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HEIGHT, GRADED RELATIVE HYPERBOLICITY AND QUASICONVEXITY

by François Dahmani & Mahan Mj

Abstract. — We introduce the notions of geometric height and graded (geometric) relative hyperbolicity in this paper. We use these to characterize quasiconvexity in hyperbolic groups, relative quasiconvexity in relatively hyperbolic groups, and convex cocompactness in mapping class groups and $\mathrm{Out}(F_n)$.

Résumé (Hauteur, hyperbolicité relative graduée, et quasiconvexité). — Nous introduisons les notions de hauteur géométrique d'un sous-groupe, et d'hyperbolicité relative graduée d'un groupe, avec une version géométrique de cette dernière. Nous utilisons ensuite ces notions pour caractériser la quasiconvexité des sous-groupes des groupes hyperboliques, la quasiconvexité relative des sous-groupes des groupes relativement hyperboliques, et le fait d'être convexe-cocompact dans un groupe modulaire de surface, ou dans un groupe d'automorphismes extérieurs de groupe libre.

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1. Introduction

It is well-known that quasiconvex subgroups of hyperbolic groups have finite height. In order to distinguish this notion from the notion of *geometric height* introduced later in this paper, we shall call the former *algebraic height*: Let G be a finitely generated group and H a subgroup. We say that a collection of conjugates $\{g_iHg_i^{-1}\}$, $i=1,\dots,n$ are *essentially distinct* if the cosets $\{g_iH\}$ are distinct. We say that H has finite *algebraic height* if there exists $n\in\mathbb{N}$ such that the intersection of any $(n+1)$ essentially distinct conjugates of H is finite. The minimal n for which this happens is called the *algebraic height* of H. Thus H has algebraic height one if and only if it is almost malnormal. This admits a natural (and obvious) generalization to a finite collection of subgroups H_i instead of one H. Thus, if G is a hyperbolic group and H a quasiconvex subgroup (or more generally if $H_1,\dots,H_n$ are quasiconvex), then H (or more generally the collection $\{H_1,\dots,H_n\}$) has finite algebraic height [GMRS98]. (See [Dah03, HW09] for generalizations to the context of relatively hyperbolic groups.) Swarup asked if the converse is true:

Question 1.1 ([Bes04]). — Let G be a hyperbolic group and H a finitely generated subgroup. If H has finite height, is H quasiconvex?

An example of an infinitely generated (and hence non-quasiconvex) malnormal subgroup of a finitely generated free group was obtained in [DM15] showing that the hypothesis that H is finitely generated cannot be relaxed. On the other hand, Bowditch shows in [Bow12] (see also [Mj08, Prop. 2.10]) the following positive result:

Theorem 1.2 ([Bow12]). — *Let G be a hyperbolic group and H a subgroup. Then G is strongly relatively hyperbolic with respect to H if and only if H is an almost malnormal quasiconvex subgroup.*

One of the motivational points for this paper is to extend Theorem 1.2 to give a characterization of quasiconvex subgroups of hyperbolic groups in terms of a notion of *graded relative hyperbolicity* defined as follows:

Definition 1.3. — Let G be a finitely generated group, d the word metric with respect to a finite generating set and H a subgroup. Let $\mathscr{H}_i$ be the collection of intersections of i essentially distinct conjugates of H, let $(\mathscr{H}_i)_0$ be a choice of conjugacy representatives, and let $\mathscr{C}\mathbf{H}_i$ be the set of cosets of elements of $(\mathscr{H}_i)_0$. Let d_i be the metric on (G,d) obtained by electrifying[(1)] the elements of $\mathscr{C}\mathbf{H}_i$. We say that G is *graded relatively hyperbolic* with respect to H (or equivalently that the pair $(G,\{H\})$ has graded relative hyperbolicity) if

(1) H has algebraic height n for some $n\in\mathbb{N}$,

[(1)]The second author acknowledges the moderating influence of the first author on the more extremist terminology *electrocution* [Mj14, Mj10]

(2) each element K of $\mathscr{H}_{i-1}$ has a finite relative generating set S_K, relative to $H \cap \mathscr{H}_i (:= \{H \cap H_i : H_i \in \mathscr{H}_i\})$; further, the cardinality of the generating set S_K is bounded by a number depending only on i (and not on K),

(3) (G, d_i) is strongly hyperbolic relative to $\mathscr{H}_{i-1}$, where each element K of $\mathscr{H}_{i-1}$ is equipped with the word metric coming from S_K.

The following is the main theorem of this paper (see Theorem 6.4 for a more precise statement using the notion of *graded geometric relative hyperbolicity* defined later) providing a partial positive answer to Question 1.1 and a generalization of Theorem 1.2:

Theorem 1.4. — *Let (G, d) be one of the following:*

(1) *G a hyperbolic group and d the word metric with respect to a finite generating set S.*

(2) *G is finitely generated and hyperbolic relative to $\mathscr{P}$, S a finite relative generating set, and d the word metric with respect to $S \cup \mathscr{P}$.*

(3) *G is the mapping class group $\mathrm{Mod}(S)$ and d a metric that is equivariantly quasi-isometric to the curve complex $\mathrm{CC}(S)$.*

(4) *G is $\mathrm{Out}(F_n)$ and d a metric that is equivariantly quasi-isometric to the free factor complex $\mathscr{F}_n$.*

Then (respectively)

(1) *if H is quasiconvex, then $(G, \{H\})$ has graded relative hyperbolicity; conversely, if $(G, \{H\})$ has geometric graded relative hyperbolicity then H is quasiconvex,*

(2) *if H is relatively quasiconvex then $(G, \{H\}, d)$ has graded relative hyperbolicity; conversely, if $(G, \{H\}, d)$ has geometric graded relative hyperbolicity then H is relatively quasiconvex,*

(3) *if H is convex cocompact in $\mathrm{Mod}(S)$ then $(G, \{H\}, d)$ has graded relative hyperbolicity; conversely, if $(G, \{H\}, d)$ has geometric graded relative hyperbolicity and the action of H on the curve complex is uniformly proper, then H is convex cocompact in $\mathrm{Mod}(S)$,*

(4) *if H is convex cocompact in $\mathrm{Out}(F_n)$ then $(G, \{H\}, d)$ has graded relative hyperbolicity; conversely, if $(G, \{H\}, d)$ has geometric graded relative hyperbolicity and the action of H on the free factor complex is uniformly proper, then H is convex cocompact in $\mathrm{Out}(F_n)$.*

Structure of the paper. — In Section 2, we will review the notions of hyperbolicity for metric spaces relative to subsets. This will be related to the notion of hyperbolic embeddedness [DGO17]. We will need to generalize the notion of hyperbolic embeddedness in [DGO17] to one of coarse hyperbolic embeddedness in order to accomplish this. We will also prove results on the preservation of quasiconvexity under electrification. We give two sets of proofs: the first set of proofs relies on assembling diverse pieces of literature on relative hyperbolicity, with several minor adaptations. We also give a more self-contained set of proofs relying on asymptotic cones.

In Section 3.1 and the preliminary discussion in Section 4, we give an account of two notions of height: algebraic and geometric. The classical (algebraic) notion of height of a subgroup concerns the number of conjugates that can have infinite intersection. The notion of geometric height is similar, but instead of considering infinite intersection, we consider unbounded intersections in a (not necessarily proper) word metric. This naturally leads us to dealing with intersections in different contexts:

(1) Intersections of conjugates of subgroups in a proper (Γ, d) (the Cayley graph of the ambient group with respect to a finite generating set).

(2) Intersections of metric thickenings of cosets in a not necessarily proper (Γ, d).

