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FROM THE KÄHLER-RICCI FLOW TO MOVING FREE BOUNDARIES AND SHOCKS

by Robert J. Berman & Chinh H. Lu

Abstract. — We show that the twisted Kähler-Ricci flow on a compact Kähler manifold X converges to a flow of moving free boundaries, in a certain scaling limit. This leads to a new phenomenon of singularity formation and topology change which can be seen as a complex generalization of the extensively studied formation of shocks in Hamilton-Jacobi equations and hyperbolic conservation laws (notably, in the adhesion model in cosmology). In particular we show how to recover the Hele-Shaw flow (Laplacian growth) of growing 2D domains from the Ricci flow. As will be explained elsewhere the scaling limit in question arises as the zero-temperature limit of a certain many particle system on X.

Résumé (Du flot de Kähler-Ricci au flot des bords libres mobiles et aux chocs)

Nous montrons que le flot de Kähler-Ricci tordu sur une variété kählérienne compacte X converge vers le flot des bords libres mobiles, dans une certaine limite d'échelle. Ceci conduit à un nouveau phénomène de formation de singularités et de changement topologique qui peut être vu comme une généralisation complexe du phénomène abondamment étudié de la formation de chocs dans la théorie des équations de Jacobi et des lois de conservation hyperboliques (notamment dans le modèle d'adhésion en cosmologie). En particulier, nous montrons comment retrouver le flot de Hele-Shaw (croissance du laplacien) dans des domaines 2D croissants à partir du flot de Ricci. Comme il sera expliqué ailleurs, la limite d'échelle en question apparaît comme la limite à température nulle de certains systèmes multi-particules sur X.

Contents

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1. Introduction

1.1. The twisted Kähler-Ricci flow and its zero-temperature limit. — The Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}\, g(t) \tag{1.1}$$

can be viewed as a diffusion type evolution equation for Riemannian metrics $g(t)$ on a given manifold X. In fact, as described in the introduction of [Ham82], this was one of the original motivations of Hamilton for introducing the flow. The point is that, locally, the principal term of minus the Ricci curvature g of a Riemannian metric is the Laplacian of the tensor g (which ensures the short-time existence of the flow). The factor 2 is just a matter of normalization as it can be altered by rescaling the time parameter by a positive number β. However, this reparametrization symmetry is broken when a source term θ is introduced in the equation:

$$\frac{\partial g(t)}{\partial t} = -\frac{1}{\beta}\,\mathrm{Ric}(g(t)) + \theta, \tag{1.2}$$

where θ is an appropriate symmetric two tensor on X and β thus plays the role of the inverse diffusion constant or equivalently, the inverse *temperature* (according to the "microscopic" Brownian motion interpretation of diffusions). In general terms the main goal of the present paper is to study the corresponding zero-temperature limit $\beta \to \infty$ of the previous equation. An important feature of the ordinary Ricci flow (1.1) is that it will typically become singular in a finite time, but in some situations (for example when X is a three manifold, as in Perelman's solution of the Poincaré conjecture) the flow can be continued on a new manifold obtained by performing a suitable topological surgery of X. In our setting it turns out that a somewhat analogous phenomenon of topology change appears at a finite time T_* in the limit $\beta \to \infty$, even if one assumes the long time existence of the flows for any finite β.

More precisely, following [Cao85, Tsu88, TZ06] we will consider the complex geometric framework where X is a complex manifold, i.e., it is endowed with a complex structure J and the initial metric g_0 is Kähler with respect to J. We will identify symmetric two-tensors and two-forms of type $(1,1)$ on X using J in the usual way - then the Kähler condition just means that the form defined by g_0 is closed. We will also assume that θ is a smooth closed (but not necessarily semi-positive) $(1,1)$-form. Then it is well-known that the corresponding flow $g^{(\beta)}(t)$ emanating from the fixed metric g_0 preserves the Kähler property as long as it exists - it is usually called the *twisted Kähler-Ricci flow* in the literature (and θ is called the *twisting form*); see [Cao85, Tsu88, TZ06, CS16, GZ17] and references therein. For simplicity we will also assume that

$$-\frac{1}{\beta}c_1(X) + [\theta] \geqslant 0 \tag{1.3}$$

as $(1,1)$-cohomology classes (where $c_1(X)$ denotes the first Chern class of X) which ensures that the flows $g^{(\beta)}(t)$ exist for all positive times [TZ06]. Our main result says that $g^{(\beta)}(t)$ admits a unique (singular) limit $g(t)$ as $\beta \to \infty$, where $g(t)$ defines a positive current with L^∞-coefficients:

Theorem 1.1. — *Let X be a compact Kähler manifold endowed with a smooth closed $(1,1)$-form θ satisfying* (1.3). *Then*

$$\lim_{\beta\to\infty} g^{(\beta)}(t) = P(g_0 + t\theta) \ (:= g(t))$$

in the weak topology of currents, where P is a (non-linear) projection operator onto the space of positive currents. Moreover, the metrics $g^{(\beta)}(t)$ are uniformly bounded on any fixed time interval $[0,T]$.

The assumption (1.3) is introduced only for the presentation purpose. In fact, the flow $g^{(\beta)}(t)$ is known to exist in a maximum time interval $[0,T_\beta)$ depending on β and as $\beta \to +\infty$, $T_\beta \to T_\infty > 0$. Then, for each $T < T_\infty$ the flow $g^{(\beta)}(t)$ converges as $\beta \to +\infty$ to $g(t)$ in the weak topology of currents and $g(t)$ has uniformly bounded coefficients for $t \in [0,T]$. In fact we will prove a more general result in Section 2 where the form θ is allowed to depend on the parameter β.

The projection operator P appearing in Theorem 1.1, which associates to a given $(1,1)$-current η on X a positive current, cohomologous to η, is defined as a (quasi) plurisubharmonic envelope on the level of potentials (see Section 2.3.2) and can be viewed as a complex generalization of the convex envelope of a function. Such envelopes play a key role in pluripotential theory (as further discussed in Section 1.4, below). In particular, the previous theorem yields a dynamic PDE construction of the envelopes in question, giving an alternative to previous dynamic constructions appearing in the real convex analytical setting [Ves99, CG12] (see the discussion in Section 3.1).

The point is that the linear curve $g_0 + t\theta$, which coincides with the limiting flow for short times will, unless $\theta \geqslant 0$, leave the space of Kähler forms at the time

$$T_* := \sup\{t \mid g_0 + t\theta \geqslant 0\} \tag{1.4}$$

and hence it cannot be the limit of the metrics $g^{(\beta)}(t)$, even in a weak sense, for $t > T_*$. In particular, this means that around the time T_* the Ricci curvatures $\operatorname{Ric} g^{(\beta)}(t)$ will become unbounded as $\beta \to \infty$. Indeed, if the Ricci term is uniformly bounded in β then one immediately sees that as $\beta \to +\infty$ the limiting flow of (1.2) will be $g(t) = g_0 + t\theta$. In particular, for every $t > 0$, $g_0 + t\theta$ must be a positive $(1,1)$-form contradicting the definition of T_*. Still we will show that the metrics $g^{(\beta)}(t)$ do remain uniformly bounded from above as $\beta \to \infty$. However, unless $\theta > 0$, the limiting L^∞-metrics $g(t)$ will, for $t \geqslant T_*$, degenerate on large portions of X, i.e., the support

$$X(t) := \operatorname{supp}(dV_{g(t)})$$

of the limiting L^∞-volume form $dV_{g(t)}$ is a proper closed subset of X evolving with t. Moreover, on the support $X(t)$ the metrics $g(t)$ do evolve linearly, or more precisely

$$g(t) = g_0 + t\theta \quad \text{on } X(t),$$

in the almost everywhere sense. As a consequence, typically the volume form $dV_{g(t)}$ has a sharp discontinuity over the boundary of $X(t)$ (i.e., the function $g(t)^n/\omega^n$ is typically discontinuous at some point on the boundary of the contact set $X(t)$) showing that the limiting (degenerate) L^∞-metrics $g(t)$ are not continuous and hence $\mathscr{C}^0$-convergence in the previous theorem cannot hold, in general. In the generic case the evolving open sets $\Omega(t) := X \smallsetminus X(t)$, where $dV_{g(t)}$ vanishes identically are increasing and may be characterized as solutions of moving free boundary value problems for the complex Monge-Ampère equation (see Section 2.3.2).

More generally, the weak convergence in Theorem 1.1 will be shown to hold even when θ is a positive closed $(1,1)$-current having continuous potentials. But then the limit $g(t)$ will, in general, not be in L^∞ (unless θ is). Moreover, the support of the corresponding measure $dV_{g(t)}$ may then be a subset of low Hausdorff dimension. For example, in the one dimensional setting appearing in the adhesion model discussed below the conjectures formulated in [SAF92] suggest an explicit formula for the Hausdorff dimension of the support, at any given time t, when θ is taken as a random Gaussian distribution with given scaling exponent (see Section 6.2).

We will pay a particular attention to the special case in Theorem 1.1 where the twisting form θ represents the trivial cohomology class, i.e.,

$$\theta = dd^c f,$$

for a function f on X. The large β-limit of the corresponding twisted Kähler-Ricci flow turns out to be intimately related to various growth processes appearing in mathematical physics (and hence the Kähler-Ricci flow can be used as a new regularization of such processes):

1.2. Hamilton-Jacobi equations, shock propagation and the adhesion model in cosmology. — In the particular case when X is an abelian variety (or more specifically $X = \mathbb{C}^n/(\Lambda + i\mathbb{Z}^n)$, for a lattice Λ in $\mathbb{R}^n$) and the potential f of the twisting form θ is invariant along the imaginary direction, we will show in Theorem 4.5 that the corresponding limiting twisted Kähler-Ricci flow $g(t)$ corresponds, under Legendre transformation in the space variables, to a viscosity solution $u(x,t)$ of the Hamilton-Jacobi equation in $\mathbb{R}^n$ with periodic Hamiltonian f [CL83, BE84, LR86]. Under this correspondence the critical time T_* (formula (1.4)) corresponds to the first moment of shock (caustic) formation in the solution $u_t(x)$, i.e., the time where u_t ceases to be differentiable. From this point of view the moving domains $\Omega(t)$ correspond, under Legendre duality, to the evolving shock hypersurfaces S_t (i.e., the non-differentiability locus of u_t). The evolution and topology change of such shocks plays a prominent role in various areas of mathematical physics (and more generally fit into the general problem of singularity formation in hyperbolic conservation laws [Ser14]). In particular, the evolving shock hypersurface S_t model the concentration of mass density in the cosmological adhesion model describing the formation of large-scale structures during the early expansion of the universe [VDFN94, GMS91, HSvdW14, HvdWV+12]. Our setting contains, in particular, the case when the initial data in the adhesion model is periodic [KPSM92, HSvdW14, HvdWV+12]. It should also be pointed out that in this picture the limit $\beta \to \infty$ can be seen as a non-linear version of the classical vanishing viscosity limit [CL83, BE84, LR86], which has the virtue of preserving convexity.

We will also study the corresponding large time limits and show in Theorem 4.7 that if the set F of absolute minima of the potential f is finite, then the support of the positive current defined by the joint large β and large t limit of the twisted Kähler-Ricci flow is a piecewise affine hypersurface whose vertices coincides with F and whose lift to $\mathbb{R}^n$ gives a Delaunay type tessellation of $\mathbb{R}^n$ (which is consistent with numerical simulations appearing in cosmology [KPSM92, HSvdW14, HvdWV+12]).

1.3. Applications to the Hele-Shaw flow (Laplacian growth). —

In another direction, in Section 5, allowing θ to be a singular current of the form

$$\theta = \omega_0 - [E],$$

where ω_0 is the initial Kähler form and $[E]$ denotes the current of integration along a given effective divisor (i.e., complex hypersurface) in X cohomologous to ω_0, we will show that the corresponding domains $\Omega(t)$, which in this setting are growing continuously with t, give rise to a higher dimensional generalization of the classical Hele-Shaw flow in a two-dimensional geometry. More precisely, the Hele-Shaw flow appears when X is a Riemann surface, ω_0 is normalized to have unit area and E is given by a point p (in the classical setting X is the Riemann sphere and p is the point at infinity; the general Riemann surface case was introduced in [HS02]). Then $\Omega(t)$ coincides, up to a time reparametrization, with the Hele-Shaw flow (also called Laplacian growth) injected at the point p in the medium X with varying permeability (encoded in the form ω_0). The latter flow was originally introduced in fluid mechanics to model the expansion of an incompressible fluid $\Omega(t)$ of high viscosity (for example oil) injected at a constant rate in another fluid of low viscosity (such as water) occupying the decreasing region $X(t)$. In more recent times the Hele-Shaw flow has made its appearance in various areas such as random matrix theory, integrable system and the Quantum Hall Effect [Zab06] to name a few (see [Vas09] for a historical overview). In particular, in the latter setting $X(t)$ represents the electron droplet. Special attention has been payed to an interesting phenomenon of topology change in the flow appearing at the time where $\Omega(t)$ becomes singular (which is different from T_* which in this singular setting vanishes). Various approaches have been proposed to regularize the Hele-Shaw flow in order to handle the singularity formation (see [Vas09, §5.3]). The present realization of the Hele-Shaw flow from the limit of the Kähler metrics $\omega^{(\beta)}(t)$ on $X \smallsetminus \{p\}$ suggest a new type of regularization scheme, for example using the corresponding thick-thin decomposition of X, as in the ordinary Ricci flow (with $X(t)$ and $\Omega(t)$ playing the role of the limiting thick and thin regions, respectively).

In a separate publication deterministic, as well as stochastic gradient flow interpretations of the present results will be explored (see the last section of the first arXiv preprint version of the present paper).

1.4. Further relations to previous results. — There is an extensive and rapidly evolving literature on the Kähler-Ricci flow (and its twisted versions) starting with [Ham82], [Cao85]; see for example [SW12] and references therein. But as far as we know the limit $\beta \to \infty$ (which is equivalent to scaling up the twisting form and rescaling time) has not been studied before. For a finite β there is no major analytical difference between the Kähler-Ricci flow and its twisted version, but in our setting one needs to make sure that the relevant geometric quantities do not blow up with β (for example, as discussed above the Ricci curvature does blow up). For a finite β the surgeries in the Kähler-Ricci flow have been related to the Minimal Model Program in algebraic geometry in [CL12, ST17], where the final complex-geometric surgery produces a minimal model of the original algebraic variety. In Section 3.3 we compare some of our results with the corresponding long time convergence results on the minimal model (which produces canonical metrics of Kähler-Einstein type) [Tsu88, TZ06, ST12]. In the algebro-geometric setting negative twisting currents θ also appear naturally, when X is the resolution of a projective variety with canonical singularities [ST17, EGZ16] (θ is then current of integration along minus the exceptional divisor). Recently, viscosity techniques were introduced in [EGZ16] to produce viscosity solutions for the twisted Kähler-Ricci flow (and in particular its singular variants appearing when θ is singular). But, again, this concerns the case when β is finite.

In the case when (X, ω) is invariant under the action of a suitable torus T (i.e., X is a toric variety or an Abelian variety) the corresponding time dependent convex envelopes (studied in Section 3.1) have recently appeared in [RZ11, RZ17] in a different complex geometric context than the Kähler-Ricci flow, namely in the study of the Cauchy problem for weak geodesic rays in the space of Kähler metrics (see Remark 4.6). Moreover, in [RWN15a, RWN15b, RWN17b] the Hele-Shaw flow and the corresponding phenomenon of topology change was exploited to study the singularities of such weak geodesic rays (and solutions to closely related homogeneous complex Monge-Ampère equations) in the general non-torus invariant setting (see Remark 5.8).

We also recall that envelope type constructions as the one appearing in the definition of the projection operator P play a pivotal role in pluripotential theory and have their origins in the classical work of Siciak and Zakharyuta on polynomial approximations in $\mathbb{C}^n$ (see [GZ05] for the global setting). Moreover, by the results in [BBGZ13] the corresponding measure $(P(\theta))^n$ on X can be characterized as the unique normalized minimizer of the (twisted) pluripotential energy (which generalizes the classical weighted logarithmic energy of a measure in $\mathbb{C}$). The L^∞-regularity of $P(\theta)$ follows from [Ber13], which can be seen as the "static" version of the present paper. See also [Rub14], for relations to time-discretizations of the Ricci flow and the Ricci continuity method.

1.5. Organization. — In Section 2 we state and prove refined versions of Theorem 1.1 (stated above). Then in Section 3 we go on to study the joint large β and large t-limits of the corresponding flows. In particular, a dynamical construction of plurisubharmonic (as well as convex) envelopes is given and a comparison with previous work on canonical metrics in Kähler geometry (concerning finite β) is made. In Sections 4 and 5 the relation to Hamilton-Jacobi equations and Hele-Shaw flows, respectively, is exhibited. The extension to twisting potentials which are merely continuous and the relation to random twistings is discussed in Section 6.2.

