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NON-DENSITY OF STABILITY FOR HOLOMORPHIC MAPPINGS ON $\mathbb{P}^k$

by Romain Dujardin

Abstract. — A well-known theorem due to Mañé-Sad-Sullivan and Lyubich asserts that J-stable maps are dense in any holomorphic family of rational maps in dimension 1. In this paper we show that the corresponding result fails in higher dimension. More precisely, we construct open subsets in the bifurcation locus in the space of holomorphic mappings of degree d of $\mathbb{P}^k(\mathbb{C})$ for every $d \geqslant 2$ and $k \geqslant 2$.

Résumé (Non-densité de la stabilité pour les applications holomorphes sur $\mathbb{P}^k$)
Un théorème célèbre dû à Mañé-Sad-Sullivan et Lyubich affirme que les paramètres J-stables forment un ouvert dense de toute famille holomorphe de systèmes dynamiques rationnels en dimension 1. Dans cet article nous montrons que ce résultat ne subsiste pas en dimension supérieure. Plus précisément nous construisons des ouverts contenus dans le lieu de bifurcation des applications holomorphes de degré d de $\mathbb{P}^k(\mathbb{C})$ pour tout $d \geqslant 2$ et $k \geqslant 2$.

Contents

1. Introduction

Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of holomorphic self-mappings on the complex projective space $\mathbb{P}^k$. When the dimension k equals 1, the stability/bifurcation theory of such families was developed in the beginning of the 1980's independently by Mañé, Sad and Sullivan [MSS83] and Lyubich [Lyu83, Lyu84], who designed the seminal notion of J-stability, that is, structural stability on the Julia set, which almost implies structural stability on $\mathbb{P}^1$ (see [MS98]). A salient feature of their work is that the set

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of J-stable mappings is (open and) dense in any such family, which is ultimately a consequence of the finiteness of the critical set.

In higher dimension we denote by J^* the "small Julia set" of f, which by definition is the support of its measure of maximal entropy. Typically, J^* is smaller than the usual Julia set J, and concentrates in a sense the repelling part of the dynamics of f, and most of the entropy.

In a remarkable recent paper, Bianchi, Berteloot and Dupont [BBD17] put forward J^*-stability as the correct generalization of the Mañé-Sad-Sullivan-Lyubich theory in higher dimension. As these authors point out, there is no reason to expect that the density of J^*-stability should hold in this setting, and leave the existence of persistent bifurcations as an open problem [BBD17, §6.2].

At this stage it is worth mentioning that for invertible dynamics in dimension 2, the classical work of Newhouse [New70] shows that persistent bifurcations can be obtained by constructing persistent (generic) homoclinic tangencies. In the holomorphic context, Buzzard [Buz97] proved that persistent homoclinic tangencies exist in the space of polynomial automorphisms of $\mathbb{C}^2$ of sufficiently high degree. Furthermore such automorphisms can easily be "embedded" inside holomorphic mappings of $\mathbb{P}^2$ (see e.g. [HP94, §6] and also [Gav98]). It seems however that for these examples, the Newhouse phenomenon is somehow unrelated to J^*-(in)stability, which has to do with repelling periodic orbits (see Section 5.1 for a more precise discussion).

Our main goal in this paper is to address the density problem for J^*-stability. We denote by $\mathcal{H}_d(\mathbb{P}^k)$ the space of holomorphic mappings of $\mathbb{P}^k$ of degree d. By choosing a set of homogeneous coordinates on $\mathbb{P}^k$, we can express $f \in \mathcal{H}_d(\mathbb{P}^k)$ in terms of a family of $(k+1)$ homogeneous polynomials in $(d+1)$ variables without common factor. In this way we get a natural identification between $\mathcal{H}_d(\mathbb{P}^k)$ and a Zariski open subset of $\mathbb{P}^N$, for $N = (k+1)\big[(d+k)!/d!k!\big] - 1$.

Main Theorem. — *The bifurcation locus has non-empty interior in the space $\mathcal{H}_d(\mathbb{P}^k)$ for every $k \geqslant 2$ and $d \geqslant 2$.*

As we shall see, our constructions are rather specific, since we start with the simplest possible mappings, namely products, and construct robustly bifurcating examples by taking small perturbations. More interestingly, we isolate some *mechanisms* leading to open sets of bifurcations. The next step would be to understand the prevalence of these phenomena in parameter space.

While he was working on this paper, the author learned about the work of Bianchi and Taflin [BT17], in which the authors construct a 1-dimensional family of rational maps on $\mathbb{P}^2$ whose bifurcation locus is the whole family, thus offering an alternative approach to the question of [BBD17].

Throughout the paper, to avoid overwhelming the ideas under undue technicalities, we focus on dimension $k = 2$ which is the first interesting case, and explain in Section 5.2 how to adapt the arguments to higher dimension.

By the analysis of Berteloot, Bianchi and Dupont, to obtain open subsets in the bifurcation locus, it is enough to create a persistent intersection between the post-critical set and a hyperbolic repeller contained in J^*. We present two mechanisms leading to such persistent intersections. The first one is based on topological considerations: the idea is to construct a kind of topological manifold contained in J^*, which must intersect the post-critical set for homological reasons. We apply this strategy for mappings that are small perturbations of product maps of the form $(p(z), w^d)$ (see Theorem 3.1). Of course what is delicate here is to manage dynamically defined sets with inherently complicated topology.

The second mechanism is based on ideas from fractal geometry. It relies on the construction of Cantor sets with a very special geometry: namely they are "fat" in a certain direction and admit persistent intersections with manifolds that are sufficiently transverse to the fat direction. The use of this property in dynamical systems originates in the work of Bonatti and Díaz [BD96] and has played an important role in C^1 dynamics since then (see the book by Bonatti, Díaz and Viana [BDV05] for a broader perspective). According to the real dynamics terminology we will refer to these Cantor sets as *blenders*.

We implement this idea for product mappings of the form $(p(z), w^d + \kappa)$, where p admits a repelling fixed point z_0 which is almost parabolic, that is $1 < |p'(z_0)| < 1.01$, and κ is large (see Theorem 4.7). The parabolic case follows by taking the limit (see Corollary 4.11).

Despite the specific nature of these examples, our thesis is that the interior of the bifurcation locus is quite large. We point out a few explicit questions and conjectures in Section 5.3. In particular since by [BBD17] bifurcations happen when a multiplier of a repelling periodic point crosses the unit circle, it is likely that blenders are often created in this process.

Note also that blenders already appear in complex dynamics in the work of Biebler [Bie16] to construct polynomial automorphisms of $\mathbb{C}^3$ with persistent homoclinic tangencies.

The plan of the paper is as follows. In Section 2 we collect a few facts from J^*-stability theory. The topological mechanism for robust bifurcations is detailed in Section 3, while Section 4 is devoted to complex blenders. Finally in Section 5 we explain how to extend the construction of Section 4 to higher dimensions and also show that J^*-stability is compatible with the Newhouse phenomenon. We also state a number of open problems and directions for future research. The main theorem ultimately follows from Theorems 3.1 and Theorem 4.7 for the case $k = 2$, $d \geqslant 3$, Theorem 4.12 for $k = 2$, $d = 2$ and Theorem 5.4 for $k \geqslant 3$.

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2. Preliminaries

Notation. — When a preferred parameter λ_0 is given in Λ, to simplify notation we denote f_{λ_0} by f_0. We use the convention to mark with a hat the objects in $\Lambda \times \mathbb{P}^k$, like $\widehat{f} : (\lambda, z) \mapsto (\lambda, f_\lambda(z))$, etc.

2.1. Stability and bifurcations for endomorphisms of $\mathbb{P}^k$. — Let f be a holomorphic map of degree d on $\mathbb{P}^k$. A classical result of Briend and Duval asserts that it admits a unique invariant measure of maximal entropy μ_f, whose (complex) Lyapunov exponents χ_i, $i=1,\dots,k$ satisfy $\chi_i \geqslant \frac{1}{2}\log d$, and which describes the asymptotic distribution of (repelling) periodic orbits [BD01]. Recall from the introduction that for a holomorphic map f on $\mathbb{P}^k$, we denote by J^* the support of μ_f.

Repelling periodic points are dense in J^*. In general the closure of repelling periodic orbits can be strictly larger than J^*, which is a source of technicalities in J^*-stability theory. Note however that this phenomenon does not happen for regular polynomial skew products of $\mathbb{C}^2$ (see [Jon99]), which are basic to all constructions in this paper.

Let now $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of holomorphic maps of degree d on $\mathbb{P}^k$ (or equivalently, a holomorphic map $\Lambda \to \mathcal{H}_d(\mathbb{P}^k)$). A notion of stability for such a family was recently introduced in [BBD17]. The relevance of this notion is justified by its number of natural equivalent definitions. The following is a combination of Theorems 1.1 and 1.6 in [BBD17], with a slightly different terminology (see below for the notion of Misiurewicz bifurcation).

Theorem 2.1 (Berteloot, Bianchi & Dupont). — *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of holomorphic maps of degree d on $\mathbb{P}^k$. Then the following assertions are equivalent:*

(i) *The function on Λ defined by the sum of Lyapunov exponents of μ_{f_λ}: $\lambda \mapsto \chi_1(\lambda)+\cdots+\chi_k(\lambda)$ is pluriharmonic on Λ.*

(ii) *The sets $(J^*(f_\lambda))_{\lambda\in\Lambda}$ move holomorphically in a weak sense.*

(iii) *There is no (classical) Misiurewicz bifurcation in Λ.*

If in addition Λ is a simply connected open subset of $\mathcal{H}_d(\mathbb{P}^k)$ or if Λ is any simply connected complex manifold and $k=2$, these conditions are equivalent to:

(iv) *Repelling periodic points contained in J^* move holomorphically over Λ.*

If these equivalent conditions hold we say that the family is *J^*-stable* over Λ. For an arbitrary family we thus have a maximal open set of stability, the *stability locus* and its complement is by definition the *bifurcation locus*. We denote by Bif the bifurcation locus.

Condition (ii) will not be used below so we don't need to explain what the "weak sense" in (ii) exactly is. By "classical Misiurewicz bifurcation" in (iii) we mean a proper intersection in $\Lambda\times\mathbb{P}^k$ between a post-critical component of $\widehat{f}$ and the graph of a holomorphically moving repelling periodic point $\gamma:\Omega\to\mathbb{P}^k$ over some open set $\Omega\subset\Lambda$. We will treat (a generalized form of) this notion in detail in Section 2.2 below (see in particular Definition 2.4 for the notion of proper intersection).

Let us also quote the following result, which is contained in Proposition 2.3 and Theorem 3.5 in [BBD17].

Proposition 2.2. — *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of holomorphic mappings of degree d on $\mathbb{P}^k$, with Λ simply connected. Assume that there exists a holomorphic map $\gamma:\Lambda\to\mathbb{P}^k$ such that for every $\lambda\in\Lambda$, $\gamma(\lambda)$ is disjoint from the post-critical set of f_λ. Then (f_λ) is J^*-stable.*

2.2. Hyperbolic repellers in J^*. — Recall that if E_0 is an invariant hyperbolic compact set for f_0, then for $f \in \mathcal{H}_d$ sufficiently close to f_0, there exists a hyperbolic set $E(f)$ for f and a conjugating homeomorphism $h = h(f) : E_0 \to E(f)$. The conjugacy can be chosen to depend continuously on f, with $h(f_0) = \mathrm{id}$, in which case it is unique and is actually a holomorphic motion. We will refer to it as the *continuation* of E_0.

According to the usual terminology, a *basic repeller* E is a locally maximal and transitive hyperbolic repelling invariant set for f_0. It is classical that if E is a basic repeller, then repelling periodic points are dense in E.

Lemma 2.3 (see also [Bia16, Lem. 2.2.15]). — *Let f_0 be a holomorphic map on $\mathbb{P}^k$ of degree $d \geqslant 2$. Let E_0 be a basic repeller for f_0 such that $E_0 \subset J^*(f_0)$. Then there exists a neighborhood Ω of f_0 in $\mathcal{H}_d$ such that for $f \in \Omega$ the continuation $E(f)$ of E_0 is a well-defined basic repeller for f, contained in $J^*(f)$.*

Proof. — The existence of $E(f)$ and the fact that it is a basic repeller follows from general hyperbolic dynamics, so we concentrate on the last conclusion. By assumption there exists a neighborhood N of E_0 such that $\bigcap_{k\geqslant 0} f^{-k}(N) = E_0$, where we restrict to the preimages remaining in N. We can assume that N is a r-neighborhood of E_0 for some $r = 2r_0 > 0$. Since E_0 is contained in $J^*(f_0)$, for every $x \in E_0$ we have that $\mu_{f_0}(B(x, r_0)) > 0$. Furthermore by compactness there exists δ_0 such that for every $x \in E_0$, $\mu_f(B(x, r_0/2)) \geqslant \delta_0$.

Now, for f close enough to f_0, $N = \bigcup_{x\in E_0} B(x, 2r_0)$ is also an isolating neighborhood for $E(f)$. In addition, the uniform continuity of the potential of the maximal entropy measure implies that if $x \in E_0$, $\mu_f(B(x, r_0/2)) \geqslant \delta_0/2$. By construction, if f is close enough to f_0, every y in E_f is the intersection of a sequence of nested preimages: $\{y\} = \bigcap_{k\geqslant 0} f_y^{-n}(B(f^n(y), r_0))$, where f_y^{-n} denotes the inverse branch of f^n mapping $f^n(y)$ to y, and $f^n(y)$ is $r_0/2$ close to E_0. Therefore $B(f^n(y), r_0) \supset B(x, r_0/2)$ for some $x \in E_0$, hence $\mu_f(B(f^n(y), r_0)) \geqslant \delta_0/2$, and finally $y \in \mathrm{Supp}(\mu_f)$. □

We now introduce a notion of proper intersection between a hyperbolic set and the post-critical set.

Definition 2.4. — Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of holomorphic maps on $\mathbb{P}^k$ of degree $d \geqslant 2$ on $\mathbb{P}^k$ and let $\lambda_0 \in \Lambda$. Let E_0 be a basic repeller for f_0. We say that a post-critical component $V_0 \subset f_0^n(\mathrm{Crit}(f_0))$ intersects E_0 properly if there exists $x_0 \in V_0 \cap E_0$ such that the graph $\widehat{x}$ of the continuation of x_0 as a point of E_0 is not contained in $\widehat{V}$, where $\widehat{V}$ is the irreducible component of $\widehat{f}^n(\mathrm{Crit}(\widehat{f}))$ containing V_0.