The first is purely group theoretic (algebraic) and the last geometric. Accordingly, we have two notions of height: algebraic and geometric. In line with this, we investigate two notions of graded relative hyperbolicity in Section 4 (cf. Definition 4.3):

(1) Graded relative hyperbolicity (algebraic).

(2) Graded geometric relative hyperbolicity.

In the fourth section, we also introduce and study a qi-intersection property, a property that ensures that quasi-convexity is preserved under passage to electrified spaces. The property exists in both variants above.

In the fifth and the sixth sections, we will prove our main results relating height and geometric graded relative hyperbolicity. On a first reading, the reader is welcome to keep the simplest (algebraic or group-theoretic) notion in mind. To get a hang of where the paper is headed, we suggest that the reader take a first look at Sections 5 and 6, armed with Section 3.3 and the statements of Proposition 4.5, Theorem 4.6, Theorem 4.10, Theorem 4.14 and Proposition 4.15. This, we hope, will clarify our intent.

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2. Relative hyperbolicity, coarse hyperbolic embeddings

We shall clarify here what it means in this paper for a geodesic space (X, d), to be hyperbolic relative to a family of subspaces $\mathscr{Y} = \{Y_i, i \in I\}$, or to cast it in another language, what it means for the family $\mathscr{Y}$ to be hyperbolically embedded in (X, d). There are slight differences from the more usual context of groups and subgroups (as in [DGO17]), but we will keep the descending compatibility (when these notions hold in the context of groups, they hold in the context of spaces).

We begin by recalling relevant constructions.

2.1. Electrification by cones. —

Given a metric space (Y, d_Y), we will endow $Y \times [0, 1]$ with the following product metric: it is the largest metric that agrees

with d_Y on $Y \times \{0\}$, and each $\{y\} \times [0,1]$ is endowed with a metric isometric to the segment $[0,1]$.

Definition 2.1 ([Far98]). — Let (X,d) be a geodesic length space, and

$$\mathscr{Y} = \{Y_i,\, i \in I\}$$

be a collection of subsets of X. The *electrification* $(X^{\mathrm{el}}_{\mathscr{Y}}, d^{\mathrm{el}}_{\mathscr{Y}})$ of (X,d) along $\mathscr{Y}$ is defined as the following coned-off space:

$$X^{\mathrm{el}}_{\mathscr{Y}} = X \sqcup \left\{\bigsqcup_{i\in I} Y_i \times [0,1]\right\}/\sim$$

where $\sim$ denotes the identification of $Y_i \times \{0\}$ with $Y_i \subset X$ for each i, and the identification of $Y_i \times \{1\}$ to a single cone point v_i (dependent on i).

The metric $d^{\mathrm{el}}_{\mathscr{Y}}$ is defined as the path metric on $X^{\mathrm{el}}_{\mathscr{Y}}$ for the natural quotient metric coming from the product metric on $Y_i \times [0,1]$ (defined as above).

Let $Y_i \in \mathscr{Y}$. The *angular metric* $\widehat{d}_{Y_i}$ (or simply, $\widehat{d}$, when there is no scope for confusion) on Y_i is defined as follows:

For $y_1, y_2 \in Y_i$, $\widehat{d}_{Y_i}(y_1,y_2)$ is the infimum of lengths of paths in $X^{\mathrm{el}}_{\mathscr{Y}}$ joining y_1 to y_2 not passing through the vertex v_i. (We allow the angular metric to take on infinity as a value).

If (X,d) is a metric space, and Y is a subspace, we write $d|_Y$ the metric induced on Y.

Definition 2.2. — Consider a geodesic metric space (X,d) and a family of subsets $\mathscr{Y} = \{Y_i, i \in I\}$. We will say that $\mathscr{Y}$ is *coarsely hyperbolically embedded* in (X,d), if there is a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is proper (i.e., $\lim_{+\infty} \psi(x) = +\infty$), and such that

(1) the electrified space $X^{\mathrm{el}}_{\mathscr{Y}}$ is hyperbolic,

(2) the angular metric at each $Y \in \mathscr{Y}$ in the cone-off is bounded from below by $\psi \circ d|_Y$.

Remark 2.3. — This notion originates from Osin's [Osi06a], and was developed further in [DGO17] in the context of groups, where one requires that each subset $Y_i \in \mathscr{Y}$ is a proper metric space for the angular metric. This automatically implies the weaker condition of the above definition. The converse is not true: if $\mathscr{Y}$ is a collection of uniformly bounded subgroups of a group X with a (not necessarily proper) word metric, it is always coarsely hyperbolically embedded, but it is hyperbolically embedded in the sense of [DGO17] only if it is finite.

As in the point of view of [Osi06a], we say that (X,d) is *strongly hyperbolic relative to* the collection $\mathscr{Y}$ (in the sense of spaces) if $\mathscr{Y}$ is coarsely hyperbolically embedded in (X,d).

As we described in the remark, unfortunately, it happens that some groups (with a Cayley graph metric) are hyperbolic relative to some subgroups in the sense of spaces, but not in the sense of groups.

Note that there are other definitions of relative hyperbolicity for spaces. Druţu introduced the following definition: a metric space is hyperbolic relative to a collection of subspaces if all asymptotic cones are tree graded with pieces being ultratranslates of asymptotic cones of the subsets.

2.2. Quasiretractions

Definition 2.4. — If (X,d) is a metric space, and $Y \subset X$ is a subset endowed with a metric d_Y, we say that d_Y is λ-undistorted in (X,d) if for all $y_1, y_2 \in Y$,

$$\lambda^{-1} d(y_1,y_2) - \lambda \leqslant d_Y(y_1,y_2) \leqslant \lambda d(y_1,y_2) + \lambda.$$

We say that d_Y is undistorted in (X,d) if it is λ-undistorted in (X,d) for some λ.

For the next proposition, define the D-coarse path metric on a subset Y of a path-metric space (X,d) to be the metric on Y obtained by taking the infimum of lengths over paths for which any subsegment of length D meets Y.

The next proposition is the translation (to the present context) of Theorem 4.31 in [DGO17], with a similar proof.

Proposition 2.5. — *Let (X,d) be a graph. Assume that $\mathscr{Y}$ is coarsely hyperbolically embedded in (X,d). Then, there exists D_0 such that for all $Y \in \mathscr{Y}$, the D_0-coarse path metric on $Y \in \mathscr{Y}$ is undistorted (or equivalently the D_0-coarse path metric on $Y \in \mathscr{Y}$ is quasi-isometric to the metric induced from (X,d)).*

We will need the following lemma, which originates in Lemma 4.29 in [DGO17]. The proof is the same; for convenience we will briefly recall it. The lemma provides quasiretractions onto hyperbolically embedded subsets in a hyperbolic space.

Lemma 2.6. — *Let (X,d) be a geodesic metric space. There exists $C > 0$ such that whenever $\mathscr{Y}$ is coarsely hyperbolically embedded (in the sense of spaces) in (X,d), then for each $Y \in \mathscr{Y}$, there exists a map $r : X \to Y$ which is the identity on Y and such that $\widehat{d}(r(x),r(y)) \leqslant C d(x,y)$.*

Proof. — Let p the cone point associated to Y, and for each x, choose a geodesic $[p,x]$ and define $r(x)$ to be the point of $[p,x]$ at distance 1 from p. Then $r(x)$ is in Y, and to prove the lemma, one only needs to check that there is C such that if $d(x,y) = 1$, then $\widehat{d}(r(x),r(y)) \leqslant C$. The constant C will be $10(\delta+1)+1$. Assume that x and y are at distance $> 5(\delta+1)$ from the cone point. By hyperbolicity, one can find two points in the triangle (p,x,y) at distance $2(\delta+1)$ from $r(x), r(y)$ at distance $\leqslant 2\delta$ from each other. This provides a path of length at most $6\delta+4$. Hence $\widehat{d}(r(x),r(y)) \leqslant 6\delta+4$. If x and y are at distance $\leqslant 5(\delta+1)$ from the cone point, then

$$\widehat{d}(r(x),r(y)) \leqslant d(r(x),x) + d(x,y) + d(y,r(y)) \leqslant 2 \times 5(\delta+1) + 1. \qquad \square$$

We can now prove the proposition.