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2. The zero-temperature limit of the Kähler-Ricci flow

2.1. Notation and setup. — Let X be an n-dimensional compact complex manifold. We will identify symmetric two-tensors with two-forms of type $(1,1)$ on X using J in the usual way: if g is a symmetric tensor, then the corresponding form $\omega := g(\cdot, J\cdot)$, is said to be *Kähler* if ω is closed and g is strictly positive (i.e., g is a Riemannian metric). We will assume that X is Kähler, i.e., it admits a Kähler metric. On a Kähler manifold the De Rham cohomology class $[\eta] \in H^2(X, \mathbb{R})$ defined by a given closed real two form η of type $(1,1)$ may (by the "$\partial\overline{\partial}$-lemma") be written as

$$[\eta] = \{\eta + dd^c u \mid u \in \mathscr{C}^\infty(X)\}, \quad dd^c := \frac{i}{2\pi}\partial\overline{\partial}. \tag{2.1}$$

In our normalization the Ricci curvature form $\mathrm{Ric}(\omega)$ of a Kähler metric ω on X is defined, locally, by

$$\mathrm{Ric}(\omega) := -dd^c \log \frac{\omega^n}{dV(z)},$$

where z are local holomorphic coordinates on X and $dV(z)$ denotes the corresponding Euclidean volume. The form $\mathrm{Ric}(\omega)$ represents, for any Kähler metric ω, minus the first Chern class $c_1(K_X) \in H^2(X, \mathbb{R})$ of the canonical line bundle $\det(T^*X)$.

2.1.1. *Setup*. — Specifically, our geometric setup is as follows: we assume given a family θ_β of closed real $(1,1)$-forms (the "twisting forms") with the asymptotics

$$\theta_\beta = \theta + o(1),$$

as $\beta \to \infty$ (in L^∞-norm). We will assume that

$$\beta^{-1} c_1(K_X) + [\theta_\beta] \geqslant 0 \tag{2.2}$$

as $(1,1)$-cohomology classes, i.e., there exists a semi-positive form χ_β in the class $c_1(K_X)/\beta + [\theta_\beta]$ (we will fix one such choice for each $\beta > 0$). This assumption ensures that the corresponding twisted Kähler-Ricci flow

$$\frac{\partial \omega(t)}{\partial t} = -\frac{1}{\beta}\,\mathrm{Ric}(\omega(t)) + \theta_\beta, \quad \omega(0) = \omega_0 \tag{2.3}$$

exist for all $t \geqslant 0$ and $\beta > 0$.[1] The extra flexibility offered by β-dependence of θ_β will turn out to be quite useful (for example, taking $\theta_\beta := \theta + \frac{1}{\beta}\operatorname{Ric}(\omega)$ for θ defining a semi-positive cohomology class, ensures that the semi-positivity condition (2.2) holds).

More precisely, we will refer to the flow above as the *non-normalized twisted Kähler-Ricci flow* (or simply the *non-normalized KRF*) to distinguish it from its normalized version:

$$\frac{\partial\omega(t)}{\partial t} = -\frac{1}{\beta}\operatorname{Ric}(\omega(t)) - \omega(t) + \theta_\beta, \quad \omega(0) = \omega_0. \tag{2.4}$$

As is well-known the two flows are equivalent under a scaling combined with a time reparametrization: denoting by $\widetilde{\omega}(s)$ the non-normalized KRF one has

$$\frac{1}{s+1}\widetilde{\omega}(s) = \omega(t), \quad e^t := s+1,$$

(the equivalence follows immediately from the fact that $\operatorname{Ric}(c\omega) = \operatorname{Ric}(\omega)$, for any given positive constant c).

In order to write the flows in terms of Kähler potentials we fix a Kähler metric ω and represent

$$\widetilde{\omega}(s) = (\omega_0 + s\chi_\beta) + dd^c\widetilde{\varphi}(s), \quad \widetilde{\varphi}(0) = 0$$

for the fixed semi-positive form χ_β in $\frac{1}{\beta}c_1(K_X) + [\theta_\beta]$ (the first term ensures that the equation holds on the level of cohomology). Then the non-normalized Kähler-Ricci flow (2.3) is equivalent to the following Monge-Ampère flow:

$$\frac{\partial\widetilde{\varphi}(s)}{\partial s} = \frac{1}{\beta}\log\frac{(\widetilde{\omega}_0 + s\chi_\beta + dd^c\widetilde{\varphi}(s))^n}{\omega^n} + f_\beta, \quad \widetilde{\varphi}(0) = 0$$

for the smooth function $\widetilde{\varphi}(s)$, which is a Kähler potential of $\widetilde{\omega}(s)$ with respect to the Kähler reference metric $\omega_0 + s\chi_\beta$, and where f_β is uniquely determined by the equation

$$\theta_\beta - \frac{1}{\beta}\operatorname{Ric}(\omega) = dd^c f_\beta + \chi_\beta \tag{2.5}$$

together with the normalization condition

$$\inf_X f_\beta = 0.$$

We will also assume that χ_β is uniformly bounded from above, i.e.,

$$\chi_\beta \leqslant C_0\omega, \tag{2.6}$$

for some constant C_0, and hence χ_β converge smoothly to $\chi \in [\theta]$ as $\beta \to +\infty$ [2]. Accordingly, f_β converge smoothly to f uniquely determined by $\chi + dd^c f = \theta$ and

[1] In fact, it is enough to assume that $c_1(K_X)/\beta + [\theta_\beta]$ is nef (i.e., a limit of positive classes), which is equivalent to the long time existence of the corresponding KRF [TZ06]. Indeed, the estimates we get will be independent of the choice of reference form χ and hence the nef case can be reduced to the semi-positive case by perturbation of the class $[\theta]$.

[2] The convergence result still holds without an upper bound on χ_β (which does not hold when θ is nef but not semi-positive), but the dependence on t in the estimates will be worse.

$\inf_X f = 0$. Since ω_0 is smooth, up to enlarging C_0 we can also assume that

$$C_0^{-1}\omega \leqslant \omega_0 \leqslant C_0\omega. \tag{2.7}$$

Finally, even when the functions f_β are not uniformly bounded, Lemma 2.11 ensures that the envelope $P_{\widehat{\omega}_t}(f_\beta)$ stays bounded from above if the functions f_β do not go uniformly to $+\infty$. After enlarging C_0 one more time we can assume that

$$P_{\widehat{\omega}_t}(f_\beta) \leqslant C_0, \quad \forall \beta > 0,\ \forall t \geqslant 0. \tag{2.8}$$

Similarly, the normalized KRF is equivalent to the Monge-Ampère flow

$$\frac{\partial \varphi(t)}{\partial t} = \frac{1}{\beta} \log \frac{(\widehat{\omega}_t + dd^c\varphi(t))^n}{\omega^n} - \varphi(t) + f_\beta,$$

where

$$\omega(t) = \widehat{\omega}_t + dd^c\varphi(t), \qquad \widehat{\omega}_t := e^{-t}\omega_0 + (1-e^{-t})\chi_\beta.$$

The corresponding scalings are now given by

$$\widetilde{\varphi}(s) = e^t\left(\varphi(t) + c_\beta(t)\right), \qquad c_\beta(t) = \frac{n}{\beta}(t - 1 + e^{-t})$$

(abusing notation slightly we will occasionally also write $\omega_t = \omega(t) = \omega_{\varphi_t}$).

Remark 2.1. — Under the scaling above a curve $\widetilde{\varphi}(s)$ of the form $\widetilde{\varphi}(s) = \varphi_0 + sf$ corresponds to a curve $\varphi(t)$ of the form $e^{-t}\varphi_0 + (1-e^{-t})f - c_\beta(t)$.

2.2. Statement of the main results.

— In the following section we will prove the following more precise version of Theorem 1.1 stated in the introduction of the paper.

Theorem 2.2. — *Let X be a compact Kähler manifold endowed with a family of twisting form θ_β as above. Denote by $\omega^{(\beta)}(t)$ the flow of Kähler metrics evolving by the (non-normalized) twisted Kähler-Ricci flow (2.3) with parameter β, emanating from a given Kähler metric ω_0 on X. Then*

$$\lim_{\beta\to\infty} \omega^{(\beta)}(t) = P(\omega_0 + t\theta)$$

in the weak topology of currents. On the level of Kähler potentials, for any fixed time-interval $[0,T]$, the functions $\varphi^{(\beta)}(t)$ converge uniformly with respect to β in the $\mathscr{C}^{1,\alpha}(X)$-topology (for any fixed $\alpha < 1$) towards the envelope $P_{\omega_0+t\theta}(0)$. More precisely, fixing a reference Kähler metric ω on X,

$$0 \leqslant \omega^{(\beta)}(t) \leqslant e^{C(1+1/\beta)t\log(t+1)}\omega$$

and

$$-C - n\log(1+t))/\beta \leqslant \frac{\partial \varphi^{(\beta)}(t)}{\partial t} \leqslant C\Big(1 + \frac{1}{\beta}\log(1+t)\Big)/t,$$

where the constant C only depends on θ through the following quantities: $\sup_X \operatorname{Tr}_\omega(\theta)$ *and on the constant C_0 defined in (2.6), (2.7) and (2.8); it also depends on a lower bound on the holomorphic bisectional curvature of the reference Kähler metric ω.*

The proof of Theorem 2.2 will be given in Section 2.4 and Section 2.5. In Section 2.6 we give an alternative proof of the uniform convergence in Theorem 2.2 without using the laplacian bound.

The definition of the non-linear projection operators P and P_{ω_0} will be recalled in Section 2.3.2. The dependence of the constants above on the potential f of θ will be crucial in the singular setting of Hele-Shaw type flows where f blows up on a hypersurface of X, but, for some constant $C' > 0$, $P_{C'\omega}(f)$ is finite (see Section 5).

Under special assumptions on X we get an essentially optimal bound on $\omega^{(\beta)}(t)$:

Theorem 2.3. — *Assume that X admits a Kähler metric ω with non-negative holomorphic bisectional curvature. Then the following more precise estimates hold*

$$\mathrm{Tr}_\omega(\omega^{(\beta)}(t)) \leqslant (t+1)\max\bigl\{\sup_X \mathrm{Tr}_\omega(\omega_0), \sup_X \mathrm{Tr}_\omega(\theta_\beta)\bigr\}, \quad \forall t \geqslant 0.$$

Moreover, for any Riemann surface (i.e., $n=1$) the previous estimate holds without any conditions on the Kähler metric ω.

The proof of Theorem 2.3 will be given in Section 2.4.5 (see Proposition 2.14). In particular, letting $\beta \to \infty$ gives that

$$\mathrm{Tr}_\omega(P(\omega_0 + t\theta)) \leqslant (t+1)\max\{\sup_X \mathrm{Tr}_\omega(\omega_0), \sup_X \mathrm{Tr}_\omega(\theta)\},$$

which is also a consequence of the estimates in the "static" situation considered in [Ber13] (of course, in the case when θ is semi-positive the latter bound follows directly from the equality $P(\omega_0 + t\theta) = \omega_0 + t\theta$).

However, it should be stressed that, in general, it is not possible to bound $\omega^{(\beta)}(t)$ by a factor $C_\beta t$, even for a fixed β (see Proposition3.6). On the other hand, as we show in Section 3.1 this is always possible if $[\theta] = [\omega_0]$ (and in particular positive).

2.3. Preliminaries

2.3.1. *Parabolic comparison/maximum principles.* — We will make repeated use of standard parabolic comparison and maximum principles for smooth sub/super solutions of parabolic problems of the form

$$\frac{\partial u}{\partial t} = \mathscr{D}u$$

for a given differential operator $\mathscr{D}$ acting on $\mathscr{C}^\infty(X)$ (or a subset thereof). We will say that u is a sub (super) solution if $(\partial/\partial t - \mathscr{D})u \leqslant 0$ ($\geqslant 0$).

Proposition 2.4 (Comparison principle). — *Let X be a compact complex manifold and consider a second order differential operator $\mathscr{D}$ on $\mathscr{C}^\infty(X)$ of the form*

$$(\mathscr{D}u)(x) = a(t,x)u(x) + F_t((dd^c u)(x)),$$

where a is a bounded function on $[0,\infty[\times X$ and $F_t(A)$ is a family of increasing functions on the set of all Hermitian matrices. If u and v are smooth sub- and super-solutions, respectively, to the corresponding parabolic problem for $\mathscr{D}$ on $X\times[0,T]$, then $u_0 \leqslant v_0$ implies that $u_t \leqslant v_t$ for all $t \in [0,T]$. In particular, the result applies to *the heat flow of the time-dependent Laplacian Δ_{g_t}, defined with respect to a family of Kähler metrics, and to the twisted KRF (normalized as well as non-normalized).*

Proof. — For completeness (and since we shall need a slight generalization; see Remark 2.7) we recall the simple proof. After replacing u with $e^{At}u$ for A sufficiently large we may as well assume that $a(t,x)<0$. Assume to get a contradiction that it is not the case, i.e., $u_t \leqslant v_t$ does not hold on $[0,T]\times X$. Then, since $u_0\leqslant v_0$, the function u_t-v_t attains its maximum at a point (t_0,x_0) with $t_0>0$. We now carry our computation at this point. By the maximum principle we have

$$u_t(x)-v_t(x)>0,\quad \frac{\partial(u_t-v_t)}{\partial t}(x)\geqslant 0,\quad (\nabla_{g_t}u)(x)=(\nabla_{g_t}v)(x)=0,$$

and $(dd^cu_t)(x)-(dd^cv_t)(x)\leqslant 0$. In particular, since $a(t,x)<0$ and F_t is increasing we have that

$$\frac{\partial(u_t-v_t)}{\partial t}(x)-(\mathscr{D}u-\mathscr{D}v)(x)>0.$$

But this contradicts that u and v are sub/super solutions (since this implies the reverse inequality $\leqslant 0$). $\square$

Remark 2.5. — The condition that X be a complex manifold (and the Kähler condition) have just been included to facilitate the formulation of the proposition. Moreover, exactly the same proof as above shows that any first order term of the $H(t,x,(\nabla u)(x))$ for H smooth can be added to $\mathscr{D}$ above (as in the setting of Hamilton-Jacobi equations considered in Section 4).

Proposition 2.6 (Maximum principle). — *Let X be a compact complex manifold and consider a second order differential operator $\mathscr{D}$ on $\mathscr{C}^\infty(X)$ of the form*

$$(\mathscr{D}u)(x)=F_t((dd^cu)(x)),$$

where $F_t(A)$ is a family of increasing functions on the set of all Hermitian matrices. Given a smooth function $u(x,t)$ on $X\times[0,T]$ we have that

– *The following dichotomy holds: either the maximum of $u(x,t)$ is attained at $X\times\{0\}$ or at a point $x\in X\times\,]0,T]$ satisfying*

$$\Big(\frac{\partial u(x,t)}{\partial t}-\mathscr{D}(u)\Big)\geqslant -F_t(0),$$

– *In particular, if $F_t(0)=0$ for all t and*

$$\Big(\frac{\partial}{\partial t}-\mathscr{D}\Big)\leqslant 0$$

on $X\times[0,T]$, then the maximum of $u(x,t)$ is attained at $X\times\{0\}$.

Proof. — The first property is proved exactly as in the beginning of the proof of the comparison principle. The second point then follows by replacing u with $u-\delta t$ for any number $\delta>0$. $\square$

Remark 2.7. — We will need a slight generalization of the comparison principle to functions $u(x,t)$ which are continuous on $X \times [0,T]$ and such that $u(\cdot,t)$ is smooth on X for any fixed $t>0$ and $u(x,\cdot)$ is quasi-concave on $[0,T]$ for x fixed, i.e., the sum of a concave and a smooth function. Then we simply define $\frac{\partial}{\partial t}u(x,t)$ on $]0,T]$ as the left derivative i.e., $\frac{\partial}{\partial t}u(x,t) := \lim_{h\to 0}(u(x,t+h)-u(x,t))/h$ for $h<0$. In particular, the notion of a subsolution still makes sense for u and the proof of the comparison principle then goes through word for word. This is just a very special case of the general notion of viscosity subsolution [CEL84] which, by definition, means that the parabolic inequality holds with respect to the super second order jet of u (which in our setting is just the ordinary jet in the space−direction and the interval between the right and the left derivative in the time-direction). See [EGZ16] for the complex setting, where very general comparison principles are established for viscosity sub/super solution (which however are not needed for our purposes).

2.3.2. *The projection operator P.* — Let η be a given closed smooth real $(1,1)$-form on X and denote by $[\eta]$ the corresponding De Rham cohomology class of currents which may be represented as in formula (2.1), in terms of functions $u\in L^1(X)$. Under this representation the subspace of all positive currents in $[\eta]$ corresponds to the space of all *η-plurisubharmonic* (η-psh for short) functions u, denoted by $\mathrm{PSH}(X,\eta)$, i.e., u is an upper semi-continuous (usc) function such that

$$\eta_u := \eta + dd^c u \geqslant 0$$

in the sense of currents. We will always assume that $\mathrm{PSH}(X,\eta)$ is non-empty (which, by definition, means that the class $[\eta]$ is *pseudo-effective*). This is the weakest notion of positivity of a class $[\eta]\in H^{1,1}(X,\mathbb{R})$, the strongest being that $[\eta]$ is a *Kähler class* (also called *positive*), which, by definition, means that it contains a Kähler metric.

