotes the quotient of $\mathscr{O}_Y(E)/\mathscr{F}$ by its torsion. If $\mathscr{F}$ is saturated, $\mathscr{S}(\mathscr{F})$ has codimension at least 2 in Y, so that the *regular locus* $Y \backslash \mathscr{S}(\mathscr{F})$ of $\mathscr{F}$ is a big open subset. Moreover a saturated subsheaf $\mathscr{F}$ is normal, so that in particular for all big open subsets $\mathscr{U}$ of Y, the restriction $\mathscr{F}(Y) \to \mathscr{F}(\mathscr{U})$ is injective.

A subset of the foliated manifold $(Y, \mathscr{T})$ is said to be *saturated under the foliation* $\mathscr{T}$ if it is a union of leaves of $\mathscr{T}$. Obviously if $Z \subset Y$ is saturated under $\mathscr{T}$, then so is its complement $Y \backslash Z$. (Unfortunately, the word "saturated" is used here with two different meanings but this shouldn't cause any confusion.)

Here is the result ([KM17, Prop. 2.2]).

Proposition 3.2. — *The harmonic Higgs bundle (E,θ) on the foliated Kähler manifold $(Y, \mathscr{T}, \Omega)$ is* weakly polystable along the leaves *in the following sense:*

(1) *it is* semistable along the leaves *of $\mathscr{T}$: if $\mathscr{F}$ is a leafwise Higgs subsheaf of (E,θ), then $\deg_{\mathscr{T}} \mathscr{F} \leqslant 0$.*

(2) *if $\mathscr{F}$ is a saturated leafwise Higgs subsheaf of (E,θ) such that $\deg_{\mathscr{T}} \mathscr{F} = 0$, then the singular locus $\mathscr{S}(\mathscr{F})$ of $\mathscr{F}$ is saturated under the foliation $\mathscr{T}$. Moreover, on $Y \backslash \mathscr{S}(\mathscr{F})$, and if F denotes the holomorphic subbundle of E such that $\mathscr{F} = \mathscr{O}_{Y \backslash \mathscr{S}(\mathscr{F})}(F)$ and $F^\perp$ its C^∞ orthogonal complement w.r.t. the harmonic metric, then $\theta(F^\perp \otimes L) \subset F^\perp$. Finally, for each leaf $\mathcal{L}$ of $\mathscr{T}$ such that $\mathcal{L} \cap \mathscr{S}(\mathscr{F}) = \varnothing$, $F^\perp_{|\mathcal{L}}$ is holomorphic on $\mathcal{L}$ and $(E,\theta)_{|\mathcal{L}} = (F, \theta_{|F})_{|\mathcal{L}} \oplus (F^\perp, \theta_{|F^\perp})_{|\mathcal{L}}$ is a holomorphic orthogonal decomposition of Higgs bundles on $\mathcal{L}$.*

3.3. The tautological foliation on the projectivized tangent bundle of a complex hyperbolic manifold. — An n-dimensional complex hyperbolic manifold X is a quotient of the complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}} = \mathrm{SU}(1,n)/\,\mathrm{U}(n)$ by a discrete torsion free subgroup Γ of $\mathrm{SU}(1,n)$. It is a Hermitian locally symmetric space of rank 1. The complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ can be realized as the subset of negative lines in $\mathbb{C}^{n+1}$ for a Hermitian form h of signature $(n,1)$ (recall our convention in Example 2.1), which is an open subset in the projective space $\mathbb{CP}^n$.

Intersections of lines of $\mathbb{CP}^n$ with $\mathbb{H}^n_{\mathbb{C}}$ are totally geodesic complex subspaces of $\mathbb{H}^n_{\mathbb{C}}$ isometric to the Poincaré disc. They are called *complex geodesics*. The space $\mathscr{G}$ of complex geodesics is the homogeneous space $\mathrm{SU}(1,n)/\,\mathrm{S}(\mathrm{U}(1,1)\times\mathrm{U}(n-1))$. It is also a complex manifold, since it can be realized as the open subset of the Grassmannian manifold $\mathrm{Gr}_2(\mathbb{C}^{n+1})$ of 2-planes in $\mathbb{C}^{n+1}$ consisting of planes on which the signature of h is $(1,1)$.

The *projectivized tangent bundle* $\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}$ of $\mathbb{H}^n_{\mathbb{C}}$ is the space of lines in the holomorphic tangent bundle $T_{\mathbb{H}^n_{\mathbb{C}}}$ of $\mathbb{H}^n_{\mathbb{C}}$. As a homogeneous space it can be identified with $\mathrm{SU}(1,n)/\,\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(n-1))$. Again, its also a complex manifold (in fact a Kähler one) since it identifies with an open subset of the manifold $\mathrm{F}_{1,2}(\mathbb{C}^{n+1})$ of incomplete flags $(\ell\subset\Pi\subset\mathbb{C}^{n+1})$ with $\dim\ell=1$, $\dim\Pi=2$. A flag $(\ell\subset\Pi\subset\mathbb{C}^{n+1})$ belong to $\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}$ if the line ℓ is negative and the plane Π has signature $(1,1)$ for the Hermitian form h.

The natural $\mathrm{SU}(1,n)$-equivariant fibration $\pi_{\mathscr{G}}:\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}\to\mathscr{G}$ which associates to a tangent line to $\mathbb{H}^n_{\mathbb{C}}$ the complex geodesic it defines is a disc bundle over $\mathscr{G}$.

By $\mathrm{SU}(1,n)$-equivariance, this fibration endows the projectivized tangent bundle $\mathbb{P}T_X=\Gamma\backslash\mathbb{P}T_{\mathbb{H}^n_{\mathbb{C}}}$ of $X=\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ with a smooth holomorphic foliation $\mathscr{T}$ by complex curves, whose leaves are given by the tangent spaces of the (immersed) complex geodesics in X. This foliation is called the *tautological foliation* of $\mathbb{P}T_X$ because the tangent line bundle L to the leaves is naturally isomorphic to the tautological line bundle $\mathscr{O}_{\mathbb{P}T_X}(-1)$ on $\mathbb{P}T_X$.

The space $\mathscr{G}$ of complex geodesics of $\mathbb{H}^n_{\mathbb{C}}$ is a pseudo-Kähler manifold: it admits an $\mathrm{SU}(1,n)$-invariant Kähler form $\omega_{\mathscr{G}}$, which is closed, of type $(1,1)$, non degenerate, but not positive definite. The form $\omega_{\mathscr{G}}$ is moreover unique up to scaling. This form defines a transverse volume form $\Omega_{\mathscr{G}}$ for the tautological foliation $\mathscr{T}$ on $\mathbb{P}T_X$, and the associated notion of foliated degree $\deg_{\mathscr{T}}$ for sheaves on $\mathbb{P}T_X$ has the following fundamental property (after a suitable normalization of the involved forms) [KM17, Prop. 3.1].

Proposition 3.3. — *Assume that $X=\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$ is compact and let $\pi:\mathbb{P}T_X\to X$ be the projectivized tangent bundle of X. If $\mathscr{F}$ is a coherent $\mathscr{O}_X$-sheaf, then $\deg_{\mathscr{T}}(\pi^*\mathscr{F})=\deg_\omega\mathscr{F}$, where $\deg_\omega\mathscr{F}$ is the usual degree of $\mathscr{F}$ computed w.r.t. the Kähler form ω on X.*

4. The Milnor-Wood inequality

Let Γ be a uniform lattice in $\mathrm{SU}(1,n)$, X be the compact quotient $\Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$, and let ρ be a representation of Γ in a simple real algebraic Hermitian Lie group $G_{\mathbb{R}}$, whose associated symmetric space is denoted by M. In this section we use the material developed or recalled in Sections 2 and 3 to prove the Milnor-Wood inequality

$$|\tau(\rho)| \leqslant r_M \operatorname{vol}(X).$$

Before going into the proof, we remark that if the lattice Γ has torsion, the measure defined by the Kähler form ω on $\mathbb{H}^n_{\mathbb{C}}$ descends to X so that $\operatorname{vol}(X)$ is well-defined. There are several ways to define the Toledo invariant of the representation ρ of Γ in this case. Since this is how we will handle lattices with torsion, we observe that by Selberg's Lemma (see [Rat06, p. 327] or [Sel60]) there exists a finite index torsion free subgroup Γ' of Γ. We then define the Toledo invariant of $\rho : \Gamma \to G_{\mathbb{R}}$ by $\tau(\rho) := \tau(\rho')/|\Gamma/\Gamma'|$, where $\rho' = \rho_{|\Gamma'}$ is the restriction of ρ to Γ' and $\tau(\rho')$ is defined as in the introduction. One checks easily that this does not depend on the choice of Γ'. Moreover, if $X' = \Gamma' \backslash \mathbb{H}^n_{\mathbb{C}}$, then $\operatorname{vol}(X) = \operatorname{vol}(X')/|\Gamma/\Gamma'|$ so that proving the Milnor-Wood inequality for $\rho : \Gamma \to G_{\mathbb{R}}$ is equivalent to proving the Milnor-Wood inequality for $\rho' : \Gamma' \to G_{\mathbb{R}}$.

In addition, the representation ρ can always be deformed to a reductive representation with the same Toledo invariant, see e.g. [KM17, Lem. 4.11]. Moreover, we may change the complex structure of M to its opposite (i.e., replace $z \in \mathfrak{t}$ by $-z \in \mathfrak{t}$, or equivalently $\mathfrak{m}^{\pm}$ by $\mathfrak{m}^{\mp}$), thereby changing the sign of the Toledo invariant of ρ.

Therefore we may assume without loss of generality that

– the lattice Γ is torsion free, so that $X = \Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ is a compact complex hyperbolic manifold of (complex) dimension n;

– the representation ρ is reductive, so that the results discussed in Section 3 are available;

– the Toledo invariant $\tau(\rho)$ is positive, so we only have to prove $\tau(\rho) \leqslant r_M \operatorname{vol}(X)$. (Observe that this implies that the harmonic map f is not antiholomorphic.)

We also assume as in Section 2 that

– the complexification G of $G_{\mathbb{R}}$ is simply connected.

See Remarks 5.14 and 5.15 where we explain why this is also not a loss of generality.

4.1. Setup. — Consider the representation $\mathbb{E}$ of G defined in Section 2.2. As explained in Section 3.1, this gives rise to a flat harmonic $G_{\mathbb{R}}$-Higgs vector bundle $(\overline{E}, \overline{\theta})$ over X, associated to the $G_{\mathbb{R}}$-Higgs principal bundle $(\overline{P}_K, \overline{\theta})$. As a representation of K, $\mathbb{E} = \bigoplus_{i=0}^{r_M} \mathbb{E}_i$, where r_M is the real rank of $G_{\mathbb{R}}$. This means that the holomorphic bundle $\overline{E}$ admits the holomorphic decomposition $\overline{E} = \bigoplus_{i=0}^{r_M} \overline{E}_i$, which is orthogonal for the harmonic metric. Moreover, the components $\overline{\beta} \in P_K(\mathfrak{m}^-) \otimes \Omega^1_X$ and $\overline{\gamma} \in P_K(\mathfrak{m}^+) \otimes \Omega^1_X$ of the Higgs field $\overline{\theta} \in \operatorname{Hom}(\overline{E}, \overline{E}) \otimes \Omega^1_X$ (see Remark 3.1) satisfy

$$\overline{\beta} \in \bigoplus_{i=0}^{r_M-1} \operatorname{Hom}(\overline{E}_i, \overline{E}_{i+1}) \otimes \Omega^1_X \quad \text{and} \quad \overline{\gamma} \in \bigoplus_{i=0}^{r_M-1} \operatorname{Hom}(\overline{E}_{i+1}, \overline{E}_i) \otimes \Omega^1_X.$$

We pull-back the harmonic Higgs bundle $(\overline{E}, \overline{\theta}) \to X$ to the projectivized tangent bundle $\mathbb{P}T_X$ of X to obtain a harmonic Higgs bundle that we denote by $(E, \theta) \to \mathbb{P}T_X$.

We restrict the Higgs field θ to the tangent space L of the tautological foliation on $\mathbb{P}T_X$, so that its components β and γ satisfy

$$\beta \in \bigoplus_{i=0}^{r_M-1} \mathrm{Hom}(E_i \otimes L, E_{i+1}) \quad \text{and} \quad \gamma \in \bigoplus_{i=0}^{r_M-1} \mathrm{Hom}(E_{i+1} \otimes L, E_i).$$

We also call P_K the pull-back of the principal bundle $\overline{P}_K$ to $\mathbb{P}T_X$.

NOTATION 4.1. — A tangent vector $\xi \neq 0$ to X at a point $x \in X$ defines a point $[\xi]$ in the projectivized tangent bundle $\mathbb{P}T_X$. It also defines an element in the fiber $\mathscr{O}_{\mathbb{P}T_X}(-1)_{[\xi]} = \{v \in T_{X,x} \mid v \in [\xi]\}$ of the tautological line bundle $\mathscr{O}_{\mathbb{P}T_X}(-1)$ at $[\xi]$, hence an element in the fiber $L_{[\xi]}$ of the tangent line bundle L to the tautological foliation $\mathscr{T}$ at $[\xi]$, which will also be denoted by ξ.

DEFINITION 4.2. — For $\xi \neq 0$ a vector tangent to X and $[\xi]$ the corresponding point in $\mathbb{P}T_X$, the *rank* $\mathrm{rk}\,\beta_{[\xi]}$ *of* $\beta_{[\xi]}$ is the rank of the corresponding element $\beta(\xi)$ in $\mathfrak{m}^-$, as defined in Definition 2.22. By Proposition 2.24, it is also the largest value of k such that $(\beta_{[\xi]})^k : E_0 \otimes L^k \to E_k$ is not zero.