Proof. — Choose $D_0 = \psi(C)$: for all $y_0, y_1 \in Y$ at distance $\leqslant D_0$, their angular distance is at most C (where C is as given by the lemma above). Consider any path in X from y_0 to y_1, call the consecutive vertices $z_0, \dots, z_n$, and project that path by r. One gets $r(z_0), \dots, r(z_n)$ in Y, two consecutive ones being at distance at most D_0. This proves the claim. □

Corollary 2.7. — *If (X, d) is hyperbolic, and if $\mathscr{Y}$ is coarsely hyperbolically embedded, then there is C such that any $Y \in \mathscr{Y}$ is C-quasiconvex in X.*

2.3. Gluing horoballs. — Given a metric space (Y, d_Y) , one can construct several models of combinatorial horoballs over it. We recall a construction (similar to that of Groves and Manning [GM08] for a graph).

We consider inductively on $k \in \mathbb{N} \smallsetminus \{0\}$ the space $\mathscr{H}_k(Y) = Y \times [1, k]$ with the maximal metric d_k that

– induces an isometry of $\{y\} \times [k-1, k]$ with $[0, 1]$ for all $y \in Y$, and all $k \geqslant 1$,
– is at most d_{k-1} on $\mathscr{H}_{k-1}(Y) \subset \mathscr{H}_k(Y)$,
– coincides with $2^{-k} \times d$ on $Y \times \{k\}$.

Then $\mathscr{H}(Y)$ is defined as the inductive limit of the $\mathscr{H}_k(Y)$'s and is called the horoball over Y. Let (X, d) be a graph, and $\mathscr{Y}$ be a collection of subgraphs (with the induced metric on each of them). The *horoballification* of (X, d) over $\mathscr{Y}$ is defined to be the space $X^h_{\mathscr{Y}} = X \sqcup \{\bigsqcup_{i \in I} \mathscr{H}(Y_i)\}/\sim_i$, where $\sim_i$ denotes the identification of the boundary horosphere of $\mathscr{H}(Y_i)$ with $Y_i \subset X$. The metric $d^h_{\mathscr{Y}}$ is defined as the path metric on $X^h_{\mathscr{Y}}$.

One can electrify a horoballification $X^h_{\mathscr{Y}}$ of a space (X, d): one gets a space quasi-isometric to the electrification $X^{\mathrm{el}}_{\mathscr{Y}}$ of X. We record this observation in the following.

Proposition 2.8. — *Let X be a graph, and $\mathscr{Y}$ be a family of subgraphs. Let $X^{\mathrm{el}}_{\mathscr{Y}}$ and $X^h_{\mathscr{Y}}$ be the electrification, and the horoballification as above. Let $(X^h)^{\mathrm{el}}_{\mathscr{H}(\mathscr{Y})}$ be the electrification of $X^h_{\mathscr{Y}}$ over the collection of horoballs $\mathscr{H}(Y_i)$ over Y_i, $i \in I$.*

Then there is a natural injective map $X^{\mathrm{el}}_{\mathscr{Y}} \hookrightarrow (X^h)^{\mathrm{el}}_{\mathscr{H}(\mathscr{Y})}$ which is the identity on X and sends the cone point of Y_i to the cone point of $\mathscr{H}(Y_i)$.

Consider the map $e : ((X^h)^{\mathrm{el}}_{\mathscr{H}(\mathscr{Y})})^{(0)} \to X^{\mathrm{el}}_{\mathscr{Y}}$ that

– *is the identity on X,*
– *sends each vertex of $\mathscr{H}(Y_i)$ of depth > 2 to $v_i \in X^{\mathrm{el}}_{\mathscr{Y}}$,*
– *sends each vertex $(y, n) \in \mathscr{H}(Y_i)$ of depth $n \leqslant 2$ to $y \in Y_i \subset X$,*
– *sends the cone point of $((X^h)^{\mathrm{el}}_{\mathscr{H}(\mathscr{Y})})$ associated to $\mathscr{H}(Y_i)$ to the cone point of $X^{\mathrm{el}}_{\mathscr{Y}}$ associated to Y_i.*

Then e is a quasi-isometry that induces an isometry on $X^{\mathrm{el}}_{\mathscr{Y}} \subset (X^h)^{\mathrm{el}}_{\mathscr{H}(\mathscr{Y})}$.

Proof. — First, note that a geodesic in $(X^h)^{\mathrm{el}}_{\mathscr{H}(\mathscr{Y})}$ between two points of X never contains an edge with a vertex of depth $\geqslant 1$. If it did, the subpath in the corresponding horoball would be either non-reduced, or would contain at least 3 edges, and could be shortened by substituting a pair of edges through the cone attached to that horoball.

Thus the image of such a geodesic under e is a path of the same length. In other words, there is an inequality on the metrics $d_{X^{\mathrm{el}}_{\mathscr{Y}}} \leqslant d_{(X^h)^{\mathrm{el}}_{\mathscr{H}(\mathscr{Y})}}$ (restricted to $X^{\mathrm{el}}_{\mathscr{Y}}$). On the other hand, there is a natural inclusion $X^{\mathrm{el}}_{\mathscr{Y}} \subset (X^h)^{\mathrm{el}}_{\mathscr{H}(\mathscr{Y})}$, and therefore on $X^{\mathrm{el}}_{\mathscr{Y}}$, $d_{(X^h)^{\mathrm{el}}_{\mathscr{H}(\mathscr{Y})}} \leqslant d_{X^{\mathrm{el}}_{\mathscr{Y}}}$. Thus e is an isometry on $X^{\mathrm{el}}_{\mathscr{Y}}$. Also, every point in $(X^h)^{\mathrm{el}}_{\mathscr{H}(\mathscr{Y})}$ is at distance at most 2 from a point in X, hence also from a point of the image of $X^{\mathrm{el}}_{\mathscr{Y}}$. □

2.4. Relative hyperbolicity and hyperbolic embeddedness. — Recall that we say that a subspace Q of a geodesic metric space (X, d) is C-quasiconvex, for some number $C > 0$, if for any two points $x, y \in Q$, and any geodesic $[x, y]$ in X, any point of $[x, y]$ is at distance at most C from a point of Q.

Definition 2.9 ([Mj10, Def. 3.5]). — A collection $\mathscr{H}$ of (uniformly) C-quasiconvex sets in a δ-hyperbolic metric space X is said to be *mutually D-cobounded* if for all $H_i, H_j \in \mathscr{H}$, with $H_i \neq H_j$, $\pi_i(H_j)$ has diameter less than D, where π_i denotes a nearest point projection of X onto H_i. A collection is *mutually cobounded* if it is mutually D-cobounded for some D.

The aim of this subsection is to establish criteria for hyperbolicity of certain spaces (electrification, horoballification), and related statements on persistence of quasiconvexity in these spaces. We also show that hyperbolicity of the horoballification implies strong relative hyperbolicity, or coarse hyperbolic embeddedness.