Given a lower semicontinuous bounded function f one obtains an η-psh function $P_\eta(f)$ as the envelope

$$P_\eta(f)(x) := \sup_{u\in \mathrm{PSH}(X,\eta)} \{u(x) \mid u \leqslant f \text{ on } X\}.$$

The operator P_η is clearly a projection operator in the sense that $P_\eta(u)=u$ if u is in $\mathrm{PSH}(X,\eta)\cap\mathscr{C}^0(X)$. We then define

$$P(\eta) := \eta + dd^c(P_\eta(0)),$$

which thus defines a positive current cohomologous to η. Equivalently, if one fixes another reference form ω in $[\eta]$, i.e.,

$$\eta = \omega + dd^c f$$

for some function f. Then

$$P(\eta) := \omega + dd^c(P_\omega(f)).$$

If the class $[\eta]$ is semi-positive, i.e., $\mathrm{PSH}(X,\eta)\cap\mathscr{C}^\infty(X)$ is non-empty, then it follows immediately from the definition that $P_\eta(f)$ is bounded if f is. However, even if f is smooth $P_\eta(f)$ will in general not be $\mathscr{C}^2$-smooth. On the other hand, by [Ber13] $P_\eta(f)$ is almost $\mathscr{C}^2$-smooth if the class $[\eta]$ is positive:

Proposition 2.8. — *Let ω be a Kähler form and f be a smooth function on X. Then the complex Hessian $dd^c(P_\omega(f))$ is in L^∞. Equivalently, given any smooth form η defining a positive class $[\eta]$ the corresponding positive current $P(\eta)$ in $[\eta]$ is in L^∞. As a consequence,*

$$P(\eta)^n = 1_C\,\eta^n, \tag{2.9}$$

in the point-wise almost everywhere sense, where C is the corresponding (closed) coincidence set:

$$C := \{x \in X \mid P_\eta(0)(x) = 0\}.$$

In fact, we will get a new proof of the previous result using the Kähler-Ricci flow (which can be seen as a dynamic version of the proof in [Ber13]); see Section 3.1.

Very recently the approximation scheme introduced in [Ber13] has been used to improve the regularity of the envelope $P_\eta(f)$ (see [Tos17], [CZ17]).

Remark 2.9. — Setting $u := P_\omega(f)$ and $\Omega := \{P_\omega(f) < f\}$ the previous proposition implies that the pair (u, Ω) can be characterized as the solution to the following free boundary value problem for the complex Monge-Ampère operator with obstacle f, i.e., $u \leqslant f$ on X and

$$(\omega + dd^c u)^n = 0 \text{ in } \Omega \quad u = f,\ du = df \text{ on } \partial\Omega$$

and $\omega + dd^c u \geqslant 0$ on X. In the case when $n = 1$ it is well-known that u is even $\mathscr{C}^{1,1}$-smooth [BK74], but the free boundary $\partial\Omega$ may be extremely irregular and even if ω is real analytic it will, in general, have singularities [Sch77].

A key role in the present paper will be played by parametrized envelopes (where f varies linearly with time).

Lemma 2.10. — *Fix a Kähler metric ω on X. Given smooth functions φ and f on X the function $t \mapsto \varphi(t,x) := P_\omega(\varphi + tf)(x)$ on $\mathbb{R}$ is concave for x fixed. Moreover, locally on $]0,\infty[$ the corresponding curve $\varphi(t)$ can be written as a uniform limit $\varphi_\varepsilon(t)$ of concave curves with values in $\mathrm{PSH}(X,\omega) \cap \mathscr{C}^\infty(X)$. Furthermore, if $\partial\varphi(t)/\partial t \leqslant g$ for a continuous function g (the time derivative here means left derivative which exists since the function is concave) then the family φ_ε can be constructed so that $\partial\varphi_\varepsilon(t)/\partial t \leqslant g$.*

Proof. — It follows immediately from its definition that the projection operator P_ω is concave and in particular locally Lipschitz continuous as a function of t. As for the approximation property it seems likely that it can be deduced in a much more general setting from an appropriate parametrized version of the approximation schemes for ω-psh function introduced by Demailly. But here we note that a direct proof can be given exploiting that $dd^c\varphi(t)$ is in L^∞ and in particular $\varphi(t)$ is in $\mathscr{C}^1(X)$. Indeed, $\varphi_\varepsilon(t)$ can be defined by using local convolutions (which gives local $\mathscr{C}^1$-convergence) together with a partition of unity and finally replacing $\varphi_\varepsilon(t)$ with $(1-\delta_1(\varepsilon))\varphi_\varepsilon(t)-\delta_2(\varepsilon)t$ for appropriate sequence $\delta_i(\varepsilon)$ tending to zero with ε. The point is that, by the $\mathscr{C}^1$-convergence the error terms coming from the first derivatives on the partition of unity are negligible and hence $\varphi_\varepsilon(t)$ is ω-psh up to a term of order $o(\varepsilon)$. Indeed, setting

$$\varphi_\varepsilon(t) := \sum_{i=1}^m \rho_i \varphi_\varepsilon^{(i)}(t), \qquad 1=\sum_{i=1}^m \rho_i, \qquad \rho_i\in\mathscr{C}_c^\infty(X)$$

and using Leibniz rule gives $dd^c\varphi_\varepsilon(t)=\sum_{i=1}^m(\rho_i dd^c\varphi_\varepsilon^{(i)}(t)+R_i(\varphi_\varepsilon^{(i)}))$ where the second term $R(\varphi_\varepsilon)$ only depends on the first order jet of ϕ. Now, by the local $\mathscr{C}^1$-convergence $R(\varphi_\varepsilon)=R(\varphi)+o(\varepsilon)$. But $R(\varphi)$ vanishes (since $\varphi=\rho_1\phi+\ldots$ and $dd^c\varphi=\rho_1 dd^c\varphi+\cdots$) and hence $dd^c\varphi_\varepsilon(t)=\sum_{i=1}^m\rho_i dd^c\varphi_\varepsilon^{(i)}(t)+o(\varepsilon)$. Finally, from the definition of convolution we have $dd^c\varphi_\varepsilon(t)+\omega\geqslant\omega-\omega\star\chi_\varepsilon\geqslant -C\varepsilon\omega$ (where χ_ε is the smoothing kernel) and $\partial\varphi_\varepsilon(t)/\partial t\leqslant g+C\varepsilon$ (for some positive constant C) and hence we may first take $\delta_1(\varepsilon)=C\varepsilon$ and then $\delta_2(\varepsilon)=C\varepsilon(1+\sup|g|)$. □

We will also have use for the following generalized envelope associated to a given compact subset K of a Kähler manifold (X,ω) and a lower semicontinuous function f on X:

$$P_{(K,\omega)}(f)(x) := \sup_{u\in\mathrm{PSH}(X,\omega)}\{u(x)\mid u\leqslant f \text{ on } K\}$$

(the function $V_{K,\omega} := P_{(K,\omega)}(0)$ is called the global extremal function of (K,ω) in [GZ05]).

We recall that a subset K in X is said to be *non-pluripolar* if it is not locally contained in the $-\infty$-set of a local psh function.

Lemma 2.11. — *Suppose that f is a lower semicontinuous function on a compact Kähler manifold (X,ω) taking values in $]0,\infty]$ such that f is bounded from above on K, where K is non-pluripolar. Then the function $P_{(K,\omega)}(f)$ is bounded from above. As a consequence, if $X=K$ and f is locally bounded on the complement of an analytic subvariety, then $P_{(X,\omega)}(f)$ is bounded from above.*

Proof. — By assumption $P_{(K,\omega)}(f)\leqslant P_{(K,\omega)}(0)+C$ for C a sufficiently large constant. But it is well-known that $P_{(K,\omega)}(0)$ is finite if and only if K is non-pluripolar [GZ05, Th. 5.2]. The last statement of the lemma then follows by fixing a coordinate ball B contained in the open subset where f is locally bounded and using that $P_{(X,\omega)}(f)\leqslant P_{(B,\omega)}(f)<\infty$. □

In general, $P_{(K,\omega)}(f)$ is not upper semicontinuous. But we recall that K is said to be *regular* (in the sense of pluripotential theory) if $P_{(K,\omega)}(f)$ is continuous (and hence ω-psh) for any continuous function f (see[BBWN11] and references therein).

2.4. A priori estimates. — The key element in the proof of Theorem 2.2 is the Laplacian estimate which provides a uniform bound on the metrics $\omega^{(\beta)}(t)$ on any fixed time interval. There are various well-known approaches for providing such an estimate for a fixed β, using parabolic versions of the classical estimate of Aubin and Yau and its variants. However, in our setting one has to make sure that all the estimates are uniform in β and that they do not rely on a uniform positive lower bound on $\omega^{(\beta)}(t)$ (which is not available).

2.4.1. *The Laplacian estimate in the one dimensional case*. — We start with the one-dimensional case where the Laplacian estimate becomes particularly explicit:

Proposition 2.12. — *When $n=1$ we have, for the normalized KRF,*

$$\operatorname{Tr}_\omega(\omega^{(\beta)}(t)) \leqslant \max\left\{\sup_X \operatorname{Tr}_\omega(\omega_0), \sup_X \operatorname{Tr}_\omega(\theta_\beta - \beta^{-1}\operatorname{Ric}(\omega))\right\}.$$

For the non-normalized KRF using the change of variables $\widetilde{\omega}(s) = (s+1)\omega(t)$, with $e^t = s+1$ we obtain the estimate

$$\widetilde{\omega}(s) \leqslant (s+1)\max\left\{\sup_X \operatorname{Tr}_\omega(\omega_0), \sup_X \operatorname{Tr}_\omega(\theta_\beta - \beta^{-1}\operatorname{Ric}(\omega))\right\},$$

as stated in Theorem 2.3.

Proof. — We write the normalized KRF as

$$h := \log\frac{\omega_{\varphi_t}}{\omega} = \beta\Big(\varphi_t - f_\beta + \frac{\partial\varphi_t}{\partial t}\Big).$$

Applying the parabolic operator $\frac{1}{\beta}\Delta_t - \partial/\partial t$, where $\Delta_t (= dd^c/\omega_{\varphi_t})$ denotes the Laplacian with respect to the metric ω_{φ_t}, to the equation above gives

$$\frac{1}{\beta}\Delta_t h - \frac{\partial}{\partial t}h = \frac{1}{\omega_{\varphi_t}}(dd^c\varphi_t - dd^c f_\beta) + \Delta_t\frac{\partial\varphi_t}{\partial t} - \frac{\partial}{\partial t}h.$$

Now

$$\frac{\partial}{\partial t}h := \frac{\partial}{\partial t}\log\Big(\frac{e^{-t}\omega_0 + (1-e^{-t})\chi_\beta + dd^c\varphi_t}{\omega}\Big) = \frac{1}{\omega_{\varphi_t}}\Big(-e^{-t}\omega_0 + e^{-t}\chi_\beta + dd^c\frac{\partial\varphi_t}{\partial t}\Big).$$

Hence, the two terms involving $\partial\varphi_t/\partial t$ cancel, giving

$$\frac{1}{\beta}\Delta_t h - \frac{\partial}{\partial t}h = \frac{\omega}{\omega_{\varphi_t}}\left(\Delta_\omega(\varphi_t - f_\beta)) - e^{-t}(\chi_\beta - \omega_0)/\omega\right), \tag{2.10}$$

i.e.,

$$\omega_{\varphi_t}\Big(\frac{1}{\beta}\Delta_t h - \frac{\partial}{\partial t}h\Big) + dd^c f_\beta + e^{-t}(\chi_\beta - \omega_0) = dd^c\varphi_t,$$

which in terms of $\omega(t)$ $(:= \omega_0 + (1-e^{-t})(\chi_\beta - \omega_0) + dd^c\varphi_t)$ becomes

$$\omega_{\varphi_t}\Big(\frac{1}{\beta}\Delta_t h - \frac{\partial}{\partial t}h\Big) + dd^c f_\beta + \chi_\beta = \omega(t).$$

Applying the parabolic maximum principle to h concludes the proof. Indeed, there are two alternatives: either h has its maximum on $X\times[0,T]$ (for T fixed) at $t=0$ which implies that $\operatorname{Tr}_\omega\omega(t) \leqslant \operatorname{Tr}_\omega\omega_0$ on $X\times[0,T]$, or the maximum of h is attained at a point (x,t) in $X\times\,]0,T]$. In the latter case $\operatorname{Tr}_\omega\omega(t) \leqslant \sup_X\operatorname{Tr}_\omega(dd^c f_\beta + \chi_\beta) \leqslant C$ (since $dd^c f_\beta + \chi_\beta = \theta_\beta - \frac{1}{\beta}\operatorname{Ric}\omega = \theta + o(1)$). $\square$

2.4.2. *The upper bound on $\varphi_t^{(\beta)}$.* — Next, we come back to the general case. For notational convenience we will drop the superscript β. Writing the normalized KRF flow as

$$-\frac{\partial(\varphi_t - f_\beta)}{\partial t} + \frac{1}{\beta}\log\frac{(\widehat{\omega}_t + dd^c f_\beta + dd^c(\varphi_t - f_\beta))^n}{\omega^n} = \varphi_t - f_\beta,$$

it follows immediately from the parabolic maximum principle that

$$\varphi_t(x) - f_\beta(x) \leqslant \max\Big\{\sup_X(0 - f_\beta), \tfrac{1}{\beta}\sup_X \log\frac{(\widehat{\omega}_t + dd^c f_\beta + 0)_+^n}{\omega^n}\Big\} \leqslant A/\beta,$$

where A only depends on the upper bounds of θ_β. Here $\eta_+ = \eta$ if $\eta \geqslant 0$ and 0 otherwise. To see how the maximum principle can be applied one can consider a point (t_0, x_0) where $\varphi_t - f_\beta$ attains a maximum. At this point $dd^c\varphi_t - f_\beta \leqslant 0$ hence the form $(\widehat{\omega}_t + dd^c f_\beta)(x_0)$ is semipositive and the computation carries out as usual. In particular,

$$\varphi_t \leqslant P_{\widehat{\omega}_t}(f_\beta) + A/\beta$$

and, as a consequence,

$$\leqslant P_{C'\omega}(f_\beta) + A/\beta,$$

where C' is any constant satisfying $\chi_\beta \leqslant C'\omega$ and $\omega_0 \leqslant C'\omega$ (thus ensuring that $\mathrm{PSH}(X, \widehat{\omega}_t) \subset \mathrm{PSH}(X, C'\omega)$).

2.4.3. *The lower bounds on $\partial\widetilde{\varphi}_s/\partial s$ and $\partial\varphi_t/\partial t$.* — Differentiating the non-normalized KRF with respect to s gives, with $g(x,s) := -\partial\widetilde{\varphi}_s(x)/\partial s$,

$$\frac{\partial g}{\partial s} - \frac{1}{\beta}\Delta_s g = -\frac{1}{\beta}\operatorname{Tr}_s(\chi_\beta) \leqslant 0.$$

Hence, by the parabolic maximum principle the sup of g is attained at $t = 0$ which gives

$$-\frac{\partial\widetilde{\varphi}_s}{\partial s} \leqslant C_1, \qquad C_1 = \sup_X\Big(-\frac{1}{\beta}\log\frac{\omega_{\varphi_0}^n}{\omega^n} - f_\beta\Big)$$

where C_1 thus only depends on the strict positive lower bound of $\omega_{\varphi_0}^n$ and on $\inf_X(f_\beta)$ (which by our normalizations vanishes).

Next, using that

$$\frac{\partial\widetilde{\varphi}_s}{\partial s} = \frac{\partial\varphi_t}{\partial t} + \varphi_t + nt/\beta$$

gives

$$\frac{\partial\varphi_t}{\partial t} \geqslant -C_1 - \varphi_t - nt/\beta \geqslant -C_1' - nt/\beta \tag{2.11}$$

using the previous upper bound on φ_t.

2.4.4. *The lower bound on φ_t.* — It follows immediately from the previous bound that

$$\varphi_t \geqslant \varphi_0 - C_1' t - nt^2/2\beta.$$

2.4.5. *The Laplacian bound*. — We will use Siu's well-known variant [Siu87, p. 98–99] of the classical Aubin-Yau Laplacian estimate.

Lemma 2.13. — *Given two Kähler forms ω' and ω such that $\omega'^n = e^F\omega^n$ we have that*

$$\Delta_{\omega'} \log \operatorname{Tr}_\omega(\omega') \geqslant \frac{\operatorname{Tr}_\omega(-\operatorname{Ric}(\omega'))}{\operatorname{Tr}_\omega \omega'} - B\operatorname{Tr}_{\omega'}\omega,$$

where $-B \leqslant 0$ is a lower bound on the holomorphic bisectional curvature of ω.

We start with the case when X admits a Kähler metric ω with non-negative holomorphic bisectional curvature. In this case we can take $B = 0$.

Proposition 2.14. — *Suppose that the Kähler metric ω has non-negative holomorphic bisectional curvature. Then*

$$\operatorname{Tr}_\omega(\omega^{(\beta)}(t)) \leqslant \max\bigl\{\sup_X \operatorname{Tr}_\omega(\omega_0), \sup_X \operatorname{Tr}_\omega(\theta_\beta)\bigr\}, \quad \forall t \geqslant 0.$$

Proof. — Setting

$$H := \log \operatorname{Tr}_\omega(\omega_t),$$

where $\omega_t = \widehat{\omega}_t + dd^c\varphi^{(\beta)}(t)$, we get, using Siu's inequality,

$$\begin{aligned} -\frac{\partial H}{\partial t} + \frac{1}{\beta}\Delta_{\omega_t} H &\geqslant \frac{\operatorname{Tr}_\omega(e^{-t}\omega_0 - e^{-t}\chi_\beta) + \operatorname{Tr}_\omega(dd^c\varphi_t^{(\beta)} - dd^c f_\beta - \beta^{-1}\operatorname{Ric}(\omega))}{\operatorname{Tr}_\omega(\omega_t)} \\ &= \frac{\operatorname{Tr}_\omega(\omega_t) - \operatorname{Tr}_\omega(\theta_\beta)}{\operatorname{Tr}_\omega(\omega_t)}. \end{aligned}$$

The rest of the proof then proceeds precisely as in the Riemann surface case. $\square$

In the general case we get the following

Proposition 2.15. — *There is a constant C such that, for $\beta > \beta_0$ large enough*

$$\omega^{(\beta)}(t) \leqslant e^{C(1+1/\beta)(1+t)e^t}\omega,$$

where C depends on the same quantities as in the statement of Theorem 2.2.