The terminology is justified by the fact that since $\widehat{f}^n(\mathrm{Crit}(\widehat{f}))$ is of codimension 1 in $\Lambda \times \mathbb{P}^k$, the assumption on $\widehat{x}$ implies that the graph $\widehat{x}$ and $\widehat{f}^n(\mathrm{Crit}(\widehat{f}))$ intersect properly at (λ_0, x_0). Note that this terminology is slightly inadequate since the notion depends on the family and not only on f_0.

The mechanism leading to robust bifurcations will be based on the following lemma, which is implicitly contained in [BBD17] (see the proof of Proposition 3.8 there).

Proposition-Definition 2.5. — *Let $(f_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of holomorphic maps on $\mathbb{P}^k$ of degree $d \geqslant 2$ on $\mathbb{P}^k$. Assume that for some $\lambda_0 \in \Lambda$, there exists a basic repeller E_0, contained in $J^*(f_0)$ and an integer $n \geqslant 1$ such that $f_0^n(\mathrm{Crit}(f_0))$ intersects E_0 properly.*

Then λ_0 belongs to the bifurcation locus of the family. If such a situation happens, we say that a Misiurewicz bifurcation occurs at λ_0.

Proof. — Fix a sequence of repelling periodic points x_j in E_0, converging to x_0. The stability of proper intersections in analytic geometry [Chi89, §12.3] implies that there exists a sequence $\lambda_j \to \lambda_0$ such that $x_j(\lambda_j)$ properly intersects $f_{\lambda_j}^n(\mathrm{Crit}(f_{\lambda_j}))$. Then by Theorem 2.1 we infer that λ_j belongs to Bif, and so does λ_0. □

It follows that in condition (iii) of Theorem 2.1 we can replace classical Misiurewicz bifurcations by Misiurewicz bifurcations in this sense.

3. Robust bifurcations from topology

As outlined above, to construct robust bifurcations, we need to find situations in which the post-critical set has a robust intersection with a basic repeller. Our first method to produce such a robust intersection is based on topological arguments.

The starting parameter is a product map in $\mathbb{C}^2$ of the form $f(z,w) = (p(z), w^d)$, where p is of degree d. Then f extends to $\mathbb{P}^2$ as a holomorphic map. The small Julia set of f is $J^*(f) = J_p \times S^1$, where S^1 denotes the unit circle in $\mathbb{C}$.

Theorem 3.1. — *Let $f(z,w) = (p(z), w^d)$ be a product map with $\deg(p) = d$. Assume that p satisfies the following properties:*

(A1) *there exists a basic hyperbolic repeller E for p that disconnects the plane;*

(A2) *p belongs to the bifurcation locus in $\mathcal{P}_d(\mathbb{C})$.*

Then f belongs to the closure of the interior of the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^2)$.

More precisely there exists a sequence of polynomials p_j of degree d converging to p, a polynomial q of degree $\leqslant 2$ and a sequence $\varepsilon_j \to 0$ such that for every j, the map $f_j \in \mathcal{H}_d(\mathbb{P}^2)$ defined by $f_j(z,w) = (p_j(z) + \varepsilon_j q(w), w^d)$ belongs to $\mathring{\mathrm{Bif}}$.

Notice that the assumption on p requires[(1)] $d \geqslant 3$, so the perturbation $\varepsilon h(w)$ does not affect the highest degree part of f, hence $f_\varepsilon \in \mathcal{H}_d(\mathbb{P}^2)$.

We denote by $\mathcal{P}_d(\mathbb{C})$ the space of polynomials of degree d in $\mathbb{C}$. Examples of polynomials $p \in \mathcal{P}_d(\mathbb{C})$ satisfying the assumptions of Theorem 3.1 are abundant. One may be tempted to think that as soon as p admits one active critical point and an attracting periodic orbit, then there is a nearby $\widetilde{p}$ satisfying the assumptions of the theorem. The next two corollaries are results in this direction. They will be proven at the end of this section.

(1) Indeed, we essentially need both an attracting orbit and an active critical point.

Corollary 3.2. — *Let $f(z,w)=(p(z),w^d)$, with $\deg(p)=d\geqslant 3$. Assume that p is a bifurcating polynomial in $\mathcal{P}_d(\mathbb{C})$, such that $(d-2)$ of its critical points are attracted by periodic sinks. Then f belongs to the closure of the interior of the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^2)$.*

When several critical points are active it is convenient to use the formalism of bifurcation currents (see [Duj14] for an introduction to this topic; the bifurcation current is denoted by T_{bif}).

Corollary 3.3. — *Let $f(z,w)=(p(z),w^d)$, with $\deg(p)=d\geqslant 3$. Assume that in $\mathcal{P}_d(\mathbb{C})$, $p\in\operatorname{Supp}(T_{\mathrm{bif}}^k)$ for some $1\leqslant k\leqslant d-1$ and that $(d-1-k)$ critical points are attracted by sinks. Then f belongs to the closure of the interior of the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^2)$.*

Notice that this holds in particular when p belongs to the Shilov boundary of the connectedness locus (corresponding to the case $k=d-1$).

Proof of Theorem 3.1. — Observe first that it is enough to prove the result under the assumption

(A2') *there exists a simple critical point c for p and an integer $k\geqslant 1$ such that $p^k(c)\in E$.*

Indeed let p_0 be a polynomial satisfying (A1) and (A2). Since p_0 belongs to the bifurcation locus, it has an active critical point c. Taking a branched cover of parameter space, we may always assume that c can be followed holomorphically as $p\mapsto c_p$. Also, E locally persists as a repelling set E_p in a neighborhood of p_0. A well-known and elementary normal families argument shows that there exists an arbitrary small perturbation p_1 of p_0 and an integer k such that $p_1^k(c_{p_1})\in E_{p_1}$. Furthermore the set of polynomials $p\in\mathcal{P}_d$ possessing a multiple critical point is algebraic. Since E is infinite, the set of polynomials $p\in\mathcal{P}_d$ such that $p(c_p)\in E$ is not locally analytic near p_1.

Altogether, it follows that there is a sequence of polynomials p_j satisfying (A1) and (A2') and converging to p_0. Thus, if the theorem has been shown to hold under the assumptions (A1) and (A2') we simply pick such a sequence and get that the result holds for p_0 as well.

In a first stage let us prove the theorem in the case where $k=1$.

Step 1: topological stability of the hyperbolic set. — For $\varepsilon=0$, $f=f_0$ admits a basic repelling set $\mathcal{E}=E\times S^1$. Notice that E has empty interior in the plane. If $g\in\mathcal{H}_d$ is sufficiently close to f, $\mathcal{E}=\mathcal{E}(f)$ admits a continuation $\mathcal{E}(g)$ as a hyperbolic set, that is, there exists a continuous (even holomorphic in g) family of equivariant homeomorphisms $h_{f,g}:\mathcal{E}(f)\to\mathcal{E}(g)$. Actually there is more:

Lemma 3.4. — *The conjugating homeomorphism $h_{f,g}$ can be extended to a homeomorphism of $\mathbb{P}^2$ (not compatible with the dynamics) depending continuously on g and such that $h_{f,f}=\mathrm{id}$.*

Proof. — The result is more or less part of the folklore. For completeness we sketch the argument. Let U be an isolating neighborhood of $\mathcal{E}$, which can be assumed to have smooth boundary, and $\mathcal{N}$ be a neighborhood of f in $\mathcal{H}_d$ such that for $g \in \mathcal{N}$, $\bigcap_{n\geqslant 0} g^{-n}(U) = \mathcal{E}(g)$ (here and in the rest of the proof only preimages remaining in U are taken into account). By the hyperbolicity assumption (and reducing U and $\mathcal{N}$ a bit if necessary), there exists $\delta > 0$ and $\lambda > 1$ so that if $x, y \in U$ are such that $f^k(x)$ and $f^k(y)$ remain in U and $d(f^k(x), f^k(y)) \leqslant \delta$ for $0 \leqslant k \leqslant n$ then $d(x, y) < \delta\lambda^{-n}$, and the same is true for every $g \in \mathcal{N}$. Reducing U and $\mathcal{N}$ again if necessary, we may assume that for every $g \in \mathcal{N}$ and $x \in U$ different preimages of x under g stay far apart, so that each such preimage can be followed unequivocally as g varies in $\mathcal{N}$. In addition we can assume that for every x, the continuation $g_{-1}(x)$ of any given preimage $f_{-1}(x)$ of x satisfies $d(f_{-1}(x), g_{-1}(x)) < \varepsilon$ for $g \in \mathcal{N}$, where $\varepsilon < \delta(1 - 1/\lambda)/3$.

Given such g we will construct the conjugacy $h = h_{f,g}$ by starting from ∂U. Fix $h = \mathrm{id}$ on ∂U, then define $h : f^{-1}(\partial U) \to g^{-1}(\partial U)$ by assigning to every $y \in f^{-1}(\partial U)$ the corresponding preimage of $f(y)$ under g, and finally extend h to a homeomorphism from $\overline{U} \smallsetminus f^{-1}(U)$ to $\overline{U} \smallsetminus g^{-1}(U)$. In addition we can ensure that h is ε-close to the identity. Now, for every $x \in \overline{U} \smallsetminus \mathcal{E}(f)$ we set $h(x) = g_{-k} \circ h \circ f^k(x)$, where k is the last integer such that $f^k(x) \in U$, and g_{-k} is the inverse branch of f^k at $f^k(x)$ obtained by continuation of $f_{-k} : f^k(x) \mapsto x$. In this way we get an equivariant homeomorphism $h : \overline{U} \smallsetminus \mathcal{E}(f) \to \overline{U} \smallsetminus \mathcal{E}(g)$.

The point is to show that this conjugacy extends continuously to $\mathcal{E}(f)$. Then the resulting extension will be a homeomorphism by reversing the roles of f and g, which simply extends to $\mathbb{P}^2$ by declaring that $h = \mathrm{id}$ outside U, thereby concluding the proof.

To prove that such an extension exists, we need to prove that for every $x \in \mathcal{E}$, if $(x_n) \in (U \smallsetminus \mathcal{E})^{\mathbb{N}}$ is any sequence converging to x then $(h(x_n))$ converges and its limit does not depend on (x_n).

As a first step towards this result, let us show that for every $x \in U \smallsetminus \mathcal{E}$, we have that $d(x, h(x)) < \delta/3$. For this, let k be the last integer such that $f^k(x) \in U$ and let us show by induction on q that $d(f^{k-q}(x), g^{k-q}(h(x))) \leqslant \varepsilon \sum_{j=0}^{q} \lambda^{-j}$. Then the result follows from our choice of ε by putting $q = k$. For $q = 0$, $g^k(h(x)) = h(f^k(x))$, and $d(f^k(x), h(f^k(x))) \leqslant \varepsilon$ by definition of h. Now assume the result has been proved for some $0 \leqslant q \leqslant k - 1$. Then

$$\begin{aligned}
&d(f^{k-q-1}(x), g^{k-q-1}(h(x)))\\
&\quad\leqslant d(f_{-1}(f^{k-q}(x)), g_{-1}(g^{k-q}(h(x)))),\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{where } f_{-1},\ g_{-1} \text{ are appropriate inverse branches,}\\
&\quad\leqslant d(f_{-1}(f^{k-q}(x)), g_{-1}(f^{k-q}(x))) + d(g_{-1}(f^{k-q}(x)), g_{-1}(g^{k-q}(h(x))))\\
&\quad\leqslant \varepsilon + \frac{1}{\lambda} d(f^{k-q}(x), g^{k-q}(h(x))),
\end{aligned}$$

where the last inequality follows from the definition of ε and the induction step. The estimate $d(x, h(x)) < \delta/3$ follows.

We are now in position to conclude the argument. If (x_n) and (y_n) are two sequences converging to x, then for large n, the orbit of x_n (resp. y_n) shadows that of x for a long time. More precisely there exists $k = k(n) \underset{n\to\infty}{\longrightarrow} \infty$ such that for $j \leqslant k$, $d(f^j(x_n), f^j(y_n)) < \delta/3$. Therefore

$$d(h(f^j(x_n)), h(f^j(y_n))) = d(g^j(h(x_n)), g^j(h(y_n))) < \delta,$$

and we conclude that $d(h(x_n), h(y_n)) < \delta\lambda^{-k(n)}$. The same argument applied to x_n and x_m for large n, m implies that $(h(x_n))$ is a Cauchy sequence. Altogether, we infer that the sequence $(h(x_n))$ converges to a limit which depends only on x. This implies that the conjugacy extends to $\mathcal{E}$ and finishes the proof of the lemma. $\square$

Step 2: analysis of f_ε. — In this paragraph we put $f_\varepsilon(z, w) = (p(z) + \varepsilon q(w), w^d)$ as in the statement of the theorem (an explicit expression for q will be given afterwards), and work only within the family (f_ε), where ε ranges in a small neighborhood of $0 \in \mathbb{C}$. For notational ease, all dynamical objects will be indexed by ε (like $\mathcal{E}_\varepsilon$, etc.). The problem is semi-local around $\mathcal{E}$ so we work in $\mathbb{C}^2$. Also, we normalize the z-coordinate so that $p(c) = 0$.

For every ε, the vertical line $C = \{c\} \times \mathbb{C}$ is a critical component of f_ε. We set $V_\varepsilon = f_\varepsilon(\{c\} \times \mathbb{C})$, which is parameterized by

$$V_\varepsilon = \left\{(\varepsilon q(w), w^d), w \in \mathbb{C}\right\}. \tag{1}$$

The solid torus $\mathbb{C} \times S^1$ is totally invariant under the dynamics and contains $\mathcal{E}_\varepsilon$. For small enough ε, the hyperbolic set $\mathcal{E}_\varepsilon$ is homeomorphic to and close to $E \times S^1$. Then from (A1) and Lemma 3.4 (applied to $f\big|_{\mathbb{C}\times S^1}$) its complement in $\mathbb{C} \times S^1$ admits both bounded connected components (the *inside* part of $\mathcal{E}_\varepsilon$) and a unique unbounded one (the *outside*).

The following lemma provides a form of "topological transversality" between V_ε and $\mathcal{E}_\varepsilon$.

Lemma 3.5. — *There exists a polynomial q of degree $\leqslant 2$ and a sequence $\varepsilon_j \to 0$ such that for $\varepsilon = \varepsilon_j$, the real analytic curve*

$$v_\varepsilon = V_\varepsilon \cap (\mathbb{C} \times S^1) = \left\{(\varepsilon q(w), w^d), |w| = 1\right\}$$

intersects both the inside and the outside of $\mathcal{E}_\varepsilon$.