Two sets of arguments are given. In the first set of arguments, the pivotal statement is of the following form: Electrification or de-electrification preserves the property of being a quasigeodesic. The arguments are essentially existent in some form in the literature, and we merely sketch the proofs and refer the reader to specific points in the literature where these may be found.

The second set of arguments uses asymptotic cones (hence the axiom of choice) and is more self-contained (it relies on Gromov-Cartan-Hadamard theorem). We decided to give both these arguments so as to leave it to the the reader to choose according to her/his taste.

2.5. Persistence of hyperbolicity and quasiconvexity

2.5.1. *The statements.* — Here we state the results for which we give arguments in the following two subsubsections.

Proposition 2.10. — *Let (X, d) be a hyperbolic geodesic space, $C > 0$, and $\mathscr{Y}$ be a family of C-quasiconvex subspaces. Then $X^{\mathrm{el}}_{\mathscr{Y}}$ is hyperbolic. If moreover the elements of $\mathscr{Y}$ are mutually cobounded, then $X^h_{\mathscr{Y}}$ is hyperbolic.*

In the same spirit, we also record the following statement on persistence of quasiconvexity.

Proposition 2.11. — *Given δ, C there exists C' such that if (X, d_X) is a δ-hyperbolic metric space with a collection $\mathscr{Y}$ of C-quasiconvex, sets. then the following holds:*

If $Q(\subset X)$ is a C-quasiconvex set (not necessarily an element of $\mathscr{Y}$), then Q is C'-quasiconvex in $(X^{\mathrm{el}}_{\mathscr{Y}}, d_e)$.

Finally, there is a partial converse. We need a little bit of vocabulary. If Z is a subset of a metric space (X,d), a (d,R)-coarse path in Z is a sequence of points of Z such that two consecutive points are always at distance at most R for the metric d.

Let H and Y two subsets of X. We will denote by $H^{+\lambda}$ the set of points at distance at most λ from H. We will say that H (Δ,ε)-meets Y if there are two points x_1, x_2 at distance $\geqslant \Delta$ from each other, and at distance $\leqslant \varepsilon\Delta$ from Y and H, and if for all pair of points at distance 20δ from $\{x_1,x_2\}$, either H or Y is at distance at least $\varepsilon\Delta - 2\delta$ from one of them.

The two points x_1, x_2 are called a pair of meeting points in H (for Y). We shall say that a subset H of X is coarsely path connected if there exists $c \geqslant 0$ such that the c-neighborhood $N_c(H)$ is path connected.

Proposition 2.12. — *Let (X,d) be hyperbolic, and let $\mathscr{Y}$ be a collection of uniformly quasiconvex subsets. Let H be a subset of X that is coarsely path connected, and quasiconvex in the electrification $X^{\mathrm{el}}_{\mathscr{Y}}$.*

Assume also that there exists $\varepsilon \in (0,1)$, and Δ_0 such that for all $\Delta > \Delta_0$, wherever H (Δ,ε)-meets an item Y in $\mathscr{Y}$, there is a path in $H^{+\varepsilon\Delta}$ between the meeting points in H that is uniformly a quasigeodesic in the metric (X,d).

Then H is quasiconvex in (X,d). The quasiconvexity constant of H can be chosen to depend only on the constants involved for $(X,d), \mathscr{Y}, \Delta_0, \varepsilon$, the coarse path connection constant, and the quasi-geodesic constant of the last assumption.

2.5.2. *Electroambient quasigeodesics.* — We recall here the concept of electroambient quasigeodesics from [Mj10, Mj14].

Let (X,d) be a metric space, and $\mathscr{Y}$ a collection of subspaces. If γ is a path in (X,d), or even in $(X^{\mathrm{el}}_{\mathscr{Y}}, d^{\mathrm{el}}_{\mathscr{Y}})$, one can define an *elementary electrification* of γ in $(X^{\mathrm{el}}_{\mathscr{Y}}, d^{\mathrm{el}}_{\mathscr{Y}})$ as follows:

For x_1, x_2 in γ, both belonging to some $Y_i \in \mathscr{Y}$, and at distance > 1, replace the arc of γ between them by a pair of edges $(x_1, v_i)(v_i, x_2)$, where v_i is the cone-point corresponding to Y_i.

A *complete electrification* of γ is a path obtained after a sequence of elementary electrifications of subarcs, admitting no further elementary electrifications.

One can *de-electrify* certain paths. Given a path γ in $(X^{\mathrm{el}}_{\mathscr{Y}}, d^{\mathrm{el}}_{\mathscr{Y}})$, a *de-electrification* of γ is a path σ in (X,d) such that

(1) γ is a complete electrification of σ,
(2) $(\sigma \smallsetminus \gamma) \cap Y_i$ is either empty or a geodesic in Y_i.

A *(λ,μ)-de-electrification* of a path γ in $(X^{\mathrm{el}}_{\mathscr{Y}}, d^{\mathrm{el}}_{\mathscr{Y}})$, is a path in X such that

(1) γ is a complete electrification of σ,
(2) $(\sigma \smallsetminus \gamma) \cap Y_i$ is either empty or a (λ,μ)-quasigeodesic in Y_i.

Observe that, given a path σ in X^{el}, there might be several ways to de-electrify it, but these ways differ only in the choice of the geodesic (or the quasi-geodesic) in the family of subspaces Y_i corresponding to the successive cone points v_{Y_i} on the path σ. It might also happen that there is no way of de-electrifying it, if the spaces in $\mathscr{Y}$ are not quasiconvex.

We say that a path γ in (X,d) is an *electroambient geodesic* if it is a de-electrification of a geodesic.

We say that it is a (λ,μ)-*electroambient quasigeodesic* if it is the (λ,μ)-de-electrification of a (λ,μ)-quasigeodesic in $(X^{\mathrm{el}}_{\mathscr{Y}}, d^{\mathrm{el}}_{\mathscr{Y}})$.

We begin by discussing Proposition 2.10.

Proof. — The first part is fairly well-known. In some other guise it appears in [Bow12, Prop. 7.4] [Szc98, Prop. 1] [Mj10]. In the first two, the electrification by cones is replaced by collapses of subspaces (identifications to points) which of course requires that the subspaces to electrify are disjoint and separated.

However this is only a technical assumption (as explicated in [Mj10]). Indeed, by replacing (or augmenting) any $Y \in \mathbb{Y}$ by $Y \times [0,D]$ glued along $Y \times \{0\}$, and replacing $\mathscr{Y}$ by the family $\{Y \times \{D\}, Y \in \mathscr{Y}\}$, we achieve a D-separated quasiconvex family. □

Next, we discuss Proposition 2.11.

Proof. — The proofs of Lemma 4.5 and Proposition 4.6 of [Far98], Proposition 4.3 and Theorem 5.3 of [Kla99] (see also [Bow12]) furnish Proposition 2.11.

The crucial ingredient in all these proofs is the fact that in a hyperbolic space, nearest point projections decrease distance exponentially. Farb proves this in the setup of horoballs in complete simply connected manifolds of pinched negative curvature. Klarreich "coarsifies" this assertion by generalizing it to the context of hyperbolic metric spaces. □

The rest of this (subsub)section is devoted to discussing Proposition 2.12. Towards doing this, we will obtain an argument for showing a variant of the second point of Proposition 2.10, namely that in a hyperbolic space (X,d), a family $\mathscr{Y}$ of uniformly quasi convex subspaces that is mutually cobounded defines a strong relative hyperbolic structure on (X,d). The second point of 2.10 as it is stated will be however proved in the next subsection.