Proof. — Recall that by abuse of notation we set $\omega_t = \widehat{\omega}_t + dd^c\varphi_t^{(\beta)}$, and $\widehat{\omega}_t := e^{-t}\omega_0 + (1-e^{-t})\chi_\beta$. By the Laplacian inequality (Lemma 2.13) we have, by setting $H := \log \operatorname{Tr}_\omega(\omega_t)$,

$$\begin{aligned} &\beta^{-1}B\operatorname{Tr}_{\omega_t}(\omega) + \bigl(\beta^{-1}\Delta_{\omega_t}H - \partial_t H\bigr) \\ &\qquad\geqslant \frac{\operatorname{Tr}_\omega(dd^c\varphi_t^{(\beta)} - dd^c f_\beta - \beta^{-1}\operatorname{Ric}(\omega)) - e^{-t}\operatorname{Tr}_\omega(\chi_\beta - \omega_0)}{\operatorname{Tr}_\omega(\omega_t)} \\ &\qquad= \frac{\operatorname{Tr}_\omega(\omega_t) - \operatorname{Tr}_\omega(\theta_\beta)}{\operatorname{Tr}_\omega(\omega_t)} \end{aligned}$$

thanks to the cancelation of the terms involving $\partial_t\varphi_t^{(\beta)}$, just as before. To handle the first term in the left-hand side above we note that

$$\omega \leqslant C_0 e^t \widehat{\omega}_t,$$

where $1/C_0$ is a positive lower bound for ω_0. Since $\operatorname{Tr}_{\omega_t}\widehat{\omega}_t = n - \Delta_{\omega_t}\varphi_t^{(\beta)}$ we thus get, by setting

$$G(x,t) := H - BC_0 e^t \varphi_t^{(\beta)} - f(t),$$

for any given function $f(t)$ of t,

$$-\partial_t f - C_0 B \partial_t (e^t \varphi_t^{(\beta)}) + n\beta^{-1} BC_0 e^t + \big(-\partial_t G + \beta^{-1}\Delta_{\omega_t} G\big) \geqslant \frac{\operatorname{Tr}_\omega(\omega_t) - \operatorname{Tr}_\omega(\theta_\beta)}{\operatorname{Tr}_\omega(\omega_t)}.$$

Next we note that, thanks to the lower bound on $\partial\varphi_t/\partial t$ (see (2.11)) we have

$$\frac{\partial(e^t\varphi_t)}{\partial t} \geqslant -e^t\Big(C_1 + \frac{nt}{\beta}\Big).$$

Hence, taking $f(t) = C_2(1+t)(1+1/\beta)e^t$ for C_2 sufficiently large (depending on C_1, C_0, B, n) gives

$$\Big(-\frac{\partial}{\partial t} + \frac{1}{\beta}\Delta_{\omega_t}\Big)G \geqslant \frac{\operatorname{Tr}_\omega(\omega_t) - \operatorname{Tr}_\omega(\theta_\beta)}{\operatorname{Tr}_\omega(\omega_t)}.$$

This shows that the estimate on $\operatorname{Tr}_\omega \omega_t$ we get from the parabolic maximum principle applied to G only depends on χ_β through the upper bound on φ_t (which in turn depends on an upper bound on χ_β and is of the order $1/\beta$). □

2.4.6. *The upper bound on $\partial\varphi_t/\partial t$ and $\partial\widetilde{\varphi}_s/\partial s$.* — From the upper bound on $\omega(t)$ and the defining equations for the KRFs one directly obtains bounds on $\partial\varphi_t/\partial t$ and $\partial\widetilde{\varphi}_s/\partial s$. However, better bounds can be obtained by a variant of the proof of the lower bounds on $\partial\varphi_t/\partial t$ and $\partial\widetilde{\varphi}_s/\partial s$. Indeed, differentiating the normalized and the non-normalized KRFs, respectively gives

$$\frac{\partial\partial\widetilde{\varphi}_s/\partial s}{\partial s} - \Delta_s \frac{\partial\partial\widetilde{\varphi}_s/\partial s}{\partial s} - \operatorname{Tr}_s \chi_\beta = 0 \tag{2.12}$$

and

$$\frac{\partial(e^t\partial\varphi_t/\partial t)}{\partial t} - \Delta_t \frac{\partial(e^t\partial\varphi_t/\partial t)}{\partial t} - \operatorname{Tr}_t(\chi_\beta - \omega_0) = 0. \tag{2.13}$$

Using that $\omega(s) = e^t\omega(t)$, $ds/d = e^{-t}dt/d$ and $\partial\widetilde{\varphi}_s/\partial s = \partial\varphi_t/\partial t + \varphi_t + nt/\beta$ the first equation above becomes

$$\frac{\partial(\partial\varphi_t/\partial t + \varphi_t + nt/\beta)}{\partial t} - \Delta_t \frac{\partial(\varphi_t + nt/\beta)}{\partial t} - \operatorname{Tr}_t \chi_\beta = 0.$$

Hence, taking the differences between equations (2.12) and (2.13) gives that $g := e^t\partial\varphi_t/\partial t - \partial\varphi_t/\partial t - \varphi_t - nt/\beta$ satisfies

$$\frac{\partial g}{\partial t} - \Delta_t \frac{\partial g}{\partial t} = -\operatorname{Tr}_t \omega_0 \leqslant 0.$$

Accordingly, the parabolic maximum principle reveals that the sup over X of $e^t\partial\varphi_t/\partial t - \partial\varphi_t/\partial t - \varphi_t - nt/\beta$ is decreasing, thanks to the upper bound on φ_t,

$$\frac{\partial\varphi_t}{\partial t} \leqslant \frac{\sup_X P_t(f_\beta) + (A+nt)/\beta)}{(e^t - 1)}$$

(this is a minor generalization of the estimate in [TZ06]). Finally, this yields

$$\frac{\partial\widetilde{\varphi}_s}{\partial s} \leqslant C'' \, \frac{1+\beta^{-1}\log(1+s)}{s}.$$

2.5. Existence and characterizations of the large β limit of the KRF. — By the previous estimates there is a subsequence of $\varphi^{(\beta)}(t)$ which converges uniformly (and even in $\mathscr{C}^{1,\alpha}$-norm) to a limiting Lipschitz curve $\varphi(t)$ with values in $\mathrm{PSH}(X,\widehat{\omega}_t)$. As we next show $\varphi(t)$ is uniquely determined, i.e., the whole family converges to $\varphi(t)$.

Proposition 2.16. — *The large β-limit of $\varphi^{(\beta)}(t)$ of the normalized KRF exists: it is equal to the curve defined as the sup over all curves $\psi(t)$ in $\mathrm{PSH}(X,\omega)$ such that $\psi(0)=\varphi(0)$ and such that $\psi(t)$ is locally Lipschitz in t (for $t>0$) and in $\mathscr{C}^1(X)$, for a fixed t and*

$$\frac{\partial\psi(t)}{\partial t} \leqslant -\psi(t)+f$$

(in the weak sense), or equivalently such that

$$(\psi(t)-f)e^t$$

is decreasing in time.

Proof. — By the second order a priori estimates we have

$$\frac{d\varphi^{(\beta)}(t)}{dt} \leqslant \frac{C(t)}{\beta} - \varphi^{(\beta)}(t) + f_\beta$$

and hence the limiting Lipschitz curve $\varphi(t)$ satisfies

$$\frac{d\varphi(t)}{dt} \leqslant -\varphi(t)+f$$

in the weak sense, i.e., $\varphi(t)$ is a candidate for the sup appearing in the statement of the proposition. Alternatively, we get

$$\frac{d\left((\varphi^{(\beta)}-f_\beta)(e^t-(C/\beta)e^t)\right)}{dt} \leqslant 0,$$

i.e.,

$$(\varphi^{(\beta)}-f_\beta)(t) \leqslant \frac{1}{(1-C/\beta)}\, e^{-t} g_\beta(x,t),$$

where $g_\beta(x,t)$ is decreasing in time. Hence, after passing to a subsequence the limit satisfies

$$(\varphi-f)(t) \leqslant e^{-t} g(x,t),$$

where $g(x,t)$ is decreasing in time.

Next, by the parabolic maximum principle $\varphi^{(\beta)}(t)$ is the sup over all smooth curves $u_\beta(t)$ with values in (the interior of) $\mathrm{PSH}(X,\widehat{\omega}_t)$ such that $u_\beta(0)=\varphi(0)$ and

$$\frac{du_\beta}{dt} \leqslant \frac{1}{\beta}\log\frac{(\widehat{\omega}_t+dd^c u_\beta(t))^n}{\omega^n} - (u_\beta(t)-f_\beta)$$

on a fixed time-interval $[0,T]$. Now take a smooth curve $v(t)$ from $[0,T]$ to $\mathrm{PSH}(X,\widehat{\omega}_t)\cap\mathscr{C}^\infty(X)$ such that and $v(0)=\varphi(0)$ and such that

$$\frac{d}{dt}v(t)\leqslant -(v(t)-f).$$

We set

$$v_\varepsilon(t):=(1-\varepsilon)v(t)-\varepsilon,$$

ensuring that

$$\frac{d}{dt}v_\varepsilon(t)\leqslant -v_\varepsilon(t)-f-\varepsilon$$

and

$$(\widehat{\omega}_t+dd^cv_\varepsilon(t))^n\geqslant \varepsilon^n\widehat{\omega}_t^n\geqslant \varepsilon^nC(T)\omega^n.$$

Hence, for β sufficiently large (depending on the lower bound $C(T)$ of the positivity of $\widehat{\omega}_t^n$ on $[0,T]$ and the convergence speed of f_β towards f),

$$\frac{dv_\varepsilon(t)}{dt}\leqslant\frac{1}{\beta}\log\frac{(\widehat{\omega}_t+dd^cv_\varepsilon(t))^n}{\omega^n}-v_\varepsilon(t)-f_\beta.$$

But then it follows from the parabolic maximum principle that $v_\varepsilon(t)\leqslant\varphi_{\beta,\varepsilon}(t)$, for $\beta\gg 1$, where $\varphi_{\beta,\varepsilon}(t)$ satisfies the same KRF as $\varphi^{(\beta)}(t)$, but with initial value $(1-\varepsilon)\varphi_0+\varepsilon v_0$. By the maximum principle we have

$$\left|\varphi_{\beta,\varepsilon}(t)-\varphi^{(\beta)}(t)\right|\leqslant C\varepsilon$$

and hence letting $\beta\to\infty$ gives, for any limit $\varphi(t)$ of $\varphi^{(\beta)}(t)$

$$v_\varepsilon(t)\leqslant\varphi(t)+C\varepsilon.$$

Since ε was arbitrary this gives $v(t)\leqslant\varphi(t)$. All that remain is thus to show that the smoothness assumption on $v(t)$ can be removed. This could be done by working with the notion of viscosity subsolutions [EGZ16], but here we will use a more direct approach by first noting that the sup above is realized by $P_{\widehat{\omega}_t}(e^{-t}\varphi_0+(1-e^{-t})f)$, as shown in the next proposition. Then we can use the regularization in Lemma 2.10 together with the slight generalization of the parabolic comparison principle formulated in Remark 2.7) to conclude. □

Proposition 2.17. — *The sup in the previous proposition coincides with $P_{\widehat{\omega}_t}(e^{-t}\varphi_0+(1-e^{-t})f)$.*

Proof. — It will be convenient to use the equivalent "non-normalized setting" which means that we replace the convex combination above with φ_0+tf and we prove that $a(t):=P_t(\varphi_0+tf)-tf$, where $P_t=P_{\omega_0+t\chi}$ (for any smooth closed semipositive $(1,1)$-form χ), is decreasing, i.e., that $a(t+s)-a(t)\leqslant 0$ for any fixed $t,s\geqslant 0$ (compare Remark 2.1). To this end we rewrite the difference above as

$$\begin{aligned}a(t+s)-a(t)&=P_{t+s}(\varphi_0+tf+sf)-P_t(\varphi_0+tf)-sf\\&=P_{t+s}((1-\lambda)\varphi_0+\lambda\psi_t)-P_t(\psi_t)-sf,\end{aligned}$$

where

$$\psi_t:=\varphi_0+tf,\quad \lambda:=(t+s)/t.$$

In particular, $\lambda \geqslant 1$ and hence it follows from the very definition of the envelope P that

$$P_{t+s}((1-\lambda)\varphi_0 + \lambda\psi_t) \leqslant (1-\lambda)\varphi_0 + \lambda P_t(\psi_t).$$

Indeed, by writing

$$\omega_0 + t\chi = \frac{\lambda-1}{\lambda}\omega_0 + \frac{1}{\lambda}(\omega_0 + (t+s)\chi)$$

it can be easily seen that the function

$$u := \frac{\lambda-1}{\lambda}\varphi_0 + \frac{1}{\lambda}P_{t+s}((1-\lambda)\varphi_0 + \lambda\psi_t)$$

is $(\omega_0 + t\chi)$-psh and $u \leqslant \psi_t$, ultimately giving

$$\begin{aligned} a(t+s) - a(t) &\leqslant (1-\lambda)\varphi_0 + \lambda P_t(\psi_t) - P_t(\psi_t) - sf \\ &\leqslant (\lambda-1)tf - sf = 0. \end{aligned}$$

A similar direct proof can be given for the normalized KRF, but using instead $\lambda = (1-e^{-(t+s)})/(1-e^{-t})$. □

Remark 2.18. — It follows from Proposition 2.17 that the parametrized non-coincidence sets $\Omega_t := \{P_{\omega_t}(\varphi + tf) < (\varphi + tf)\}$ are, in fact, increasing in t.

2.6. Alternative proof of the uniform convergence in Theorem 2.2.

— It is possible to prove the uniform large β-convergence of the flows $\varphi^{(\beta)}(t)$ directly without the Laplacian estimate and without going through the characterization in terms of curves appearing in Proposition 2.16.

For this purpose we recall that $\varphi^{(\beta)}(t)$ solves the non-normalized Kähler-Ricci flow

$$(\omega_0 + t\chi_\beta + dd^c\varphi^{(\beta)}(t))^n = e^{\beta(\partial_t\varphi^{(\beta)}(t) - f_\beta)}\omega^n,$$

and we want to prove that $\varphi^{(\beta)}(t)$ converge as $\beta \to +\infty$ uniformly to $P_{\omega_t}(\varphi_0 + tf)$, where $\omega_t := \omega_0 + t\chi$, $\chi = \lim_{\beta\to+\infty}\chi_\beta$, $f = \lim_{\beta\to+\infty} f_\beta$.

2.6.1. *The upper bound.*

— We first establish an upper bound for $\varphi^{(\beta)}(t)$. Consider

$$\psi_t := \varphi_0 + tf_\beta + \frac{Ct + nt\log(t+1)}{\beta},$$

where $C > 0$ is a constant to be chosen later. If the function $\varphi_t^{(\beta)} - \psi_t$ attains a maximum at some $(t_0, x_0) \in [0,T] \times X$ with $t_0 > 0$ then, at (t_0, x_0),

$$\begin{aligned} 0 \leqslant (\omega_0 + t\chi_\beta + dd^c\varphi_t^{(\beta)}) &\leqslant (\omega_0 + t\chi_\beta + dd^c\varphi_0 + tdd^cf_\beta) \\ &= (\omega_0 + dd^c\varphi_0 + t\theta_\beta - t\beta^{-1}\operatorname{Ric}(\omega)) \\ &\leqslant C'(1+t)\omega, \end{aligned}$$

for some uniform constant C' (since we assumed that $\theta_\beta = \theta + o(1)$). Hence if $C > 0$ is large enough (independent of β), we have, at (t_0, x_0),

$$(\omega_0 + t\chi_\beta + dd^c\varphi_t^{(\beta)})^n < e^C(1+t)^n\omega^n.$$

Also, the maximum principle reveals that, at (t_0, x_0),

$$\partial_t \varphi_t^{(\beta)} \geqslant \partial_t \psi_t \geqslant f_\beta + \frac{C + n\log(t+1)}{\beta},$$

which gives a contradiction. Thus if $C > 0$ is large enough (independent of β) then $\varphi_t^{(\beta)} \leqslant \psi_t$, for all $t \geqslant 0$ giving the upper bound.