– The *generic rank* $\mathrm{rk}\,\beta$ *of* β is the maximum of the ranks of $\beta_{[\xi]}$ for $[\xi] \in \mathbb{P}T_X$.
– The *singular locus of* β is the following subset of $\mathbb{P}T_X$:

$$\begin{aligned}\mathscr{S}(\beta) :&= \{[\xi] \in \mathbb{P}T_X \mid \mathrm{rk}\,\beta_{[\xi]} < \mathrm{rk}\,\beta\} \\ &= \{[\xi] \in \mathbb{P}T_X \mid (\beta_{[\xi]})^{\mathrm{rk}\,\beta} : E_0 \otimes L^{\mathrm{rk}\,\beta} \to E_{\mathrm{rk}\,\beta} \text{ vanishes}\}.\end{aligned}$$

– The *regular locus of* β is $\mathscr{R}(\beta) := \mathbb{P}T_X \backslash \mathscr{S}(\beta)$.
– The *singular locus* $\mathscr{S}(\overline{\beta})$ *of* $\overline{\beta}$ is the projection to X of $\mathscr{S}(\beta)$.
The *regular locus* $\mathscr{R}(\overline{\beta})$ *of* $\overline{\beta}$ is $X \backslash \mathscr{S}(\overline{\beta})$ (it is not the projection to X of $\mathscr{R}(\beta)$).

Our assumption that $\tau(\rho) > 0$ implies that $\overline{\beta} \neq 0$ and $\beta \neq 0$, therefore $1 \leqslant \mathrm{rk}\,\beta \leqslant r_M$. Observe that while $\mathscr{S}(\beta)$ is an analytic subset of $\mathbb{P}T_X$ of codimension at least 1, its projection $\mathscr{S}(\overline{\beta})$, although it is an analytic subset of X (because $\pi : \mathbb{P}T_X \to X$ is a proper map), might be equal to the whole base X. A very important consequence of the weak polystability along the leaves (Proposition 3.2 (2)), is that this does not happen when there is equality in the refined Milnor-Wood inequality of Theorem 4.6, see Proposition 4.7. This will indeed be a crucial point when dealing with maximal representations.

4.2. REWORDING OF THE INEQUALITY. — Since the Hermitian symmetric space M associated with $G_{\mathbb{R}}$ is a Kähler-Einstein manifold, the first Chern form $c_1(T_M)$ of its tangent bundle is a constant multiple of the $G_{\mathbb{R}}$-invariant Kähler form ω_M: $c_1(T_M) = -\frac{1}{4\pi}\, c\,\omega_M$ for some positive constant c ($c = c_1(\check{M})$ is the first Chern number of the compact dual $\check{M}$ of M). On the other hand, the line bundle $\mathscr{L}$ associated with the K-representation $\mathbb{E}_0$ is a generator of the Picard group of the compact dual $\check{M}$ of M and it can be checked that the canonical bundle $K_{\check{M}}$ of $\check{M}$ is precisely given by $\mathscr{L}^{-c}$, see e.g. [KM10, §2]. Therefore the pull-back $f^{\star}\omega_M$ is 4π times the first Chern form of the line bundle $f^\star\mathscr{L} = \overline{E}_0$, so that the Toledo invariant of ρ is

$$\tau(\rho) = 4\pi \deg(\overline{E}_0) = 4\pi \deg_{\mathscr{T}}(E_0), \tag{4.1}$$

where the last equality follows from Proposition 3.3. Similarly, we get that

$$\deg(K_X) = \frac{n+1}{4\pi}\operatorname{vol}(X). \tag{4.2}$$

Recall that L is the tangential line bundle to the tautological foliation $\mathscr{T}$ on the projectivized tangent bundle $\mathbb{P}T_X$, and let $L^\star$ be its dual line bundle. One can compute as explained in [KM17, §4.3.1] that

$$\deg_{\mathscr{T}}(L^\star) = \frac{1}{2\pi}\operatorname{vol}(X). \tag{4.3}$$

Therefore, the Milnor-Wood inequality can be rephrased as an inequality between the foliated degrees of the line bundles E_0 and $L^\star$ on $\mathbb{P}T_X$ and we need to prove:

$$\deg_{\mathscr{T}}(E_0) \leqslant \frac{r_M}{2}\deg_{\mathscr{T}}(L^\star), \tag{4.4}$$

where r_M is the rank of the symmetric space M.

4.3. Leafwise Higgs subsheaves associated with the holomorphic component of the Higgs field. — We now define a subsheaf $\mathscr{V}$ of $\mathscr{E} := \mathscr{O}(E)$ associated with β (recall that $\beta \neq 0$) in the same way we defined the submodule $\mathbb{V}^r$ of $\mathbb{E}$ associated with the nilpotent element $y \in \mathfrak{m}^-$ of rank r in Definition 2.25 (for all $\xi \in L$, $\beta(\xi) \in P_K(\mathfrak{m}^-)$ is a nilpotent endomorphism of the bundle E). This subsheaf will be shown to be a leafwise Higgs subsheaf of the Higgs bundle (E,θ) on $\mathbb{P}T_X$. In Section 4.4 this will be used to prove the Milnor-Wood inequality.

More precisely, mimicking Definition 2.25 and for $-\operatorname{rk}\beta \leqslant k \leqslant \operatorname{rk}\beta$, we define the following subsheaves of $\mathscr{E}$:

$$\mathscr{W}_k := \sum_{\substack{\ell\geqslant 0\\ k+\ell+1\geqslant 0}} \operatorname{Ker}\beta^{k+\ell+1} \cap \operatorname{Im}\beta^\ell,$$

where in order to define $\operatorname{Ker}\beta^j$ we consider β^j as a sheaf morphism from $\mathscr{E}$ to $\mathscr{E}\otimes(L^\star)^j$ and to define $\operatorname{Im}\beta^j$ we consider β^j as a sheaf morphism from $\mathscr{E}\otimes L^j$ to $\mathscr{E}$.

For $k = 0,1,\dots,\operatorname{rk}\beta$, let $\mathscr{V}_k$ be the saturation in $\mathscr{E}_k := \mathscr{O}(E_k)$ of the subsheaf $\mathscr{E}_k \cap \mathscr{W}_{\operatorname{rk}\beta-2k}$ and set

$$\mathscr{V} = \bigoplus_{0\leqslant k\leqslant \operatorname{rk}\beta} \mathscr{V}_k.$$

Since the sheaves $\mathscr{V}_k$ are saturated subsheaves of $\mathscr{O}(E_k)$, there exist a big open subset $\mathscr{U}$ of $\mathbb{P}T_X$ and subbundles V_k of E_k defined on $\mathscr{U}$ such that the restriction of the $\mathscr{V}_k$'s to $\mathscr{U}$ are the sheaves of sections of the V_k's. On $\mathscr{U}$ we let V be the subbundle $\oplus_{0\leqslant k\leqslant r} V_k$, so that $\mathscr{V}_{|\mathscr{U}} = \mathscr{O}_{\mathscr{U}}(V)$.

Observe that on the regular locus $\mathscr{R}(\beta)$ of β, the rank of β^k, as a vector bundle morphism from $E\otimes L^k$ to E, is constant for all k. Indeed, by Definition 2.22, the rank of $\beta(\xi)$ viewed as an element of $\mathfrak{m}^-$ determines its K-orbit. Hence on this open subset the formulas used above to define the subsheaves $\mathscr{W}_k$ of $\mathscr{E}$ in fact define subbundles W_k

of E such that $\mathscr{W}_{k|\mathscr{R}(\beta)} = \mathscr{O}_{\mathscr{R}(\beta)}(W_k)$. Therefore, on $\mathscr{R}(\beta)$, the subbundles V_k such that $\mathscr{V}_{k|\mathscr{R}(\beta)} = \mathscr{O}(V_k)$ are given by $V_k = E_k \cap W_{\operatorname{rk}\beta-2k}$. Thus, we may assume that $\mathscr{R}(\beta)$ is contained (and therefore dense) in $\mathscr{U}$.

We view an element p of the K-principal bundle $P_K \xrightarrow{\pi_K} \mathbb{P}T_X$ above $\xi \in \mathbb{P}T_X$ as an isomorphism between the fiber E_ξ of $E = P_K(\mathbb{E})$ and the model space $\mathbb{E}$. The component β of the Higgs field is a section of $P_K(\mathfrak{m}^-) \otimes L^\star \subset P_K(\operatorname{End}(\mathbb{E})) \otimes L^\star$.

Let $y = y_{\operatorname{rk}\beta}$ and $Q_{\operatorname{rk}\beta}$ be defined as in Section 2.4.

Lemma 4.3. — *On the big open set $\mathscr{U}$, the subsheaf $\mathscr{V}$ defines a reduction $P_{K\cap Q_{\operatorname{rk}\beta}}$ of the structure group of P_K to the subgroup $K \cap Q_{\operatorname{rk}\beta} \subset K$.*

Proof. — We begin by working on $\mathscr{R}(\beta) \subset \mathscr{U}$. Since for all $\xi \in \mathscr{R}(\beta)$ and all $\eta \in L_\xi$, $\eta \neq 0$, we have that $\beta_\xi(\eta)$ has rank $\operatorname{rk}\beta$, there exists $p \in (P_K)_\xi$ such that $p \circ \beta_\xi(\eta) \circ p^{-1} = y \in \mathfrak{m}^- \subset \operatorname{End}(\mathbb{E})$, so that $p(V_\xi) = \mathbb{V}^{\operatorname{rk}\beta}$. Therefore, on $\mathscr{R}(\beta)$, by Proposition 2.33 (4), the subbundle V of E defines a (holomorphic) reduction $P_{K\cap Q_{\operatorname{rk}\beta}}$ of the structure group of P_K to the subgroup $K \cap Q_{\operatorname{rk}\beta}$ of K ($Q_{\operatorname{rk}\beta}$ is the normalizer in G of the parabolic subalgebra $\mathfrak{q}_{\operatorname{rk}\beta}$, see Definition 2.30). Explicitly $P_{K\cap Q_{\operatorname{rk}\beta}} = \{p \in P_K \mid p(V_{\pi_K(p)}) = \mathbb{V}^{\operatorname{rk}\beta}\}$.

We now work on $\mathscr{U}$. Enlarge the structure group of P_K to $\operatorname{GL}(\mathbb{E})$. The subbundle $V = \bigoplus_{k=0}^{\operatorname{rk}\beta} V_k$ of E defines a reduction P_S of the structure group of $P_{\operatorname{GL}(\mathbb{E})} \xrightarrow{\pi_{\operatorname{GL}(\mathbb{E})}} \mathbb{P}T_X$ to the stabilizer S of $\mathbb{V}^{\operatorname{rk}\beta}$ in $\operatorname{GL}(\mathbb{E})$ by setting $P_S = \{p \in P_{\operatorname{GL}(\mathbb{E})} \mid p(V_{\pi_{\operatorname{GL}(\mathbb{E})}(p)}) = \mathbb{V}^{\operatorname{rk}\beta}\}$.

Let $B \subset \mathscr{U}$ be an open ball on which P_K is trivial. Then the reductions $P_{K\cap Q_{\operatorname{rk}\beta}}$ of P_K on $B \cap \mathscr{R}(\beta)$ and P_S of $P_{\operatorname{GL}(\mathbb{E})}$ on B are respectively given by holomorphic maps $\sigma : B \cap \mathscr{R}(\beta) \to K/(K \cap Q_{\operatorname{rk}\beta})$ and $s : B \to \operatorname{GL}(\mathbb{E})/S$. Moreover, if ι denotes the natural map $K/(K\cap Q_{\operatorname{rk}\beta}) \to \operatorname{GL}(\mathbb{E})/S$, which is injective, then we have $s = \iota \circ \sigma$ on $B \cap \mathscr{R}(\beta)$. Since $K/(K \cap Q_{\operatorname{rk}\beta})$ is compact, its image by ι is closed in $\operatorname{GL}(\mathbb{E})/S$. Therefore, since $B \cap \mathscr{R}(\beta)$ is dense in B, s maps B to $\iota(K/(K \cap Q_{\operatorname{rk}\beta}))$. This means that the reduction $P_{K\cap Q_{\operatorname{rk}\beta}}$ initially defined on $\mathscr{R}(\beta)$ extends to $\mathscr{U}$. □

We deduce the following result.

Proposition 4.4. — *The subsheaf $\mathscr{V}$ is a leafwise Higgs subsheaf of the Higgs bundle (E, θ) on $\mathbb{P}T_X$.*

Proof. — By Proposition 2.29, we know that y and $\mathfrak{m}^+$ stabilize $\mathbb{V}^{\operatorname{rk}\beta}$. Therefore, on $\mathscr{R}(\beta)$, the two components β and γ of the Higgs field stabilize the subsheaf $\mathscr{V}$ since it is the sheaf of sections of the subbundle $V = P_{K\cap Q_{\operatorname{rk}\beta}}(\mathbb{V}^{\operatorname{rk}\beta})$ of $E = P_{K\cap Q_{\operatorname{rk}\beta}}(\mathbb{E})$. By continuity, this still holds on $\mathscr{U}$ since on this big open set $\mathscr{V}$ is also the sheaf of section of $V = P_{K\cap Q_{\operatorname{rk}\beta}}(\mathbb{V}^{\operatorname{rk}\beta})$. Now, $\mathscr{V}$ is a saturated, hence normal, subsheaf of $\mathscr{O}(E)$ by definition. Hence the restriction map $\mathscr{V}(\mathbb{P}T_X) \to \mathscr{V}(\mathscr{U})$ is an isomorphism since $\mathscr{U}$ is big. Therefore $\mathscr{V}$ is indeed a leafwise Higgs subsheaf of (E, θ) on $\mathbb{P}T_X$. □

4.4. Proof of the Milnor-Wood inequality. — Recall that the slope of an H-module was introduced in Section 2.5 and is an element in $X(H) \otimes \mathbb{Q}$. To relate it to the slope of the associated vector bundle, we use the following fact which follows immediately from the definitions.