Proof. — Take q of the form $q(w) = (w - 1)\widetilde{q}(w)$. Then the fiber $\{w = 1\}$ is invariant under f_ε. Since in addition $f_\varepsilon(z, 1) = (p(z), 1)$ does not depend on ε, we get that $\mathcal{E}_\varepsilon \cap \{w = 1\} = E \times \{1\}$ (indeed, if $x \in E \times \{1\}$ the f_ε-orbit of x trivially shadows its f-orbit). The curve v_ε intersects $\{w = 1\}$ in d points $\varepsilon q(\zeta^k)$, where $\zeta = e^{2i\pi/d}$ (here we identify $\{w = 1\}$ with $\mathbb{C}$ and drop the second coordinate). Notice that for $k = 0$, $\varepsilon q(1) = 0$ belongs to E. Put $\zeta^\pm = \zeta^{\pm 1}$.

Denote by $\mathrm{Out}(E)$ (resp. $\mathrm{Inn}(E)$) the unbounded component of (resp. the union of bounded connected components of) the complement of E in $\mathbb{C}$, which by assumption are both non-empty. Remark that the uniform hyperbolicity and transitivity of E imply that every $x \in E$ is accumulated both by $\mathrm{Out}(E)$ and $\mathrm{Inn}(E)$.

Claim. — *There exists α so that for $q(w)=(w-1)(w-\alpha)$, there exist $\varepsilon_j\to 0$ so that $\varepsilon q(\zeta^-)$ (resp. $\varepsilon q(\zeta^+)$) belongs to* $\mathrm{Inn}(E)$ *(resp.* $\mathrm{Out}(E)$*).*

Indeed let $\alpha\in\mathbb{R}$ be such that $(\zeta^+-\alpha)/(\zeta^--\alpha)=e^{i\theta}$ with $\theta\notin\pi\mathbb{Q}$ and assume that the claim is false. Then for every small enough ε, $\varepsilon q(\zeta^-)\in\mathrm{Out}(E)$ implies that $\varepsilon q(\zeta^+)\notin\mathrm{Inn}(E)$. Using the above remark we reformulate this as

$$\varepsilon q(\zeta^-)\in\mathrm{Out}(E)\Longrightarrow\varepsilon q(\zeta^+)\in\mathbb{C}\smallsetminus\mathrm{Inn}(E)=\mathrm{Out}(E)\cup E=\overline{\mathrm{Out}(E)},$$

so by continuity we conclude that

$$\varepsilon q(\zeta^-)\in\overline{\mathrm{Out}(E)}\Longrightarrow\varepsilon q(\zeta^+)\in\overline{\mathrm{Out}(E)}.$$

Of course the reverse implication holds by symmetry, so we get that

$$\varepsilon q(\zeta^-)\in\overline{\mathrm{Out}(E)}\iff\varepsilon q(\zeta^+)\in\overline{\mathrm{Out}(E)}.$$

Taking the complement and putting $\delta=\varepsilon q(\zeta^-)$, we conclude that for every small enough $\delta\in\mathbb{C}$,

$$\delta\in\mathrm{Inn}(E)\iff\delta\,\frac{q(\zeta^+)}{q(\zeta^-)}\in\mathrm{Inn}(E).\tag{2}$$

Observe that

$$\frac{q(\zeta^+)}{q(\zeta^-)}=\frac{\zeta^+-1}{\zeta^--1}\,\frac{\zeta^+-\alpha}{\zeta^--\alpha}=-e^{2i\pi/d}\,\frac{\zeta^+-\alpha}{\zeta^--\alpha}=-e^{2i\pi/d}e^{i\theta}=e^{i\theta'},\quad\text{with }\theta'\notin\pi\mathbb{Q},$$

therefore by (2), $\mathrm{Inn}(E)$ is invariant under an irrational rotation about 0. Now, if Ω is an open ball in $\mathrm{Inn}(E)$ close to 0, its orbit under this irrational rotation contains a circle about 0, which contradicts the fact that $0\in\overline{\mathrm{Out}(E)}$, and concludes the proof of the claim.

Note that when E is a Jordan curve, which is often the case in practice (see Lemma 3.6), the argument can be simplified a bit. In particular E cannot be invariant under multiplication by $e^{2i\pi/d}$ at 0 so we can simply choose $q(w)=w-1$.

Summing up, we have shown that for $\varepsilon=\varepsilon_j$, $v_\varepsilon\cap\{w=1\}$ intersects different connected components of $\{w=1\}\smallsetminus E$. Thus to conclude the proof of the lemma it is enough to prove that for ε small enough the map $x\mapsto\mathrm{Component}(x)$ induces a 1-1 correspondence between the connected components of $\{w=1\}\smallsetminus E$ and the connected components of $(\mathbb{C}\times S^1)\smallsetminus\mathcal{E}_\varepsilon$.

Indeed this is obvious for $\varepsilon=0$. Now, when we vary ε, Lemma 3.4 shows that there is a homeomorphism h_ε of $\mathbb{P}^2$ such that $h_\varepsilon(\mathcal{E})=\mathcal{E}_\varepsilon$, which is homotopic to the identity. Since $f_\varepsilon\big|_{\{w=1\}}$ does not depend on ε, the construction of h_ε shows that we can choose $h_\varepsilon=\mathrm{id}$ on this fiber. Thus for every $x\in\{w=1\}\smallsetminus E$, h_ε sends the connected component of $(\mathbb{C}\times S^1)\smallsetminus\mathcal{E}$ containing x to the connected component of $(\mathbb{C}\times S^1)\smallsetminus\mathcal{E}_\varepsilon$ containing x, which was the desired claim. This finishes the proof of Lemma 3.5. ◻

Step 3: robustness of the Misiurewicz phenomenon in $\mathcal{H}_d(\mathbb{P}^2)$. — Consider an arbitrary holomorphic map $f' \in \mathcal{H}_d(\mathbb{P}^2)$ close to f_0. Let us first analyze how the post-critical component V_0 can be continued for f' (this has been done explicitly before for the family (f_ε)). Again, the problem is semi-local around $\mathcal{E}$ so for convenience we work in coordinates in $\mathbb{C}^2$. The vertical line $\{c\} \times \mathbb{C}$ is a component of multiplicity 1 of $\mathrm{Crit}(f_0)$, and moreover $\mathrm{Crit}(f_0)$ is smooth along $\{c\} \times \mathbb{C}$ outside $\{w = 0\}$. Therefore every compact piece of $(\{c\} \times \mathbb{C}) \smallsetminus \{w = 0\}$ can be followed as a part of the critical set for f' close to f_0. More specifically, if r is so small that $D(c, r)$ contains no other critical point of p, and if we let $U = D(c, r) \times \{1/10 < |w| < 10\}$, there exists a neighborhood $N(f_0)$ such that for every $f' \in N(f_0)$, $\mathrm{Crit}(f') \cap U$ is of multiplicity 1 and consists in an annulus $A(f')$ which is a graph over $A = \{c\} \times \{1/10 < |w| < 10\}$, and converges to A as $f' \to f_0$. Finally we define $V(f')$ to be the component of $f'(\mathrm{Crit}(f'))$ containing $f'(A(f'))$.

Fix a continuous function $\varphi_0 : \mathbb{C} \times S^1 \to \mathbb{R}$ (depending only on $z \in \mathbb{C}$) such that $\mathcal{E}_0 = \{\varphi_0 = 0\}$, $\mathrm{Inn}(\mathcal{E}_0) = \{\varphi_0 < 0\}$ and $\mathrm{Out}(\mathcal{E}_0) = \{\varphi_0 > 0\}$. Extend it to $\mathbb{C}^2$ by putting $\varphi_0(z, w) := \varphi_0(z)$. Define also ψ_0 by $\psi_0(z, w) = |w| - 1$, and let $\Phi_0 = (\varphi_0, \psi_0) : \mathbb{C}^2 \to \mathbb{R}^2$, so that $\mathcal{E}_0 = \{\Phi_0 = 0\}$. By Lemma 3.4, for f' close to f_0, there exists a continuous map $\Phi_{f'} = (\varphi_{f'}, \psi_{f'}) : \mathbb{C}^2 \to \mathbb{R}^2$ such that $\mathcal{E}(f') = \{\Phi_{f'} = 0\}$. Notice that for $f' = f_\varepsilon$, since $\mathcal{E}_\varepsilon \subset \mathbb{C} \times S^1$, we can choose $\psi_{f_\varepsilon} =: \psi_\varepsilon = \psi_0$.

Let now $f' = f_{\varepsilon_j}$ be as in Lemma 3.5. We will use elementary algebraic topology to show that $V(f')$ intersects $\mathcal{E}(f')$ for f' close to f_{ε_j}. Working in the natural parameterization of V_ε given in (1), the real valued function φ_{ε_j} changes sign along the segment $[\zeta^-, \zeta^+]$ of the real curve v_{ε_j}, say $\varphi_{\varepsilon_j}(\zeta^-) < 0 < \varphi_{\varepsilon_j}(\zeta^+)$. In other words, $\varphi_{\varepsilon_j} \circ f_{\varepsilon_j}(c, \cdot)$ changes sign along the segment $[\zeta^-, \zeta^+]$ of the unit circle.

Consider a simple loop ℓ in the complex plane, starting at ζ^-, then joining it to ζ^+ outside the unit disk, and then returning back to ζ^- inside the unit disk, and staying in the annulus A (e.g. we can take the exponential of the oriented boundary of the rectangle $[1 - \rho, 1 + \rho] \times \left[-\frac{2\pi}{d}, \frac{2\pi}{d}\right]$ for some small $\rho > 0$.) Then the loop $f_{\varepsilon_j}(\{c\} \times \ell)$ is disjoint from $\mathcal{E}_{\varepsilon_j}$ and the winding number of Φ_{ε_j} along $f_{\varepsilon_j}(\{c\} \times \ell)$ is equal to 1.

For f' sufficiently close to f_{ε_j}, we can lift $\{c\} \times \ell$ to a loop ℓ' contained in $A(f') \subset \mathrm{Crit}(f')$. By continuity, the winding number of $\Phi_{f'}$ along $f' \circ \ell'$ is 1. Since ℓ' bounds a disk in $A(f')$, we infer that $\Phi_{f'}$ must vanish on $f'(A(f'))$. In other words, $V(f')$ intersects $\mathcal{E}(f')$, which was the desired result.

To conclude that a robust Misiurewicz bifurcation occurs at f_{ε_j}, it remains to check that the intersection between $V(f')$ and $\mathcal{E}(f')$ that we have produced is proper. For this, it is enough to observe that there are product maps $(p_1(z), w^d)$, with p_1 arbitrary close to p, such that if $c_1 \in \mathrm{Crit}(p_1)$ is the critical point continuing c, then $p_1(c_1) \notin E(p_1)$. Indeed c is active at p in $\mathcal{P}_d(\mathbb{C})$ so under arbitrary small perturbations, it can be sent into the basin of a sink. This finishes the proof of the theorem in the case where $k = 1$.

It remains to treat the case where k is arbitrary. Observe that we can assume that $p^j(c)$ is not critical for $1 \leqslant j \leqslant k$, otherwise we replace c by the last appearing critical point in this orbit segment. The structure of the proof is the same. Step 1 doesn't need to be modified.

For Step 2, since in the case $k = 1$ we have used the explicit parameterization of $f_\varepsilon(\{c\} \times \mathbb{C})$, here rather than working with $f_\varepsilon^k(\mathrm{Crit}(f_\varepsilon))$, we show that $f_\varepsilon(\mathrm{Crit}(f_\varepsilon))$ intersects $f_\varepsilon^{-(k-1)}(\mathcal{E}_\varepsilon)$ in topologically transverse manner for a well-chosen sequence (ε_j). The only slight difficulty to keep in mind when pulling back is that $\mathcal{E}_\varepsilon$ can intersect other components of the post-critical set.

We put $F = p^{-(k-1)}(E)$ and normalize the z coordinate so that $p(c) = 0 \in F$. As in the previous case let q be of the form $q(w) = (w-1)(w-\alpha)$. Let $\mathcal{F}_\varepsilon = f_\varepsilon^{-(k-1)}(\mathcal{E}_\varepsilon)$ which is contained in $\mathbb{C} \times S^1$. For $\varepsilon = 0$, $\mathcal{F}_0 = F \times S^1$ but now $\mathcal{F}_\varepsilon$ needn't be homeomorphic to $\mathcal{F}_0$. Since f is proper and dominant $\mathcal{F}_\varepsilon$ is a compact set with empty interior. Thus if Ω is a bounded connected component of $(\mathbb{C} \times S^1) \smallsetminus \mathcal{E}_\varepsilon$, $f_\varepsilon^{-(k-1)}(\Omega)$ is a union of bounded components of $(\mathbb{C} \times S^1) \smallsetminus \mathcal{F}_\varepsilon$. It follows that in $\mathbb{C} \times S^1$, $\mathrm{Inn}(\mathcal{F}_\varepsilon) = f_\varepsilon^{-(k-1)}(\mathrm{Inn}(\mathcal{E}_\varepsilon))$ and likewise for $\mathrm{Out}(\mathcal{F}_\varepsilon)$.

We have that $\mathcal{F}_\varepsilon \cap \{w = 1\} = F$, and if $(z, 1) \in \mathrm{Inn}(F)$ in $\mathbb{C}$, then $(z, 1) \in \mathrm{Inn}(\mathcal{F}_\varepsilon)$ in $\mathbb{C} \times S^1$, and likewise for the outer part. Indeed $f^{k-1}(z, 1) = (p^{k-1}(z), 1)$ and $p^{k-1}(z) \in \mathrm{Inn}(E)$ so $(p^{k-1}(z), 1) \in \mathrm{Inn}(\mathcal{E}_\varepsilon)$ and the result follows by pulling back the corresponding inner component. So we can argue exactly as in Lemma 3.5 to conclude that there exists a sequence $\varepsilon_j \to 0$ such that the real curve $v_e = f_\varepsilon(\{c\} \times \mathbb{C}) \cap (\mathbb{C} \times S^1)$ intersects both $\mathrm{Inn}(\mathcal{F}_\varepsilon)$ and $\mathrm{Out}(\mathcal{F}_\varepsilon)$.

This analysis being done, we can push forward by $f_{\varepsilon_j}^{k-1}$ to get a segment of the real analytic curve $f_{\varepsilon_j}^k(\{c\} \times \mathbb{C}) \cap (\mathbb{C} \times S^1)$ whose endpoints belong to $\mathrm{Inn}(\mathcal{E}_{\varepsilon_j})$ and $\mathrm{Out}(\mathcal{E}_{\varepsilon_j})$ respectively and apply the argument of Step 3 without modification. The proof is complete. $\square$

Corollary 3.2 is a direct consequence of the following lemma.