We shall have need for the following Lemma [Mj10, Lem. 3.9] (see also [Kla99, Prop. 4.3] [Mj14, Lem. 2.5]).

Lemma 2.13. — *Suppose (X,d) is δ-hyperbolic. Let $\mathscr{H}$ be a collection of C-quasiconvex D-mutually cobounded subsets. Then for all $P \geqslant 1$, there exists $\varepsilon_0 = \varepsilon_0(C,P,D,\delta)$ such that the following holds:*

Let β be an electric (P,P)-quasigeodesic without backtracking (i.e., β does not return to any $H_1 \in \mathscr{H}$ after leaving it) and γ a geodesic in (X,d), both joining x,y. Then, given $\varepsilon \geqslant \varepsilon_0$ there exists $D = D(P,\varepsilon)$ such that

(1) Similar Intersection Patterns 1: *if precisely one of* $\{\beta, \gamma\}$ *meets an* ε*-neighborhood* $N_\varepsilon(H_1)$ *of an electrified quasiconvex set* $H_1 \in \mathscr{H}$*, then the length (measured in the intrinsic path-metric on* $N_\varepsilon(H_1)$ *) from the entry point to the exit point is at most* D*,*

(2) Similar Intersection Patterns 2: *if both* $\{\beta, \gamma\}$ *meet some* $N_\varepsilon(H_1)$ *then the length (measured in the intrinsic path-metric on* $N_\varepsilon(H_1)$ *) from the entry point of* β *to that of* γ *is at most* D*; similarly for exit points.*

Note that Lemma 2.13 above is quite general and does not require X to be proper. The two properties occurring in Lemma 2.13 were introduced by Farb [Far98] in the context of a group G and a collection $\mathscr{H}$ of cosets of a subgroup H. The two together are termed 'bounded coset penetration' in [Far98].

Remark 2.14. — In [Mj10], the extra hypothesis of separatedness was used. However, this is superfluous by the same remark on augmentations of elements of $\mathscr{Y}$ that we made in the beginning of the proof of Proposition 2.10. Lemma 2.13 may be stated equivalently as the following (compare with 2.23 below) .

If X is a hyperbolic metric space and $\mathscr{H}$ a collection of uniformly quasiconvex mutually cobounded subsets, then X is strongly hyperbolic relative to the collection $\mathscr{H}$.

We give a slightly modified version of [Mj10, Lem. 3.15] below by using the equivalent hypothesis of strong relative hyperbolicity (i.e., Lemma 2.13).

Lemma 2.15. — *Let* (X, d) *be a* δ*-hyperbolic metric space, and* $\mathscr{H}$ *a family of subsets such that* X *is strongly hyperbolic relative to* $\mathscr{H}$*. Then for all* $\lambda, \mu > 0$*, there exist* λ', μ' *such that any electroambient* (λ, μ)*-quasi-geodesic is a* (λ', μ')*-quasi-geodesic in* (X, d)*.*

The proof of Lemma 2.15 goes through mutatis mutandis for strongly relatively hyperbolic spaces as well, i.e., hyperbolicity of X may be replaced by relative hyperbolicity in Lemma 2.15 above. We state this explicitly below:

Corollary 2.16. — *Let* (X, d) *be strongly relatively hyperbolic relative to a collection* $\mathscr{Y}$ *of path connected subsets. Then, for all* $\lambda, \mu > 0$*, there exists* λ', μ' *such that any electroambient* (λ, μ)*-quasi-geodesic is a* (λ', μ')*-quasi-geodesic in* (X, d)*.*

We include a brief sketch of the proof-idea following [Mj10]. Let γ be an electroambient quasigeodesic. By lemma, its electrification $\widehat{\gamma}$ is a quasi-geodesic in $X^{\mathrm{el}}_{\mathscr{Y}}$. Let σ be the electric geodesic joining the end-points of γ. Hence σ and $\widehat{\gamma}$ have similar intersection patters with the sets Y_i [Far98], i.e., they enter and leave any Y_i at nearby points. It then suffices to show that an electroambient representative of σ is in fact a quasigeodesic in X. A proof of this is last statement is given in [McM01, Th. 8.1] in the context of horoballs in hyperbolic space (see also Lemmas 4.8, 4.9 and their proofs in [Far98]). The same proof works after horoballification for an arbitrary relatively hyperbolic space. □

The proof of Proposition 2.12 as stated will be given in the next subsubsection. We shall provide here a proof that suffices for the purposes of this paper. We assume, in addition to the hypothesis of the proposition that there exists an integer $n > 0$, and $D_0 \geqslant 0$ such that for all distinct $Y_1, \dots, Y_n \in \mathscr{Y}$, $\bigcap_i Y_i^{+\varepsilon}$ has diameter at most D. The existence of such a number n will translate into the notion of finite geometric height later in the paper.

Proof. — We prove the statement by inducting on n. For $n = 1$ there is nothing to show; so we start with $n = 2$. Note that in this case, the hypothesis is equivalent to the assumption that the Y_i's are cobounded. Assume therefore that the elements of $\mathscr{Y}$ are uniformly quasiconvex in (X, d); and that they are uniformly mutually cobounded. We shall show that H is also quasiconvex for a uniform constant.

First, since (X, d) is hyperbolic it follows by Proposition 2.10 that $(X^{\mathrm{el}}_{\mathscr{Y}}, d^{\mathrm{el}})$ is hyperbolic.

Let $x, y \in H$. By assumption, there exists $C_0 \geqslant 0$ such that H is (C_0, C_0)-qi embedded in (G, d^{el}). Denote by $\mathscr{P}$ the set of cone points corresponding to elements of $\mathscr{Y}$ and let γ be a (C, C)-quasi-geodesic without backtracking in (X, d^{el}) with vertices in $H \cup \mathscr{P}$ joining $x, y \in H$. By assumption, the collection $\mathscr{Y}$ is uniformly C-quasiconvex. Further, by assumption, there exists $\varepsilon \in (0, 1)$, and Δ_0 such that for all $\Delta > \Delta_0$, wherever H (Δ, ε)-meets an item Y in $\mathscr{Y}$, there is a path in $H^{+\varepsilon\Delta}$ between the meeting points in H that is uniformly a quasigeodesic in the metric (X, d). Hence, for some uniform constants λ, μ, we may (coarsely) (λ, μ)-de-electrify γ to obtain a (λ, μ)-electroambient quasigeodesic γ' in (X, d), that lies close to H. [Note that the meeting points of H with elements of $\mathscr{Y}$ are only coarsely defined. So we are actually replacing pieces of γ by quasigeodesics in $H^{+\varepsilon\Delta}$ rather than in H itself.]

Note that by assumption $\mathscr{Y}$ are uniformly quasiconvex in (X, d); and further that they are uniformly mutually cobounded. Hence the space X is actually strongly hyperbolic relative to $\mathscr{Y}$. By Corollary 2.16, it follows that γ' is a quasi-geodesic in (X, d), for a uniform constant.

Since this was done for arbitrary $x, y \in H_{i,\ell}$, we obtain that H is D-quasiconvex in (X, d). This finishes the proof of Proposition 2.12 for $n = 2$.