Fact 4.5. — Let H be a complex reductive group and let $P_H \to Z$ a holomorphic H-principal bundle over a complex manifold Z. Let $\mathbb{V}_i, i \in \{1,2\}$, be H-modules and let $V_i = P_H(\mathbb{V}_i)$ be the associated vector bundles. Assume that $\mu_H(\mathbb{V}_1) = \mu_H(\mathbb{V}_2)$. Then

$$\frac{\det V_1}{\operatorname{rk} V_1} = \frac{\det V_2}{\operatorname{rk} V_2},$$

as elements of $\operatorname{Pic}(Z) \otimes_{\mathbb{Z}} \mathbb{Q}$. In other words, the line bundles $(\det V_1)^{\otimes \operatorname{rk} V_2}$ and $(\det V_2)^{\otimes \operatorname{rk} V_1}$ are isomorphic.

Therefore, if one is given a reasonable way to compute slopes of vector bundles on Z, e.g. the usual one if Z is X with its Kähler form ω, or the foliated one if Z is $\mathbb{P}T_X$ with its tautological foliation $\mathscr{T}$ and transverse volume form $\Omega_{\mathscr{G}}$, then using the notation of Fact 4.5, $\mu_H(\mathbb{V}_1) = \mu_H(\mathbb{V}_2)$ implies that the slopes of V_1 and V_2 are equal.

This observation, together with the computation of the slopes of the $H_{\operatorname{rk}\beta}$-submodules $\mathbb{V}_k^{\operatorname{rk}\beta}$ of $\mathbb{E}$ and the construction of the leafwise Higgs subsheaf $\mathscr{V}$ of (E,θ), gives the Milnor Wood inequality.

Theorem 4.6. — *We have the inequalities*

$$\deg_{\mathscr{T}}(E_0) + \frac{\operatorname{rk}\beta}{2}\deg_{\mathscr{T}}(L) \leqslant \mu_{\mathscr{T}}(\mathscr{V}) \leqslant 0.$$

Therefore the Milnor-Wood inequality $\deg_{\mathscr{T}}(E_0) \leqslant (r_M/2)\deg_{\mathscr{T}}(L^\star)$ *holds.*

Proof. — Let $n_k := \dim \mathbb{V}_k^{\operatorname{rk}\beta}$. First, recall that Proposition 2.33 (2) states that the unipotent radical of $Q_{\operatorname{rk}\beta}$ acts trivially on $\mathbb{V}^{\operatorname{rk}\beta}$. Hence so does the unipotent radical of $K \cap Q_{\operatorname{rk}\beta}$. Thus, in fact, $\mathbb{V}^{\operatorname{rk}\beta}$ is a $(K\cap Q_{\operatorname{rk}\beta})/R_u(K\cap Q_{\operatorname{rk}\beta})$-module, and $(K\cap Q_{\operatorname{rk}\beta})/R_u(K\cap Q_{\operatorname{rk}\beta}) \simeq H_{\operatorname{rk}\beta}$ is reductive.

Therefore, by Lemma 4.3, on the big open set $\mathscr{U}$, the vector bundles V_k such that $\mathscr{V}_k = \mathscr{O}(V_k)$ are given by $V_k = P_{K\cap Q_{\operatorname{rk}\beta}}(\mathbb{V}_k^{\operatorname{rk}\beta})$. By Proposition 2.45 and Fact 4.5 applied to $\mathbb{V}_k^{\operatorname{rk}\beta}$ and $\mathbb{V}'_k := \mathbb{V}_0^{\operatorname{rk}\beta} \otimes (\mathbb{V}_1^{\operatorname{rk}\beta} \otimes \mathbb{V}_0^{\operatorname{rk}\beta\star})^{\otimes k}$, we have on $\mathscr{U}$

$$(\det V_k)^{\operatorname{rk} V'_k} \simeq (\det V'_k)^{\operatorname{rk} V_k},$$

where $V'_k = P_{K\cap Q_{\operatorname{rk}\beta}}(\mathbb{V}'_k)$. Letting $\mathscr{V}'_k := \mathscr{V}_0 \otimes (\mathscr{V}_1 \otimes \mathscr{V}_0{}^\star)^{\otimes k}$, we deduce that the line bundles $(\det \mathscr{V}_k)^{\operatorname{rk}\mathscr{V}'_k}$ and $(\det \mathscr{V}'_k)^{\operatorname{rk}\mathscr{V}_k}$ on $\mathbb{P}T_X$ are isomorphic on $\mathscr{U}$, hence on $\mathbb{P}T_X$, because $\mathscr{U}$ is a big open set of $\mathbb{P}T_X$. Therefore

$$\mu_{\mathscr{T}}(\mathscr{V}_k) = \mu_{\mathscr{T}}(\mathscr{V}'_k) = \deg_{\mathscr{T}}(\mathscr{V}_0) + k\,\mu_{\mathscr{T}}(\mathscr{V}_0^\star \otimes \mathscr{V}_1).$$

Since $\beta^{\operatorname{rk}\beta} : \mathscr{V}_0 \otimes L^{\operatorname{rk}\beta} \to \mathscr{V}_{\operatorname{rk}\beta}$ is not zero and $n_0 = n_{\operatorname{rk}\beta} = 1$, we also have $\mu_{\mathscr{T}}(\mathscr{V}_{\operatorname{rk}\beta}) \geqslant \mu_{\mathscr{T}}(\mathscr{V}_0) + \operatorname{rk}\beta\,\mu_{\mathscr{T}}(L)$ so that $\mu_{\mathscr{T}}(\mathscr{V}_0^\star \otimes \mathscr{V}_1) \geqslant \deg_{\mathscr{T}}(L)$.

Finally, remembering that $n_k = n_{\operatorname{rk}\beta - k}$ by Proposition 2.35 and that $\mathscr{V}_0 = E_0$, we get

$$\begin{aligned}
2\deg_{\mathscr{T}}(\mathscr{V}) &= \sum_{k=0}^{\operatorname{rk}\beta} \deg_{\mathscr{T}}(\mathscr{V}_k) + \sum_{k=0}^{\operatorname{rk}\beta} \deg_{\mathscr{T}}(\mathscr{V}_{\operatorname{rk}\beta-k}) \\
&= \sum_{k=0}^{\operatorname{rk}\beta} \left(n_k\, \mu_{\mathscr{T}}(\mathscr{V}_k) + n_{\operatorname{rk}\beta-k}\, \mu_{\mathscr{T}}(\mathscr{V}_{\operatorname{rk}\beta-k})\right) \\
&\geqslant \sum_{k=0}^{\operatorname{rk}\beta} n_k \left(\deg_{\mathscr{T}}(\mathscr{V}_0) + k\deg_{\mathscr{T}}(L) + \deg_{\mathscr{T}}(\mathscr{V}_0) + (\operatorname{rk}\beta - k)\deg_{\mathscr{T}}(L)\right) \\
&= (\dim \mathbb{V}^{\operatorname{rk}\beta})\left(2\deg_{\mathscr{T}}(E_0) + \operatorname{rk}\beta \deg_{\mathscr{T}}(L)\right).
\end{aligned}$$

The inequality $\mu_{\mathscr{T}}(\mathscr{V}) \leqslant 0$ follows from Propositions 4.4 and 3.2, and the conclusion from the fact that $\operatorname{rk}\beta \leqslant r_M$. □

In case of equality in Theorem 4.6, we have

Proposition 4.7. — *Assume that* $\deg_{\mathscr{T}}(E_0) + \frac{\operatorname{rk}\beta}{2}\deg_{\mathscr{T}}(L) = 0$. *Then*

(1) *on the regular locus* $\mathscr{R}(\beta) = \mathbb{P}T_X \backslash \mathscr{S}(\beta)$ *of* β, *the orthogonal complement* $V^\perp = (\bigoplus V_i)^\perp$ *of the subbundle* $V = \bigoplus V_i$ *of* E *w.r.t. the harmonic metric is stable under the Higgs field* $\theta : E \otimes L \to E$;

(2) *the regular locus* $\mathscr{R}(\overline{\beta}) \subset X$ *of* $\overline{\beta}$ *is (open and) dense in* X.

Proof. — The first point follows from the discussion after the definition of the subsheaves $\mathscr{V}_k$ and the polystability property (2) in Proposition 3.2, since our hypothesis implies that $\deg_{\mathscr{T}} \mathscr{V} = 0$ by Theorem 4.6.

Proposition 3.2 (2) also implies that the singular locus $\mathscr{S}(\beta)$ of β, which is a closed proper subset of $\mathbb{P}T_X$, is saturated under the tautological foliation $\mathscr{T}$, see the proof of [KM17, Lem. 4.5]. Since the leaves of the tautological foliation on $\mathbb{P}T_X$ are projections of $\mathrm{SU}(1,1)$-homogeneous subsets of $\Gamma\backslash \mathrm{SU}(1,n)$, M. Ratner's results on unipotent flows and the fact that $\mathbb{H}^n_{\mathbb{C}} = \mathrm{SU}(1,n)/\mathrm{U}(n)$ is a rank 1 symmetric space, then imply by [KM17, Prop. 3.6] that the singular locus $\mathscr{S}(\overline{\beta}) := \pi(\mathscr{S}(\beta))$ of $\overline{\beta}$ is a proper analytic subset of $X = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$, hence the second point of the proposition. □

5. Maximal representations

Maximal representations $\rho : \Gamma \to G_{\mathbb{R}}$, where Γ is a uniform lattice in $\mathrm{SU}(1,n)$ with $n \geqslant 2$ and $G_{\mathbb{R}}$ is a *classical* Hermitian group, were classified in [KM17]. Therefore we focus here on *exceptional* targets, namely $G_{\mathbb{R}}$ is either $\mathrm{E}_{6(-14)}$, which is not of tube type, or $\mathrm{E}_{7(-25)}$, which is.

In Section 5.1 we exclude the possibility of maximal representations in $\mathrm{E}_{7(-25)}$. In fact, our uniform approach allows to easily prove that maximal representations in tube type target groups $G_{\mathbb{R}}$ do not exist (Proposition C). The case of $\mathrm{E}_{6(-14)}$ is treated in Section 5.2, Theorem A and Corollary B are established in Section 5.3.

5.1. Tube type targets. — We prove Proposition C, namely that whenever $G_{\mathbb{R}}$ has tube type and $n \geqslant 2$, the Toledo invariant of a representation from Γ to $G_{\mathbb{R}}$ satisfies an inequality stronger than the Milnor-Wood inequality, preventing a representation in such a group to be maximal.

Proposition 5.1. — *Let Γ be a uniform lattice in $\mathrm{SU}(1,n)$, and let $X = \Gamma\backslash\mathbb{H}^n_{\mathbb{C}}$. Assume that the simple real algebraic Hermitian Lie group $G_{\mathbb{R}}$ has tube type and let r_M be the real rank of $G_{\mathbb{R}}$. Let ρ be a representation $\Gamma \to G_{\mathbb{R}}$. Then*

$$|\tau(\rho)| \leqslant \max\Big\{r_M - 1, \frac{r_M}{2}\cdot\frac{n+1}{n}\Big\}\,\mathrm{vol}(X).$$

Remark 5.2. — Observe that $\max\big\{r_M - 1, \frac{r_M}{2}\cdot\frac{n+1}{n}\big\} = r_M - 1$ unless $r_M \leqslant 2$ or $r_M = 3$ and $n = 2$.

Proof. — We may assume that Γ is torsion free, that $\tau(\rho) > 0$, that ρ is reductive (see the beginning of Section 4), and that G is simply connected (see Remarks 5.14 and 5.15). Then, the constructions of Sections 2, 3 and 4 are valid and the inequality of the proposition is equivalent to the inequality

$$\deg_{\mathscr{T}}(E_0) \leqslant \max\Big(\frac{r_M - 1}{2}, \frac{r_M}{2}\cdot\frac{n+1}{2n}\Big)\deg_{\mathscr{T}}(L^\star).$$

We use freely the notation of Section 4. If the generic rank of β on the projectivized tangent bundle $\mathbb{P}T_X$ of X is $\leqslant r_M - 1$ then we are done by Theorem 4.6. Therefore we may assume that the generic rank of β on $\mathbb{P}T_X$ is r_M.