2.6.2. *The lower bound*. — The lower bound is then proved as before using regularization and the parabolic comparison principle. Alternatively, the lower bound can also be proved directly (without regularization) using the monotonicity property (Proposition 2.17) and the viscosity comparison principle

$$\varphi^{(\beta)}(t) \geqslant v_t := (1-\delta)P_t(\varphi_0 + tf_\beta) + \frac{Ctn\log\delta}{\beta}, \quad \delta \in (0,1),$$

for some uniform constant C. Indeed, assume that the function $\varphi_t^{(\beta)} - v_t$ attains a minimum on $[0,T] \times X$ at some point (t_0, x_0) with $t_0 > 0$. Then using the notion of viscosity subsolutions [EGZ11] we see that, at (t_0, x_0),

$$(\omega_0 + t\chi_\beta + dd^c\varphi_t^{(\beta)})^n \geqslant \delta^n \omega_0^n.$$

On the other hand the function $v_t(x)$ is uniformly semiconcave in t and its left derivative satisfies $\partial_t v_t \leqslant (1-\delta)f_\beta + Cn\beta^{-1}\log\delta$ (as follows from Proposition 2.17). Also, by the maximum principle the left derivative in t of $\varphi_t^{(\beta)} - v_t$ at (t_0, x_0) is $\leqslant 0$, hence

$$\partial_t \varphi_t^{(\beta)} - f_\beta \leqslant Cn\beta^{-1}\log\delta.$$

Thus choosing $C > 0$ large enough (dependent only on ω_0, ω) we see that the minimum of $\varphi_t^{(\beta)} - v_t$ is attained at $t_0 = 0$ giving the lower bound.

Finally, since χ_β converges smoothly to χ and f_β converge smoothly to f, the bounds above allow us to let $\beta \to +\infty$ to conclude the proof of the uniform convergence $\varphi_t^{(\beta)} \to P_{\omega_0 + t\chi}(\varphi_0 + tf)$ in each fixed interval $[0,T]$.

Remark 2.19. — More generally, the uniform convergence holds even if f is merely continuous (see Section 6).

3. Large time asymptotics of the flows

In order to study the joint large t and large β-limit of the non-normalized Kähler-Ricci flows $\omega^{(\beta)}(t)$ introduced in the previous section we consider, as usual, the normalized Kähler forms $\omega^{(\beta)}(t)/(t+1)$ (which have uniformly bounded volume) evolving according to the normalized Kähler-Ricci flow (2.4). Our first observation is that the following double limit always exists:

$$\lim_{t\to\infty}\lim_{\beta\to\infty} \omega^{(\beta)}(t)/(t+1) = P(\theta), \tag{3.1}$$

for any initial Kähler metric ω_0 (where the large t-limit holds in the weak topology of currents). This follows immediately from Theorem 2.2 combined with the following lemma.

Lemma 3.1. — *Assume that $\chi \geqslant 0$ and set $\widehat{\omega}_t := e^{-t}\omega_0 + (1-e^{-t})\chi$. Then, for any given smooth functions φ_0 and f on X,*

$$P_{\widehat{\omega}_t}(e^{-t}\varphi_0 + (1-e^{-t})f) \longrightarrow P_\chi(f)$$

as $t \to \infty$, in the L^1-topology. In particular, if $[\theta] \geqslant 0$ then $P(e^{-t}\omega_0 + (1-e^{-t})\theta) \to P(\theta)$ in the weak topology of currents.

Proof. — Set $\psi_t := P_{\widehat{\omega}_t}(e^{-t}\varphi_0 + (1-e^{-t})f) =: P_{\widehat{\omega}_t}(f(t))$. Since $\psi_t \in \mathrm{PSH}(X, C\omega)$ for C sufficiently large and ψ_t is uniformly bounded the family ψ_t is relatively compact in the L^1-topology. We denote by ψ_∞ a given limit point of ψ_t, which clearly is in $\mathrm{PSH}(X,\chi)$. Moreover, $\psi_t \leqslant f(t)$ implies $\psi_\infty \leqslant f$ and hence $\psi_\infty \leqslant P_\chi f$. To prove the converse we set $\psi := P_\chi f$ and fix $\delta > 0$. Observe that

$$dd^c(1-\delta)\psi + \widehat{\omega}_t \geqslant (1-\delta)dd^c\psi + (1-e^{-t})\chi \geqslant (1-\delta)(dd^c\psi + \chi) \geqslant 0, \quad \text{for } t \gg 1.$$

Hence, since ψ is bounded we get

$$(1-\delta)\psi \leqslant P_{\widehat{\omega}_t}(f + C\delta) \leqslant P_{\widehat{\omega}_t}(f_t) + C\delta + C'e^{-t}.$$

Hence, letting first $t \to \infty$ gives $(1-\delta)\psi \leqslant \psi_\infty + C\delta$. Finally, letting $\delta \to 0$ concludes the proof. □

In the following two sections we will look closer at the situation appearing in the two extreme cases, where $\frac{1}{\beta}c_1(K_X) + [\theta_\beta]$ is positive and trivial, respectively. Then we will make some comments on the intermediate cases and the relations to previous results in complex geometry concerning the case when β is fixed.

3.1. The case when $c_1(K_X)/\beta + [\theta_\beta] = [\omega_0]$: a dynamic construction of envelopes

In this section we will consider the situation when the normalized KRF preserves the initial cohomology class. Given a volume form dV on X and a smooth function f on X, setting $\theta = dd^c f + \omega_0$ and

$$\theta_\beta = \theta + \frac{1}{\beta}\operatorname{Ric}(\omega)$$

for a fixed choice of Kähler metric $\omega \in [\omega_0]$ the normalized KRF in $[\omega_0]$ on the level of Kähler potentials becomes

$$\frac{\partial\varphi^{(\beta)}(t)}{\partial t} = \frac{1}{\beta}\log\frac{(\omega_0 + dd^c\varphi^{(\beta)}(t))^n}{\omega^n} - \varphi^{(\beta)}(t) + f. \tag{3.2}$$

Theorem 3.2. — *Let (X,ω) be a compact Kähler manifold of dimension n and fix a volume form dV on X. Given a smooth function f we denote by $\varphi^{(\beta)}(x,t)$ the solution of the evolution equation* (3.2) *with initial data φ_0 and set*

$$\varphi_t^{(\infty)} := P_{\omega_0}(e^{-t}\varphi_0 + (1-e^{-t})f).$$

Then there is a uniform constant $C > 0$ such that

$$\sup\nolimits_X \bigl|\varphi_t^{(\beta)} - \varphi_t^{(\infty)}\bigr| \leqslant C\,\frac{\log\beta}{\beta} \tag{3.3}$$

and

$$\Bigl|\frac{\partial\varphi}{\partial t}\Bigr| \leqslant Ce^{-t}, \quad |dd^c\varphi_t|_{\omega_0} \leqslant C. \tag{3.4}$$

Proof. — Note that $g := e^t \partial_t \varphi_t^{(\beta)}$ satisfies

$$\frac{\partial g}{\partial t} - \frac{1}{\beta}\,\Delta_t g = 0$$

and hence, by the parabolic maximum principle,

$$|g(x,t)| \leqslant \sup_X |g(x,0)| = \frac{1}{\beta}\log((\omega_0 + dd^c\varphi_0)^n/\omega^n) + f \leqslant C,$$

for some uniform constant C (independent of $\beta > 1$). Hence $\bigl|\partial_t \varphi_t^{(\beta)}\bigr| \leqslant Ce^{-t}$. Now the proof of Proposition 2.15 can be repeated word by word (using the auxiliary function $F(t) := -Ce^{-t}$) to give the desired bound for $dd^c\varphi_t^{(\beta)}$.

Next, the rate of convergence in (3.3) can be proved by repeating the arguments in Section 2.6 (for the lower bound one can take δ in Section 2.6 to be $1/\beta$, for $\beta > 1$). The details are left to the readers as an exercise. □

In particular, by (3.3)

$$\sup_X \bigl|\varphi_t^{(\beta)} - P_{\omega_0}(f)\bigr| \leqslant C\Bigl(\frac{\log\beta}{\beta} + e^{-t}\Bigr)$$

and hence the envelope $P_{\omega_0}(f)$ can be constructed from the joint large β and large t-limit of the Monge-Ampère flow (3.2):

$$P_{\omega_0}(f) := \lim_{t\to\infty} \varphi_t^{(\beta_t)}$$

in the $\mathscr{C}^0(X)$-norm for any family of t-dependent β_t such that $\beta_t \to \infty$ as $t \to \infty$. Interpreting β_t as the "inverse temperature" this construction is thus analogous to the method of simulated annealing algorithms used in numerics to find nearly optimal global minima of a given energy type function by cooling down a thermodynamical system (and decreasing the corresponding free energy). The analogy can be made more precise using the gradient flow picture (which will be explored in a separate publication) where the energy functional in question is the pluricomplex energy introduced in [BBGZ13]. It would be interesting to see whether this is numerically useful in concrete situations, for example by adapting the numerical implementations for the Kähler-Ricci flow on a toric manifold introduced in [DHH$^+$08] (concerning a finite β).

It may be illuminating to compare the dynamic construction of the envelope $P_\omega(f)$ above with the dynamic PDE construction of the *convex* envelope of a given smooth function f on $\mathbb{R}^n$ introduced in [Ves99]:

$$\frac{\partial\psi(t)}{\partial t} = \sqrt{1 + |\partial_x\psi(t)|^2}\,\min\{0, \lambda_1(\partial_x^2\psi(t))\} \quad \psi(0) = f,$$

i.e., the graph of the solution ψ_t evolves in the normal direction at each point, with the speed $\min\{0, \lambda_1(\partial_x^2\psi(t))\}$ (expressed in terms of the first eigenvalue of the real Hessian $\partial_x^2\psi(t))$); here $\psi(t)$ is a solution in the viscosity sense. A variant of the latter construction, obtained by removing the first factor in the right-hand side of the evolution equation above, was studied in [CG12] using stochastic calculus, where exponential convergence was established with a uniform control bound on $\partial_x^2\psi(t))$, which is thus analogous to the result in Theorem 3.2 above. Our approach can also be applied to convex envelopes by imposing invariance in the imaginary directions (as in Section 4.3). But the main difference in our setting is that we start with an arbitrary convex function $\psi(0)$ and the dependence on f instead appears in the evolution equation itself. Moreover, the large parameter β appears as a regularization parameter ensuring that the solution remains smooth for positive times.

3.2. The case when the class $\frac{1}{\beta}c_1(K_X)+[\theta_\beta]$ is trivial. — Next we specialize to the case when $\frac{1}{\beta}c_1(K_X)+[\theta_\beta]$ is trivial, which is the one relevant for the applications to Hele-Shaw type flows and Hamilton-Jacobi equation (in the latter case K_X is even trivial). Equivalently, this means that the non-normalized KRF preserves the initial cohomology class. In particular, letting $\beta\to\infty$ reveals that $[\theta]$ is trivial and hence we can write

$$\theta = dd^c f, \quad \inf_X f = 0$$

for a unique function f and then take

$$\theta_\beta := dd^c f_\beta + \frac{1}{\beta}\operatorname{Ric}\omega, \quad f_\beta := f.$$

for a fixed Kähler form ω, i.e., by imposing the equation (2.5).

In this setting, the normalized flow always tends to zero as $t\to\infty$ (as the volume of the class does). But, by the result in [Cao85], the non-normalized KRF flow converges to a Kähler form ω_β:

Proposition 3.3. — *For a fixed $\beta>0$ the non-normalized Kähler-Ricci flow $\omega^{(\beta)}(t)$ emanating from any given form ω_0 converges (in the $\mathscr{C}^\infty$-topology), as $t\to\infty$, to the unique solution $\omega_\beta\in[\omega_0]$ of the Calabi-Yau equation*

$$\frac{1}{V_0}\,\omega_\beta^n = \frac{e^{-\beta f}\omega^n}{\int_X e^{-\beta f}\omega^n}, \tag{3.5}$$

where V_0 is the volume of ω_0. More precisely, under the normalizations above the convergence holds on the level of Kähler potentials.

Remark 3.4. — By definition the volume form of the limiting Kähler metric is the Boltzmann-Gibbs measure associated to the Hamiltonian function f, at inverse temperature β, which gives a hint of the statistical mechanical interpretation of the large β-limit.

It should be stressed that by the estimates in [Cao85] one has in this setting that

$$\omega^{(\beta)}(t) \leqslant C_\beta\omega_0 \tag{3.6}$$

independently of t, which seemingly improves on the bounds in Theorem 2.2 and Theorem 2.3 for large t (the proof uses a different application of the Laplacian estimate, along the lines of Yau's original argument, which needs a two-sided bound on the potential). But the point of the estimates in Theorem 2.3, where one gets a linear growth in t is to get a multiplicative constant that is independent of β (at least when $t\to\infty$). In fact, for a generic f, it is impossible to get a constant C_β in formula (3.6) which is independent of β. Indeed, unless f vanishes identically the Gibbs measure in the right-hand side of the Calabi-Yau equation (3.5) blows up as $\beta \to \infty$, concentrating on the subset of X where f attains its absolute minimum ($= 0$ with our normalizations). Hence, for a generic f any limit point of $\omega^{(\beta)}(\infty)$ is a sum of Dirac measures. Accordingly, the convergence in the previous proposition, motivates (by formally interchanging the large t and large β-limits) the following

Proposition 3.5. — *Let f be a smooth function on X and ω_0 a Kähler form on X. Then any limit point of the family $(P(\omega_0+tdd^c f))^n$ (in the weak topology) is supported in the closed set F where f attains its absolute minimum. In particular,*

– *if f admits a unique absolute minimum x_0 then*

$$\lim_{t\to\infty} (P(\omega_0 + tdd^c f))^n = V_0\delta_{x_0}$$

weakly. Hence, the corresponding non-normalized Kähler-Ricci flows $\omega^{(\beta)}(t)$ emanating from ω_0 satisfy

$$\lim_{t\to+\infty} \lim_{\beta\to+\infty} \omega^{(\beta)}(t)^n = V_0\delta_{x_0}$$

in the weak topology.

– *In general, under the normalization* $\inf_X f = 0$,

$$\lim_{t\to+\infty} P_\omega(\varphi_0 + tf) = P_{(\omega,F)}(\varphi_0)$$

(increasing pointwise) for any initial continuous ω-psh function φ_0. In particular, if F is not pluripolar, then

$$\lim_{t\to\infty} \lim_{\beta\to\infty} \omega^{(\beta)}(t) = \omega_\infty$$

in the weak topology, where ω_∞ is the positive current defined by $\omega_0 + dd^c P_{(\omega_0,F)}(0)$.

Proof. — First observe that, under the normalization $f(x_0) = 0$ and $f > 0$ we have $P_\chi(f) = 0$, as in this case $\chi = 0$ and all psh functions on a compact manifold X are constants. Next, by Lemma 3.1 $P_{\widehat{\omega}_t}((1 - e^{-t})f) := P_t(f_t) \to P_\chi f$ in the L^1-topology. Since $P_t(f_t)$ is in $\mathrm{PSH}(X, C\omega)$ for C sufficiently large it follows from basic properties of psh functions that $\sup_X P_t(f_t) \to \sup_X P_\chi(f) = 0$. Fixing $\varepsilon > 0$ this means that for $t \geqslant t_\varepsilon$, $\sup_X P_t(f_t) \leqslant \varepsilon/2$ and hence the non-coincidence sets Ω_t satisfy $\{f > \varepsilon\} \subset \Omega_t$ for $t \geqslant t_\varepsilon$. In particular, $(\widehat{\omega}_t + dd^c P_{\widehat{\omega}_t}((1 - e^{-t})f))$ and hence its non-normalized version $(\omega_0 + dd^c P_{\omega_0}(tf))^n$ is supported in $\{f \leqslant \varepsilon\}$ for $t \geqslant t_\varepsilon$, which concludes the proof of the first statement. The first point then follows immediately.

To prove the second point we may assume that $\inf_X f = 0$, hence the family $P_\omega(\varphi_0 + tf)$ is increasing in t. By assumption $P_\omega(\varphi_0 + tf) \leqslant \varphi_0 + tf = \varphi_0$ on F and hence $P_\omega(\varphi_0 + tf) \leqslant P_{(\omega,F)}(\varphi_0)$. To prove the reversed inequality we fix $\varepsilon > 0$ and $u \in \mathrm{PSH}(X, \omega)$ such that $u \leqslant \varphi_0$ on F. Since the sets $\{f \leqslant c\}$ decrease to the compact set F as $c \downarrow 0$ and φ_0 is continuous, there exists $c > 0$ small enough such that $\{f \leqslant c\} \subset \{u - \varepsilon < \varphi_0\}$. Now, for $t > c^{-1} \sup_X(u - \varphi_0)$ we have $u - \varepsilon \leqslant \varphi_0 + tf$, giving that the limit φ_∞ of the increasing family $P_\omega(\varphi_0+tf)$ is greater than $u-\varepsilon$. As u and ε were chosen arbitrarily the conclusion follows.[3] □

3.3. Comparison with convergence properties for a finite β and canonical metrics

3.3.1. *The big case*. — Let us start by considering the case when $\theta=0$. Then, up to a scaling, we may as well also assume that $\beta=1$. When K_X is nef and big, which equivalently means that K_X is semi-positive (by the base point freeness theorem) and with non-zero volume, $K_X^n>0$, it is well-known that the normalized Kähler-Ricci flow emanating from any given Kähler metric ω_0 on X converges, weakly in the sense of currents, to the unique (possibly singular) Kähler-Einstein metric (or rather current) ω_{KE} on X [Tsu88, TZ06]. This fact implies the following

Proposition 3.6. — *Assume that K_X is nef and big, but not ample. Then it is not possible to have an upper bound of the form $\omega^{(\beta)}(t)\leqslant C_\beta t$ along the non-normalized Kähler-Ricci flow, for t large.*

Proof. — Fixing a semi-positive form χ in $c_1(K_X)$ and representing $\omega_{\mathrm{KE}}=\chi+dd^c\varphi_{\mathrm{KE}}$ the potential φ_{KE} may be characterized as the unique continuous solution in $\mathrm{PSH}(X,\chi)$ to the equation

$$(\chi+dd^c\varphi)^n=e^\varphi dV_\chi$$

(in the sense of pluripotential theory) where dV_χ is the normalized volume form determined by χ (i.e., $\mathrm{Ric}\, dV_\chi=\chi$). In particular, if K_X is not positive (i.e., not ample) then ω_{KE} is not a bounded current. Indeed, assuming to get a contradiction that $\omega_{\mathrm{KE}}\leqslant C\omega_0$ the previous equation gives that $\omega_{\mathrm{KE}}^n\geqslant\delta\omega_0^n$ for some positive constant δ. But this means that, up to enlarging the constant C we get $\omega_0/C\leqslant\omega_{\mathrm{KE}}\leqslant C\omega_0$ which forces K_X to be ample (for example, by the Nakai-Moishezon criterion or by a direct regularization argument). □

More generally, essentially the same arguments apply to any smooth twisting form θ and parameter β as long as $c_1(K_X)/\beta+[\theta]$ is nef and big.