The induction step is now easy. Assume that the statement is true for $n = m$. We shall prove it for $n = m + 1$. Electrify all pairwise intersections of $Y_i^{+\varepsilon}$ to obtain an electric metric d_2. Then the collection $\{Y_i\}$ is cobounded with respect to the electric metric d_2. Here again, the space (X, d_2) is strongly hyperbolic relative to the collection $\{Y_i\}$. By the argument in the case $n = 2$ above, H is quasiconvex in (X, d_2). The collection of pairwise intersections of the $Y_i^{+\varepsilon}$'s in X satisfies the property that an intersection of any m of them is bounded. We are then through by the induction hypothesis. □

2.5.3. *Proofs through asymptotic cones.* — The repeated use of different references coming from different contexts in the previous subsubsection might call for more systematic self-contained proofs of the statements of subsection 2.5.1. This is our purpose in this subsubsection.

In this part we will use the structure of an argument originally due to Gromov, and developed by Coulon amongst others (see for instance [Cou14, Prop. 5.28]), which uses asymptotic cones in order to show hyperbolicity or quasiconvexity of constructions.

We fix a non-principal ultrafilter ω and will use the construction of asymptotic cones with respect to this ultrafilter ω. A few observations are in order here.

In all the following, (X_N, x_N) is a sequence of pointed δ_N-hyperbolic spaces, with δ_N converging to 0. Recall then that the asymptotic cone $\lim_\omega(X_N, x_N)$ is an $\mathbb{R}$-tree, with a base point.

If $\mathscr{Y}_N$ is a family of c_N-quasiconvex subsets of X_N (for c_N tending to 0), we want to consider the asymptotic cone $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N}, x_N)$ and relate it to $\lim_\omega(X_N, x_N)$.

Let us define the following equivalence relation on the set of sequences in X_N. Two sequences $(u_N), (v_N)$ are equivalent if $d_{X_N}(u_N, v_N) = O(1)$ for the ultrafilter ω (more precisely, if there exists a constant C such that for ω-almost all values of N, $d_{X_N}(u_N, v_N) \leqslant C$). Let us consider the set of equivalence classes of sequences, and let us only keep those that have some (hence all) representative (u_N) such that the electric distance $d_{X_N^{\mathrm{el}}}(x_N, u_N)$ is $O(1)$. Let us call $\mathfrak{C}$ this set of equivalence classes. For a sequence $u = (u_N)$ we write $\sim_u$ for its class in $\mathfrak{C}$. We also allow ourselves to write $\lim_\omega(X_N, \sim_u)$ for $\lim_\omega(X_N, u_N)$ to avoid cluttered notation.

Lemma 2.17. — *There is a natural inclusion from the disjoint union*

$$\bigsqcup_{\sim_u \in \mathfrak{C}} \lim_\omega(X_N, \sim_u)$$

into $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N}, x_N)$.

Proof. — By definition of $\mathfrak{C}$ we have a well defined map

$$\lim_\omega(X_N, \sim_u) \longrightarrow \lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N}, x_N)$$

for each class $\sim_u$ in $\mathfrak{C}$. Given two sequences (y_N), and (z_N) in the same class $\sim_u \in \mathfrak{C}$, if $d_{X_N}(z_N, y_N)$ is not $o(1)$ for ω, then in the electric metric, it is not $o(1)$, since the added edges all have length 1. Thus this map is injective. If (y_N), and (z_N) are not in the same class in $\mathfrak{C}$, then $d_{X_N}(z_N, y_N)$ is not $o(1)$ for ω, and again, in the electric metric, it is not $o(1)$. The map of the lemma is thus injective. □

Note that the inclusion is continuous, as inclusions along the sequence are distance non-increasing. But it is not isometric. We need to describe what happens with the cone off.

Consider a sequence (Y_N) of subsets of $\mathscr{Y}$. One says that the sequence is visible in $\lim_\omega(X_N, u_N)$ if $d_{X_N}(u_N, Y_N) \leqslant O(1)$. In that case, $\lim_\omega(Y_N, u_N)$ is a subset of $\lim_\omega(X_N, u_N)$, consisting of the images of all the sequences of elements of Y_N that remain at $O(1)$-distance from u_N. Note that given a sequence (Y_N), it can be visible in several limits $\lim_\omega(X_N, u_N)$ (for several non-equivalent (u_N)). In those classes where (Y_N) is not visible, $\lim_\omega(Y_N, u_N)$ is empty. Let us define $\lim_\omega(Y_N, *)$ to be $\bigsqcup_{\sim_u \in \mathfrak{C}} \lim_\omega(Y_N, u_N)$ in $\bigsqcup_{\sim_u \in \mathfrak{C}} \lim_\omega(X_N, u_N)$. By the previous lemma, this is naturally a subset of $\lim_\omega((X_N)^{el}_{\mathscr{Y}_N}, x_N)$.

We define $\mathscr{Y}^\omega$ to be the collection of all sets $\lim_\omega(Y_N,*)$, for all possible sequences $(Y_N)\in\prod_{N>0}Y_N$. This is a family of subsets of $\lim_\omega((X_N)^{el}_{\mathscr{Y}_N},x_N)$.

Let us define $\left[\bigsqcup_{\sim_u\in\mathfrak{C}}\lim_\omega(X_N,u_N)\right]^{\mathrm{el}}$ to be the cone-off (of parameter 1) of $\left[\bigsqcup_{\sim_u\in\mathfrak{C}}\lim_\omega(X_N,\sim_u)\right]$ over each $\lim_\omega(Y_N,*)$. Note that there is a natural copy of $[\lim_\omega(X_N,\sim_u)]^{\mathrm{el}}$ in $\left[\bigsqcup_{\sim_u\in\mathfrak{C}}\lim_\omega(X_N,\sim_u)\right]^{\mathrm{el}}$.

Lemma 2.18. — *$\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N},x_N)$ is isometric to $\left[\bigsqcup_{\sim_u\in\mathfrak{C}}\lim_\omega(X_N,\sim_u)\right]^{\mathrm{el}}$.*

Proof. — First there is a natural bijection

$$\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N},x_N)\xrightarrow{\sim}\left[\bigsqcup_{\sim_u\in\mathfrak{C}}\lim_\omega(X_N,\sim_u)\right]^{\mathrm{el}}.$$

Indeed, for any sequence (y_N) with $y_N\in X_N$ at distance $O(1)$ from x_N in the electric metric, Lemma 2.17 provides an image in $\left[\bigsqcup_{\sim_u\in\mathfrak{C}}\lim_\omega(X_N,\sim_u)\right]^{\mathrm{el}}$. For any sequence (c_N) with c_N a cone-point in $X_N^{el}\smallsetminus X_N$, at distance $O(1)$ from x_N in the electric metric, c_N is in the cone electrifying a certain Y_N, which is therefore at distance $O(1)$ from x_N for the electric metric. Choose $u_N\in Y_N$, then the equivalence class of (u_N) is in $\mathfrak{C}$, and of course Y_N is visible in this class. Thus, there is a cone point $c\in\left[\bigsqcup_{\sim_u\in\mathfrak{C}}\lim_\omega(X_N,\sim_u)\right]^{\mathrm{el}}$ at distance 1 from $\lim_\omega(Y_N,*)$. We choose this point as the image of the sequence (c_N) in $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N},x_N)$. This is well defined, and injective, for if c'_N is another sequence of cone-points ω-almost everywhere different from c_N, then it defines an ω-almost everywhere different sequence Y'_N, and a different set $\lim_\omega(Y'_N,*)$. We also can extend our map to all $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N},x_N)$ linearly on the cone-edges. This produces a bijection $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N},x_N)\to\left[\bigsqcup_{\sim_u\in\mathfrak{C}}\lim_\omega(X_N,\sim_u)\right]^{\mathrm{el}}$.