We come back to the Higgs bundle $(\overline{E}, \overline{\theta})$ on X and we consider $\overline{\beta} : \overline{E}\otimes T_X \to \overline{E}$. The fact that $\mathrm{rk}\,\beta = r_M$ implies that $\overline{\beta}^{r_M}$, seen as a morphism from $\overline{E}_0 \otimes \overline{E}^\star_{r_M}$ to the r_M-th symmetric power $S^{r_M}\Omega^1_X$ of Ω^1_X, is not zero. Since Ω^1_X is a semistable bundle over X (X is Kähler-Einstein), so is $S^{r_M}\Omega^1_X$. On the other hand, $\overline{E}_0 \otimes \overline{E}^\star_{r_M}$ is also semistable because it is a line bundle by Section 2.4.4. Therefore $\mu_\omega(\overline{E}_0 \otimes \overline{E}^\star_{r_M}) \leqslant \mu_\omega(S^{r_M}\Omega^1_X)$ (recall that ω is our Kähler form on X), so that $\deg_\omega \overline{E}_0 - \deg_\omega \overline{E}_{r_M} \leqslant r_M\,\mu_\omega(\Omega^1_X)$. Now, as explained in Section 2.4.4, the K-modules $\mathbb{E}_{r_M}$ and $\mathbb{E}^\star_0$ are isomorphic, so that $\deg_\omega \overline{E}_{r_M} = -\deg_\omega \overline{E}_0$ by Fact 4.5. We get the result, since by Equations (4.2) and (4.3) in Section 4.2 and the isomorphism $K_X \simeq \det\Omega^1_X$, we have $\deg_\omega(\Omega^1_X) = \frac{n+1}{2}\deg_{\mathscr{T}}(L^\star)$. □

5.2. Target group $\mathrm{E}_{6(-14)}$

5.2.1. *Algebraic preliminaries*. — In the case $G_{\mathbb{R}} = \mathrm{E}_{6(-14)}$, the cominuscule representation of $G = \mathrm{E}_6$ is the defining representation of E_6 on the 27-dimensional complex exceptional Jordan algebra $\mathbb{E} = \mathfrak{J}_{\mathbb{C}}$. The real rank of $\mathrm{E}_{6(-14)}$ is 2 and $\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_1 \oplus \mathbb{E}_2$ with $\mathbb{E}_0$, $\mathbb{E}_1$, and $\mathbb{E}_2$ of dimension 1, 16 and 10 respectively. The semi-simple part of K is isomorphic to Spin_{10} and as Spin_{10}-representations, $\mathbb{E}_0$, $\mathbb{E}_1$, and $\mathbb{E}_2$ are respectively a trivial, half-spin, and vector representation.

Given $x \in \mathfrak{m}^-$ (resp. $y \in \mathfrak{m}^+$), x (resp. y) defines linear maps $\mathbb{E}_0 \to \mathbb{E}_1$ and $\mathbb{E}_1 \to \mathbb{E}_2$ (resp. $\mathbb{E}_1 \to \mathbb{E}_0$ and $\mathbb{E}_2 \to \mathbb{E}_1$). We denote these maps by $\lambda_1(x), \lambda_2(x)$

resp. $\mu_1(y), \mu_2(y)$. We thus have maps $\lambda_1(x) : \mathbb{E}_0 \to \mathbb{E}_1$, $\lambda_2(x) : \mathbb{E}_1 \to \mathbb{E}_2$ and $\mu_1(y) : \mathbb{E}_1 \to \mathbb{E}_0$, $\mu_2(y) : \mathbb{E}_2 \to \mathbb{E}_1$.

We start with a description of the Spin_{10}-representations $\mathbb{E}_1$ and $\mathbb{E}_2$ in terms of octonions. More precisely, by Proposition 2.15(d), there is a Spin_{10}-equivariant isomorphism $\mathbb{E}_1 \simeq \mathbb{E}_0 \otimes \mathfrak{m}^-$. Choosing a non-zero vector in $\mathbb{E}_0$, this yields an isomorphism $\alpha : \mathbb{E}_1 \to \mathfrak{m}^-$. We consider the quadratic map $\kappa : \mathbb{E}_1 \to \mathbb{E}_2$ defined by $\kappa(x) = \lambda_2(\alpha(x)) \cdot x$. This is a Spin_{10}-equivariant quadratic map $\mathbb{E}_1 \to \mathbb{E}_2$.

As the next proposition shows, there is, up to a scale, only one such map, and this map can be described in terms of octonions. We denote by $\mathbb{O}$ the (non associative) algebra of octonions (see [Che97, §4.5] or [SV00]). We denote by $N : \mathbb{O} \to \mathbb{C}$ the norm, a quadratic map such that $N(z_1 z_2) = N(z_1)N(z_2)$ for all octonions z_1, z_2. The following is certainly well-known to specialists, however we could not find an adequate reference.

Proposition 5.3. — *There are* $(\mathrm{Spin}_8 \times \mathrm{Spin}_2)$*-equivariant identifications of* $\mathbb{E}_1$ *with* $\mathbb{O} \oplus \mathbb{O}$ *and* $\mathbb{E}_2$ *with* $\mathbb{C} \oplus \mathbb{O} \oplus \mathbb{C}$ *such that* $\kappa(u, v) = (N(u), uv, N(v))$.

Proof. — We consider the Spin_{10} half-spin representation $\mathbb{E}_1$. According to [Che97], when we restrict to Spin_8, this representation splits as $\mathscr{S}^+ \oplus \mathscr{S}^-$, where $\mathscr{S}^\pm$ denote the two half-spin representations of Spin_8. Similarly, the Spin_{10} vector representation $\mathbb{E}_2$ splits as $\mathbb{C} \oplus \mathbb{V} \oplus \mathbb{C}$, where $\mathbb{V}$ denotes the 8-dimensional vector representation.

Now, the quadratic map κ is given by a Spin_{10}-equivariant linear morphism $S^2\mathbb{E}_1 \to \mathbb{E}_2$ which is itself induced by an injection

$$\mathbb{E}_2 \subset \mathbb{E}_1 \otimes \mathbb{E}_1 = \mathscr{S}^+ \otimes \mathscr{S}^+ \oplus \mathscr{S}^+ \otimes \mathscr{S}^- \oplus \mathscr{S}^- \otimes \mathscr{S}^+ \oplus \mathscr{S}^- \otimes \mathscr{S}^-.$$

Since there are Spin_8-equivariant maps $\mathscr{S}^+ \otimes \mathscr{S}^- \to \mathbb{V}$, $\mathscr{S}^+ \otimes \mathscr{S}^+ \to \mathbb{C}$ and $\mathscr{S}^- \otimes \mathscr{S}^- \to \mathbb{C}$, and no Spin_8-equivariant maps from other factors in the tensor product $\mathbb{E}_1 \otimes \mathbb{E}_1$ to $\mathbb{E}_2$, κ is of the form $\kappa(s_+, s_-) = (\psi_+(s_+), \varphi(s_+, s_-), \psi_-(s_-))$, for some equivariant quadratic maps $\psi_\pm : \mathscr{S}^\pm \to \mathbb{C}$, and for a bilinear map $\varphi : \mathscr{S}^+ \times \mathscr{S}^- \to \mathbb{V}$. None of these maps can vanish, otherwise the image of κ would be degenerate. Moreover, there are, up to scale, only one such map, as it follows from the decomposition of the tensor product of two spin representations [Che97, §3.3]. It is given, once $\mathscr{S}^+, \mathscr{S}^-$ and $\mathbb{V}$ are identified with the space of octonions $\mathbb{O}$, by the formulas: $\psi_+(s_+) = N(s_+), \varphi(s_+, s_-) = s_+ s_-$ and $\psi_-(s_-) = N(s_-)$ (see again [Che97, §4.5]). The proposition follows. □

We can deduce from the explicit formula above some information about maps $\lambda_2(x)$.

Proposition 5.4. — *Let* $x, y \in \mathbb{E}_1 \simeq \mathfrak{m}^-$.

(a) *x has rank one if and only if $x \neq 0$ and $\kappa(x) = 0$.*

(b) *x has rank one if and only if* $\dim(\mathrm{Im}\,\lambda_2(x)) = 5$.

(c) *x has rank two if and only if* $\dim(\mathrm{Im}\,\lambda_2(x)) = 9$ *if and only if* $\kappa(x) \neq 0$.

(d) *Assume that all non trivial linear combinations of x and y have rank* 2 *(in particular, x and y are not colinear). Then* $\dim(\mathrm{Ker}\,\lambda_2(x) \cap \mathrm{Ker}\,\lambda_2(y)) \leqslant 3$.

(e) *Assume that x and y have rank* 1 *and* $\dim(\operatorname{Im}\lambda_2(x)\cap\operatorname{Im}\lambda_2(y))\geqslant 4$. *Then x and y are proportional.*

Proof. — We use the above isomorphism $\mathbb{E}_1\simeq\mathbb{O}\oplus\mathbb{O}$. According to [Igu70, Prop. 2], there are exactly 3 orbits in $\mathbb{E}_1$ under $\mathrm{Spin}_{10}\times\mathbb{C}^*$. Let $u\in\mathbb{O}$ such that $N(u)=0$. We have $\kappa(u,0)=(0,0,0)$ and $\kappa(1,0)=(1,0,0)$. Thus, $(u,0)$ and $(1,0)$ cannot be in the same orbit. It follows that $(u,0)$ has rank 1 and $(1,0)$ has rank 2 and statement (a) of the proposition is proved.

Let $\widetilde{\kappa}:\mathbb{E}_1\times\mathbb{E}_1\to\mathbb{E}_2$ be the polarization of κ, namely, the unique symmetric bilinear map such that $\widetilde{\kappa}(x,x)=\kappa(x)$ for all x in $\mathbb{E}_1$. We have $\lambda_2(x)=\widetilde{\kappa}(x,\cdot)$. Thus the image of $\lambda_2(u,0)$ is the set of triples $(t,z,0)$ with $t\in\mathbb{C}$ arbitrary and z of the form uw for some w in $\mathbb{O}$: this space has dimension 5. On the other hand, the image of $\lambda_2(1,0)$ is the set of triples $(t,z,0)$ with t and z arbitrary. It has dimension 9. Points (b) and (c) are proved.

For point (d), let us assume that any non trivial linear combination of x and y has rank 2. Thanks to the result of Igusa, we may assume that $x=(1,0)$. Writing $y=(a,b)$, the assumption implies that $b\neq 0$ (in fact, if $y=(a,0)$, then some linear combination of x and y will be of the form $(u,0)$ with $N(u)=0$). The kernel of $\lambda_2(x)$ is the space of elements of the form $(u,0)$ with $\langle u,1\rangle=0$ (here, $\langle\cdot,\cdot\rangle$ denotes the symmetric bilinear form associated with N). If such an element is in the kernel of $\lambda_2(y)$ then $bu=0$ and so $N(u)=0$. In fact, the octonion algebra is *alternative*, meaning that for all elements a,b one has $(ab)b=a(b^2)$. Since $\overline{b}$ is a linear combination of 1 and b, we also have $(ab)\overline{b}=a(b\overline{b})$. From the equation $0=bu$ it thus follows $0=(bu)\overline{u}=b(u\overline{u})=bN(u)$ so $N(u)=0$. Thus, the intersection of the kernels of $\lambda_2(x)$ and $\lambda_2(y)$ is isomorphic to an isotropic subspace of the space of octonions u with $\langle u,1\rangle=0$. Such an isotropic subspace can have at most dimension 3.

Finally, let us assume that x and y have rank 1 and that

$$\dim(\operatorname{Im}\lambda_2(x)\cap\operatorname{Im}\lambda_2(y))\geqslant 4.$$

Then we may assume that $x=(u,0)$ with $N(u)=0$ as above. The image of $\lambda_2(x)$ is then the set of triples $(t,z,0)$ with t arbitrary and z of the form uw for some octonion w. Thus, this space is an isotropic subspace of $\mathbb{E}_2$ of maximal dimension 5. Using the Spin_{10}-action, it follows that for any $x\in\mathbb{E}_1$ of rank 1, the image of $\lambda_2(x)$ is an isotropic subspace of dimension 5. Since two maximal isotropic subspaces in the same family can intersect only in odd dimension, it follows from the hypothesis on x and y that the images of $\lambda_2(x)$ and $\lambda_2(y)$ are equal. Thus x and y are pure spinors representing the same maximal isotropic subspace, so they are proportional by [Che97, III.1.4]. It is also possible to draw this conclusion by a direct computation, using the fact that for $u,v\in\mathbb{O}$ such that $N(u)=N(v)=0$, the equality $\{uz\mid z\in\mathbb{O}\}=\{vz\mid z\in\mathbb{O}\}$ is equivalent to the fact that u and v are proportional. □

Recall that we constructed a quadratic Spin_{10}-equivariant map $\kappa:\mathbb{E}_1\to\mathbb{E}_2$ by setting $\kappa(x)=\lambda_2(\alpha(x))\cdot x$, where $\alpha:\mathbb{E}_1\to\mathfrak{m}^-$ is a Spin_{10}-equivariant isomorphism. Similarly, let $\beta:\mathbb{E}_1^\star\to\mathfrak{m}^+$ be a Spin_{10}-equivariant isomorphism. Let $\iota:\mathbb{E}_1^\star\to\mathbb{E}_2^\star$ be the quadratic equivariant map obtained setting $\iota(w) = {}^t\mu_2(\beta(w)) \cdot w$. With the same proof, we get information about $\mathfrak{m}^+$ and the linear maps $\mu_2(w)$.

Proposition 5.5. — *Let $w, z \in \mathbb{E}_1^* \simeq \mathfrak{m}^+$.*

(a) *w has rank one if and only if $w \neq 0$ and $\iota(w) = 0$.*

(b) *w has rank one if and only if $\dim(\operatorname{Im}\mu_2(w)) = 5$.*

(c) *w has rank two if and only if $\dim(\operatorname{Im}\mu_2(w)) = 9$ if and only if $\iota(w) \neq 0$.*

(d) *Assume that all non trivial linear combinations of w and z have rank 2 (in particular, w and z are not colinear). Then $\dim(\operatorname{Im}\mu_2(w) \cap \operatorname{Im}\mu_2(z)) \leqslant 3$.*

(e) *Assume that w and z have rank 1 and $\dim(\operatorname{Ker}\mu_2(w) \cap \operatorname{Ker}\mu_2(z)) \geqslant 4$. Then w and z are proportional.*

5.2.2. *Maximal representations.* — Let Γ be a torsion free uniform lattice in $\mathrm{SU}(1, n)$ and $\rho : \Gamma \to \mathrm{E}_{6(-14)}$ be a reductive representation. We may therefore consider the the Higgs bundle $(\overline{E}, \overline{\theta})$ on X and its pull-back (E, θ) on $\mathbb{P}T_X$ associated with ρ and the representation of E_6 on $\mathbb{E} = \mathbb{E}_0 \oplus \mathbb{E}_1 \oplus \mathbb{E}_2$ as in Section 4.