3.3.2. *The non-big case*. — Again we start with the case when $\theta=0$ with K_X nef, but now not big. Assuming that the abundance conjecture holds, i.e., that K_X is semi-ample it was shown in [ST12] that the normalized Kähler-Ricci flow, emanating from any given Kähler metric ω_0, on X converges, weakly in the sense of currents, to a canonical current ω_X on X defined as follows: by the semi-ampleness assumption there exists a holomorphic map F from X to a variety Y such that $K_X=F^*A$ where A is an ample line bundle on Y. In case Y is zero-dimensional the limit ω_X vanishes identically (as in Section 3.2). Otherwise, denoting by κ the dimension of Y (which equals the Kodaira dimension of X), picking a Kähler form ω_A in $c_1(A)$ and taking $\chi:=F^*\omega_A$, the limiting current ω_X obtained in [ST12] can be realized as

[3] The same result holds even when F is non-pluripolar and φ_0 is unbounded (using the domination principle in finite energy classes due to Dinew [BL12]).

$F^*(\omega_A + dd^c\psi)$, where ψ is the unique continuous solution in $\mathrm{PSH}(Y, \omega_A)$ of the equation

$$(\omega_A + dd^c\psi)^\kappa = e^\psi F_*(dV_\chi).$$

Next, we make some heuristic remarks about the connection to the double limit in formula (3.1). We assume that K_X is semi-ample and fix a smooth form θ in $c_1(K_X)$, a Kähler metric ω on X and define θ_β and f by

$$\theta_\beta := \theta - \frac{1}{\beta}\operatorname{Ric}\omega, \quad \theta = dd^c f + \chi.$$

In particular, $c_1(K_X)/\beta + [\theta_\beta] = c_1(K_X)$ for all β. In the light of the result in [ST12] one would expect that the corresponding twisted normalized KRF $\omega^{(\beta)}(t)$ converges, as $t \to \infty$, to the current $F^*(\omega_A + dd^c\psi_\beta)$, where ψ_β is the unique continuous solution in $\mathrm{PSH}(X, \chi)$ of the equation

$$(\omega_A + dd^c\psi_\beta)^\kappa = e^{\beta\psi_\beta} F_*(e^{-\beta f}\omega^n).$$

We will make the hypothesis that this is the case. It can be shown that as $\beta \to \infty$ there exist a (mildly singular) volume form μ_Y on Y such that

$$F_*(e^{-\beta f} dV_\chi) = e^{-\beta(\overline{f} + o(1))}\mu_Y,$$

where $\overline{f}(y) := \inf_{F^{-1}(\{y\})} f$ (using that the push forward F_* amounts to integration along the fibers of F which thus picks out the infimum of f over the fibers as $\beta \to \infty$; the error term $o(1)$ is uniform away from the branching locus of the map F). But then a variant of Theorem 3.2 (see [Ber13]) shows that $\psi_\beta \to \psi_\infty := P_{\omega_A}(\overline{f})$ and hence, under the hypothetical convergence above,

$$\lim_{t\to\infty}\lim_{\beta\to\infty} \omega^{(\beta)}(t) = \chi + dd^c P_{F^*\omega_A}(F^*\overline{f}).$$

Finally, since $K_X = F^*A$ we have $\mathrm{PSH}(X, \chi) = F^*\,\mathrm{PSH}(Y, \omega_A)$, forcing $P_{F^*\omega_A}(F^*\overline{f}) = P_\chi(f)$, i.e., the right-hand side above is equal to the current obtained by interchanging the limits in the lhs (as in Lemma 3.1), i.e., the two limits may be interchanged under the hypothesis above.

4. Applications to Hamilton-Jacobi equations and shocks

4.1. Background. — Let H be a smooth function on $\mathbb{R}^n$. The corresponding *Hamilton-Jacobi equation* (with *Hamiltonian H*) is the following evolution equation

$$\frac{\partial\psi_t(y)}{\partial t} + H(\nabla\psi_t(y)) = 0, \quad \psi_{|t=0} = \psi_0 \tag{4.1}$$

for a function $\psi(x,t)$ on $\mathbb{R}^n \times [0,\infty[$. It is a classical fact that, even if the initial function ψ_0 is smooth a solution ψ_t typically develops shock singularities at a finite time T_*, i.e., it ceases to be differentiable in the space-variable (due to the crossing of characteristics). In order to get a solution defined for any positive time the notion of *viscosity solution* was introduced in [CL83, CEL84]. The momentary *shock locus S_t* of such a solution ψ_t is defined by

$$S_t := \{x \mid \psi_t \text{ is not differentiable at } x\}.$$

When H is convex the classical *Hopf-Lax formula* provides an explicit envelope expression for a viscosity solution of the Cauchy problem for the HJ-equation (4.1) with any given smooth initial data ψ_0 (which, for example, appears naturally in optimal control problems):

$$\psi_t(y) = \inf_{x\in\mathbb{R}^n} \left\{\psi_0(x) + tH^*\big((x-y)/t\big)\right\},$$

expressed in terms of the Legendre transform:

$$g^*(y) := \sup_{x\in\mathbb{R}^n} (x\cdot y - g(x)).$$

On the other hand, in the case when H is non-convex, but the initial data ψ_0 is assumed convex, the *second Hopf formula* [BE84, LR86] provides a viscosity solution which may be represented as

$$\psi_t = (\psi_0^* + tH)^*. \tag{4.2}$$

This was shown in [BE84] using the theory of differential games and in [CL83] by a more direct approach. In particular the viscosity solution ψ_t above remains convex for all positive times and as a consequence its shock locus S_t has Hausdorff codimension one (unless it is empty, as is the case for small t). More precisely, S_t is a union of $\mathscr{C}^2$-manifolds up to set of vanishing $(n-1)$-dimensional Hausdorff measure [Alb94].

We recall that the viscosity terminology can be traced back to the fact that viscosity solutions may often be realized as limits of smooth solutions $\psi^{(\beta)}$ of the following perturbed (viscous) HJ-equations (where the constant β^{-1} plays the role of the viscosity constant in fluid and gas dynamics):

$$\frac{\partial\psi_t(y)}{\partial t} + H(\nabla\psi_t(y)) = \frac{1}{\beta}\,\Delta\psi_t(u) \tag{4.3}$$

as $\beta\to\infty$. For example, the following result holds:

Theorem 4.1 (Vanishing viscosity limit [CL83, Th. 3.1], [CEL84])
Assume that $\psi_t^{(\beta)}$ are smooth solutions to the previous equation and that a subsequence converges uniformly to ψ_t. Then ψ_t is a viscosity solution to the HJ-equation (4.1).

In particular, under suitable growth assumptions, ensuring that the viscosity solution ψ_t is uniquely determined, the whole family converges to ψ_t. Note however that, in general, $\Delta\psi_t^{(\beta)}$ will not be uniformly bounded (even locally), as this would entail that the limit ψ_t is differentiable on $\mathbb{R}^n$.

Next, we make the observation that in the case when the initial data ψ_0 above is taken to be $|y|^2/2$ the second Hopf formula is equivalent to the Hopf-Lax formula for the convex Hamiltonian $|x|^2/2$:

Lemma 4.2. — *Let Φ_0 be a given function on $\mathbb{R}^n$ and denote by Φ_t the Hopf-Lax viscosity solution to the HJ-equation with convex Hamiltonian $|x|^2/2$ and initial data Φ_0. Then*

$$\psi_t(y) := \big(-\Phi_t(y)t + |y|^2/2\big)$$

gives the viscosity solution to the HJ-equation with non-convex Hamiltonian $H := \Phi_0$ and initial data $\psi_0(y) := |y|^2/2$ provided by the second Hopf formula (and conversely). In particular, the shock loci of Φ_t and ψ_t coincide.

Proof. — This follows immediately from comparing the Hopf-Lax formula and the second Hopf formula. □

The previous lemma is consistent (as it must) with the fact that when $\psi_0(y) = |y|^2/2$ the Hamiltonian H can, by the definition of the HJ-equation, be recovered as minus the derivative at $t = 0$ of the corresponding viscosity solution ψ_t.

4.2. The adhesion model in cosmology. — The convex case where $H(x) = |x|^2/2$ is ubiquitous in mathematical physics and appears, in particular, in the adhesion model for the formation of the large-scale structure in the early universe (known as the "cosmic web") where Φ_0 is proportional to the gravitational potential of the initial fluctuations of the density field and the shock region S_t corresponds to emerging regions of localized mass concentration (the adhesion model is an extension of the Zel'dovich approximation beyond $t \geqslant T_*$) [GMS91, VDFN94, HSvdW14]. The corresponding singularities of S_t and their metamorphosis as t evolves have been classified in dimensions $n \leqslant 3$, for generic initial data, using the catastrophe theory of Lagrangian singularities initiated by Arnold [AZS81, GMS91, Bog99, HSvdW14, KPSM92]. We refer to the papers [GMS91, VDFN94, HSvdW14] for background on the adhesion model. Here we just point out that in the setting of the adhesion model the Legendre transform $\phi_t := \psi_t^*$ of the corresponding function ψ_t appearing in the previous lemma is given by

$$\phi_t(x) = x + t\Phi_0(x)$$

and the corresponding map

$$x \longmapsto \nabla_x \phi_t(x)$$

describes, in the Zel'dovich approximation, the displacement of a particle with initial coordinate x to the position y a time t (in the physics literature the initial coordinate space x is called the Lagrangian space and the position space y at time t is called the Euler space; accordingly ϕ_t is often called the Lagrangian potential). The map above is injective precisely for $t < T_*$. In the next section we will show that the adhesion model can be realized as the zero-temperature limit of the twisted Kähler-Ricci flow (using Lemma 4.2).

Remark 4.3. — When $H(x) = |x|^2/2$ the vector field $v_t(y) := \nabla u_t(y)$ determined by a solution u_t of the corresponding HJ-equation satisfies Burgers' equation:

$$\frac{\partial v_t(y)}{\partial t} + \frac{1}{2}\nabla \left|v_t(y)\right|^2 = 0,$$

which is the prototype of a hyperbolic conservation law [Ser14] and non-linear wave phenomena [GMS91].

4.3. Relation to the Kähler-Ricci flow and Theorem 2.2. — The relation between the Hamilton-Jacobi equation and the Kähler-Ricci flow, which does not seem to have been noted before, arises when the linear viscosity term in the perturbed HJ-equation (4.3) is replaced by the following non-linear one:

$$\frac{\partial\psi_t(y)}{\partial t}+H(\nabla\psi_t(y))=\frac{1}{\beta}\log(\partial^2\psi_t(y)) \tag{4.4}$$

for ψ_0 strictly convex (for example, $\psi_0(y)=|y|^2/2$, as in the adhesion model above). Here, for a real-valued $\mathscr{C}^2$ function u in $\mathbb{R}^n$ we let $\partial^2 u$ denote the real Hessian matrix of u (hence $\partial^2\psi_t$ is the Hessian matrix of ψ_t in the space variable x). One virtue of the latter evolution equation is that, as will be shown below, the smooth solution $\psi_t^{(\beta)}$ remains convex (and even strictly so) for positive times.

We will consider the case when the Hamiltonian H is periodic, i.e., invariant under the action of a lattice Λ on $\mathbb{R}^n$ by translations.[4] Without loss of generality we may and will assume that a fundamental domain for Λ has unit volume. Since there are no non-constant periodic convex functions on $\mathbb{R}^n$ the natural condition on the initial function ψ_0 is that it is in the class of all convex functions u which are *quasi-periodic* in the sense that $\psi(y)-|y|^2/2$ is Λ-periodic on $\mathbb{R}^n$. We denote by $\mathscr{C}_\Lambda$ the space of all quasi-periodic convex functions on $\mathbb{R}^n$. The point is that for any $\psi\in\mathscr{C}_\Lambda$ the Hessian $\partial^2\psi$ is periodic and the gradient map $\partial\psi$ is Λ-equivariant and hence all terms appearing in the equation (4.4) are Λ-periodic.

Lemma 4.4. — *Equip the space $\mathscr{C}_\Lambda$ with the sup-norm. Then the Legendre transform $\phi\mapsto\psi:=\phi^*$ induces an isometry on $\mathscr{C}_\Lambda$ and for any quasi-periodic function f*

$$\sup_{\phi\leqslant f}\{\phi\}=f^{**},$$

where the sup, that we shall denote by $P(f)$, can be taken either over all convex functions ϕ or over all quasi-periodic convex functions. Moreover, the subspace of all ϕ in $\mathscr{C}_\Lambda$ such that $\sup_{x\in\mathbb{R}}(\phi(x)-|x|^2/2)=0$ *is compact.*

Proof. — The isometry property follows directly from the relation $(\phi+c)^*=\phi^*-c$. Next, if $P'f$ denotes the sup over all convex ϕ below f then, by the extremal property, the function $P'f-|x|^2/2$ has to be Λ-periodic, as $f-|x|^2/2$ is, i.e., $P'f$ is quasi-periodic, as desired. Finally, it is well-known that if f is convex then $P'f=f^{**}$.

The compactness is a consequence of the Arzelà-Ascoli theorem and the fact that if ϕ is in $\mathscr{C}_\Lambda$ then the periodic function $(\phi(x)-|x|^2/2)$ is L-Lipschitz for a constant L only depending on the diameter of a fundamental domain of Λ; see [Hul16, Lem. 3.14] where further properties of the space $\mathscr{C}_\Lambda$ are also established. □

In this setting Theorem 2.2 admits the following dual formulation:

[4] It seems likely that the general case could be studied by extending our results to (appropriate) non-compact manifolds X or by approximation.

Theorem 4.5. — *Consider the perturbed HJ-equation* (4.4) *with* Λ*-periodic smooth Hamiltonian* H *and strictly convex and quasi-periodic initial data* ψ_0*. Denote by* $\psi^{(\beta)}$ *the unique solution of the corresponding Cauchy problem such that* $\psi_t^{(\beta)}$ *is quasi-periodic and strictly convex. Then* $\psi_t^{(\beta)}$ *converges, as* $t \to \infty$*, uniformly in space, to* ψ_t *given by the second Hopf formula* (4.2)*, which is the unique viscosity solution of the HJ-equation* (4.1) *with initial data* ψ_0*. Moreover,* $\psi_t^{(\beta)}$ *is strictly convex for any* $t > 0$*, uniformly in* β*:*

$$\operatorname{Tr}\bigl(\partial^2 \psi_t^{(\beta)}\bigr) \geqslant \frac{1}{t+1} \min \left\{ \sup_{\mathbb{R}^n} \operatorname{Tr}(\partial^2 \psi_0), \frac{1}{\sup_{\mathbb{R}^n} \operatorname{Tr}(\partial^2 H)} \right\}.$$

The trace operator in the theorem above is the usual trace of a $n \times n$-matrix.