Lemma 3.6. — *Let p be a polynomial of degree $d \geqslant 3$ with marked critical points $c_1, \dots, c_{d-1}$. Assume that $c_2, \dots, c_{d-1}$ are attracted by attracting cycles and that c_1 is active. Then there exists $\widetilde{p}$ arbitrary close to p satisfying assumptions (A1) and (A2).*

Proof. — Observe first that if p is an arbitrary polynomial and $\mathcal{B}$ is the immediate basin of attraction of an attracting cycle of period ℓ, such that $\partial\mathcal{B}$ contains no critical point nor parabolic periodic points, then by Mañé's lemma [Mañ93] $\partial\mathcal{B}$ is a hyperbolic set, which obviously disconnects the plane. We claim that it is basic, that is, transitive and locally maximal. Local maximality follows from general arguments: namely by [PU10, Lem. 6.1.2] it follows from the fact that $f\big|_{\partial\mathcal{B}}$ is an open map (due to the absence of critical points).

To prove transitivity, let Ω be a component of $\mathcal{B}$ ($f^\ell(\Omega) = \Omega$). It is enough to show that $f^\ell\big|_{\partial\Omega}$ is transitive. Uniform hyperbolicity implies that $\partial\Omega$ is locally connected, therefore the Riemann map $\phi : \mathbb{D} \to \Omega$ extends by continuity to $\phi : \overline{\mathbb{D}} \to \overline{\Omega}$. In addition $\phi\big|_{\mathbb{D}}$ must be injective otherwise contradicting the polynomial convexity of Ω. Thus $\partial\mathcal{B}$ is a Jordan curve. (This result actually holds in much greater generality, see [RY08]).

The formula $\phi f^\ell \phi^{-1}$ defines a self-map of the unit disk, which can be extended by Schwarz reflexion to a rational map on $\mathbb{P}^1$ preserving the unit circle, hence a Blaschke product. Now the Julia set of a Blaschke product is either the unit circle or a Cantor set inside the unit circle, and the second case occurs only if there is an attracting or parabolic fixed point on the circle, which is not the case here. We conclude that the Julia set is the circle, hence topologically transitive (even topologically mixing). Coming back to f we conclude that $f^\ell\big|_{\partial\Omega}$ is topologically mixing, which yields the desired claim.

Under the assumptions of the lemma, let $\mathcal{B}$ be the immediate attracting basin of the cycle attracting, say, c_2, which is robust under perturbations, and consider the active critical point c_1. Since p is a bifurcating polynomial, one can make a cycle of period ℓm bifurcate close to p, so there exists p_1 close to p with a parabolic point α of period ℓm and multiplier different from ± 1. Then:

(1) $\alpha \notin \partial\mathcal{B}$: indeed $p_1^{\ell m}$ behaves like a rational rotation of order greater than 2 at α, so there cannot exist an invariant Jordan curve through α. On the other hand by [RY08] $\partial\mathcal{B}$ is a Jordan curve invariant under p^ℓ.

(2) α must attract c_1 under iteration of $p_1^{\ell m}$, therefore $c_1 \notin \partial\mathcal{B}$.

This implies that for this polynomial p_1, $\partial\mathcal{B}$ is a basic repeller disconnecting the plane so (A1) holds. Since in addition p_1 has a parabolic cycle, (A2) holds and the corollary follows. ☐

Proof of Corollary 3.3. — We freely use the formalism of bifurcation currents, see [Duj14] for an account. By passing to a branched cover, it is no loss of generality to assume that the critical points are marked as $c_1, \dots, c_{d-1}$. Let T_i, $1 \leqslant i \leqslant d-1$, be the associated bifurcation currents. Since $p \in \operatorname{Supp}(T_{\mathrm{bif}}^k)$, reordering the critical points if necessary, we can assume that $p \in \operatorname{Supp}(T_1 \wedge \cdots \wedge T_k)$ and that $c_{k+1}, \dots, c_{d-1}$ are attracted by sinks. Then, using the continuity of the potentials of the T_i, we infer that there exists a sequence of subvarieties $[W_n]$ of codimension $k-1$ in $\mathcal{P}_d$ along which $c_2, \dots, c_k$ are periodic and a sequence of integers (d_n) such that $T_1 \wedge \cdots \wedge T_k = \lim_{n\to\infty} T_1 \wedge d_n^{-1}[W_n]$ (see [DF08, Th. 6.16]). Therefore arbitrary close to p there are polynomials with one active critical point and the remaining ones attracted by periodic sinks. The result then follows from Corollary 3.2. ☐

4. Robust bifurcations from fractal geometry

In this section we consider product mappings of the form $f(z,w) = (p(z), w^d + \kappa)$, where p admits a repelling fixed point z_0 that is almost parabolic, i.e., the multiplier $p'(z_0)$ satisfies $1 < |p'(z_0)| < 1.01$, and κ is large. We show that certain perturbations of f exhibit *blenders*, in the sense of Bonatti and Díaz [BD96], after proper rescaling. The degree 2 case requires a slightly different treatment so we handle the cases $d \geqslant 3$ and $d = 2$ separately. Also we strive for a uniformity in κ which allows to cover the parabolic case as well (see Corollary 4.11).

4.1. Estimates for Iterated Functions Systems in $\mathbb{C}$. — It is well-known [DK88, HS16] that the limit set of the IFS generated by two affine maps of the form $\ell_\pm(z) = \mu z \pm 1$ has empty interior when μ is non real and $|\mu|$ is sufficiently close to 1. Since we will need precise estimates as well as variants of this result, we give a detailed treatment. It will be convenient for us to work with mappings that are contractions on the unit disk, so we write them in a slightly different form.

Let $\mathcal{L} = (\ell_j)_{j=1}^d$ be a family of holomorphic maps $\ell_j : \mathbb{D} \to \ell_j(\mathbb{D}) \Subset \mathbb{D}$. Since the ℓ_j contract the Poincaré metric they define an IFS whose *limit set* is

$$E = \bigcap_{n\geqslant 0} \bigcup_{|J|=n} \ell_{j_n} \circ \cdots \circ \ell_{j_1}(\mathbb{D})$$

(we use the classical multi-index notation $J = (j_1, \dots, j_n)$). We say that the IFS $\mathcal{L}$ satisfies the *covering property* on an open set $\Delta \subset \mathbb{D}$ if $\bigcup_{j=1}^d \ell_j(\Delta) \supset \overline{\Delta}$. It then follows that $\Delta \subset E$ and this property obviously persists for any small (C^1) perturbation of (ℓ_j).

Our first model situation concerns an IFS with $d \geqslant 3$ branches roughly directed by d^{th} roots of unity.

Lemma 4.1. — *Let $d \geqslant 3$ and $\mathcal{L} = (\ell_j)_{j=1}^d$ be the IFS generated by the affine maps*

$$\ell_j : z \longmapsto mz + \alpha_j(1 - |m|)e^{(2\pi i/d)j},$$

where $0.98 < |m| < 1$ and α_j is such that $3/5 < |\alpha_j| < 1$ and $|\arg \alpha_j| < \pi/20$. Then $\mathcal{L}$ satisfies the covering property on the disk $D(0, 1/10)$.

Proof. — Since $|\alpha| < 1$ we have that $\ell_j(\mathbb{D}) \Subset \mathbb{D}$ so we are in the above situation. Let now z in $\overline{D}(0, 1/10)$, we have to prove that $\ell_j^{-1}(z) \in D(0, 1/10)$ for some j. We will show that if $\arg(z) \in [-\pi/3, \pi/3]$ then $j = d$ (or equivalently $j = 0$) is convenient. Then the other cases follow, since the unit circle is covered by translates of $[-\pi/3, \pi/3]$ by d^{th} roots of unity.

We compute $\ell_0^{-1}(z) = \frac{1}{m}(z - \alpha_0(1 - |m|))$. We have to show that for $z = \rho e^{i\theta}$, with $\rho \leqslant 1/10$ and $|\theta| \leqslant \pi/3$, then $\left|\ell_0^{-1}(z)\right| < 1/10$, or equivalently $|z - \alpha_0(1 - |m|)| < |m|/10$. Now,

$$\begin{aligned}
|z - \alpha_0(1 - |m|)|^2 &= \rho^2 + |\alpha_0|^2 (1 - |m|)^2 - 2\Re\left(z\overline{\alpha_0}(1 - |m|)\right) \\
&\leqslant \rho^2 + (1 - |m|)^2 - 2\rho\,|\alpha_0| \cos(\arg(z) - \arg(\alpha_0))(1 - |m|) \\
&\leqslant \rho^2 + (1 - |m|)^2 - \frac{2}{5}\rho(1 - |m|),
\end{aligned}$$

where in last line we use $|\alpha_0| > 3/5$ and $\cos(\arg(z) - \arg(\alpha_0)) \geqslant 1/3$. So we are left to showing that

$$\rho^2 + (1 - |m|)^2 - \frac{2}{5}\rho(1 - |m|) < \frac{|m|^2}{10^2}, \quad \text{for } \rho \leqslant \frac{1}{10} \text{ and } 0.98 < |m| < 1.$$

For the sake of this computation, put $1-|m|=\delta<1/50$. The previous equation can be rewritten as

$$\rho^2+\delta^2-\frac{2}{5}\rho\delta<\frac{(1-\delta)^2}{10^2}\iff\rho\Big(\rho-\frac{2}{5}\delta\Big)<\frac{(1-\delta)^2}{10^2}-\delta^2.$$

Since $\rho\leqslant 1/10$, to prove the last inequality it is enough to show that

$$\rho-\frac{2}{5}\delta<\frac{(1-\delta)^2}{10}-10\delta^2\iff\rho-\frac{1}{10}<\frac{1}{5}\delta-\frac{99}{10}\delta^2=\frac{1}{5}\delta\Big(1-\frac{99}{2}\delta\Big),$$

and the latter is true because the left hand side is non-positive while the right hand side is positive. □

Remark 4.2. — Notice that conjugating by a rotation implies that for every fixed θ, the same result holds when $\alpha_j(1-|m|)$ is replaced by $e^{i\theta}\alpha_j(1-|m|)$ in the expression of ℓ_j.

In view of 2-dimensional applications, it is useful to point out a slight reformulation of this result. Let A be the range of allowed values for α in Lemma 4.1, i.e.,

$$A=\Big\{\alpha\in\mathbb{C},\ \frac{3}{5}<|\alpha|<1,\ |\arg(\alpha)|<\frac{\pi}{20}\Big\}. \tag{3}$$

Lemma 4.3. — *Let $d\geqslant 3$ and $\mathcal{L}=(\ell_j)_{j=1}^d$ be the IFS generated by the affine maps*

$$\ell_j: z\longmapsto mz+\alpha_j(1-|m|)e^{(2\pi i/d)j}.$$

where $0.98<|m|<1$. Fix $\eta>0$ and assume that for every $j=1,\dots,d$, the disk $D(\alpha_j,\eta)$ is contained in A.

Then for every $z_0\in D(0,1/10)$, there exists j such that

$$\ell_j^{-1}(D(z_0,\eta(1-|m|)))\subset D(0,1/10).$$

Proof. — Notice that when η tends to 0 this is precisely the statement of Lemma 4.1. In the general case, we simply observe that for every z, when α varies in a disk of radius η, $\ell_j^{-1}(z)$ ranges in a disk of radius $\eta(1-|m|)/|m|$. For $\eta>0$, put $A_\eta=\{\alpha\in A, D(\alpha,\eta))\subset A\}$ (beware that this is the opposite of a η-neighborhood). If for some z,

$$\ell_j^{-1}(z)\in D(0,1/10)\ \text{ for every }\alpha\in A$$

(this is what we have proved for z in an angular sector of width $2\pi/3$ and $j=0$) then

$$\ell_j^{-1}(z)\in D\left(0,\frac{1}{10}-\eta\frac{1-|m|}{|m|}\right)\ \text{ for every }\alpha\in A_\eta.$$

Now, since $\ell_j^{-1}(D(z,r))=D\left(\ell_j^{-1}(z),r/|m|\right)$, we infer that for $\alpha\in A_\eta$, the subset $\ell_j^{-1}(D(z,\eta(1-|m|)))$ is contained in $D(0,1/10)$, which was the result to be proved. □

For an IFS $\mathcal{L}$, we define $\mathcal{L}^n$ to the IFS generated by n-fold compositions of the generators of $\mathcal{L}$. It has the same limit set as $\mathcal{L}$. Here is an analogue of Lemma 4.1 for an IFS with two branches. In this case the multiplier needs to be chosen close to the imaginary axis.

Lemma 4.4. — *Let $\mathcal{L} = \ell_\pm$ be the IFS generated by the affine maps*

$$\ell_\pm : z \longmapsto mz \pm \alpha(1-|m|),$$

where $0.99 < |m| < 1$, $|\arg(m) - \pi/2| < \pi/50$ *and α is a complex number with* $0.9 < |\alpha| < 1$. *Then $\mathcal{L}^2$ satisfies the covering property on the disk $D(0, 1/10)$.*

Proof. — To establish the result, we show that $\mathcal{L}^2 = (\ell_+^2, \ell_+\circ\ell_-, \ell_-\circ\ell_+, \ell_-^2)$ satisfies the assumptions of Lemma 4.1 (up to conjugating by an appropriate rotation). Indeed, observe first that conjugating by a rotation, we can assume that α is a positive real number. Now, let us compute the expression of ℓ_+^2:

$$\ell_+^2(z) = m^2 z + (m+1)\alpha(1-|m|) = m^2 z + \beta_{++}(1-|m|^2),\ \text{ where } \beta_{++} = \frac{(m+1)\alpha}{|m|+1}.$$

Observe first that $0.98 < |m|^2 < 1$. Next, we see that β_{++} is approximately equal to $\frac{1+i}{2}\,\alpha$. More precisely, given the assumptions on m it can be shown that:

- $1.8 < |m+1|^2 < 2.2$, therefore $0.6 < |(m+1)\alpha|/(|m|+1) < 0.8$;
- $|\arg(m+1) - \pi/4| < \pi/20$.

For $\ell_+\circ\ell_-$, $\ell_-\circ\ell_+$ and ℓ_-^2, we have analogous estimates, with $\pi/4$ replaced by $-\pi/4$, $3\pi/4$ and $5\pi/4$ respectively. Thus, conjugating by a rotation of angle $\pi/4$, we are in position to apply Lemma 4.1, and the result follows. ☐

4.2. Blenders in $\mathbb{C}^2$. — We now study a 2-dimensional version of the phenomenon studied in Section 4.1. Similarly to (3) we define the angular sector

$$A' = \left\{\alpha\in\mathbb{C},\ 0.7 < |\alpha| < 0.9,\ |\arg(\alpha)| < \frac{\pi}{40}\right\}. \tag{4}$$

It is easily shown that for every $\alpha \in A'$, the disk $D\big(\alpha, 1/40\big)$ is contained in A, therefore if $\alpha_j \in A'$, the conclusion of Lemma 4.3 holds for any $\eta < 1/40$.