Recall that the components of the Higgs field $\overline{\theta}$ are

$$P_K(\mathfrak{m}^-) \ni \overline{\beta} =: (\overline{\beta}_1, \overline{\beta}_2) \in \big(\operatorname{Hom}(\overline{E}_0, \overline{E}_1) \otimes \Omega^1_X\big) \oplus \big(\operatorname{Hom}(\overline{E}_1, \overline{E}_2) \otimes \Omega^1_X\big)$$

and

$$P_K(\mathfrak{m}^+) \ni \overline{\gamma} =: (\overline{\gamma}_1, \overline{\gamma}_2) \in \big(\operatorname{Hom}(\overline{E}_1, \overline{E}_0) \otimes \Omega^1_X\big) \oplus \big(\operatorname{Hom}(\overline{E}_2, \overline{E}_1) \otimes \Omega^1_X\big).$$

To lighten the notation, the fibers of the bundles $\overline{E}$, $\overline{E}_0$, $\overline{E}_1$ and $\overline{E}_2$ above some $x \in X$ will also be denoted by $\overline{E}$, $\overline{E}_0$, $\overline{E}_1$ and $\overline{E}_2$.

Propositions 5.4 and 5.5 immediately imply the following.

Corollary 5.6. — *Let $x \in X$ and ξ, η be tangent vectors at x.*

As an element of $\operatorname{Hom}(\overline{E}_1, \overline{E}_2)$, $\overline{\beta}_2(\xi)$ has rank 0, 5 or 9. Moreover:

(a) *If $\overline{\beta}(\xi)$ has rank 1, i.e., $\overline{\beta}_1(\xi) \neq 0$ but $\overline{\beta}_2(\xi)\overline{\beta}_1(\xi) = 0$, then $\overline{\beta}_2(\xi) : \overline{E}_1 \to \overline{E}_2$ has rank 5;*

(b) *If $\overline{\beta}(\xi)$ has rank 2, i.e., if $\overline{\beta}_2(\xi)\overline{\beta}_1(\xi) \neq 0$, then $\overline{\beta}_2(\xi) : \overline{E}_1 \to \overline{E}_2$ has rank 9;*

(c) *If all non trivial linear combinations of $\overline{\beta}(\xi)$ and $\overline{\beta}(\eta)$ have rank 2 (in particular, ξ and η are not colinear), then we have $\dim(\operatorname{Ker}\overline{\beta}_2(\xi) \cap \operatorname{Ker}\overline{\beta}_2(\eta)) \leqslant 3$.*

Similarly, as an element of $\operatorname{Hom}(\overline{E}_2, \overline{E}_1)$, $\overline{\gamma}_2(\xi)$ has rank 0, 5 or 9. Moreover:

(a′) *If $\overline{\gamma}(\xi)$ has rank 1, i.e., $\overline{\gamma}_2(\xi) \neq 0$ but $\overline{\gamma}_1(\xi)\overline{\gamma}_2(\xi) = 0$, then $\overline{\gamma}_2(\xi) : \overline{E}_2 \to \overline{E}_1$ has rank 5;*

(b′) *If $\overline{\gamma}(\xi)$ has rank 2, i.e., if $\overline{\gamma}_1(\xi)\overline{\gamma}_2(\xi) \neq 0$, then $\overline{\gamma}_2(\xi) : \overline{E}_2 \to \overline{E}_1$ has rank 9;*

(c′) *If $\overline{\gamma}(\xi)$ and $\overline{\gamma}(\eta)$ have rank 1 and $\dim(\operatorname{Ker}\overline{\gamma}_2(\xi) \cap \operatorname{Ker}\overline{\gamma}_2(\eta)) \geqslant 4$, then $\overline{\gamma}(\xi)$ and $\overline{\gamma}(\eta)$ are colinear.*

Thanks to this corollary, in case of equality in the Milnor-Wood inequality, we may prove

Proposition 5.7. — *Let $(\overline{E},(\overline{\beta},\overline{\gamma}))$ be the harmonic Higgs bundle on X associated with the reductive representation $\rho:\Gamma\to \mathrm{E}_{6(-14)}$ of the torsion free uniform lattice $\Gamma\subset \mathrm{SU}(1,n)$ and assume $n\geqslant 2$.*

If $\deg_{\mathscr{T}}(E_0)=\deg_{\mathscr{T}}(L^\star)$ and $x\in\mathscr{R}(\overline{\beta})$, then for all $\xi\in T_{X,x}$, $\overline{\gamma}(\xi)=0$.

Proof. — The letters ξ and η will denote tangent vectors at x.

The equality $\deg_{\mathscr{T}}(E_0)=\deg_{\mathscr{T}}(L^\star)$ and Theorem 4.6 imply that the generic rank of β on $\mathbb{P}T_X$ is 2. Therefore, since x belongs to the regular locus $\mathscr{R}(\overline{\beta})$ of $\overline{\beta}$, for all $\xi\neq 0$ in $T_{X,x}$, the rank of $\overline{\beta}(\xi)$ is 2. This implies by Corollary 5.6 (b) that the rank of $\overline{\beta}_2(\xi)$ is 9 for all $\xi\neq 0$, and by Corollary 5.6 (c) and linearity of $\overline{\beta}:T_{X,x}\to\mathrm{Hom}(\overline{E},\overline{E})$, that if ξ and η are not colinear, $\dim(\operatorname{Ker}\overline{\beta}_2(\xi)\cap\operatorname{Ker}\overline{\beta}_2(\eta))\leqslant 3$.

We will make a crucial use of the integrability relation $[\overline{\theta},\overline{\theta}]=0$ of the Higgs field $\overline{\theta}$. This relation is equivalent to the following three conditions:

$$\begin{cases}\overline{\gamma}_1(\xi)\overline{\beta}_1(\eta)=\overline{\gamma}_1(\eta)\overline{\beta}_1(\xi) & \text{in }\ \mathrm{End}(\overline{E}_0)\\ \overline{\beta}_1(\xi)\overline{\gamma}_1(\eta)+\overline{\gamma}_2(\xi)\overline{\beta}_2(\eta)=\overline{\beta}_1(\eta)\overline{\gamma}_1(\xi)+\overline{\gamma}_2(\eta)\overline{\beta}_2(\xi) & \text{in }\ \mathrm{End}(\overline{E}_1)\\ \overline{\beta}_2(\xi)\overline{\gamma}_2(\eta)=\overline{\beta}_2(\eta)\overline{\gamma}_2(\xi) & \text{in }\ \mathrm{End}(\overline{E}_2)\end{cases}$$

which hold for all ξ,η.

Suppose first that there exists ξ such that $\overline{\gamma}_2(\xi):\overline{E}_2\to\overline{E}_1$ has rank 9. Consider the subspace $W:=\operatorname{Ker}\overline{\gamma}_1(\xi)\cap\operatorname{Ker}\overline{\beta}_2(\xi)\subset\overline{E}_1$. Since $\dim\overline{E}_1=16$, $\dim\operatorname{Ker}\overline{\gamma}_1(\xi)=15$ and $\dim\operatorname{Ker}\overline{\beta}_2(\xi)=7$, we have $\dim W\geqslant 6$. On this subspace, the second integrability condition reads $\overline{\beta}_1(\xi)\overline{\gamma}_1(\eta)+\overline{\gamma}_2(\xi)\overline{\beta}_2(\eta)=0$ for all η. Therefore $\overline{\gamma}_2(\xi)\overline{\beta}_2(\eta)(W)\subset\overline{E}_1$ is 1-dimensional. Because of our assumption on the rank of $\overline{\gamma}_2(\xi)$, $\overline{\beta}_2(\eta)(W)$ is of dimension at most 2, and this implies that $\dim W\cap\operatorname{Ker}\overline{\beta}_2(\eta)\geqslant 4$, hence that $\dim\operatorname{Ker}\overline{\beta}_2(\xi)\cap\operatorname{Ker}\overline{\beta}_2(\eta)\geqslant 4$. Since $n\geqslant 2$, we may choose ξ and η to be linearly independent and we get a contradiction with Corollary 5.6 (c).

Suppose now that for all $\xi\neq 0$, $\overline{\gamma}_2(\xi)$ has rank 5. Fix $\xi\neq 0$, and let $[\xi]$ be the class of ξ in the fiber of $\mathbb{P}T_X$ above x. Let $V(\xi)=V_0(\xi)\oplus V_1(\xi)\oplus V_2(\xi)$ be the fiber above $[\xi]$ of the subbundle V of the Higgs bundle (E,θ) on $\mathbb{P}T_X$. We have

$$\begin{cases}V_0(\xi)=E_0=\overline{E}_0\\ V_1(\xi)=E_1\cap F_0=E_1\cap(\operatorname{Ker}\beta_{[\xi]}\oplus\operatorname{Ker}\beta^2_{[\xi]}\cap\operatorname{Im}\beta_{[\xi]})=\operatorname{Ker}\overline{\beta}_2(\xi)\oplus\operatorname{Im}\overline{\beta}_1(\xi)\\ V_2(\xi)=E_2\cap F_{-2}=E_2\cap(\operatorname{Ker}\beta_{[\xi]}\cap\operatorname{Im}\beta^2_{[\xi]})=\operatorname{Im}\overline{\beta}_2(\xi)\overline{\beta}_1(\xi)\end{cases}$$

and we know by Proposition 4.7 (1) that the orthogonal complement $V_1(\xi)^\perp\oplus V_2(\xi)^\perp$ of $V_0(\xi)\oplus V_1(\xi)\oplus V_2(\xi)$ is invariant by $\overline{\theta}(\xi)$, in particular that $\overline{\gamma}_2(\xi)$ maps $V_2(\xi)^\perp$ to $V_1(\xi)^\perp$. Here and after, for $i\in\{1,2\}$, we denote by $V_i(\xi)^\perp$ the orthogonal complement of $V_i(\xi)$ in E_i with respect to the harmonic metric.

By the third integrability condition, $\overline{\gamma}_2(\xi)$ maps $\operatorname{Ker}\overline{\gamma}_2(\eta)$ in $\operatorname{Ker}\overline{\beta}_2(\eta)$. Hence $\overline{\gamma}_2(\xi)$ maps $V_2(\xi)^\perp\cap\operatorname{Ker}\overline{\gamma}_2(\eta)$ to $\operatorname{Ker}\overline{\beta}_2(\eta)\cap V_1(\xi)^\perp$.

But $\overline{\beta}_2(\xi)$ is injective on $V_1(\xi)^\perp$ because $\operatorname{Ker}\overline{\beta}_2(\xi)\subset V_1(\xi)$. Hence for η close to ξ, $\overline{\beta}_2(\eta)$ is also injective on $V_1(\xi)^\perp$, so that $\operatorname{Ker}\overline{\beta}_2(\eta)\cap V_1(\xi)^\perp=0$ and hence $V_2(\xi)^\perp\cap\operatorname{Ker}\overline{\gamma}_2(\eta)\subset\operatorname{Ker}\overline{\gamma}_2(\xi)$ by the previous paragraph. Now, $\dim V_2(\xi)^\perp=9$

and $\operatorname{rk}\overline{\gamma}_2(\eta) = 5$, thus $V_2(\xi)^\perp \cap \operatorname{Ker}\overline{\gamma}_2(\eta)$ is at least 4-dimensional, and so is $\operatorname{Ker}\overline{\gamma}_2(\xi) \cap \operatorname{Ker}\overline{\gamma}_2(\eta)$. This implies by Corollary 5.6 (c') that $\overline{\gamma}_2(\xi)$ and $\overline{\gamma}_2(\eta)$ are colinear, a contradiction since $n \geqslant 2$ and $\overline{\gamma}_2$ is injective by our assumption that all the $\overline{\gamma}_2(\zeta)$, $\zeta \neq 0$, have rank 5.

We conclude that there exists $\xi \neq 0$ such that $\overline{\gamma}_2(\xi) = 0$. Then also $\overline{\gamma}_1(\xi) = 0$ and by the second integrability condition, for all η, $\overline{\beta}_1(\xi)\overline{\gamma}_1(\eta) = \overline{\gamma}_2(\eta)\overline{\beta}_2(\xi)$. Therefore $\overline{\gamma}_2(\eta)$ has rank at most 1 on $\operatorname{Im}\overline{\beta}_2(\xi)$ which is 9-dimensional in $\overline{E}_2$, so that $\overline{\gamma}_2(\eta)$ has rank at most 2, hence vanishes. Therefore $\overline{\gamma}_2 = 0$ and $\overline{\gamma} = 0$ identically on $T_{X,x}$. $\square$

Theorem 5.8. — *Let Γ be a torsion free uniform lattice in $\mathrm{SU}(1,n)$ with $n \geqslant 2$ and ρ be a reductive maximal representation of Γ in $\mathrm{E}_{6(-14)}$. Then there exists a holomorphic or antiholomorphic ρ-equivariant map from $\mathbb{H}^n_{\mathbb{C}}$ to the symmetric space M associated with $\mathrm{E}_{6(-14)}$.*

Proof. — As explained at the beginning of Section 4, we may and do assume that $\tau(\rho) > 0$. By Proposition 5.7, $\overline{\gamma}$ vanishes on the regular locus $\mathscr{R}(\overline{\beta})$ of $\overline{\beta}$. By Proposition 4.7 (2), $\mathscr{R}(\overline{\beta})$ is dense in X, so that $\overline{\gamma}$ vanishes identically on X. This means that the ρ-equivariant harmonic map $f : \mathbb{H}^n_{\mathbb{C}} \to M$ used to define the Higgs bundle $(\overline{E}, \overline{\theta})$ is holomorphic. $\square$

5.3. Proof of the main results. — In this subsection, we give detailed proofs of Theorem A and Corollary B stated in the introduction, although some of the arguments might be well-known.