Proof. — To make the connection to the complex geometric setting we let X be the abelian variety $X := \mathbb{C}^n/(\Lambda + i\mathbb{Z}^n)$ and consider the following holomorphic T-action on X:

$$([x+iy],[a]) \longmapsto [x+iy+a],$$

where T denotes the real n-torus $T := \mathbb{R}^n/\mathbb{Z}^n$ and $\pi(z) := [z]$ denotes the corresponding quotient map. Let ω be the standard flat Kähler metric on X induced from the Euclidean metric ω_0 on $\mathbb{C}^n$ normalized so that $\omega_0 = dd^c|x|^2/2$ and fix a closed T-invariant $(1,1)$- form θ which is exact, i.e.,

$$\theta = dd^c f$$

for a T-invariant function f on X (uniquely determined up to an additive constant). Now we can identify T-invariant elements in $\operatorname{PSH}(X,\omega)$ with convex functions $\phi(x)$ on $\mathbb{R}^n$ in the space $\mathscr{C}_\Lambda$ (by setting $\phi := |x|^2/2 + \pi^*\varphi$ and using that $dd^c(|x|^2/2 + \pi^*\varphi) = \omega_0 + dd^c\pi^*\varphi \geqslant 0$). Accordingly, the non-normalized KRF in the class $[\omega]$ with twisting form θ thus gets identified with the following parabolic equation on $\mathbb{R}^n$:

$$\frac{\partial \phi_t^{(\beta)}(x)}{\partial t} = \frac{1}{\beta} \log(\partial^2 \phi_t^{(\beta)}(x)) + H(x), \tag{4.5}$$

where H is the Λ-periodic function on $\mathbb{R}^n$ corresponding to f and $\phi_t^{(\beta)} \in \mathscr{C}_\Lambda$. More precisely, $\phi_t^{(\beta)}$ is smooth and strictly convex. The key observation now is that setting

$$\psi_t^{(\beta)}(y) := \phi_t^{(\beta)*}(y)$$

gives a solution in $\mathscr{C}_\Lambda$ to the perturbed HJ-equation (4.4). Indeed, this follows from the following well-known properties of the (involutive) Legendre transform between smooth and strictly convex functions (say with quadratic growth at infinity):

$$\partial^2\phi(x) = (\partial^2\psi(y))^{-1}, \quad \frac{\partial(\phi + tv)(x)}{\partial t}\Big|_{t=0} = -v(\partial_y \psi(y)), \quad y := \partial_x \phi(x)$$

(see for example the appendix in [BB13] for a proof of the latter formula). Now, by Theorem 2.2 and the previous lemma

$$\lim_{\beta \to \infty} \phi_t^{(\beta)} = P_\Lambda(\phi_0 + tH) = (\phi_0 + tH)^{**}$$

in $\mathscr{C}_\Lambda$. Since the Legendre transform is an isometry on $\mathscr{C}_\Lambda$ and in particular continuous this equivalently means that $\lim_{\beta\to\infty}\psi_t^{(\beta)}=(\phi_0+tH)^*$, which coincides with the viscosity solution of the HJ-equation provided by the second Hopf formula. Finally, the proof of the previous theorem is concluded by noting that the uniqueness of viscosity solutions in $\mathscr{C}_\Lambda$ follows from the standard uniqueness argument [CL83, CEL84], using that for any two functions in $\mathscr{C}_\Lambda$ the difference $u-v$ is continuous and attains its maximum and minimum (since it is periodic). □

In fact, in this way Theorem 2.2 could be used to give an alternative proof of the fact that the second Hopf formula defines a viscosity solution to the HJ-equation (4.1), by adapting the proof of Theorem 4.1 to the present non-linear setting.

Remark 4.6. — Convex envelopes of the form $\psi_t:=(\psi_0^*+tH)^*(=\phi_t^*)$ and the corresponding sets $X(t)$ also appear in a different Kähler-geometric setting in [RWN17a, RZ17], where it is shown that ψ_t defines a torus invariant (weak) Kähler geodesic precisely on $[0,T_*[$ (what we call T_* is called the "convex life span" in [RWN17a, RZ17]). By definition, such a Kähler geodesic ϕ_t is characterized by the homogeneous Monge-Ampère equation $\mathrm{MA}(\phi)=0$ on the product $X\times]0,T[$. The relation to ($\mathscr{C}^1$-smooth) solutions of Hamilton-Jacobi equations was also pointed out in Section 6 in [RZ17]. In the light of the results in [RWN17a, RZ17] it seems notable that in our setting ϕ_t has a natural complex geometric interpretation also for $t>T_*$ (namely, as a limiting Kähler-Ricci flow).

4.4. Remarks on convex duality in the present setting.

— By a well-known duality principle in convex analysis differentiability of a convex functions ψ corresponds, loosely speaking, to strict convexity of its Legendre transform $\phi:=\psi^*$. To make this precise we will assume that both ϕ and ψ are defined on all of $\mathbb{R}^n$ and have super-linear growth (which is the case when any, and hence both, of the functions are in $\mathscr{C}_\Lambda$). This ensures that the sub gradient maps $\partial\phi$ and $\partial\psi$ are both surjective. We recall that a convex function ϕ is differentiable at x if and only if the subgradient $(\partial\phi)(x)$ is single valued and then we will write $(\partial\phi)(x)=(\nabla\phi)(x)$. The starting point for the duality in question is the following fact (which follows directly from the definitions):

$$x\in\partial\psi(y)\iff y\in\partial\phi(x)\iff x\cdot y=\phi(x)+\psi(y).$$

In our setting $\phi:=\phi_t$ (for a fixed time t) is $\mathscr{C}^{1,1}$-smooth, i.e., $\partial\phi(=\nabla\phi)$ defines a surjective Lipschitz map $\mathbb{R}^n\to\mathbb{R}^n$. As a consequence, a point y is in the shock locus S_t of ψ_t if and only if $y\in\partial\phi_t(U)$, for an open set U where the Lipschitz map $\partial\phi_t$ is not injective (which can be interpreted as a local strict convexity of ϕ_t). Let now X_t be the support of the Monge-Ampère measure $\det(\partial^2\phi_t)dx$ and denote by Ω_t its complement. For simplicity we assume that the locus where ϕ_t is in C^2_{loc} is dense in $\mathbb{R}^n$ (which presumably holds for a generic H using the arguments in [AZS81, Bog99]). In that case the continuous map $\partial\phi_t$ maps the interior of X_t injectively to $\mathbb{R}^n\smallsetminus S_{\psi_t}$ and $\overline{\Omega}_t$ non-injectively to S_{ψ_t} (since a $\mathscr{C}^2$-convex function u has an invertible gradient if and only if $\det(\partial^2 u) > 0$). Conversely, $\nabla\psi_t$ maps $\mathbb{R}-S_{\psi_t}$ to X_t. See for example [VDFN94, Fig. 5] for an illustration of this duality.

It may also be illuminating to consider the case when ψ is piece-wise affine (which, as we will show in the next section, happens when $t=\infty$). Then $(\nabla\phi)(\mathbb{R}^n \smallsetminus S_\phi)$ is contained in the 0-dimensional stratum $S_\psi^{(0)}$ of S_ψ (i.e., in the vertex set). Indeed, if $y_0 := (\nabla\phi)(x_0)$ is not in $S_\psi^{(0)}$ then there is an open affine segment L passing through y_0 along which ψ is affine. One then gets a contradiction to the differentiability of ϕ at x_0 by noting that $L \subset \partial\phi(x_0)$. Indeed, since $x_0 \in \partial\psi(y_0)$ one gets $\psi(y) = \psi(y_0) + x_0 \cdot (y-y_0)$ along L. But this means that $x_0 \cdot y = \phi(x_0) + \psi(y)$ and hence $y \in \partial\phi(x_0)$.

In fact, this argument also shows that ϕ is piecewise affine if and only if its Legendre transform ψ is. Indeed, if ψ is piecewise affine then by the growth assumptions the sup defining ϕ is always attained. Hence, for any $x \in \mathbb{R}^n \smallsetminus S_\phi$ we have that $\phi = (\chi_{S_\psi^{(0)}}\psi)^*$. Since the right-hand side is also a convex function and the complement of $\mathbb{R}^n \smallsetminus S_\phi$ is a null set it then follows that $\phi = (\chi_{S_\psi^{(0)}}\psi)^*$ everywhere, showing that ϕ is also piece-wise affine, as desired.

4.5. The large time limit and Delaunay/Voronoi tessellations. — Next, we specialize the large time convergence result in Proposition3.5 to the present setting, showing, in particular, that the Hessian of the limiting solution vanishes almost everywhere:

Theorem 4.7. — *Denote by F_Λ the closed set in $\mathbb{R}^n$ where the Λ-periodic Hamiltonian H attains its minimum, normalized to be 0 and assume that F_Λ is discrete. Then, for any given initial data ψ_0 in the space $\mathscr{C}_\Lambda$ the unique viscosity solution ψ_t in $\mathscr{C}_\Lambda$ of the corresponding Hamilton-Jacobi equation converges uniformly to the following convex piecewise affine function:*

$$\psi_\infty(y) := \sup_{x\in F_\Lambda} (x\cdot y - \psi_0^*(x)). \tag{4.6}$$

Equivalently, the large β-limit ϕ_t of the Kähler-Ricci flow (4.5) *converges to the convex piecewise affine function $\phi_\infty(x)$ whose graph is the convex hull of the discrete graph of the function ϕ_0 restricted to F_Λ.*

Proof. — By the second point in Proposition 3.5

$$\phi_\infty(x) := \sup_{\phi\in\mathscr{C}_\Lambda} \{\phi(x) \mid \phi \leqslant \phi_0 \text{ on } F_\Lambda\}.$$

Indeed, recall that the limit in Proposition3.5 is the supremum over all ω-psh functions lying below $\chi_{F_\Lambda}\phi_0$. But as F_Λ is non-pluripolar (which follows from the classical fact in pluripotential theory that $\mathbb{R}^n$ is non-pluripolar in $\mathbb{C}^n$), the function ϕ_∞ is convex bounded in $\mathbb{R}^n$. This together with the maximality property yields that ϕ_∞ is T-invariant and hence the corresponding function in $\mathscr{C}_\Lambda$ equals the supremum taken over $\mathscr{C}_\Lambda$ as above. Alternatively, the boundedness can also be seen directly in the present setting using the compactness property in Lemma 4.4. Writing this as $\phi_\infty = P(\chi_{F_\Lambda}\phi_0)$, where $\chi_{F_\Lambda} = 0$ on F_Λ and $+\infty$ on the complement of F_Λ (compare Lemma 4.4) reveals that the previous sup coincides with the relaxed sup ϕ' obtained by simply requiring that ϕ be convex (but not quasi-periodic), i.e., the graph of $\phi_\infty(x)$ is the convex hull of the discrete graph of the function ϕ_0 restricted to F_Λ, as desired. By Lemma 4.4 this means that $\phi_\infty = P(\chi_{F_\Lambda}\phi_0) = ((\chi_{F_\Lambda}\phi_0)^*)^*$ and hence $\phi_\infty^* = (\chi_{F_\Lambda}\phi_0)^*$. Moreover, since the Legendre transform is a continuous operator on $\mathscr{C}_\Lambda$ it follows from the second Hopf formula that $\psi_\infty := \lim_{t\to\infty}\psi_t = \phi_\infty^*$, which proves formula (4.6). As a consequence $\psi_\infty(y)$ is locally the max of a finite number of affine functions (indeed, since F_Λ is locally finite and ϕ has quadratic growth the sup defining $(\chi_{F_\Lambda}\phi_0)^*(y)$ can, locally with respect to y, be taken over finitely points in F_Λ). Hence, $\psi := \psi_\infty$ is piecewise affine and hence so is ϕ_∞ (compare Remark 4.4). □

In particular, if $\psi_0(y) = |y|^2/2$, then we can complete the square and rewrite

$$\psi_\infty(y) = \frac{1}{2}|y|^2 - \inf_{x\in F_\Lambda}\frac{1}{2}|x-y|^2.$$

Accordingly the non-differentiability S_{ψ_∞} locus of ψ_∞ coincides with the subset of all points y in $\mathbb{R}^n$ where the corresponding minimum is non-unique (compare Remark 4.4). The latter set is the honeycomb like connected $(n-1)$-dimensional piecewise linear manifold obtained as the union of the boundaries of the open sets $\{O_y\}_{y\in F_\Lambda}$ consisting of points in $\mathbb{R}^n$ for which y is the unique closest point in F_Λ. In the computational geometry literature the sets O_y are called Voronoi cells (attached to the point set F_Λ) and the corresponding tessellation of $\mathbb{R}^n$ by convex polytopes is called the *Voronoi tessellation* (or Voronoi diagram) [OBSC00]. Similarly, the non-differentiability locus S_{ϕ_∞} of ϕ_∞ is the $(n-1)$-dimensional stratum in the *Delaunay tessellation* of $\mathbb{R}^n$ whose 0-dimensional stratum is given by the point set F_Λ. The Delaunay tessellation can be defined as the dual tessellation of the Voronoi tessellation, in a suitable sense. For example, when $n = 2$ this simply means that S_{ϕ_∞} is obtained by connecting any two points in F_Λ which are neighbors in the corresponding Voronoi tessellation by a segment [OBSC00].

Remark 4.8. — Under suitable generality assumptions it is well-known that the corresponding Delaunay tessellation consists of simplices giving a triangulation of F with remarkable optimality properties [OBSC00].

The previous proposition give a rigorous mathematical justification of the Voronoi tessellations appearing in numerical simulations in cosmology, which use periodic boundary conditions [KPSM92, HSvdW14, HvdWV$^+$12]: for large times Voronoi polytopes form around points where H has its absolute minimum (the Voronoi polytopes in question are called voids in the cosmology literature, since the mass in the universe is localized on the shock locus S_{ψ_∞} between voids). The dual Delaunay tessellation is also frequently used for the numerics [KPSM92, HSvdW14, HvdWV$^+$12].

Remark 4.9. — When H has a unique minimum x_m (modulo Λ), the corresponding convex piecewise affine function ψ_∞ appears naturally in tropical geometry as a tropical theta function with characteristics (in the case when x_m and Λ are defined over the integers). The tropical subvariety defined by its non-differentiability locus is called the tropical theta divisor and seems to first have appeared in complex geometry in the compactification of the moduli space of abelian varieties (see [MZ08] and references therein).

5. Application to Hele-Shaw type flows

5.1. Background. — The Hele-Shaw flow was originally introduced in fluid mechanics in the end of the 19th century to model the expansion of an incompressible fluid of high viscosity (for example oil) injected at a constant rate in another fluid of low viscosity (such as water) in a two dimensional geometry. Nowadays the Hele-Shaw flow, also called *Laplacian growth,* is ubiquitous in engineering, as well as in mathematical physics where it appears in various areas ranging from diffusion limited aggregation (DLA) to integrable systems (the dispersionless limit of the Toda lattice hierarchy), random matrix theory and quantum gravity; see [Vas09, GV06] and references therein.

To explain the general geometric setup, introduced in [HS02], we let X be a compact Riemann surface and fix a point p (the injection point) together with an area form ω_0 of total area one (whose density models the inverse permeability of the medium). The classical situation appears when X is the Riemann sphere and p is the point at infinity so that $X \smallsetminus \{p\}$ may be identified with the complex plane $\mathbb{C}$. A family of increasing domains $\Omega^{(\lambda)}$ with time parameter $\lambda \in [0,1]$ is said to be a classical solution to the Hele-Shaw flow corresponding to (p,ω_0) if $\Omega^{(0)} = \varnothing$ and the closure of $\Omega^{(\lambda)}$ is diffeomorphic to the unit-disc in $\mathbb{C}$ for $\lambda > 0$, the point p is contained in the interior of $\Omega^{(\lambda)}$, the area grows linearly:

$$\int_{\Omega^{(\lambda)}} \omega_0 = \lambda$$

and the velocity of the boundary $\partial\Omega^{(\lambda)}$ equals minus the gradient (with respect to ω_0) of the Green function g_p for $\Omega^{(\lambda)}$ with a logarithmic pole at p (i.e., Darcy's law holds). Such a solution exists for λ sufficiently small (see [HS02] for the case when ω_0 is real analytic and [RWN15c] for the general case). However, typically the boundary of the expanding domains $\Omega^{(\lambda)}$ develop a singularity for some time $\lambda < 1$ and then changes its topology so that the notion of a classical solution breaks down. Still, there is a well-known notion of weak solution of the Hele-Shaw flow, defined in terms of subharmonic envelopes (obstacles) and which exists for any $\lambda \in [0,1]$ (where $\Omega^{(1)} = X$); see [HS02] and references therein. In our notations the envelopes in question may be defined as

$$\phi_\lambda := \sup_{\phi \in \mathrm{PSH}(X,\omega_0)} \left\{ \phi \mid \phi \leqslant 0,\ \phi \leqslant \lambda \log|z-p|^2 + O(1) \right\}, \tag{5.1}$$

which, for λ fixed, is thus a restrained version of the envelope $P_{\omega_0}(0)$ defined in Section 2.3.2, where one imposes a logarithmic singularity of order λ at the given point p. The *weak Hele-Shaw flow* is then defined as the evolution of the corresponding increasing non-coincidence sets:

$$\Omega^{(\lambda)} := \{\phi_\lambda < 0\} \subset X,$$

(which thus is empty for $\lambda = 0$, as it should). We will write

$$X^{(\lambda)} := X \smallsetminus \Omega^{(\lambda)}$$

for the corresponding decreasing "water domains". When ω_0 is real analytic it follows from the results in [HS02, Sak91] (applied to the pull-back of ω_0 to the universal covering $\widetilde{X}$ of X) that the boundary of $\Omega^{(\lambda)}$ is a piecewise real analytic curve having a finite number of cusp and double points (if moreover ω_0 has negative Ricci curvature then the lifted Hele-Shaw on $\widetilde{X}$ exists for any $t > 0$).

Example 5.1. — The classical situation in fluid mechanics appears when X is the Riemann sphere and p is the point at infinity, so that $X \smallsetminus \{p\}$ may be identified with the complex plane $\mathbb{C}$. Writing $\omega_0 = dd^c\Phi_0$ in $\mathbb{C}$ (where the condition Φ_0 has logarithmic growth, since $\int \omega_0 = 1$), the function ϕ_λ may be identified with the subharmonic function $\Phi_\lambda := \Phi_0 + \phi_\lambda$ with the property that $\Phi_\lambda = (1-\lambda)\log|z|^2 + O(1)$ as $z \to \infty$. Accordingly, $X^{(\lambda)}$ may, for $\lambda > 0$, be identified with a decreasing family of compact domains in $\mathbb{C}$.