Lemma 4.5. — *Let $d \geqslant 3$ and $\mathcal{L} = (L_j)_{j=1}^d$ be an IFS in $\mathbb{D}^2$ generated by biholomorphic contractions of the form*

$$L_j(z,w) = (\ell_j(z), \varphi_j(z,w)),$$

and let $\mathcal{E}$ be its limit set. Assume that ℓ_j is of the form

$$\ell_j : z \longmapsto mz + \alpha_j(1-|m|)e^{(2\pi i/d)j},$$

where $0.98 < |m| < 1$ and $\alpha_j \in A'$, and that $\varphi_j : \mathbb{D}^2 \to \mathbb{D}$ is a holomorphic map such that

$$\left|\frac{\partial\varphi_j}{\partial z}\right| < 1 \quad \text{and} \quad \left|\frac{\partial\varphi_j}{\partial w}\right| < \frac{1}{2}.$$

Then any vertical graph Γ intersecting $D(0, 1/10)\times\mathbb{D}$, whose slope is bounded by $\frac{1}{100}(1-|m|)$ must intersect $\mathcal{E}$, that is $\Gamma\cap\mathcal{E}\neq\varnothing$.

Furthermore, the same holds for any IFS $\overline{\mathcal{L}}$ generated by $(\overline{L}_j)_{j=1}^d$, whenever

$$\big\|L_j - \overline{L}_j\big\|_{C^1} < \frac{1}{1000}(1-|m|). \tag{5}$$

Note that by the Cauchy estimates a control on the C^0 norm of $L_j - \overline{L}_j$ in a slightly larger domain is enough to achieve (5).

Proof. — Let us work directly with the perturbed situation. Let $\mathcal{G}$ be the set of vertical graphs Γ in $\mathbb{D}^2$ of the form $z=\gamma(w)$ with $|\gamma'| < \frac{1}{100}\,(1-|m|)$ and $|\gamma(w_0)|<1/10$ for some $w_0\in\mathbb{D}$.

We want to show that if $\Gamma\in\mathcal{G}$ then $\Gamma\cap\mathcal{E}(\overline{\mathcal{L}})\neq\varnothing$. For this we must prove that there exists an infinite sequence (j_k) such that[2] $\Gamma\cap\big(\overline{L}_{j_1}\circ\cdots\circ\overline{L}_{j_k}(\mathbb{D}^2)\big)\neq\varnothing$, or in other words, that $\overline{L}_{j_k}^{-1}\circ\cdots\circ\overline{L}_{j_1}^{-1}$ is well-defined on some piece of Γ.

Arguing by induction, to establish this property it is enough to show that if $\Gamma\in\mathcal{G}$ there exists $j\in\{1,\dots,d\}$ such that $\overline{L}_j^{-1}(\Gamma)\cap\mathbb{D}^2\in\mathcal{G}$. Write $\overline{L}_j = L_j+\varepsilon_j$. We first observe that the assumption on the slope implies that the diameter of the first projection of Γ is smaller than $\frac{1}{50}(1-|m|)$, hence contained in a disk $D\big(z_0,\frac{1}{50}(1-|m|)\big)$ for some $z_0\in D\,(0,1/10)$. Let $j\in\{1,\dots,d\}$ be as provided by Lemma 4.3 for this value of z_0, and consider $\overline{L}_j^{-1}(\Gamma)$ (for notational ease we drop the restriction to $\mathbb{D}^2$). This is a subvariety contained in $\overline{L}_j^{-1}\big(D\big(z_0,\frac{1}{50}(1-|m|)\big)\times\mathbb{D}\big)$. We claim that it is contained in $D\,(0,1/10)\times\mathbb{D}$.

Indeed, writing in coordinates $\overline{L}_j = (\ell_j+\varepsilon_1,\varphi_j+\varepsilon_2)$, we see that if (z,w) is such that $\overline{L}_j(z,w)$ belongs to $D\big(z_0,\frac{1}{50}(1-|m|)\big)\times\mathbb{D}$, then $\ell_j(z)+\varepsilon_1(z,w)\in D\big(z_0,\frac{1}{50}(1-|m|)\big)$. Since by assumption, $|\varepsilon_1(z,w)|<10^{-3}(1-|m|)$ it follows that $\ell_j(z)\in D\big(z_0,\frac{1}{45}(1-|m|)\big)$. By Lemma 4.3 this implies that $z\in D\,(0,1/10)$, thereby establishing our claim.

The equation of $\overline{L}_j^{-1}(\Gamma)$ is of the form

$$\ell_j(z)+\varepsilon_1(z,w) = \gamma\left(\varphi_j(z,w)+\varepsilon_2(z,w)\right), \tag{6}$$

which can be rewritten as

$$z = \ell_j^{-1}\left(\gamma\left(\varphi_j(z,w)+\varepsilon_2(z,w)\right)-\varepsilon_1(z,w)\right). \tag{7}$$

The z-derivative of the right hand side of this equation is smaller than 1 so the contraction mapping principle tells us that it has a unique solution for each w. In other words $\overline{L}_j^{-1}(\Gamma)$ is a vertical graph. Finally, the implicit function theorem applied to (6) implies that the slope of this graph is bounded by

$$\frac{|\gamma'|\left(\frac12+|\partial\varepsilon_2/\partial w|\right)+|\partial\varepsilon_1/\partial w|}{|m|-|\gamma'|\,(1+|\partial\varepsilon_2/\partial z|)-|\partial\varepsilon_1/\partial z|} < \frac{1}{100}\,(1-|m|)$$

and we are done. □

Remark 4.6. — A similar result holds for an IFS with two branches of the form $(\ell_\pm(z),\varphi_\pm(z,w))$, for $\ell_\pm(z)=mz\pm\alpha(1-|m|)$ and m close to the imaginary axis, as in Lemma 4.4. Indeed, exactly as in the proof of that lemma, it is enough to take two iterates of the IFS to get back to the setting of Lemma 4.5.

[2] Notice the order of compositions.

4.3. Blenders for endomorphisms of $\mathbb{P}^2$ and the post-critical set

Theorem 4.7. — *Let $d \geqslant 3$ and $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a product map of the form*

$$f(z,w) = (p(z), q(w)) = (p(z), w^d + \kappa),$$

with $\deg(p) = d$. Assume that the polynomial p belongs to the bifurcation locus in $\mathcal{P}_d$ and admits a repelling fixed point z_0 of low multiplier: $1 < |p'(z_0)| < 1.01$.

Then there exists a constant κ_0 depending only on d such that if $|\kappa| > \kappa_0$, f belongs to the closure of the interior of the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^2)$.

It is easy to construct polynomials p satisfying the assumptions of the theorem for every $d \geqslant 3$. A variant of this construction is given in Theorem 4.12 below which allows to treat the case $d = 2$ as well.

Proof. — We translate the z coordinate so that $z_0 = 0$, and put $m = (p'(0))^{-1}$, so that m is a complex number such that $0.99 < |m| < 1$. To fix the ideas we work in the case $d = 3$, the adaptation to the general case is easy and left to the reader. Furthermore to ease notation we replace κ by $-\kappa$ so that the second coordinate becomes $q(w) = w^3 - \kappa$.

Step 1: analysis of J_q. — When κ is large, the Julia set of q is a Cantor set; here we describe its geometry. Consider the disk $D_\kappa := D(0, 2|\kappa|^{1/3})$. The critical point 0 satisfies $q(0) = \kappa \notin D_\kappa$ so $q^{-1}(D_\kappa)$ is the union of 3 topological disks, invariant under a rotation of angle $2\pi/3$. Fix a cube root $\kappa^{1/3}$ of κ. If $x \in D_\kappa$, solving the equation $w^3 - \kappa = x$ yields

$$w^3 = \kappa + x = \kappa\left(1 + \frac{x}{\kappa}\right), \text{ hence } w = \kappa^{1/3}\zeta\Big(1 + \frac{x}{3\kappa} + O\big(|\kappa|^{-4/3}\big)\Big),$$

where $\zeta \in \{1, j, j^2\}$ ranges over the set of cube roots of unity. Therefore we see that $q^{-1}(D_\kappa)$ is approximately the union of 3 disks of radius $2/3\,|\kappa|^{1/3}$ centered at the cube roots of κ. Define q_ζ^{-1} to be the inverse branch of q on D_κ such that $q_\zeta^{-1}(0) = \zeta\kappa^{1/3}$.

We introduce the new coordinate $\widetilde{w} = \frac{1}{2}\kappa^{-1/3}w$. After coordinate change, the expression of q becomes $\widetilde{q}(\widetilde{w}) = \frac{1}{2}\kappa^{2/3}(8\widetilde{w}^3 - 1)$, so that for large enough κ, $\widetilde{q}^{-1}(\mathbb{D})$ is made of 3 components respectively contained in $D\big(\frac{1}{2}\zeta, \frac{1}{2}|\kappa|^{-2/3}\big)$, $\zeta \in \{1, j, j^2\}$. Similarly, from the Cauchy estimate we infer that $|(\widetilde{q_\zeta^{-1}})'| \leqslant |\kappa|^{-2/3}$ on $\mathbb{D}$.

Step 2: rescaling and construction of a blender. — Close to the origin, p behaves like the multiplication by $p'(0)$, more precisely we have $p(z) = p'(0)z + O(z^2)$. In particular there is a unique inverse branch p_0^{-1} defined in a neighborhood of 0 of size $\delta_0(p)$ and such that $p_0^{-1}(0) = 0$.

For $0 < \delta \ll \delta_0(p)$ we rescale the disk $D(0,\delta)$ to unit size by introducing the new coordinate $\widetilde{z} = \delta^{-1}z$. In the new coordinate, p becomes $\widetilde{p}(\cdot) = \delta^{-1}p(\delta\cdot)$, thus $\widetilde{p}(\widetilde{z}) = p'(0)\widetilde{z} + O(\delta)$ where the $O(\cdot)$ is uniform for $z \in \mathbb{D}$, and likewise for the inverse map $\widetilde{p_0}^{-1}(\widetilde{z}) = m\widetilde{z} + O(\delta)$. Similar results hold for the derivatives by applying the Cauchy estimates in a slightly larger disk.

In the new coordinate system $(\widetilde{z},\widetilde{w}) = \big(\delta^{-1}z, \frac{1}{2}\kappa^{-1/3}w\big) =: \phi(z,w)$, we have an IFS with 3 branches on $\mathbb{D}^2$ defined by the $(\widetilde{p_0}^{-1}, \widetilde{q_\zeta}^{-1})$ for $\zeta\in\big\{1,j,j^2\big\}$. Its limit set $\widetilde{\mathcal{E}}$ is equal to $\{0\}\times J_q$. Now we perturb $\widetilde{f}$ by introducing

$$\widetilde{f_\bullet}(\widetilde{z},\widetilde{w}) = (\widetilde{p}(\widetilde{z}) - 2\alpha(1-|m|)\widetilde{w}, \widetilde{q}(\widetilde{w})),$$

where α is a real number in the interval $(0.7, 0.9)$: once for all we choose $\alpha = 0.8$. We denote by $\widetilde{\mathcal{E}_\bullet}$ the corresponding limit set. Notice that in the original coordinates, the corresponding expression is

$$\phi^{-1}\circ\widetilde{f_\bullet}\circ\phi : (z,w)\longmapsto (p(z)+\varepsilon w, q(w)) = f_\varepsilon(z,w) \text{ for } \varepsilon = -\frac{\delta\alpha(1-|m|)}{\kappa^{1/3}}. \tag{8}$$

The IFS induced by $\widetilde{f_\bullet}$ on $\mathbb{D}^2$ is induced by 3 inverse branches for $\widetilde{f_\bullet}$ of the form

$$\begin{aligned}(\widetilde{z},\widetilde{w})\longmapsto \big(\widetilde{f_\bullet}\big)_\zeta^{-1}(\widetilde{z},\widetilde{w}) &= \Big(\widetilde{p_0}^{-1}\Big(\widetilde{z}+2\alpha(1-|m|)\widetilde{q_\zeta}^{-1}(\widetilde{w})\Big), \widetilde{q_\zeta}^{-1}(\widetilde{w})\Big)\\ &= \Big(m\widetilde{z}+\alpha m(1-|m|)\zeta\big(1+O\big(|\kappa|^{-2/3}\big)\big)+O(\delta), \frac{1}{2}\zeta+O\big(|\kappa|^{-2/3}\big)\Big).\end{aligned} \tag{9}$$

Further conjugating the first coordinate by a rotation of angle $\arg(m)$, we can assume that the translation part in the first component of $\big(\widetilde{f_\bullet}\big)_\zeta^{-1}$ equals $\alpha|m|(1-|m|)$ (see also Remark 4.2). Hence if δ is so small and κ so large that (5) is satisfied, applying Lemma 4.5 (with $\alpha|m|$ instead of α, which is licit since $\alpha|m|\in(0.7,0.9)$), we deduce that if V is any vertical graph contained in $D(0,1/10)\times\mathbb{D}$ and small enough slope, then $\widetilde{V}\cap\widetilde{\mathcal{E}_\bullet}\neq\varnothing$.

Notice that δ and κ can be chosen independently from each other. By Step 1 the terms $O\big(|\kappa|^{-2/3}\big)$ are actually smaller than $|\kappa|^{-2/3}$, hence choosing $|\kappa|\geqslant 2000^{3/2}$ is enough.(3) Likewise, we have to choose δ so that the term $O(\delta)$ in the first component of (9) is roughly bounded by $\frac{1}{2000}(1-|m|)$.

Step 3.0: conclusion in a particular case. — Let f be as in the statement of the theorem, and fix κ large enough so that the previous requirements are satisfied. We will first prove the result under the simplifying assumption that there exists a simple critical point c such that $p(c)=0$ (recall that 0 is the moderately repelling fixed point).

The basic set $\mathcal{E}(f) = \{0\}\times J_q$ is contained in J^*, so by Lemma 2.3 this property persists in a neighborhood of f. Notice that arbitrary close to f there are maps f_1 for which $\mathcal{E}(f_1)$ is disjoint from the post-critical set: it is enough to consider product maps of the form $(p_1(z), q(w))$ for which p_1 has a repelling fixed point at 0 not belonging to the post-critical set. It follows that any intersection between $\mathcal{E}(g)$ and $g(\mathrm{Crit}(g))$ for g close to f is proper in $\mathcal{H}_3(\mathbb{P}^2)$ so by Proposition 2.5 it gives rise to bifurcations.