So let $n \geqslant 2$, $\Gamma \subset \mathrm{SU}(1,n)$ be a uniform lattice and ρ a maximal representation of Γ in a real algebraic Hermitian Lie group $G_{\mathbb{R}}$. Recall that $G_{\mathbb{R}}$ is semisimple with no compact factors and that $G_{\mathbb{R}}$ is the connected component $\boldsymbol{G}(\mathbb{R})^\circ$ of the group of real point of a real algebraic group $\boldsymbol{G}$.

At first we will assume that Γ is torsion free and, as before, that $G = \boldsymbol{G}(\mathbb{C})$ is simply connected. Remarks 5.14 and 5.15 explain how to deal with non simply connected G's and Remark 5.16 with the presence of torsion.

We may moreover assume that $G_{\mathbb{R}}$ is simple, because a representation in a product of simple groups is maximal if and only if the induced representations in the simple factors are maximal.

Furthermore, we may and will assume that ρ is reductive, since by [BIW09, Cor. 4], maximal representations are always reductive. Alternatively, one may proceed as in the following, but first with reductive maximal representations and prove Theorem A and Corollary B for them. Once the compactness of the centralizer $Z_{\mathbb{R}}$ for reductive maximal representations has been proved, one can prove that all maximal representations are reductive as in [KM17, §4.5].

Finally, as explained at the beginning of Section 4, we may and do assume that $\tau(\rho) > 0$.

Let then $f : \mathbb{H}^n_{\mathbb{C}} \to M = G_{\mathbb{R}}/K_{\mathbb{R}}$ be a harmonic ρ-equivariant map (such a map exists by [Cor88]). By Proposition 5.1, $G_{\mathbb{R}}$ is not of tube type, hence $G_{\mathbb{R}}$ is isogenous to either $\mathrm{SU}(p,q)$ with $p < q$, $\mathrm{SO}^\star(2m)$ with $m \geqslant 5$ odd, or $\mathrm{E}_{6(-14)}$ (whose real ranks are respectively p, $(m-1)/2$, and 2). Then [KM17, Th. 4.1 & Prop. 5.6] and Theorem 5.8 imply that f is holomorphic. By the Ahlfors-Schwarz lemma (cf. [Roy80]), since the holomorphic sectional curvature is -1 on $\mathbb{H}^n_{\mathbb{C}}$ and bounded above by $-1/r_M$ on M, we have the pointwise inequality $f^\star\omega_M \leqslant r_M\,\omega$. The maximality of ρ then implies that $f^\star\omega_M = r_M\,\omega$. Since there is equality in the Ahlfors-Schwarz lemma, f is totally geodesic (see e.g. [Roy80]).

Our results so far and [KM17, Prop. 4.10 & 5.7] also imply that $G_{\mathbb{R}}$ is either $\mathrm{SU}(p,q)$ with $q \geqslant pn$ (in which maximal representations exist indeed) or $\mathrm{E}_{6(-14)}$, in particular $G_{\mathbb{R}}$ is not isogenous to $\mathrm{SO}^\star(2m)$ with m odd. In the $\mathrm{SU}(p,q)$ case, we also know from [KM08, §3.1] (see also [KM17, Prop. 4.10]) that a holomorphic map $f : \mathbb{H}^n_{\mathbb{C}} \to M$ such that $f^\star\omega_M = r_M\,\omega$ is unique up to composition by an element of $\mathrm{SU}(p,q)$ and equivariant w.r.t a morphism $\varphi : \mathrm{SU}(1,n) \to \mathrm{SU}(p,q)$.

We need another argument to handle the $\mathrm{E}_{6(-14)}$ case. Following [BIW09, p. 690] and [Ham13, p.640], a totally geodesic map $g : \mathbb{H}^n_{\mathbb{C}} \to M$ is called *tight* if $\sup_{\Delta\subset\mathbb{H}^n_{\mathbb{C}}} \int_\Delta g^*\omega_M = \sup_{\Delta_M\subset M} \int_{\Delta_M} \omega_M$, where Δ, resp. Δ_M, denote a geodesic triangle in $\mathbb{H}^n_{\mathbb{C}}$, resp. M (recall that ω_M is the Kähler form on M). It follows from the fact that our map f is holomorphic and such that $f^\star\omega_M = r_M\,\omega$ that it is tight. Tight holomorphic maps between Hermitian symmetric spaces were classified in [Ham13] using [Iha67]. As we assumed that $G_{\mathbb{R}}$ is simple, M is irreducible. In this case, and since $n \geqslant 2$, tight holomorphic totally geodesic maps $\mathbb{H}^n_{\mathbb{C}} \to M$ exist if and only $G_{\mathbb{R}} = \mathrm{SU}(p,q)$ with $q \geqslant pn$ or $G_{\mathbb{R}} = \mathrm{E}_{6(-14)}$ if $n=2$. Moreover if $G_{\mathbb{R}} = \mathrm{E}_{6(-14)}$ they are also deduced one from another by composition by an element of $G_{\mathbb{R}}$.

Remark 5.9. — There is a small inaccuracy in [Ham13], where it is said that there are two "tight regular" (in the terminology of this paper) maximal noncompact subalgebras of $\mathfrak{e}_{6(-14)}$ in which $\mathfrak{su}(1,2)$ can be tightly embedded. In fact, only $\mathfrak{su}(2,4) \subset \mathfrak{e}_{6(-14)}$ is a maximal subalgebra for these properties. This was confirmed to us by the author.

We assume that $G_{\mathbb{R}} \simeq \mathrm{E}_{6(-14)}$ and we now show that there is a morphism of groups $\varphi : \mathrm{SU}(1,2) \to G_{\mathbb{R}}$ such that f is φ-equivariant. The application f is given by a morphism of Lie algebras $\mathfrak{su}(1,2) \to \mathfrak{su}(2,4) \to \mathfrak{e}_{6(-14)}$. As explained in [Iha67, §4.5], the inclusion $\mathfrak{su}(2,4) \to \mathfrak{e}_{6(-14)}$ is the inclusion of the Lie algebra $\mathfrak{u}$ whose root system is generated by $\beta_1, \beta_3, \beta_4, \beta_2, \gamma$, where the β_i's are the simple roots (we use Bourbaki's notations for the simple roots, whereas Ihara has different notations) and $\gamma = \beta_2 + \beta_3 + 2\beta_4 + 2\beta_5 + \beta_6$.

To show the existence of our morphism φ, it is enough to show that the subgroup $U_{\mathbb{R}} \subset G_{\mathbb{R}}$ whose Lie algebra is $\mathfrak{u}$ is isomorphic to $\mathrm{SU}(2,4)$. Since, by [Iha67, §4.5], $\mathfrak{u} \simeq \mathfrak{su}(2,4)$, it is enough to show that the complexification U of $U_{\mathbb{R}}$ is simply connected. To prove this, we show that for $T_U \subset U$ a maximal torus, the characters of T_U are all the weights of the root system of U.

Since G is simply connected, all the fundamental weights of its root system define characters of T, and those characters restrict to some characters of T_U. It is thus enough to show that the fundamental weights for U can be expressed as linear combinations of the restrictions of the fundamental weights for G. The fundamental weights corresponding to the roots $\beta_1, \beta_3, \beta_4, \beta_2, \gamma$ are the restriction of the weights $\varpi_1, \varpi_3 - 2\varpi_6, \varpi_4 - 3\varpi_6, \varpi_2 - 2\varpi_6, \varpi_6$ respectively. This proves that $U_{\mathbb{R}} \simeq \mathrm{SU}(2,4)$.

This proves all the assertions of Theorem A except the uniqueness of the harmonic map $\mathbb{H}^n_{\mathbb{C}} \to M$ that is ρ-equivariant. To prove it, we need to have a closer look at f. We know that f is equivariant with respect to a morphism of Lie groups $\varphi : \mathrm{SU}(1,n) \to G_{\mathbb{R}}$ and that up to conjugacy of ρ, we may assume that f and φ are as follows:

– for $G_{\mathbb{R}} = \mathrm{SU}(p,q)$ with $q \geqslant pn$, by [KM08, §3.1] (see also [KM17, Prop. 4.10], or [Ham13]), φ is the composition

$$\mathrm{SU}(1,n) \xhookrightarrow{\text{diag}} \mathrm{SU}(1,n)^p \hookrightarrow \mathrm{SU}(p,pn) \hookrightarrow \mathrm{SU}(p,q)\,;$$

– for $G_{\mathbb{R}} = \mathrm{E}_{6(-14)}$ and $n = 2$, by [Ham13] and the discussion above, φ is the composition

$$\mathrm{SU}(1,2) \xhookrightarrow{\text{diag}} \mathrm{SU}(1,2)^2 \hookrightarrow \mathrm{SU}(2,4) \hookrightarrow \mathrm{E}_{6(-14)}.$$

In both cases, the image N of f in $M = G_{\mathbb{R}}/K_{\mathbb{R}}$ is the orbit of $o = K_{\mathbb{R}}$ under $H_{\mathbb{R}} := \varphi(\mathrm{SU}(1,n)) \subset G_{\mathbb{R}}$.

We now describe the centralizer $Z_{\mathbb{R}}$ of $H_{\mathbb{R}}$ in $G_{\mathbb{R}}$. In case $G_{\mathbb{R}} = \mathrm{SU}(p,q)$, let $C_{\mathbb{R}}$ denote the group $\mathrm{U}(p) \times \mathrm{U}(q-pn)$ and let $\chi : C_{\mathbb{R}} \to \mathrm{U}(1)$ be the character defined by $\chi(x,y) = \det(x)^{n+1} \cdot \det(y)$. In case $G_{\mathbb{R}} = \mathrm{E}_{6(-14)}$, let $C_{\mathbb{R}} = \mathrm{U}(2) \times \mathrm{U}(2)$ and let $\chi : C_{\mathbb{R}} \to \mathrm{U}(1)$ be the character defined by $\chi(x,y) = \det(x)^{21} \cdot \det(y)^6$. Then:

Lemma 5.10. — *The centralizer $Z_{\mathbb{R}}$ of $H_{\mathbb{R}}$ in $G_{\mathbb{R}}$ is a subgroup of $K_{\mathbb{R}}$ (hence it is compact). It is isomorphic to the kernel of χ in $C_{\mathbb{R}}$.*

Proof. — In the case of $\mathrm{SU}(p,q)$, the description of φ given above shows that the standard representation $\mathbb{C}^{p+q}$ of $\mathrm{SU}(p,q)$, when seen as a representation of $\mathrm{SU}(1,n)$ via φ, splits as

$$\mathbb{C}^{p+q} = \mathbb{C}^{p+pn} \oplus \mathbb{C}^r = \mathbb{C}^{1+n} \otimes \mathbb{C}^p \oplus \mathbb{C}^r,$$

where $\mathbb{C}^{1+n}$ is the standard representation of $\mathrm{SU}(1,n)$, $r = q - pn$, and $\mathbb{C}^p$ and $\mathbb{C}^r$ are trivial representations of $\mathrm{SU}(1,n)$. To conclude, we argue as follows. Let $g \in Z_{\mathbb{R}}$. Then g yields an endomorphism of the $H_{\mathbb{R}}$-module $\mathbb{C}^{1+n} \otimes \mathbb{C}^p \oplus \mathbb{C}^r$. Since by Schur's lemma such an endomorphism will preserve isotypic factors, we see that g must preserve the factors $\mathbb{C}^{1+n} \otimes \mathbb{C}^p$ and $\mathbb{C}^r$. Moreover its restriction to $\mathbb{C}^{1+n} \otimes \mathbb{C}^p$ belongs to $\mathrm{U}(p,pn)$ and so by Schur lemma again it must act by an element of $\mathrm{U}(p)$, so that it belongs to $C_{\mathbb{R}}$. Now, the determinant of an element $(x,y) \in \mathrm{U}(p) \times \mathrm{U}(q-pn)$ considered as an element in $\mathrm{U}(\mathbb{C}^{1+n} \otimes \mathbb{C}^p \oplus \mathbb{C}^r)$ is $\det(x)^{n+1} \cdot \det(y)$, which implies that $\chi(x,y) = 1$.