To show that it is an isometry, consider two sequences $(y_N),(z_N)$ both in X_N, such that the distance in $X^{\mathrm{el}}_{\mathscr{Y}_N}$ converges (for ω) to ℓ. Then there is a path of length ℓ_N (converging to ℓ) in $X^{\mathrm{el}}_{\mathscr{Y}_N}$ with, eventually, at most $\ell/2$ cone points on it. It follows from the construction that their images in $\left[\bigsqcup_{\sim_u\in\mathfrak{C}}\lim_\omega(X_N,\sim_u)\right]^{\mathrm{el}}$ are at distance at most ℓ. Conversely, assume (y_N) and (z_N) are sequences in X_N giving points in $\lim_\omega(X_N,\sim_u)$ and $\lim_\omega(X_N,\sim_v)$, for $\sim_u,\sim_v\in\mathfrak{C}$, and take a path γ between these points in $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N},x_N)\simeq\left[\bigsqcup_{\sim_u\in\mathfrak{C}}\lim_\omega(X_N,\sim_u)\right]^{\mathrm{el}}$, of length $\ell>0$. It has finite length, so it contains finitely many cone points c_i $(i=1,\dots,r)$, coning $\lim_\omega(Y_N^{(i)},*)$, for which $Y_N^{(i)}$ is visible in both $\sim_{u_i},\sim_{u_{i+1}}$. This easily produces a path in $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N},x_N)$ of length ℓ, by using the corresponding cone points and the path between the spaces $Y_N^{(i)}$ given by the restriction of γ.

We have thus observed that the bijection we started with is 1-Lipschitz as is its inverse. It is therefore an isometry. □

We finally describe a tree-like structure on $[\bigsqcup_{\mathfrak{C}}\lim_\omega(X_N,\sim_u)]^{\mathrm{el}}$ where the pieces are the subspaces $[\lim_\omega(X_N,\sim_u)]^{\mathrm{el}}$ for $\sim_u\in\mathfrak{C}$, which are electrifications of $\lim_\omega(X_N,\sim_u)$ over the subsets of the form $\lim_\omega(Y_N,\sim_u)$, for all sequences $(Y_N)\in\prod(\mathscr{Y}_N)$ that are visible in $\sim_u$.

Let us say that two classes $\sim_u$ and $\sim_v$ are joined by a sequence (Y_N) if the latter sequence is visible in both of them.

First we describe a simpler case of this tree-like structure.

Lemma 2.19. — *Assume that $\mathscr{Y}_N$ consists of c_N-quasiconvex subsets of X_N with c_N tending to 0.*

For any pair of different classes $\sim_u, \sim_v$ in $\mathfrak{C}$, the subspaces $[\lim_\omega(X_N, \sim_u)]^{\mathrm{el}}$ and $[\lim_\omega(X_N, \sim_v)]^{\mathrm{el}}$ of $\left[\bigsqcup_{\sim_w \in \mathfrak{C}} \lim_\omega(X_N, \sim_w)\right]^{\mathrm{el}}$ have intersection of diameter at most 2.

Proof. — Note that by Lemma 2.17 the intersection consists of cone points. Thus consider two sequences $(Y_N), (Y'_N)$ both visible in $\sim_u$ and $\sim_v$. Consider then $y_N(u), y_N(v) \in Y_N$ such that $y_N(u)$ is visible in $\sim_u$ (hence not in $\sim_v$) and symmetrically $y_N(v)$ is visible in $\sim_v$ (hence not in $\sim_u$), and take $y'_N(u), y'_N(v) \in Y'_N$ similarly. The distances $d_{X_N}(y_N(u), y'_N(u))$ and $d_{X_N}(y_N(v), y'_N(v))$ both are $O(1)$ whereas $d_{X_N}(y_N(u), y_N(v))$ and $d_{X_N}(y'_N(u), y'_N(v))$ both go to infinity (for ω). The space X_N being a δ_N-hyperbolic space (for $\delta_N \to 0$), the quadrilateral with these four vertices have their sides $[y_N(u), y_N(v)]$ and $[y'_N(u), y'_N(v)]$ getting $o(1)$-close to each other, on sequences that are visible for $\sim_u$ and sequences that are visible for $\sim_v$. But these sides are close to Y_N and Y'_N respectively. It follows that in $\lim_\omega(X_N, \sim_u)$ and in $\lim_\omega(X_N, \sim_v)$, the limit of Y_N and of Y'_N share a point. Thus the cone point of their electrifications are at distance 2. □

Note that if there is a bound on the diameter of the projection of Y_N on Y'_N, then there is only one point in the intersection.

Lemma 2.20. — *If there is a cycle of classes $\sim_{u_1}, \sim_{u_2}, \dots, \sim_{u_k}, \sim_{u_{k+1}} = \sim_{u_1}$, where $\sim_{u_i}$ is joined to $\sim_{u_{i+1}}$ by a sequence $(Y_N^{(i)})$, then there is $1 < i_0 < k+1$ such that $(Y_N^{(i_0)})$ is visible in $\sim_{u_1}, \sim_{u_2}$ and the cone points of $(Y_N^{(i_0)})$ and of $(Y_N^{(1)})$ are at distance 2 from each other in $[\lim_\omega(X_N, \sim_{u_1})]^{\mathrm{el}}$, and in $[\bigsqcup_{\mathfrak{C}} \lim_\omega(X_N, \sim_u)]^{\mathrm{el}}$.*

As a corollary, $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N}, x_N)$ is a quasi-tree of the spaces $[\lim_\omega(X_N, \sim)]^{\mathrm{el}}$, and more precisely, all paths from $[\lim_\omega(X_N, \sim_u)]^{\mathrm{el}}$ to $[\lim_\omega(X_N, \sim_v)]^{\mathrm{el}}$, if $\sim_u \neq \sim_v$ have to pass through the 2-neighborhood of a certain cone point of $[\lim_\omega(X_N, \sim_u)]^{\mathrm{el}}$.

Proof. — The number k is fixed, and the argument will generalize the one of the previous lemma. For each i, let $(y_N^{(i)})$ and $(z_N^{(i)})$ be sequences of points of $Y_N^{(i)}$ respectively visible in $\sim_{u_i}$ and in $\sim_{u_{i+1}}$. One has $d_{X_N}(y_N^{(i)}, z_N^{(i)})$ unbounded, and $d_{X_N}(z_N^{(i)}, y_N^{(i+1)}) = O(1)$. Therefore in the $2k$-gon $(y_N^{(1)}, z_N^{(1)}, y_N^{(2)}, z_N^{(2)}, \dots, y_N^{(k)}, z_N^{(k)})$, using the approximation by a finite tree (for hyperbolic spaces), we see that one of the segments $[y_N^{(i)}, z_N^{(i)}]$ ($i \neq 1$) must come $k\delta_N$-close to $[y_N^{(1)}, z_N^{(1)}]$, and at distance $O(1)$ from $y_N^{(1)}$.

After extracting a subsequence, one can assume that i is constant in N, and we choose it to be our i_0. It follows that the sequence $(Y_N^{(i_0)})$ is visible for $\sim_{u_1}$ and the limit of $(Y_N^{(i_0)})$ and of (Y_N^1) share a point in $\lim_\omega(X_N, \sim_{u_1})$. The conclusion that the cone points of $(Y_N^{(i_0)})$ and of $(Y_N^{(1)})$ are at distance 2 from each other in

$[\lim_\omega(X_N,\sim_{u_1})]^{\mathrm{el}}$ follows, and this also implies that they are at distance at most 2 in $[\bigsqcup_{\mathfrak{C}}\lim_\omega(X_N,\sim_u)]^{\mathrm{el}}$. □

Note that if the subsets of $\mathscr{Y}_N$ are c_n-mutually cobounded, with c_n going to 0 (or even bounded) then one can improve the lemma by saying that eventually $Y_N^{(i_0)}=Y_N^{(1)}$.