Consider the perturbation $f_\varepsilon(z,w) = (p(z)+\varepsilon w, q(w))$ as in (8), and let us show that for small enough $\varepsilon\neq 0$, f_ε lies in the interior of the bifurcation locus in $\mathcal{H}_3(\mathbb{P}^2)$. If ε is small enough then $\delta = |\kappa|^{1/3}|\varepsilon|/0.8(1-|m|)$ satisfies the requirements of Step 2, so $\mathcal{E}(f_\varepsilon)$ is a blender-type Cantor set contained in $D(0,\delta)\times D_\kappa$ (recall that $D_\kappa = D(0,2|\kappa|^{1/3})$).

(3)There is no attempt to optimize the bound on κ here.

In the rescaled coordinates $(\widetilde{z},\widetilde{w})$, the vertical line $\{c/\delta\}\times\mathbb{C}$ is a component of the critical set of $\widetilde{f}_\bullet$, and its image is

$$\widetilde{V}_\bullet := \{(-2\alpha(1-|m|)\widetilde{w},\widetilde{q}(\widetilde{w})),\widetilde{w}\in\mathbb{C})\}\,.$$

The intersection $\widetilde{V}_\bullet\cap\mathbb{D}^2$ is the union of 3 vertical graphs of the form

$$\widetilde{z}=-2\alpha(1-|m|)\widetilde{q}_\zeta^{-1}(\widetilde{w}),\ \widetilde{w}\in\mathbb{D},\ \zeta\in\left\{1,j,j^2\right\},$$

in particular these graphs are contained in $D\,(0,1/10)\times\mathbb{D}$ and their slope is smaller than $2(1-|m|)|\kappa|^{-2/3}$. Thus by Step 2 we deduce that $\widetilde{V}_\bullet\cap\widetilde{\mathcal{E}}_\bullet\neq\varnothing$, therefore $f_\varepsilon(\mathrm{Crit}(f_\varepsilon))\cap\mathcal{E}_\varepsilon\neq\varnothing$.

If now g is close to f_ε, as in the proof of Theorem 3.1 (see the beginning of Step 3 there), every compact piece of $\{c\}\times(\mathbb{C}\smallsetminus\{0\})$ can be followed as part of $\mathrm{Crit}(g)$, so $g(\mathrm{Crit}(g))$ contains three vertical graphs in $D(0,\delta)\times D_\kappa$ which are close to the corresponding ones for f_ε. Likewise, the basic set $\mathcal{E}(g)$ is induced by three inverse branches g_ζ^{-1} close to the corresponding ones for f_ε, thus Lemma 4.5 implies that $\mathcal{E}(g)\cap g(\mathrm{Crit}(g))\neq\varnothing$. This shows that for small $\varepsilon\neq0$, $f_\varepsilon\in\mathring{\mathrm{Bif}}$, hence $f\in\overline{\mathring{\mathrm{Bif}}}$, as desired.

Step 3.1: conclusion in the general case. — Let us now assume that p is an arbitrary polynomial satisfying the assumptions of the theorem. Replacing p by an arbitrary close perturbation, we may assume that there exists a simple critical point c and an integer $\ell\geqslant1$ such that $p^\ell(c)=0$. Indeed p belongs to the bifurcation locus so it admits an active critical point. We may suppose that all critical points are simple because the locus of polynomials with a multiple critical point is a proper subvariety and the bifurcation locus is not pluripolar. Now, either 0 is already the image of a critical point and we are done, or we can make a conjugacy depending holomorphically on p such that $p(1)=0$ and 0 is fixed. Since c is active, Montel's theorem implies that perturbing p slightly it can be mapped under iteration onto 0 or 1, and we are done.

Notice that we can also assume that the orbit segment $p(c),\dots,p^\ell(c)$ contains no other critical point: otherwise we replace c by the last appearing critical point in this orbit.

As in the case $\ell=1$, for small ε we consider the map defined by $f_\varepsilon(z,w)=(p(z)+\varepsilon w,q(w))$, which admits a blender-type Cantor set $\mathcal{E}(f_\varepsilon)$ in $D(0,\delta)\times D_\kappa$, for $\delta=|\kappa|^{1/3}\,|\varepsilon|/0.8(1-|m|)$.

We need to show that this Cantor set intersects the post-critical set. Then as before this intersection will be robust under further perturbations and we infer that $f_\varepsilon\in\mathring{\mathrm{Bif}}$, hence $f\in\overline{\mathring{\mathrm{Bif}}}$. The difficulty is that we cannot control $f_\varepsilon^\ell(\{c\}\times\mathbb{C})$ precisely enough to guarantee that it contains an almost flat vertical graph in the rescaled bidisk.[4] Instead we will use Proposition 2.2 together with a graph transform argument.

We start with a lemma, which will be proven afterwards.

[4] Note that already for $\ell=1$ the variation of $f_\varepsilon^\ell(\{c\}\times\mathbb{C})\cap(D(0,\delta)\times D_\kappa)$ with ε is of the same order of magnitude as δ, and this quantity tends to increase exponentially with ℓ.

Lemma 4.8. — *For small ε there exists a holomorphically varying fixed point x_ε for f_ε^3, contained in $\mathcal{E}(f_\varepsilon)\cap(D\,(0,\delta/10)\times D_\kappa)$.*

Let us show that there exist arbitrary small non-zero values of ε such that x_ε belongs to the post-critical set of f_ε. Indeed for small $\varepsilon_0>0$, consider the family $(f_\varepsilon)_{\varepsilon\in D(0,\varepsilon_0)}$. For $\varepsilon=0$, x_0 belongs to $\{0\}\times\mathbb{C}=f_0^\ell(\{c\}\times\mathbb{C})$. So either for every small ε, $x_\varepsilon\in f_\varepsilon^\ell(\{c\}\times\mathbb{C})$ and we are done, or the family $(f_\varepsilon)_{\varepsilon\in D(0,\varepsilon_0)}$ admits bifurcations. Within this family, the bifurcation locus cannot have isolated points because by [BBD17, Th. 1.6] it supports the measure dd^cL which has continuous potential. It follows that $0\in D(0,\varepsilon_0)$ is an accumulation point of the bifurcation locus, so by Proposition 2.2 there exists a sequence of parameters $0\neq\varepsilon_k\to0$, such that x_{ε_k} belongs to the post-critical set.

In addition $\mathrm{Crit}(f_\varepsilon)=(\mathrm{Crit}(p)\times\mathbb{C})\cup(\mathbb{C}\times\mathrm{Crit}(q))$ is a union of vertical and horizontal lines. The unique horizontal component $\{0\}\times\mathbb{C}$ escapes to infinity so we conclude that for such parameters ε_k, there exists a critical point c' for p and an integer N such that $x_\varepsilon\in f_\varepsilon^N(\{c'\}\times\mathbb{C})$. The image of a graph over D_κ in the second coordinate is the union of 3 such graphs: indeed write $\Gamma=\{z=\varphi(w),\ w\in D_\kappa\}$, and observe that

$$f(\Gamma)=\bigcup_{\zeta\in\{1,j,j^2\}}\big\{z=p(\varphi(q_\zeta^{-1}(w))+\varepsilon q_\zeta^{-1}(w),\ w\in D_\kappa\big\}.$$

Thus we infer that the irreducible component of $f_\varepsilon^n(\{c'\}\times\mathbb{C})$ through x_ε is a vertical graph. In particular it is smooth and its tangent vector at x_ε is not parallel to the horizontal axis.

The following lemma then allows to conclude the proof.

Lemma 4.9. — *The periodic point x_ε of Lemma 4.8 admits a strong unstable manifold $W^{uu}(x_\varepsilon)$ that is a graph over the second coordinate in $D\,(0,\delta)\times D_\kappa$ with slope smaller than $\delta(1-|m|)/200\,|\kappa|^{1/3}$. Furthermore if Γ is any any germ of holomorphic disk through x_ε, not tangent to the horizontal direction, the sequence of cut-off iterates $f^{3n}(\Gamma)\big|_{D(0,\delta)\times D_\kappa}$ converges to $W^{uu}(x_\varepsilon)$.*

For $\varepsilon=\varepsilon_k$, starting from the component V of $f_\varepsilon^N(\{c'\}\times\mathbb{C})$ through x_ε, we iterate under f_ε^3, and the lemma says that for large n, $f_\varepsilon^{3n}(V)$ contains a graph in $D\,(0,\delta)\times D_\kappa$ of slope less than $\delta(1-|m|)/200\,|\kappa|^{1/3}$, intersecting $D\,(0,\delta/10)\times D_\kappa$. In the rescaled coordinates, this corresponds to the requirements of Lemma 4.5 so we get an intersection between $f_\varepsilon^{N+3n}(\mathrm{Crit}(f_\varepsilon))$ and $\mathcal{E}(f_\varepsilon)$. Notice that the vertical graph producing this intersection is the image under f_ε^{N+3n} of a disk of the form $\{c'\}\times\Delta$, where Δ is a small topological disk close to a cube root of κ. Finally, as in the particular case of Step 3.0, if g is a small perturbation of f_ε, $\{c'\}\times\Delta$ can be lifted to a disk $\Delta(g)\subset\mathrm{Crit}(g)$, and $g^{N+3n}(\Delta(g))$ is C^1-close to the corresponding component of $f_\varepsilon^{N+3n}(\mathrm{Crit}(f_\varepsilon))$, therefore it intersects $\mathcal{E}(g)$. This completes the proof of the theorem. □

Before proving Lemma 4.8 let us state a Rouché-like fixed point theorem in two variables:

Proposition 4.10. — *Let $h : N(\overline{\mathbb{B}}) \to \mathbb{C}^2$ be a holomorphic map defined in a neighborhood of a bidisk in $\mathbb{C}^2$. Assume that h admits a unique simple fixed point in $\mathbb{B}$. If $\eta : N(\overline{\mathbb{B}}) \to \mathbb{C}^2$ is a holomorphic map such that $\|\eta\| < \|h - \mathrm{id}\|$ on $\partial\mathbb{B}$ then $h + \eta$ admits a unique fixed point in $\mathbb{B}$.*

Proof. — Consider the continuous family of holomorphic mappings $F_t = F + t\eta$ for $t \in [0, 1]$. For $t = 0$, the equation $F_t(z) = z$ admits a unique simple solution in $\mathbb{B}$. The assumption on η implies that for every η in $[0, 1]$, $\|F_t(z) - z\| > 0$ on $\partial\mathbb{B}$. Thus the continuity of intersection indices of properly intersection varieties of complementary dimensions implies the result. □

Proof of Lemma 4.8. — To understand the argument, let us come back to the linear IFS studied in Lemma 4.1 (for $d = 3$). Each ℓ_j admits a unique fixed point, and there are two cases. Either m is far away from 1 and this fixed point is close to the origin. Going to two dimensions, we will choose x_ε corresponding to one of these fixed points. When m is close to 1 the fixed point is close to the boundary of the unit disk, so it is not convenient for us. On the other hand if we look at $\ell_1 \circ \ell_2 \circ \ell_3$, the translation terms almost compensate and we get a fixed point close to the origin, now corresponding to a period 3 point for f_ε.

For the details it is convenient to work in the rescaled coordinate system $(\widetilde{z}, \widetilde{w})$: the expression of $\widetilde{f}_\bullet^{-1}$ is given in (9) (hereafter we drop the subscript for notational convenience), and we look for a fixed point (resp. a period 3 point) in $D(0, 1/10) \times \mathbb{D}$. Assume first m is far away from 1: $|m - 1| > 1/10$ and consider the inverse branch $\widetilde{f}_1^{-1}$. We write $\widetilde{f}_1^{-1}(\widetilde{z}, \widetilde{w})$ as the sum of an affine term and a perturbation:

$$\widetilde{f}_1^{-1}(\widetilde{z}, \widetilde{w}) = h(\widetilde{z}, \widetilde{w}) + \eta(\widetilde{z}, \widetilde{w}), \text{ where } h(\widetilde{z}, \widetilde{w}) = (m\widetilde{z} + \alpha m(1 - |m|), 1/2)\,.$$

Solving $h(\widetilde{z}, \widetilde{w}) = (\widetilde{z}, \widetilde{w})$ yields the solution $\left(\frac{\alpha m(1-|m|)}{m-1}, \frac{1}{2}\right)$ which belongs to $D(0, 1/10) \times \mathbb{D}$ because

$$\left|\frac{\alpha m(1 - |m|)}{m - 1}\right| \leqslant \frac{0.8/100}{1/10} = \frac{8}{100}$$

(recall $\alpha = 0.8$ and $1 - |m| < 0.01$). On the other hand if $|\widetilde{z}| = 1/10$ we get

$$\|h(\widetilde{z}, \widetilde{w}) - (\widetilde{z}, \widetilde{w})\| \geqslant |(m - 1)\widetilde{z} + \alpha m(1 - |m|)| \geqslant \frac{1}{10}\,|m - 1| - \frac{0.8}{100} \geqslant \frac{2}{1000},$$

and when $|\widetilde{w}| = 1$ considering the second component gives $\|h(\widetilde{z}, \widetilde{w}) - (\widetilde{z}, \widetilde{w})\| \geqslant 1/3$, so

$$\|h(\widetilde{z}, \widetilde{w}) - (\widetilde{z}, \widetilde{w})\| \geqslant \frac{2}{1000} \quad \text{on } \partial\big(D(0, 1/10) \times \mathbb{D}\big).$$

Finally the choices already made for κ and δ imply that $\|\eta(\widetilde{z}, \widetilde{w})\| < 2/1000$ and Proposition 4.10 yields the desired fixed point. (Recall that Theorem 4.7 claims a uniformity in κ; on the other hand δ can be freely reduced).