5.2. A canonical regularization of the Hele-Shaw flow using the Kähler-Ricci flows.

— To make the link to the present setting of Kähler-Ricci flows we set

$$\theta = \omega_0 - \delta_p, \tag{5.2}$$

where δ_p denotes the Dirac measure at p, which defines a trivial cohomology class (this is thus a singular version of the setting in Section 3.2). The corresponding Kähler-Ricci flows will be defined as follows: first fixing a Kähler form ω on X we set

$$\theta_\beta := \theta + \frac{1}{\beta}\operatorname{Ric}\omega,$$

for a fixed Kähler form ω, i.e., by imposing the equation (2.5). Moreover, we will use ω_0 as the initial data in the corresponding Kähler-Ricci flows. We then get the following theorem saying that the corresponding Kähler-Ricci flows concentrate, as $\beta \to \infty$, precisely on the complement $X^{(\lambda)}$ of $\Omega^{(\lambda)}$ (i.e., on the "water domain") up to a time reparametrization:

Theorem 5.2. — *Consider the non-normalized Kähler-Ricci flow $\omega^\beta(t)$ with twisting current θ_β as above and initial condition ω_0. Then*

$$\lim_{\beta\to\infty} \omega^{(\beta)}(t) = 1_{X\smallsetminus\Omega^{(\lambda(t))}}(t+1)\omega_0,$$

weakly on X, where Ω^λ is the weak Hele-Shaw flow corresponding to (p, ω_0) and $\lambda(t) = t/(t+1)$. Moreover,

$$\sup_X \frac{\omega^{(\beta)}(t)}{\omega} \leqslant (t+1)\sup_X \frac{\omega_0}{\omega}.$$

Remark 5.3. — If one instead let $\omega^\beta(t)$ denote the corresponding *normalized* Kähler-Ricci flow, which has total area $e^{-t}(=1-\lambda)$, then the corresponding limiting measure is given by $1_{X\smallsetminus\Omega^{(\lambda(t))}}\omega_0$ and the last estimate above holds without the factor $(t+1)$. In particular, in the canonical case, where ω is taken as ω_0, setting $\eta_t:=\omega_0-\omega^{(\beta)}(t)$ then yields a family of semi-positive forms of increasing area $1-e^{-t}$ concentrating on the "oil-domains" Ω_t.

To prove the previous theorem we first need to make the link between the envelopes (5.1) and the ones appearing in our setting. To this end we introduce, as before, the potential f of θ (with respect to the reference semi-positive form $\chi=0$ in $[\theta]$, satisfying

$$\theta=dd^cf,$$

which defines a lower semicontinuous function $f: X\to\,]0,\infty]$ which is smooth on $X\smallsetminus\{p\}$ and such that $-f$ has a logarithmic singularity of order one at p.

Lemma 5.4. — *The following holds*

$$\phi_\lambda:=(1-\lambda)P_{\omega_0}\Big(\frac{\lambda}{(1-\lambda)}f\Big)-\lambda f,$$

Equivalently, setting $t=\lambda/(1-\lambda)$ (i.e., $\lambda:=t/(t+1)$) gives

$$\Omega^{(\lambda)}:=\{P_{\omega_0}(tf)<tf\}:=\Omega_t.$$

Proof. — By a simple scaling argument it will be enough to prove that

$$\phi_\lambda=P_{\omega_0(1-\lambda)}(\lambda f)-\lambda f.$$

But the latter identity follows immediately from the fact that a given function $\phi\in\mathrm{PSH}(X,\omega_0)$ has a logarithmic pole of order at least λ at a point p, i.e., it satisfies

$$\phi+\lambda f\leqslant C$$

on X if and only if the $\omega_0(1-\lambda)$-psh function $\phi+\lambda f$ on $X\smallsetminus\{p\}$ extends to a unique $\omega_0(1-\lambda)$-psh function on all of X (as follows from the basic local fact that a psh function has a unique psh extension over an analytic subvariety, or more generally over a pluripolar subset). □

Finally, we need to extend Theorem 2.2 to the present setting. To this end we first recall that, by [ST17, Th. 3.2], there is, for β fixed, a notion of weak Kähler-Ricci flows on X which applies to any twisting current θ which is smooth away from a (suitable) divisor D in X. In particular, the result applies to any current θ of the form

$$\theta=\theta_0-[E],$$

where θ_0 is smooth and $[E]$ denotes the current of integration along an effective divisor, i.e.,

$$D=-E:=-\sum_i c_iE_i$$

for $c_i>0$ and E_i are irreducible hypersurfaces in X. The result in [ST17, Th. 3.2] yields a unique flow $\omega^{(\beta)}(t)$ of currents in $[\omega_0+t\theta]$ which are smooth on $X\smallsetminus D$ and such that the corresponding Kähler potentials are in $L^\infty(X)$ (as shown in [EGZ16, §4.2] this flow coincides with the unique viscosity solution constructed in [EGZ16, §4.2]).

Theorem 5.5. — *Let θ be a current of the form $\theta = \theta_0 - [E]$, with θ_0 smooth and E an effective divisor. Then the conclusion in Theorem 2.2 still applies and the constant C only depends on upper bounds on θ_0 (and the oscillation of its potential) and on the divisor E. Moreover, the sharp bounds in Theorem 2.3 still hold with θ replaced by θ_0.*

Proof. — We recall that the weak KRF defined in [ST17, Th. 3.2] is constructed by approximating θ with a suitable sequence θ_ε of smooth forms. In the present setting this can be done so that $\theta_\varepsilon \leqslant C\omega$ and θ_ε converges to θ in $\mathscr{C}^\infty_{\mathrm{loc}}(X \smallsetminus E)$. Indeed, decomposing $f = f_0 + f_E$ in terms of potentials for θ_0 and $-[E]$, respectively, we have that up to a smooth function f can be written as $-\log \|s_E\|^2$, where s_E is a holomorphic section of the line bundle $\mathscr{O}(E)$ cutting out E and $\|\cdot\|$ is a fixed smooth Hermitian metric on $\mathscr{O}(E)$. Then the form θ_ε is simply obtained by replacing $\log \|s_E\|^2$ with $\log(\|s_E\|^2 + \varepsilon)$. The proof of the theorem then follows immediately from Theorem 2.2 applied to θ_ε by noting that that $P(f) \leqslant \sup_X f_0 + P(f_E)$, where the second term thus only depends on the divisor E, as desired (and is finite, by Lemma 2.11). ☐

Example 5.6. — Coming back to the classical setting when E is the point p and $X \smallsetminus \{p\} = \mathbb{C}$ considered in the previous example, the density $\rho^{(\beta)}(t)$ with respect to Lebesgue measure on $\mathbb{C}$ of the Kähler form $\omega^{(\beta)}(t)$ on $X \smallsetminus \{p\}$ is a solution of the following logarithmic diffusion equation for the smooth and strictly positive probability densities $\rho(t)$ on $\mathbb{C}$

$$\frac{\partial \rho(t)}{\partial t} = \frac{1}{\pi\beta}\,\partial_{\overline{z}}\,\frac{\partial_z \rho(t)}{\rho(t)} + \rho_0 + O(1/\beta), \quad \rho(0) = \rho_0,$$

where the last term is equal to $\frac{1}{\beta}\Delta \log \rho_0(t)$ (but it could be removed at the expense of slightly worse estimates in t and β). The equivalence between Ricci flow on Riemann surfaces and logarithmic diffusion is well-known [VER96], but as far as we know the limit $\beta \to \infty$ has not been investigated before.

5.3. Monge-Ampère growth. — There is also a natural higher dimensional generalization of the Hele-Shaw flow/Laplacian growth on a compact Kähler manifold (X, ω_0) where the higher dimensional viscous "fluid" is injected along a given effective divisor E on X. Indeed, one simply defines ϕ_λ as before, but imposing a singularity of order λ along E (i.e., $z - p$ is in formula (5.1) replaced by a local defining equation for E). Then one obtains a sequence of increasing domains Ω_λ as before for which the name *Monge-Ampère growth* was proposed in [Ber14]. The terminology is motivated by the fact that Ω_λ can be characterized as the solution of a free boundary problem for the complex Monge-Ampère operator on (X, ω_0) with singular obstacle λf (see Remark 2.9), where f is defined by

$$\theta = \omega_0 - [E], \quad \theta = dd^c f,$$

as before. By the recent results in [RWN17b], for λ sufficiently small, Ω_λ is diffeomorphic to a ball (and admits a regular foliation, transversal to E, by holomorphic discs along which ϕ_λ is ω_0-harmonic).

Now, by Theorem 5.5, the volume forms $\omega_\beta^n(t)$ of the Kähler-Ricci flows with twisting form θ as above concentrate on $X^{(\lambda)}(:= X \smallsetminus \Omega^{(\lambda(t))})$:

$$\lim_{\beta\to\infty} \omega_\beta^n(t) = \frac{1_{X^{(\lambda)}}\omega_0^n}{\int_{X^{(\lambda)}} \omega_0^n}$$

with uniform upper bounds on the normalized Kähler forms $\omega_\beta(t)/(t+1)$ on $X \smallsetminus E$, as before (in this setting $\int_{X^{(\lambda)}} \omega_0^n = [\omega_0 - \lambda(t)E]^n$).

Example 5.7. — In the case when $X = \mathbb{P}^n$ equipped with a Kähler form ω_0 of unit volume and E is the hyperplane at infinity the corresponding sets $X(t)$ yield, for $t > 0$, a decreasing family of compact domains in $\mathbb{C}^n$ of volume $1/(t+1)^n$.

Remark 5.8. — As shown in [RWN17a] performing a Legendre transform of ϕ_λ with respect to λ produces a weak geodesic ray $\widehat{\phi}_\tau$ in the space of Kähler metrics (compare Remark 4.6). Moreover, topology change in the corresponding Hele-Shaw flow $\Omega^{(\lambda)}$ corresponds (in a certain sense) to singularities of the geodesic $\widehat{\phi}_\tau$ [RWN15b, RWN15a]. In a nutshell, this stems from the fact (shown in [RWN17a]) that $\Omega^{(\lambda)} = \{h < \lambda\}$ where $h(x) := d\widehat{\phi}_\tau/d\tau|_{\tau=0^+}$.

6. Twisting currents with merely continuous potentials

6.1. Continuous potentials. — Without loss of generality we may and will in this section, assume that $\varphi_0 = 0$.

As will be explained in this section the weak convergence in Theorem 1.1 can be extended to any twisting form (or rather current) with continuous potentials.

To illustrate this we start with the case $n = 1$ and assume that $\frac{1}{\beta}c_1(K_X) + [\theta_\beta]$ is trivial, i.e., that the non-normalized KRF preserves the initial cohomology class. To simplify the notation we will drop the subscript β in the notation f_β for the corresponding twisting potential.

Proposition 6.1. — *Assume that $n = 1$ and f is Hölder continuous. Then there is a unique solution $\varphi^{(\beta)}(t)$ to the corresponding non-normalized KRF which is in $\mathscr{C}^{2,\alpha}(X)$ for some $\alpha > 0$.*

Proof. — In the following β will be fixed and we will not pay attention to the dependence on β. First assume that f is smooth. Differentiating the non-normalized KRF with respect to t reveals that $d\varphi^{(\beta)}(t)/dt$ evolves by the heat equation for the metric $\omega_\beta(t)$ and hence, by the parabolic maximum principle, $|d\varphi^{(\beta)}(t)/dt| \leqslant C$, where the constant only depends on $\sup_X |f|$. The defining equation for the KRF then gives that $C'^{-1} \leqslant \omega_\beta(t) \leqslant C'$ for a positive constant C' only depending on $\sup_X |f|$. But then applying the parabolic Krylov-Safonov Hölder estimate to the heat equation with respect to $\omega_\beta(t)$ gives that there exists a Hölder exponent α' such that $\left\|d\varphi^{(\beta)}(t)/dt\right\|_{\mathscr{C}^{\alpha'}} \leqslant C''$. Using again the defining equation for $\varphi^{(\beta)}(t)$, we deduce that

$1+\Delta_\omega\varphi^{(\beta)}(t)=e^{\beta g_\beta(t)}$, where the Hölder norm of $g_\beta(t)$ is under control, for some Hölder exponent. But then the proof is concluded by invoking the classical Schauder estimates for the Laplacian Δ_ω and approximating f with smooth functions (note that the limit of the approximate solutions is unique, by the comparison principle). □

Given a twisting potential f we denote by $P_t^{(\beta)}f$ the solution of the corresponding KRF at time t and set $P_tf:=P_{\omega_0}(tf)$.

Lemma 6.2. — *The operator $P_t^{(\beta)}$ is increasing, i.e., if $f\leqslant g$, then $P_t^{(\beta)}f\leqslant P_t^{(\beta)}g$. Moreover, $P_t^{(\beta)}(f+c)=P_t^{(\beta)}(f)+ct$ for any $c\in\mathbb{R}$ and hence*

$$\left\|P_t^{(\beta)}f-P_t^{(\beta)}g\right\|_{L^\infty(X)}\leqslant t\left\|f-g\right\|_{L^\infty(X)}, \tag{6.1}$$

and similarly for the operator P_t.

Proof. — The increasing property follows directly from the comparison principle and the scaling property from the very definitions of the flows. □

Theorem 6.3. — *Let X be a Riemann surface endowed with the twisting current $\theta=dd^cf$, where f is Hölder continuous. Then the corresponding non-normalized KRFs $\omega_\beta(t)$ defines a family of Hölder continuous Kähler metrics satisfying the weak convergence in Theorem* 1.1*, as $\beta\to\infty$ (more precisely, the convergence holds in $\mathscr{C}^0(X)$) on the level of Kähler potentials).*

Proof. — In the following t will be fixed once and for all. Let f_ε be a family of smooth functions such that $\|f_\varepsilon-f\|_\infty\leqslant\varepsilon$. By the previous lemma

$$\left\|P_t^{(\beta)}f-P_tf\right\|_{L^\infty(X)}\leqslant\left\|P_t^{(\beta)}f_\varepsilon-P_tf_\varepsilon\right\|_{L^\infty(X)}+2\varepsilon t.$$

Hence, letting first $\beta\to\infty$ (using Theorem 2.2) and then $\varepsilon\to0$ concludes the proof.

Of course, even if $\omega_\beta(t)$ is bounded for a fixed β the limiting current $\omega_\infty(t)$ will, in general, not be bounded unless f has a bounded Laplacian. The previous theorem also holds when f is assumed to be merely continuous, but then the corresponding evolution equations have to be interpreted in a generalized sense. More generally, when f is continuous and the dimension n of X is arbitrary the corresponding KRFs are well-defined in the sense of viscosity solutions and satisfy the comparison principle, by [EGZ16]. Accordingly, the $\mathscr{C}^0$-convergence in the previous theorem still holds. However, even if f is Hölder continuous it does not seem to follow, in general, from existing regularity theory that $\omega_\beta(t)$ is even bounded, for β fixed. □

6.2. An outlook on random twistings. — Hölder continuous potentials f appear naturally when f is taken to be an appropriate random Gaussian function. For example, in the setting described in Section 4.3, when $n=1$ and $X=\mathbb{R}/\mathbb{Z}+i\mathbb{R}/\mathbb{Z}$ and the potential f is assumed invariant along the imaginary direction, we can identify the potential f with a 1-periodic function $f(x)$ on $\mathbb{R}$ and expand $f(x)$ in a Fourier series:

$$f(x)=\sum_{k\in\mathbb{Z}}A_k\cos(2\pi kx)+B_k\sin(2\pi kx).$$

Taking the coefficients A_k and B_k to be independent Gaussian random numbers with mean zero and variance proportional to k^{-3-2h}, for a given number $h \in [-1,1]$, it is well-known that f is almost surely in the Hölder class $\mathscr{C}^{1,h}$. Indeed, the derivative f' is a Brownian fractional bridge, whose sample paths are well-known to be almost surely in $\mathscr{C}^h$ (recall that a Brownian bridge is defined as a Brownian motion B conditioned by $B(0) = B(1)$ and similarly in the fractional case, with $h = 1/2$ corresponding to ordinary Brownian motion). The corresponding limiting convex envelopes $\phi_t(x)$ have been studied extensively in the mathematical physics literature in the setting of Burgers' equation and the adhesion model, where f' represents the random initial velocity function (compare Section 4). According to a conjecture in [SAF92], for any fixed positive time t, the support X_t of the distribution second derivative of the corresponding random function $\phi_t(x)$ on $\mathbb{R}$ is almost surely of Hausdorff dimension h when $h \in [0,1]$ and 0 when $h \in [-1,0]$ (which, when $h = 1$ is consistent with the uniform bound in Theorem 1.1 and formula (2.9) which, in this real setting, holds as long as $f \in \mathscr{C}^{1,1}$). See [Gir03] for the case when $h = -1/2$ and [Sin92] for a proof of the conjecture in the case $h = 1/2$ in a non-periodic setting.

In view of the connections to the Kähler-Ricci flow and the Hele-Shaw flow exhibited in Sections 4 and 5 it would be interesting to extend this picture to any complex manifold, or at least to Riemann surfaces. For example, in the latter case one would, at least heuristically, get conformally invariant processes of random metrics $\omega_\beta(t)$ by taking f to be a Gaussian free field on X. Heuristically, this means that f is taken as random function in the corresponding Dirichlet Hilbert space $H^1(X)/\mathbb{R}$. However, the situation is complicated by the fact that, almost surely, f only exists as a distribution in a certain Banach completion of $H^1(X)/\mathbb{R}$. [She07]. On the other hand the formal random measure appearing in the static version of the non-normalized KRF, i.e., in the Laplace equation

$$\omega_0 + dd^c\varphi_\beta(t) = e^{-\beta f}\omega_0$$

appears as the Liouville measure of quantum gravity and has been rigorously defined, for $\beta \in]0,2[$ in [DS11] using a regularization procedure. But as far as we know the corresponding stochastic parabolic problem has not been investigated.

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