In the case of $\mathrm{E}_{6(-14)}$, we use a model given by Manivel in [Man06, Ex. 3 p. 464] of the 27-dimensional representation $\mathbb{E}$. There is a subgroup in $\mathrm{E}_{6(-14)}$ isomorphic to $\mathrm{SU}(2,4) \times \mathrm{SU}(2)$ and $\mathbb{E}$ splits as $\wedge^2\mathbb{U} \oplus \mathbb{U} \otimes \mathbb{A}$, where $\mathbb{U}$ (resp. $\mathbb{A}$) is the natural representations of $\mathrm{SU}(2,4)$ (resp. $\mathrm{SU}(2)$) of complex dimension 6 (resp. 2). Here we restrict further to $\mathrm{SU}(1,2)\times\mathrm{SU}(2)$, where the first factor $\mathrm{SU}(1,2)$ is diagonally embedded in $\mathrm{SU}(2,4)$, meaning that the representation $\mathbb{U}$ splits as $\mathbb{V}\otimes\mathbb{B}$, with $\mathbb{V}$ the natural 3-dimensional representation of $\mathrm{SU}(1,2)$ and $\mathbb{B}$ the trivial 2-dimensional representation. We get

$$\mathbb{E}\simeq\wedge^2(\mathbb{V}\otimes\mathbb{B})\oplus\mathbb{V}\otimes\mathbb{A}\otimes\mathbb{B}\simeq\wedge^2\mathbb{V}\otimes S^2\mathbb{B}\oplus S^2\mathbb{V}\otimes\wedge^2\mathbb{B}\oplus\mathbb{V}\otimes\mathbb{A}\otimes\mathbb{B}.$$

As in the case of $\mathrm{SU}(p,q)$, an element g in the centralizer of $H_{\mathbb{R}}$ will yield an $H_{\mathbb{R}}$-equivariant endomorphism, and will preserve each of these factors, so that it belongs to $\mathrm{GL}(\mathbb{A})\times\mathrm{GL}(\mathbb{B})$. Since it is an element of the group $E_{6(-14)}$, one sees that it must be given by an element in $\mathrm{U}(\mathbb{A})\times\mathrm{U}(\mathbb{B})$.

The computation of the character χ is done as follows. If $f=(x,y)\in\mathrm{U}(\mathbb{A})\times\mathrm{U}(\mathbb{B})$, then the determinant of the action of f on $\mathbb{E}$ is the product of the determinants of the actions of f on $\wedge^2(\mathbb{V}\otimes\mathbb{B})$ and on $\mathbb{V}\otimes\mathbb{A}\otimes\mathbb{B}$. The action on $\mathbb{V}\otimes\mathbb{A}\otimes\mathbb{B}$ has determinant $\det(x)^6\det(y)^6$, and the action on $\wedge^2(\mathbb{V}\otimes\mathbb{B})$ has determinant $\det(y)^{15}$. □

Remark 5.11. — The compactness of $Z_{\mathbb{R}}$ is proved in greater generality in [BIW09, Cor. 4].

Lemma 5.12. — *The pointwise stabilizer of $N=f(\mathbb{H}^n_{\mathbb{C}})$ in $G_{\mathbb{R}}$ is exactly $Z_{\mathbb{R}}$. The stabilizer of N in $G_{\mathbb{R}}$ is the almost direct product $H_{\mathbb{R}}\cdot Z_{\mathbb{R}}$.*

Proof. — Let $o=K_{\mathbb{R}}\in N$ be the base point of M. Let us denote by $\mathrm{Fix}(N)\subset G_{\mathbb{R}}$ the subgroup of elements which fix all the elements in N. We want to prove that $\mathrm{Fix}(N)=Z_{\mathbb{R}}$. We have an inclusion $Z_{\mathbb{R}}\subset\mathrm{Fix}(N)$. Indeed, if $h\in H_{\mathbb{R}}$ and $z\in Z_{\mathbb{R}}$, then $z\cdot o=o$ since $Z_{\mathbb{R}}\subset K_{\mathbb{R}}$. Thus, since z and h commute, $z\cdot(h\cdot o)=h\cdot(z\cdot o)=h\cdot o$.

The subgroup $H_{\mathbb{R}}$ may be defined referring only to N as follows. Let $\mathfrak{g}_{\mathbb{R}}=\mathfrak{k}_{\mathbb{R}}\oplus\mathfrak{m}_{\mathbb{R}}$ be the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$. The tangent space T_oN identifies with a subspace of $\mathfrak{m}_{\mathbb{R}}$ that we denote by $\mathfrak{n}_{\mathbb{R}}$. The space $\mathfrak{n}_{\mathbb{R}}$ defines a Lie triple system, so that $\mathfrak{h}_{\mathbb{R}}:=[\mathfrak{n}_{\mathbb{R}},\mathfrak{n}_{\mathbb{R}}]\oplus\mathfrak{n}_{\mathbb{R}}\subset\mathfrak{g}_{\mathbb{R}}$ is a Lie subalgebra. Then, $H_{\mathbb{R}}$ is the connected Lie group of $G_{\mathbb{R}}$ with Lie algebra $\mathfrak{h}_{\mathbb{R}}$.

For the reverse inclusion we need to prove that $\mathrm{Fix}(N)\subset Z_{\mathbb{R}}$. It follows from the given description of $H_{\mathbb{R}}$ that $H_{\mathbb{R}}$ is normalized by $\mathrm{Fix}(N)$. Let $g\in\mathrm{Fix}(N)$ and $h\in H_{\mathbb{R}}$. Then the commutator $ghg^{-1}h^{-1}$ belongs to $H_{\mathbb{R}}$ and acts trivially on N. Thus, it belongs to the center of $H_{\mathbb{R}}$. Since this center is finite, the connectedness of $H_{\mathbb{R}}$ implies that $ghg^{-1}h^{-1}$ is the neutral element. Therefore g centralizes $H_{\mathbb{R}}$, so it belongs to $Z_{\mathbb{R}}$.

Since the automorphism group of N is isogenous to $H_{\mathbb{R}}$, the second assertion of the Lemma follows from the first. □

Remark 5.13. — Lemmas 5.10 and 5.12 fix a mistake in the description given in [KM08, §3.1.2] of the stabilizer of N in the $\mathrm{SU}(p,q)$ case.

Proof of Corollary B. — The facts that ρ is discrete and faithful and that $\rho(\Gamma)$ acts cocompactly on N follow from the ρ-equivariance of the totally geodesic embedding f.

The reductivity of ρ has been already asserted and the compactness of $Z_{\mathbb{R}}$ was established in Lemma 5.10. Now, given $\gamma \in \Gamma$, the equivariance of f w.r.t. ρ and φ means that $\rho(\gamma)$ and $\varphi(\gamma)$ have the same action on N. We let $\rho_{\mathrm{cpt}}(\gamma) = \rho(\gamma)\varphi(\gamma)^{-1}$. This is an element of the pointwise stabilizer of N, which is equal to the centralizer $Z_{\mathbb{R}}$ of $H_{\mathbb{R}}$ by Lemma 5.12. Since $\varphi(\gamma) \in H_{\mathbb{R}}$ by definition of φ, the elements $\varphi(\gamma)$ and $\rho_{\mathrm{cpt}}(\gamma)$ commute. It follows that $\varphi(\gamma)$ and $\rho(\gamma)$ commute, and that ρ_{cpt} is a morphism of groups. □

Proof of the uniqueness of f. — By the uniqueness statement for tight holomorphic totally geodesic maps $\mathbb{H}^n_{\mathbb{C}} \to M$, we know that if $f' : \mathbb{H}^n_{\mathbb{C}} \to M$ is another ρ-equivariant harmonic map, then there exists $g \in G_{\mathbb{R}}$ such that $f' = g \circ f$. By ρ-equivariance of f and f', we have that

$$\rho(\gamma) \circ g(x) = g \circ \rho(\gamma)(x), \quad \forall\, \gamma \in \Gamma \text{ and } \forall\, x \in N.$$

It follows that $g \cdot N$ is $\rho(\Gamma)$-stable. Thus the map

$$\begin{aligned} d_{g\cdot N} : N &\longrightarrow \mathbb{R} \\ x &\longmapsto \inf_{y \in g\cdot N} d(x,y), \end{aligned}$$

where d denotes the distance in M, is invariant under the cocompact action of $\rho(\Gamma)$ on N. It is therefore bounded. Since it is moreover convex ([BH99, p. 178]), it is constant, equal to a, say. In the same way, the map

$$\begin{aligned} d_N : g\cdot N &\longrightarrow \mathbb{R} \\ x &\longmapsto \inf_{y \in N} d(x,y) \end{aligned}$$

is also constant equal to a.

If $a > 0$ it follows from the sandwich lemma ([BH99, p. 182]) that the convex hull of $N \cup g\cdot N$ in M is isometric to the product $N \times [0,a]$. This implies that there exists a tangent vector $v \in T_oM \simeq \mathfrak{m}_{\mathbb{R}}$, orthogonal to $T_oN \simeq \mathfrak{n}_{\mathbb{R}}$ such that $[v,u] = 0$ for all $u \in \mathfrak{n}_{\mathbb{R}}$. Indeed the norm (for the Killing form) of $[v,u] \in \mathfrak{g}_{\mathbb{R}}$ is up to a constant the sectional curvature of the plane generated by the tangent vectors u and v, which is 0 since they belong to different factors of a Riemannian product. In this case the 1-parameter group of transvections along the geodesic defined by v is included in the centralizer $Z_{\mathbb{R}}$ of $H_{\mathbb{R}}$, a contradiction since $Z_{\mathbb{R}}$ is compact.

Hence $a = 0$ and $g \cdot N = N$. Therefore there exist $h \in H_{\mathbb{R}}$ and $z \in Z_{\mathbb{R}}$ such that $g = hz = zh$. The above commutation relation between $\rho(\gamma) = \varphi(\gamma)\rho_{\mathrm{cpt}}(\gamma)$ and g on N means that $\rho(\gamma)g\rho(\gamma)^{-1}g^{-1}$ fixes N pointwise and hence belongs to $Z_{\mathbb{R}}$ by Lemma 5.12. Hence for all $\gamma \in \Gamma$ we obtain that $\varphi(\gamma)h\varphi(\gamma)^{-1}h^{-1}$ belongs to $Z_{\mathbb{R}} \cap H_{\mathbb{R}}$ (recall that $\rho_{\mathrm{cpt}}(\gamma) \in Z_{\mathbb{R}}$). Now Γ is Zariski dense in $\mathrm{SU}(1,n)$ by the Borel density theorem and we deduce that $\varphi(x)h\varphi(x)^{-1}h^{-1} \in Z_{\mathbb{R}} \cap H_{\mathbb{R}}$ for all $x \in \mathrm{SU}(1,n)$. Since $Z_{\mathbb{R}} \cap H_{\mathbb{R}}$ is finite and $\mathrm{SU}(1,n)$ is connected, $h \in Z_{\mathbb{R}}$. Therefore $g \in Z_{\mathbb{R}}$ and $f' = f$. □

Proof of the uniqueness of φ. — Since f is well-defined, it makes sense to claim that there is a unique morphism $\varphi : \mathrm{SU}(1,n) \to G_{\mathbb{R}}$ such that f is equivariant with respect to φ. We prove that this statement holds: let φ_1, φ_2 be two such morphisms. We get by equivariance, for $x \in \mathbb{H}^n_{\mathbb{C}}$ and $g \in \mathrm{SU}(1,n)$:

$$f(g \cdot x) = \varphi_1(g) \cdot f(x) = \varphi_2(g) \cdot f(x).$$

Thus, setting $\psi(g) = \varphi_2(g)^{-1}\varphi_1(g)$, $\psi(g)$ is in the pointwise stabilizer of N. It follows from Lemma 5.12 that $\psi(g)$ commutes with $\varphi_1(g')$ and $\varphi_2(g')$, for all $g' \in \mathrm{SU}(1,n)$. Thus, $\psi : \mathrm{SU}(1,n) \to G_{\mathbb{R}}$ is a morphism of groups, and $\psi(g_1)\psi(g_2) = \psi(g_2)\psi(g_1)$ for all g_1, g_2 in $\mathrm{SU}(1,n)$. It follows from the fact that $\mathrm{SU}(1,n)$ is simple that ψ is trivial. $\square$

Remark 5.14. — If we drop the assumption that the group of complex points $G = \boldsymbol{G}(\mathbb{C})$ of the algebraic group $\boldsymbol{G}$ is simply connected, then $\mathbb{E}$ might no longer be a representation of G and our constructions cannot be made. However, in this case, letting $\widetilde{G}$ be the simply connected cover of G and $\mathbb{E}$ the cominuscule representation of $\widetilde{G}$ that we have been considering, there is an integer k such that $\mathbb{E}^{\otimes k}$ is a representation of G. The arguments given in the article can be adapted with the representation $\mathbb{E}^{\otimes k}$ instead of $\mathbb{E}$, and the main results (Theorem 4.6, Theorem A, Corollary B and Proposition C) remain true without the simple connectedness assumption.