Suppose now that $|m-1|\leqslant 1/10$, and consider the inverse branch for $\widetilde{f}^3$ given by $\widetilde{f}_{j^2}^{-1}\circ\widetilde{f}_j^{-1}\circ\widetilde{f}_1^{-1}$. After computation, it expresses in coordinates as

$$\widetilde{f}_{j^2}^{-1}\circ\widetilde{f}_j^{-1}\circ\widetilde{f}_1^{-1}(\widetilde{z},\widetilde{w})=\Big(m^3\widetilde{z}+\alpha m(1-|m|)(j^2+mj+m^2)+\eta_1(\widetilde{z},\widetilde{w}),\frac{j^2}{2}+\eta_2(\widetilde{z},\widetilde{w})\Big),$$

with

$$|\eta_1(\widetilde{z},\widetilde{w})|\leqslant 3(1-|m|)\,|\kappa|^{-2/3}+3O(\delta)\ \text{and}\ |\eta_2(\widetilde{z},\widetilde{w})|\leqslant|\kappa|^{-2/3}. \tag{10}$$

In this case we show directly that $\widetilde{f}_{j^2}^{-1}\circ\widetilde{f}_j^{-1}\circ\widetilde{f}_1^{-1}$ sends the bidisk $D\big(0,1/10\big)\times\mathbb{D}$ strictly into itself, so it admits a unique fixed point by contraction of the Kobayashi metric. By the maximum principle it is enough to show that the boundary of the bidisk is mapped into its interior, and clearly we only need to focus on the first coordinate. For $|\widetilde{z}|=1/10$ the first component in (4.3) is bounded by

$$\begin{aligned}\frac{|m|^3}{10}+\alpha(1-|m|)\left|\frac{m^3-1}{m-j}\right|+|\eta_1(\widetilde{z},\widetilde{w})|&\leqslant\frac{|m|^3}{10}+2(1-|m|)^2+\frac{1}{100}(1-|m|)\\&\leqslant\frac{|m|}{10}+\frac{3}{100}(1-|m|)<\frac{1}{10},\end{aligned}$$

where in the first line we use

$$\begin{aligned}\left|\frac{m^3-1}{m-j}\right|\leqslant\left|\frac{(1-m)(m^2+m+1)}{m-j}\right|&\leqslant\frac{3(1-|m|)}{|1-j|-|m-1|}\\&\leqslant\frac{3}{\sqrt{3}-1/10}(1-|m|)\leqslant 2(1-|m|)\end{aligned}$$

and the bound for $|\eta_1(\widetilde{z},\widetilde{w})|$ coming from our choice of κ and δ in Step 2. Thus the desired contraction property is established and the result follows. ☐

Proof of Lemma 4.9. — The existence and the graph transform property of the strong unstable manifold are classical. In our case the specific geometric features of f_ε make the construction rather easy so we sketch it for convenience. Since f_ε preserves the foliation $\{w=\mathrm{C}^{\mathrm{st}}\}$, corresponding to the least repelling direction, x_ε admits a weak unstable manifold contained in a horizontal leaf. On the other hand, if g denotes the inverse branch of f_ε^3 such that $x_\varepsilon=g(x_\varepsilon)$, then $g^n(D(0,\delta)\times D_\kappa)$ is a sequence of topological bidisks converging to $\{x_\varepsilon\}$, which are asymptotically stretched in the horizontal direction. Therefore, if Γ is a germ of holomorphic disk through x_ε transverse to the horizontal leaf, for large enough n it crosses $g^n(D(0,\delta)\times D_\kappa)$ vertically, so iterating forward, the cut-off iterate

$$f_\varepsilon^{3n}(\Gamma)\big|_{D(0,\delta)\times D_\kappa}=f_\varepsilon^{3n}(\Gamma\cap g^n(D(0,\delta)\times D_\kappa))$$

is a vertical graph in $D(0,\delta)\times D_\kappa$. Furthermore, it Γ and Γ' are two such graphs, then the C^0 distance between Γ and Γ' in $g^n(D(0,\delta)\times D_\kappa)$ is $O(|\kappa|^{-n})$, and it gets multiplied by a factor $O(|m|^{-3n})$ under f_ε^{3n}. Applying this to Γ and $f_\varepsilon^3(\Gamma)$ shows that $f_\varepsilon^{3n}(\Gamma)\big|_{D(0,\delta)\times D_\kappa}$ is Cauchy, hence converges, and its limit is by definition the strong unstable manifold $W^{uu}(x_\varepsilon)$.

It remains to establish the estimate on the slope of $W^{uu}(x_\varepsilon)$. For this it is more convenient to work in the rescaled coordinates $(\widetilde{z},\widetilde{w})$, in which case the expected bound on the slope is $\frac{1}{100}(1-|m|)$. Since in these coordinates, the strong unstable manifold is the only vertical graph invariant under the graph transform, it is enough to show that the set of vertical graphs through x_ε with slope bounded by $\frac{1}{100}(1-|m|)$ is graph transform invariant. As already seen, if $\widetilde{z}=\varphi(\widetilde{w})$ is a vertical graph in $\mathbb{D}\times\mathbb{D}$, its forward image under $\widetilde{f_\varepsilon}$, restricted to $\mathbb{C}\times\mathbb{D}$ is the union of 3 vertical graphs of equation

$$\widetilde{z}=\widetilde{p}(\varphi(\widetilde{q_\zeta^{-1}}(\widetilde{w})))-2\alpha(1-|m|)q_\zeta^{-1}(\widetilde{w}).$$

Assuming $\|\varphi'\|_{\mathbb{D}}\leqslant\frac{1}{100}(1-|m|)$, each of them has a slope bounded by

$$\big\|(q_\zeta^{-1})'\big\|_{\mathbb{D}}\,\|\widetilde{p}'\|_{\mathbb{D}}\,\|\varphi'\|_{\mathbb{D}}+2(1-|m|)\big\|(q_\zeta^{-1})'\big\|_{\mathbb{D}}\leqslant(1-|m|)\Big(\frac{1}{100}\,|\kappa|^{-2/3}\,\|\widetilde{p}'\|_{\mathbb{D}}+2\,|\kappa|^{-2/3}\Big).$$

Since for small δ, $\|\widetilde{p}'\|_{\mathbb{D}}\leqslant 2$ and $|\kappa|^{2/3}\geqslant 2000$, this quantity is bounded by $\frac{1}{100}(1-|m|)$.

For a general graph Γ we cannot iterate this reasoning because its forward iterates may leave the bidisk. However in our case we start with $\Gamma\ni x_\varepsilon$ which is either fixed or of period 3 (with its orbit contained in $\mathbb{D}^2$), so we indeed get an invariant set of vertical graphs and we are done. □

The uniformity in κ in Theorem 4.7 allows to let the multiplier tend to 1.

Corollary 4.11. — *Let $d\geqslant 3$ and $f:\mathbb{P}^2\to\mathbb{P}^2$ be a product map of the form*

$$f(z,w)=(p(z),w^d+\kappa).$$

Assume p_0 admits a neutral fixed point. Then there exists a constant $\kappa_0=\kappa_0(d)$ such that if $|\kappa|>\kappa_0$, f belongs to the closure of the interior of the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^2)$.

Proof. — Using the uniformity with respect to κ in Theorem 4.7 it is enough to show that in every neighborhood of p in $\mathcal{P}_d$ there exists a polynomial p_1 with a repelling fixed point of low multiplier, belonging to the bifurcation locus.

Without loss of generality we may assume that z_0 is rationally indifferent, that is $p'(z_0)=e^{2i\pi p/q}$. Taking a branched cover of $\mathcal{P}_d$ if necessary (this is needed only if $p'(z_0)=1$), we can follow the fixed point z_0 holomorphically and normalize the coordinates so that $z_0=0$. Fix a one-dimensional holomorphic family of polynomials $(p_\lambda)_{\lambda\in\mathbb{D}}$ with $p_0=p$ and such that $\frac{d}{d\lambda}\left(p_\lambda'(0)\right)\big|_{\lambda=0}\neq 0$. Put $\rho_\lambda=p_\lambda'(0)$. Then a classical computation shows that there exists a local change of coordinates $x=\varphi_\lambda(z)$ depending holomorphically on λ such that in the new coordinates f_λ^q expresses as

$$f_\lambda^q(x)=\rho_\lambda^q x+x^{\nu q+1}+x^{\nu q+2}g_\lambda(x)$$

for some integer $\nu\geqslant 1$ (see [DS85, Prop. 1] or [DL15, Prop. 8.1]). Write $\rho_\lambda^q=1+b\lambda+O(\lambda^2)$, where $b=q\,d\rho_\lambda/d\lambda\big|_{\lambda=0}$. Then

$$f_\lambda^q(x)-x=x\big((\rho_\lambda^q-1)+x^{\nu q}+x^{\nu q+1}g_\lambda(x)\big),\tag{11}$$

hence f_λ^q admits $\nu q+1$ fixed points near the origin: one at 0 with multiplier ρ_λ^q and the remaining νq ones approximately equal to the $(\nu q)^{\text{th}}$ roots of $(1-\rho_\lambda^q)$. More precisely plugging into the right hand side of (11), we get that these fixed points $(\alpha_i)_{i=1,\dots,\nu q}$ satisfy

$$(\alpha_i)^{\nu q} = (1-\rho_\lambda^q) - (\alpha_i)^{\nu q+1} g_\lambda(\alpha_i) = -b\lambda + O\big(|\lambda|^{1+1/\nu q}\big),$$

and their multipliers are of the form

$$(f_\lambda^q)'(\alpha_i) = \rho_\lambda^q + (\nu q+1)\alpha_i^{\nu q} + O\left((\alpha_i)^{\nu q+1}\right) = 1 - \nu q b\lambda + O\big(|\lambda|^{1+1/\nu q}\big).$$

We see that arbitrary close to the origin in parameter space, in the regime where 0 is repelling under f_λ^q (i.e., $|\rho_\lambda^q|>1$), we can arrange so that $|(f_\lambda^q)'(\alpha_i)|<1$ (e.g. when $b\lambda$ is real and positive) or $|(f_\lambda^q)'(\alpha_i)|>1$ (by taking $b\lambda$ on the imaginary axis). Therefore bifurcations happen and the result follows. $\square$

We now state a version of Theorem 4.7 in degree 2.

Theorem 4.12. — *There exists a parameter c in the Mandelbrot set such that for every large enough $\kappa\in\mathbb{C}$, the product map $f(z,w)=(z^2+c,w^2+\kappa)$ belongs to the closure of the interior of the bifurcation locus in $\mathcal{H}_2(\mathbb{P}^2)$.*

Proof (sketch). — Pick $c\in M$ such that z^2+c has a fixed point z_0 whose multiplier $2z_0 =: m^{-1}$ satisfies $0.99<|m|<1$ and $|\arg(m)-\pi/2|<\pi/50$. (The existence of such a parameter follows exactly as in the previous corollary, by starting from the boundary of the main cardioid and perturbing in the appropriate direction.) Shifting c slightly we can further arrange that the critical point 0 falls onto z_0 under iteration. Arguing as in Step 2 of the proof of Theorem 4.7 (using Lemma 4.4 and Remark 4.6) shows that the perturbation $f_\varepsilon(z,w)=(z^2+c+\varepsilon w, w^2+\kappa)$ admits a blender in $D(z_0,\delta)\times D(0,2|\kappa|^{1/2})$ for $\delta = |\kappa|^{1/2}|\varepsilon|/0.95(1-|m|)$, when ε is small and κ is larger than some absolute constant. Finally the argument given in the third step of the proof implies the existence of a sequence $\varepsilon_j\to 0$ such that the post-critical set of f_{ε_j} robustly intersects this blender, and the result follows. We leave the reader fill the details. $\square$

5. Further considerations and open problems

5.1. Bifurcations of saddle sets.

— In this paragraph we show that persistent homoclinic tangencies, hence robust homoclinic bifurcations, can coexist with J^*-stability. This implies that the robust bifurcations constructed in this paper are *not* induced by the Newhouse phenomenon. This also highlights the large gap between J^*-stability and structural stability on $\mathbb{P}^2$.

Theorem 5.1. — *There exists a J^*-stable family of holomorphic endomorphisms of $\mathbb{P}^2$ whose members possess a horseshoe with a generic tangency between the stable and unstable laminations.*

Proof. — The idea is to embed a polynomial automorphism h of $\mathbb{C}^2$ with a robust homoclinic tangency in an endomorphism of $\mathbb{P}^2$ (this idea already appears in Buzzard [Buz97] and Gavosto [Gav98]). What we need to do is to arrange so that J^*-stability holds.

The first observation is that we can choose h to be a product of two complex Hénon mappings of the same degree. Indeed, the automorphisms constructed by Buzzard are of the form $F = F_3 \circ F_2 \circ F_1$, where $F_1(z,w) = (z + f(w), w)$, $F_2(z,w) = (z, w + g(z))$ and $F_3(z,w) = (cz, c^{-1}w)$ (see [Buz97, p. 394]). At the initial stage of Buzzard's argument, f and g are holomorphic mappings defined on certain open subsets of $\mathbb{C}$, which are later approximated by polynomials. Thus we can assume that f and g are polynomials of the same (large) degree d. Let ι be the involution $(z,w) \mapsto (w,z)$ and write F as $(F_3 \circ F_2 \circ \iota) \circ (\iota \circ F_1)$. Then $\iota \circ F_1$ is a conservative Hénon map, but $F_3 \circ F_2 \circ \iota$ is not of the right form. So we introduce the linear map $\ell : (z,w) \mapsto (z, cw)$, and write

$$F = (F_3 \circ F_2 \circ \iota \circ \ell^{-1}) \circ (\ell \circ \iota \circ F_1) := h^- \circ h^+.$$

We leave the reader check that h^- and h^+ are of the form $(z,w) \mapsto (w, c^{\pm 1} z + p^{\pm}(w))$, with $p^{\pm}$ of degree d, as desired.

With notation as above introduce $F_\varepsilon = h_\varepsilon^- \circ h_\varepsilon^+$, where

$$h_\varepsilon^{\pm}(z,w) = (w + \varepsilon z^d, c^{\pm 1} z + p^{\pm}(w)).$$

This is a holomorphic endomorphism of $\mathbb{P}^2$ which on every given compact subset in $\mathbb{C}^2$ can be seen as a small perturbation of F. In particular if we fix ε small enough and let c and $p^{\pm}$ vary we get a holomorphic family of endomorphisms of $\mathbb{P}^2$ with a saddle set exhibiting a persistent generic tangency between its stable and unstable laminations.

What remains to do is to show that this family is J^*-stable. This will be a consequence of the following lemma.

Lemma 5.2. — *Let $h_\varepsilon(z,w) = (w + \varepsilon z^d, cz + p(w))$, where p is a polynomial of degree d. Let $\beta = 1/(d-1)$, α be a real number such that $\beta/d < \alpha < \beta$ and*

$$V_\varepsilon = \Big\{ (z,w) \in \mathbb{C}^2, \frac{1}{2}\varepsilon^{-\beta} < |z| < \frac{3}{2}\varepsilon^{-\beta},\ |w| < \varepsilon^{-\alpha} \Big\}.$$

Then for ε sufficiently small (locally uniformly with respect to the parameters c and p), $h_\varepsilon^{-1}(V_\varepsilon) \Subset V_\varepsilon$ and $h_\varepsilon : h_\varepsilon^{-1}(V_\varepsilon) \to V_\varepsilon$ is an unbranched covering.