Proof of Remark 5.14. — We denote by $Z_{\mathtt{max}}$ the maximum value $\langle \Lambda, z\rangle$ for a weight Λ in $X(\mathbb{E}^{\otimes k})$. Since such a weight is of the form $\chi_1 + \cdots + \chi_k$ for some weights $\chi_i \in X(\mathbb{E})$, we have $Z_{\mathtt{max}} = k z_{\mathtt{max}}$. Similarly the highest value of $\langle \Lambda, h\rangle$ for $\Lambda \in X(\mathbb{E}^{\otimes k})$ is kr. We define as in Definition 2.13 and 2.25 the subspaces $\mathbb{E}_i^{\otimes k}$ and $\mathbb{V}_i^{kr}$ by the relations

$$\mathbb{E}_i^{\otimes k} = \bigoplus_{\Lambda : \langle \Lambda, z\rangle = k z_{\mathtt{max}} - 2i} \mathbb{E}_\Lambda^{\otimes k} \quad \text{and} \quad \mathbb{V}_i^{kr} = \bigoplus_{\substack{\Lambda \in X(\mathbb{E}_i^{\otimes k}):\\ \langle \Lambda, h\rangle = kr - 2i}} \mathbb{E}_\chi.$$

We set $\mathbb{V}^{kr} = \bigoplus_i \mathbb{V}_i^{kr}$. If $\Lambda = \chi_1 + \cdots + \chi_k \in X(\mathbb{E}_i^{\otimes k})$, then for all ℓ we have $\chi_\ell \in \mathbb{E}_{j_\ell}$ with integers j_ℓ such that $i = j_1 + \cdots + j_k$. We have the inequality $\langle \chi_\ell, h\rangle \geqslant r - 2j_\ell$ by Proposition 2.26. Thus, if $\mathbb{E}_\Lambda^{\otimes k} \subset \mathbb{V}_i^{kr}$, this implies that for each ℓ, $\mathbb{E}_{\chi_\ell} \subset \mathbb{V}_\ell^r$. We deduce that $\mathbb{V}^{kr} = (\mathbb{V}^r)^{\otimes k}$. It follows that at the level of sheaves, we will be able to define a subsheaf of $\mathscr{E}^{\otimes k}$ by the same trick as in (2.2). Since passing from $\mathbb{E}$ to $\mathbb{E}^{\otimes k}$ and from $\mathbb{V}^r$ to $(\mathbb{V}^r)^{\otimes k}$ just multiplies the slopes by k, we get the same Milnor-Wood inequality. In the case of equality, the arguments of Section 5.2 are still valid and we get that $\gamma = 0$, proving that the map f is holomorphic. $\square$

Remark 5.15. — Alternatively, it is possible to deal with non simply connected groups G as follows. Let $\widetilde{\boldsymbol{G}}$ be the universal cover of $\boldsymbol{G}$, $\widetilde{G}_{\mathbb{R}} = \widetilde{\boldsymbol{G}}(\mathbb{R})^\circ$, and $\widetilde{G} = \widetilde{\boldsymbol{G}}(\mathbb{C})$. Then $\widetilde{G}$ is simply connected ($\widetilde{G}_{\mathbb{R}}$ might not be). By Selberg's lemma, there is a finite index sublattice $\Gamma' \subset \Gamma$ such that the representation $\rho : \Gamma \to G_{\mathbb{R}}$ restricted to Γ' lifts to $\widetilde{G}_{\mathbb{R}}$, see e.g. [KM10, Lem. 2.2]: we have $\rho' : \Gamma' \to \widetilde{G}_{\mathbb{R}}$. Since $\tau(\rho') = |\Gamma/\Gamma'|\,\tau(\rho)$ and $\mathrm{vol}(\Gamma'\backslash\mathbb{H}^n_{\mathbb{C}}) = |\Gamma/\Gamma'|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{C}})$, we may use our results for ρ' and deduce the Milnor-Wood inequality for ρ, which is maximal if and only if ρ' is. Moreover, in the maximal case, we have by [Cor88] a ρ-equivariant harmonic map $f : \mathbb{H}^n_{\mathbb{C}} \to M$ and, by our results, a unique ρ'-equivariant harmonic map $f' : \mathbb{H}^n_{\mathbb{C}} \to M$. Since f is also ρ'-equivariant, by uniqueness, we have $f = f'$ and our results follow.

Remark 5.16. — If Γ has torsion, then by Selberg's Lemma (see [Rat06, p. 327] or [Sel60]) there exists a finite index torsion free normal subgroup Γ' of Γ. Let ρ' denotes the restriction of ρ to Γ'. Since $\tau(\rho') = |\Gamma/\Gamma'|\,\tau(\rho)$ and $\mathrm{vol}(\Gamma'\backslash\mathbb{H}^n_{\mathbb{C}}) = |\Gamma/\Gamma'|\,\mathrm{vol}(\Gamma\backslash\mathbb{H}^n_{\mathbb{C}})$, $\rho : \Gamma \to G_{\mathbb{R}}$ is maximal if and only if $\rho' : \Gamma' \to G_{\mathbb{R}}$ is. Then, in view of the foregoing, Theorem A and Corollary B hold for ρ'. Therefore there exist a unique ρ'-equivariant map $f' : \mathbb{H}^n_{\mathbb{C}} \to M$. Now if $\gamma \in \Gamma$, the map $f'_\gamma := \rho(\gamma)^{-1} \circ f' \circ \gamma$ is also harmonic and the fact that Γ' is normal in Γ implies that it is also ρ'-equivariant. Hence $f' = f'_\gamma$ for all $\gamma \in \Gamma$, i.e., f' is in fact ρ-equivariant and Theorem A and Corollary B also hold for ρ.

References

[AMRT10] A. Ash, D. Mumford, M. Rapoport & Y.-S. Tai – *Smooth compactifications of locally symmetric varieties*, 2nd ed., Cambridge University Press, Cambridge, 2010.

[BGPR17] O. Biquard, O. García-Prada & R. Rubio – "Higgs bundles, the Toledo invariant and the Cayley correspondence", *J. Topology* **10** (2017), no. 3, p. 795–826.

[Bor69] A. Borel – *Linear algebraic groups*, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[Bou68] N. Bourbaki – *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, vol. 1337, Hermann, Paris, 1968.

[BGPG03] S. B. Bradlow, O. Garcia-Prada & P. B. Gothen – "Surface group representations and $\mathrm{U}(p,q)$-Higgs bundles", *J. Differential Geom.* **64** (2003), p. 111–170.

[BGPG06] ______, "Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces", *Geom. Dedicata* **122** (2006), p. 185–213.

[BH99] M. R. Bridson & A. Haefliger – *Metric spaces of non-positive curvature*, Grundlehren Math. Wiss., vol. 319, Springer-Verlag, Berlin, 1999.

[BM15] A. S. Buch & L. C. Mihalcea – "Curve neighborhoods of Schubert varieties", *J. Differential Geom.* **99** (2015), no. 2, p. 255–283.

[BI07] M. Burger & A. Iozzi – "Bounded differential forms, generalized Milnor-Wood inequality and an application to deformation rigidity", *Geom. Dedicata* **125** (2007), p. 1–23.

[BIW09] M. Burger, A. Iozzi & A. Wienhard – "Tight homomorphisms and Hermitian symmetric spaces", *Geom. Funct. Anal.* **19** (2009), no. 3, p. 678–721.

[BIW10] ______, "Surface group representations with maximal Toledo invariant", *Ann. of Math. (2)* **172** (2010), p. 517–566.

[Che97] C. Chevalley – *The algebraic theory of spinors and Clifford algebras*, Collected works, vol. 2, Springer-Verlag, Berlin, 1997.

[Cor88] K. Corlette – "Flat G-bundles with canonical metrics", *J. Differential Geom.* **28** (1988), p. 361–382.

[Del80] P. Deligne – "La conjecture de Weil. II", *Publ. Math. Inst. Hautes Études Sci.* **52** (1980), p. 137–252.

[FH91] W. Fulton & J. Harris – *Representation theory. A first course*, Graduate Texts in Math., vol. 129, Springer-Verlag, New York, 1991.

[Gro94] B. H. Gross – "A remark on tube domains", *Math. Res. Lett.* **1** (1994), no. 1, p. 1–9.

[GW12] O. Guichard & A. Wienhard – "Anosov representations: domains of discontinuity and applications", *Invent. Math.* **190** (2012), p. 357–438.

[Ham13] O. Hamlet – "Tight holomorphic maps, a classification", *J. Lie Theory* **23** (2013), no. 3, p. 639–654.

[HC56] Harish-Chandra – "Representations of semisimple Lie groups. VI. Integrable and square-integrable representations", *Amer. J. Math.* **78** (1956), p. 564–628.

[Hel01] S. Helgason – *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Math., vol. 34, American Mathematical Society, Providence, RI, 2001, Corrected reprint of the 1978 original.

[Her91] L. Hernández – "Maximal representations of surface groups in bounded symmetric domains", *Trans. Amer. Math. Soc.* **324** (1991), p. 405–420.

[Hit87] N. Hitchin – "The self-duality equations on a Riemann surface", *Proc. London Math. Soc.* **55** (1987), p. 59–126.

[Hit92] ———, "Lie groups and Teichmüller space", *Topology* **31** (1992), p. 449–473.

[Hum75] J. E. Humphreys – *Linear algebraic groups*, Graduate Texts in Math., vol. 21, Springer, New York, 1975.

[Igu70] J.-i. Igusa – "A classification of spinors up to dimension twelve", *Amer. J. Math.* **92** (1970), p. 997–1028.

[Iha67] S.-i. Ihara – "Holomorphic imbeddings of symmetric domains", *J. Math. Soc. Japan* **19** (1967), p. 261–302.

[Kna02] A. W. Knapp – *Lie groups beyond an introduction*, 2nd ed., Progress in Math., vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002.

[Kos12] B. Kostant – "The cascade of orthogonal roots and the coadjoint structure of the nilradical of a Borel subgroup of a semisimple Lie group", *Moscow Math. J.* **12** (2012), no. 3, p. 605–620.

[KM08] V. Koziarz & J. Maubon – "Representations of complex hyperbolic lattices into rank 2 classical Lie groups of Hermitian type", *Geom. Dedicata* **137** (2008), p. 85–111.

[KM10] ———, "The Toledo invariant on smooth varieties of general type", *J. reine angew. Math.* **649** (2010), p. 207–230.

[KM17] ———, "Maximal representations of uniform complex hyperbolic lattices", *Ann. of Math. (2)* **185** (2017), p. 493–540.

[Man06] L. Manivel – "Configurations of lines and models of Lie algebras", *J. Algebra* **304** (2006), no. 1, p. 457–486.

[MX02] E. Markman & E. Z. Xia – "The moduli of flat $\mathrm{PU}(p,p)$-structures with large Toledo invariants", *Math. Z.* **240** (2002), p. 95–109.

[McG02] W. M. McGovern – "The adjoint representation and the adjoint action", in *Algebraic quotients. Torus actions and cohomology. The adjoint representation and the adjoint action*, Encyclopaedia Math. Sci., vol. 131, Springer, Berlin, 2002, p. 159–238.

[Mok89] N. Mok – *Metric rigidity theorems on Hermitian locally symmetric manifolds*, Series in Pure Mathematics, vol. 6, World Scientific, Teaneck, NJ, 1989.

[Mur59] S. Murakami – "Sur certains espaces fibrés principaux différentiables et holomorphes", *Nagoya Math. J.* **15** (1959), p. 171–199.

[NT76] H. Nakagawa & R. Takagi – "On locally symmetric Kaehler submanifolds in a complex projective space", *J. Math. Soc. Japan* **28** (1976), no. 4, p. 638–667.

[PS69] I. Piatetski-Shapiro – *Automorphic functions and the geometry of classical domains*, Mathematics and its applications, vol. 8, Gordon and Breach Science Publishers, New York-London-Paris, 1969.

[Rat06] J. Ratcliffe – *Foundations of hyperbolic manifolds*, Springer, New York, 2006.

[RRS92] R. Richardson, G. Röhrle & R. Steinberg – "Parabolic subgroups with abelian unipotent radical", *Invent. Math.* **110** (1992), no. 3, p. 649–671.

[Roy80] H. L. Royden – "The Ahlfors-Schwarz lemma in several complex variables", *Comment. Math. Helv.* **55** (1980), no. 4, p. 547–558.

[Sam78] J. H. Sampson – "Some properties and applications of harmonic mappings", *Ann. Sci. École Norm. Sup. (4)* **11** (1978), p. 211–228.

[Sat80] I. Satake – *Algebraic structures of symmetric domains*, Kanô Memorial Lectures, vol. 4, Princeton University Press, Princeton, NJ, 1980.

[Sel60] A. Selberg – "On discontinuous groups in higher-dimensional symmetric spaces", in *Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960)*, Tata Institute of Fundamental Research, Bombay, 1960, p. 147–164.

[SZ10] M. Sheng & K. Zuo – "Polarized variation of Hodge structures of Calabi-Yau type and characteristic subvarieties over bounded symmetric domains", *Math. Ann.* **348** (2010), no. 1, p. 211–236.

[Sim88] C. Simpson – "Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization", *J. Amer. Math. Soc.* **1** (1988), p. 867–918.

[Sim92] ———, "Higgs bundles and local systems", *Publ. Math. Inst. Hautes Études Sci.* **75** (1992), p. 5–95.

[Siu80] Y.-T. Siu – "The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds", *Ann. of Math. (2)* **112** (1980), p. 73–111.

[SV00] T. A. Springer & F. D. Veldkamp – *Octonions, Jordan algebras and exceptional groups*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000.

[SZ85] J. Steenbrink & S. Zucker – "Variation of mixed Hodge structure. I", *Invent. Math.* **80** (1985), p. 489–542.

[Wol72] J. A. Wolf – "Fine structure of Hermitian symmetric spaces", in *Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970)*, Pure and App. Math., vol. 8, Dekker, New York, 1972, p. 271–357.

[Xia00] E. Z. Xia – "The moduli of flat PU(2, 1) structures on Riemann surfaces", *Pacific J. Math.* **195** (2000), p. 231–256.

Manuscript received 28th April 2017
accepted 17th April 2019

Pierre-Emmanuel Chaput, Institut Élie Cartan de Lorraine, Université de Lorraine
Site de Nancy, B.P. 70239, F-54506 Vandoeuvre-lès-Nancy Cedex
E-mail : pierre-emmanuel.chaput@univ-lorraine.fr
Url : http://www.iecl.univ-lorraine.fr/~Pierre-Emmanuel.Chaput/

Julien Maubon, Institut Élie Cartan de Lorraine, Université de Lorraine
Site de Nancy, B.P. 70239, F-54506 Vandoeuvre-lès-Nancy Cedex
E-mail : julien.maubon@univ-lorraine.fr