Applying this lemma successively to h_ε^+ and h_ε^-, we deduce that for small ε, $F_\varepsilon^{-1}(V_\varepsilon) \Subset V_\varepsilon$ and $F_\varepsilon : F_\varepsilon^{-1}(V_\varepsilon) \to V_\varepsilon$ is a covering of degree $(2d)^2$. Since $\deg(F_\varepsilon) = 2d$, we infer that $J^*(F_\varepsilon) \subset V_\varepsilon$, and since in addition preimages in V_ε can be followed locally holomorphically with the parameters, it follows that the family obtained by fixing ε and varying c and $p^{\pm}$ is locally J^*-stable. Indeed pick any point $\gamma \in V_\varepsilon$, and view it as a (constant) graph from the parameter space to $\mathbb{P}^2$. Then, since $F_\varepsilon^{-1}(V_\varepsilon) \subset V_\varepsilon$ and $F_\varepsilon : F_\varepsilon^{-1}(V_\varepsilon) \to V_\varepsilon$ is a covering, γ is persistently disjoint from the post-critical set, so Proposition 2.2 applies and we are done. $\square$

Proof of Lemma 5.2. — To ease notation, without loss of generality we assume that $c=1$. We rescale the coordinates by putting $\widetilde{z}=\varepsilon^{-1/(d-1)}z=\varepsilon^{-\beta}z$ and $\widetilde{w}=w$. In the new coordinates (still denoted by (z,w) for convenience) h_ε becomes

$$h_\varepsilon:(z,w)\longmapsto\big(\varepsilon^\beta w+z^d,\varepsilon^{-\beta}z+p\,(w)\big)$$

and V_ε becomes $A(1/2,3/2)\times D(0,\varepsilon^{-\alpha})$, where $A(1/2,3/2)$ is the open annulus bounded by the circles of radii 1/2 and 3/2.

Let now $(u,v)\in V_\varepsilon$ and (z,w) be such that $h_\varepsilon(z,w)=(u,v)$, that is:

$$(i)\ \varepsilon^\beta w+z^d=u \text{ and } (ii)\ \varepsilon^{-\beta}z+p\,(w)=v.$$

Then we claim that $\varepsilon^\beta\,|w|\ll|z|^d$. Indeed otherwise since $|u|\approx 1$ we get that $\varepsilon^\beta\,|w|\gtrsim 1$ hence $|w|\gtrsim\varepsilon^{-\beta}$, therefore $|p(w)|\gtrsim\varepsilon^{-d\beta}$. Since $d\beta>\alpha$, this implies that $|p(w)|\gg|v|$ hence by (ii) $\varepsilon^{-\beta}\,|z|\approx|p(w)|\approx|w|^d\gtrsim\varepsilon^{-d\beta}$, so $|z|\gtrsim\varepsilon^{-\beta(d-1)}=\varepsilon^{-1}$. Plugging this back into (i), we see that $\varepsilon^\beta\,|w|\approx|z|^d$. Together with $\varepsilon^{-\beta}\,|z|\approx|w|^d$, this yields $|z|\approx\varepsilon^{\beta/(d+1)}=\varepsilon^{1/(d^2-1)}=o(1)$. This is contradictory, so the claim is proved.

Since $\varepsilon^\beta\,|w|\ll|z|^d$, the equation (i) admits d unramified solutions in z, close to the d^{th} roots of u. Therefore $\varepsilon^{-\beta}\,|z|\approx\varepsilon^{-\beta}\gg|v|$ so for each such z, solving (ii) in the variable w gives d solutions satisfying $|w|^d\approx\varepsilon^{-\beta}$, that is $|w|\approx\varepsilon^{-\beta/d}$, and since $\beta/d<\alpha$ these solutions belong to $D(0,\varepsilon^{-\alpha})$.

Finally, the critical set is the curve of equation $dz^{d-1}p'(w)=1$. If $(z,w)\in\operatorname{Crit}(h_\varepsilon)\cap V_\varepsilon$ we have $|z|\approx 1$ so we get that $|w|=O(1)$. Thus the second coordinate of $h_\varepsilon(z,w)$ is of order of magnitude $\varepsilon^{-\beta}\gg\varepsilon^{-\alpha}$. This means that critical points escape V_ε after one iteration (i.e., $h_\varepsilon(\operatorname{Crit}(h_\varepsilon)\cap V_\varepsilon)\cap V_\varepsilon=\varnothing$) and we conclude that h_ε is unbranched in $h_\varepsilon^{-1}(V_\varepsilon)$. □

Remark 5.3. — Another consequence of Theorem 5.1 is that there exists a J^*-stable family whose generic members have with infinitely many repelling periodic points outside J^*. Indeed since the polynomial automorphism F is conservative, by a small perturbation we can choose the Jacobian to be either smaller of larger than 1. In the latter case, it is well known that the persistent homoclinic tangency gives rise to a residual set of parameters with infinitely many sources.

5.2. Higher dimension. — So far we have concentrated on complex dimension 2. The results of Section 4 can actually be adapted to an arbitrary number of dimensions. Let us only state one result, which guarantees the existence of open sets in the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^k)$ for every $d\geqslant 2$ and every $k\geqslant 3$.

Theorem 5.4. — *Let f be a polynomial mapping in $\mathbb{C}^2$ of the form $f(z,w)=(p(z),w^d+\kappa)$, where $p\in\mathcal{P}_d$ admits a fixed point z_0 of multiplier $1<|p'(z_0)|<1.01$ and belongs to the bifurcation locus in $\mathcal{P}_d$ (if $d=2$ we further require that $|\arg(p'(z_0))-\pi/2|<\pi/50$).*

Let g be a regular polynomial mapping in $\mathbb{C}^{k-2}$ admitting a repelling fixed point in J^* with eigenvalues larger than 2.

Then if $|\kappa| > \kappa_0(d,k)$ is sufficiently large, the product map (f,g) belongs to the closure of the interior of the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^k)$.

Proof. — We focus on the case $d = 3$ and leave the adaptation to degree 2 to the reader. We keep notation as in the proof of Theorem 4.7 and explain how to generalize the argument. Put $x = (z,w)$ and denote by y the variable in $\mathbb{C}^{k-2}$. A preliminary observation is that $F(x,y) := (f(x), g(y))$ defines a regular polynomial mapping in $\mathbb{C}^k$ so it extends holomorphically to $\mathbb{P}^k$. Let y_0 be the repelling fixed point for g considered in the statement of the theorem, and recall that $\mathcal{E}(f) = \{0\} \times J_{w^d+\kappa}$ is a basic repeller contained in $J^*(f)$. Therefore $\mathcal{E}(F) := \mathcal{E}(f) \times \{y_0\}$ is a basic repeller contained in $J^*(F)$ and this property persists for its continuation after a small perturbation of F in $\mathcal{H}_d(\mathbb{P}^k)$ by Lemma 2.3.

Let $f_\varepsilon(x) = f_\varepsilon(z,w) = (p(z) + \varepsilon w, w^d + \kappa)$ as before, and put $F_\varepsilon(x,y) = (f_\varepsilon(x), g(y))$. Recall from Theorem 4.7 that after translation and rescaling of the (z,w) coordinate, when ε is small and κ is large, the continuation of $\{0\} \times J_q$ defines a blender type repeller for f_ε in the unit bidisk. By assumption, there exists $\eta > 0$ such that g admits a contracting inverse branch mapping y_0 to itself, of derivative norm smaller than $1/2$ in the polydisk centered at y_0 and of radius η. We translate and rescale the y coordinate by putting $\widetilde{y} = \eta^{-1}(y - y_0)$. In the new coordinates $(\widetilde{x}, \widetilde{y})$, the resulting map $\widetilde{F}_\varepsilon$ defines an IFS on $\mathbb{D}^k$ with d branches for every small enough ε.

The mechanism leading to robust intersections between $\mathcal{E}(F)$ and the post-critical set will be based on the following higher dimensional version of Lemma 4.5, which is worth stating precisely.

Lemma 5.5. — *Let $d \geqslant 3$ and $\mathcal{L} = (L_j)_{j=1}^d$ be an IFS in $\mathbb{D}^k$ generated by biholomorphic contractions of the form*

$$L_j(z,\omega) = (\ell_j(z), \varphi_j(z,\omega)),$$

with $(z,\omega) \in \mathbb{D} \times \mathbb{D}^{k-1}$, and let $\mathcal{E}$ be its limit set. Assume that ℓ_j is of the form

$$\ell_j(z) = mz + \alpha_j(1 - |m|)e^{(2\pi i/d)j}, \ \textit{where } 0.98 < |m| < 1 \textit{ and } \alpha_j \in A' \ \textit{(see (4))},$$

and $\varphi_j : \mathbb{D}^k \to \mathbb{D}^{k-1}$ is a holomorphic map such that $\|\partial_z \varphi_j\| < 1$ and $\|\partial_\omega \varphi_j\| < 1/2$.

Then any vertical graph Γ of the form $z = \gamma(\omega)$ intersecting $D(0, 1/10) \times \mathbb{D}^{k-1}$, whose slope satisfies $\|d\gamma\| \leqslant \frac{1}{100\sqrt{k-1}}(1 - |m|)$ intersects $\mathcal{E}$.

Furthermore, the same holds for any IFS $\overline{\mathcal{L}}$ generated by $(\overline{L}_j)_{j=1}^d$, whenever the C^1 norm of $L_j - \overline{L}_j$ is bounded by $\frac{1}{1000\sqrt{k-1}}(1 - |m|)$.

The proof is identical to that of Lemma 4.5. The appearance of the factor $\sqrt{k-1}$ in the constants comes from the fact that the diameter of the first projection of Γ is now bounded by $\operatorname{diam}(\mathbb{D}^{k-1}) \times \operatorname{slope}(\Gamma) = 2\sqrt{k-1}\,\|d\gamma\|$. Under the above assumptions on ε, κ and η, in the rescaled coordinates (and possibly after rotation in z), the IFS induced by F_ε on $\mathbb{D}^k$ satisfies the hypotheses of the lemma (see (9)).

The third step of Theorem 4.7 shows that, possibly after a preliminary perturbation of p, there exists a sequence of parameters $\varepsilon_j \to 0$ and for every k an integer $N' = N + 3n$ and a component of multiplicity 1 of $\operatorname{Crit} f_{\varepsilon_j}$ of the form $\{c'\} \times \mathbb{C}$, such that (after coordinate rescaling) $f_{\varepsilon_j}^{N'}(\{c'\} \times \mathbb{C})$ contains a vertical graph Γ in the unit bidisk, satisfying the assumptions of Lemma 4.5. Remark that by Lemma 4.9, by increasing κ we can make the slope of this graph smaller than $\frac{1}{100\sqrt{k-1}}(1-|m|)$.

Going to dimension k, $\{c'\} \times \mathbb{C}^{k-1}$ is a component of multiplicity 1 of F_ε and the previous discussion shows that (after rescaling) $F_{\varepsilon_j}^{N'}(\{c'\} \times \mathbb{C}^{k-1}) \cap \mathbb{D}^k$ admits a component of the form $\Gamma \times \mathbb{C}^{k-2}$, hence satisfying the requirements of Lemma 5.5. In addition, this vertical graph is the image of a piece of $\Gamma \times \mathbb{C}^{k-2}$ contained in $\{c'\} \times \Delta \times U$, where Δ is a disk in $\mathbb{C}$ close to a cube root of κ and U is a small neighborhood of the repelling point y_0. Increasing N' further if necessary, we can assume that U is disjoint from $\operatorname{Crit}(g)$. Since $\operatorname{Crit}(F_\varepsilon) = \pi_1^{-1}(\operatorname{Crit}(f)) \cup \pi_2^{-1}(\operatorname{Crit}(g))$, we infer that $\operatorname{Crit}(F_\varepsilon)$ is smooth along $\{c'\} \times \mathbb{C} \times U$.

We are now ready to conclude that $F_{\varepsilon_j} \in \mathring{\operatorname{Bif}}$. Indeed if G is a small perturbation of F in $\mathcal{H}_d(\mathbb{P}^k)$, we can lift $\{c'\} \times \Delta \times U$ to an open cell in $\operatorname{Crit}(G)$, whose image $\Gamma(G)$ under $G^{N'}$ is close to $\Gamma \times \mathbb{C}^{k-2}$. So Lemma 5.5 implies that $\Gamma(G) \cap \mathcal{E}(G) \neq \varnothing$, and we are done. □

5.3. The interior of the bifurcation locus. — The constructions of Section 4 raise a number of interesting problems. A natural question after Theorem 4.7 (and Corollary 4.11) is whether in these statements the fixed point z_0 can be assumed to be periodic instead of fixed. Then, using the technique of Corollary 4.11, this suggests that the hypothesis on p in Theorem 4.7 could simply be replaced by "p belongs to the bifurcation locus".

While this paper was on revision, this result was achieved by Taflin by using a related—but different—method.[5]

Theorem 5.6 (Taflin [Taf17]). — *Let $f(z,w) = (p(z), q(w))$ be a product of two polynomial maps of degree d in $\mathbb{C}$. Then if p or q belongs to the bifurcation locus in $\mathcal{P}_d(\mathbb{C})$, f belongs to the closure of the interior of the bifurcation locus in $\mathcal{H}_d(\mathbb{P}^2)$.*

In other words, the bifurcation locus restricted to the subfamily $\mathcal{P}_d \times \mathcal{P}_d$ has empty interior, nevertheless it should be accumulated at every point by robust bifurcations in the larger space of holomorphic mappings on $\mathbb{P}^2$.

In dimension 1, Shishikura's theorem on the dimension of the boundary of the Mandelbrot set [Shi98], as well its generalizations to arbitrary spaces of rational mappings by Tan Lei and McMullen [Lei98, McM00] are based on the construction of Cantor repellers of large dimension (i.e., close to 2). In dimension 2, bifurcations are created by collisions between the post-critical set (which has Hausdorff dimension 2) and hyperbolic repellers. Intuition from fractal geometry and real dynamics leads us

[5]The result was explicitly stated as a conjecture in the first version of this paper.

to think that generically, a collision between the post-critical set and a hyperbolic repeller of dimension *larger than* 2 should yield massive bifurcation sets.

Here is a specific situation where we expect such a phenomenon to happen.

Conjecture 5.7. — *Let f be a Lattès map on $\mathbb{P}^2$. Then f belongs to the closure of the interior of the bifurcation locus.*

Indeed, a Lattès example is semi-conjugate to a multiplication on a complex 2-torus, so it admits hyperbolic repellers of dimension arbitrary close to 4. In addition, it is conformal so the geometry of the perturbations of these repellers is expectedly easier to understand than in the general case. Notice that Lattès mappings indeed belong to the bifurcation locus since the sum of the Lyapunov exponents of μ_f is minimal there (see [BBD17, Th. 6.3]).

Finally, on a more ambitious note, one may ask whether the bifurcation locus is the closure of its interior in $\mathcal{H}_d(\mathbb{P}^2)$ or if on the contrary there are regions where the density of stability still holds. As outlined in the introduction, bifurcations are created when a multiplier of a repelling periodic orbit crosses the unit circle, so an interesting approach to this problem would be to understand when blenders are created in this process.

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