From the previous lemmas we get:

Corollary 2.21. — *The space* $[\bigsqcup_{\mathfrak{C}}\lim_\omega(X_N,\sim_u)]^{\mathrm{el}}$ *is the union of spaces of the form* $[\lim_\omega(X_N,\sim_u)]^{\mathrm{el}}$ *for* $\sim_u\in\mathfrak{C}$*, with some cone points identified.*

Moreover, if $\sim_u\neq\sim_v$*, and if* γ_1,γ_2 *are any (finite) paths from* $[\lim_\omega(X_N,\sim_u)]^{\mathrm{el}}$ *to* $[\lim_\omega(X_N,\sim_v)]^{\mathrm{el}}$*, then for each* $i\in\{1,2\}$*, there exists a cone point* $c_i\in[\lim_\omega(X_N,\sim_u)]^{\mathrm{el}}$ *in* γ_i *such that the distance between* c_1 *and* c_2 *is at most* 2*.*

Indeed, such a pair of paths provides us with a certain finite cycle of classes, starting with $\sim_u$, and we may apply the previous lemma.

We say that the space $[\bigsqcup_{\mathfrak{C}}\lim_\omega(X_N,\sim_u)]^{\mathrm{el}}$ is a 2-quasi-tree of spaces of the form $[\lim_\omega(X_N,\sim_u)]^{\mathrm{el}}$ for $\sim_u\in\mathfrak{C}$.

Let us prove Proposition 2.10.

Proof. — We claim that, for all ρ, there exists $\delta_0<\rho/10^{14}$ and $C_0<\rho/10^{14}$ such that if X is δ_0-hyperbolic, and if $\mathscr{Y}$ is a collection of C_0-quasiconvex subsets, then every ball of radius ρ of $X^{\mathrm{el}}_{\mathscr{Y}}$ is 10-hyperbolic.

For proving the claim, assume it false, and consider a sequence of counterexamples $(X_N,\mathscr{Y}_N)$ for $\delta_0=C_0=1/N$, $N=1,2,\dots$. This means that $(X_N)^{\mathrm{el}}_{\mathscr{Y}_N}$ fails to be 10-hyperbolic. There are four points x_N,y_N,z_N,t_N, all at distance at most 2ρ from x_N, such that $(x_N,z_N)_{t_N}\leqslant\inf\{(x_N,y_N)_{t_N},(y_N,z_N)_{t_N}\}-10$. We pass to the ultralimit for ω. In $\lim_\omega(X_N^{\mathrm{el}},x_N)$, each sequence x_N,y_N,z_N,t_N converges, since these points stay at bounded distance from x_N, and the inequality persists. Hence one gets four points falsifying the 10-hyperbolicity condition in a pointed space $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N},x_N)$.

But the asymptotic cone $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N},x_N)$ is, by Corollary 2.21, a 2-quasi-tree of spaces that are electrifications (of parameter 1) of real trees $\lim_\omega(X_N,x'_N)$, for some base point x'_N over the family $\mathscr{Y}^\omega$, consisting of *convex* subsets (i.e., of subtrees). This space $(\lim_\omega(X_N,x'_N))^{\mathrm{el}}_{\mathscr{Y}^\omega}$ has 2-thin geodesic triangles, therefore $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}_N},x_N)$ itself is 10-hyperbolic, a contradiction. The claim hence holds: $X^{\mathrm{el}}_{\mathscr{Y}}$ is ρ-locally 10-hyperbolic.

We now claim that, under the same hypothesis, it is $(2+10C_0+10\delta_0)$-coarsely simply-connected, that is to say that any loop in it can be homotoped to a point by a sequence of substitutions of arcs of length $<(2+10C_0+10\delta_0)$ by its complement in a loop of length $<(2+10C_0+10\delta_0)$. Indeed, any time such a loop passes through a cone point associated to some $Y\in\mathscr{Y}$, one can consider a geodesic in X between its entering and exiting points in Y, which stays in the C_0 neighborhood of Y. Therefore, a $(2+10C_0)$-coarse homotopy of the loop transforms it into a loop in X, which is

δ-hyperbolic. Since a δ-hyperbolic space is 10δ-coarsely simply-connected, the second claim follows.

The final ingredient is the Gromov-Cartan-Hadamard theorem [Cou14, Th. A.1], stating that, if ρ is sufficiently large compared to μ, any ρ-locally 10-hyperbolic space which is μ-coarsely simply connected is (globally) δ'-hyperbolic, for some δ'. We thus get that there exists $\delta_0 < \rho/10^{14}$ and $C_0 < \rho/10^{14}$ such that if X is δ_0-hyperbolic, and if $\mathscr{Y}$ is a collection of C_0-quasiconvex subsets, then $X^{\mathrm{el}}_{\mathscr{Y}}$ is δ'-hyperbolic.

Now let us argue that this implies the first point of the proposition. If X and $\mathscr{Y}$ are given as in the statement, one may rescale X by a certain factor $\lambda > 1$, so that it is δ_0-hyperbolic, and such that $\mathscr{Y}$ is a collection of C_0-quasiconvex subsets. Let us define $X^{el_\lambda}_{\mathscr{Y}}$ to be

$$X^{el_\lambda}_{\mathscr{Y}} = X \sqcup \left\{\bigsqcup_{i\in I} Y_i \times [0,\lambda]\right\}/\sim,$$

where $\sim$ denotes the identification of $Y_i \times \{0\}$ with $Y_i \subset X$ for each i, and the identification of $Y_i \times \{1\}$ to a single cone point v_i (dependent on i), and where $Y_i \times [0,\lambda]$ is endowed with the product metric as defined in the first paragraph of 2.1 except that $\{y\} \times [0,n]$ is isometric to $[0,\lambda]$. The claim ensures that $X^{el_\lambda}_{\mathscr{Y}}$ is hyperbolic. However, it is obviously quasi-isometric to $X^{\mathrm{el}}_{\mathscr{Y}}$. We have the first point.

For the second part, one can proceed with a similar proof, with horoballs. The claim is then that for all ρ, there exist δ_0, C_0 and D_0 such that if X is δ_0-hyperbolic, and if $\mathscr{Y}$ is a collection of C_0-quasiconvex subsets, D_0-mutually cobounded, then any ball of radius ρ of the horoballification $X^h_{\mathscr{Y}}$ is 10-hyperbolic.

The proof of the claim is similar. Consider a sequence of counterexamples $X_N, \mathscr{Y}_N$, for the parameters $\delta = C = D = 1/N$ for N going to infinity, with the four points x_N, y_N, z_N, t_N in $(X_N)^h_{\mathscr{Y}}$, in a ball of radius ρ, falsifying the hyperbolicity condition.

There are two cases. Either x_N (which is in $(X_N)^h_{\mathscr{Y}}$) escapes from X_N, i.e., its distance from some basepoint in X_N tends to ∞ for the ultrafilter ω, or it does not. In the case that it escapes from X_N, then, when it is larger than ρ all four points x_N, y_N, z_N, t_N are in a single horoball, but such a horoball is 10-hyperbolic hence a contradiction.

The other case is when there is $x'_N \in X_N$ whose distance to x_N remains bounded (for the ultrafilter ω). Note that $\{x_N, y_N, z_N, t_N\}$ converge in the asymptotic cone $\lim_\omega((X_N)^h_{\mathscr{Y}_N}, x'_N)$ of the sequence of pointed spaces $((X_N)^h_{\mathscr{Y}_N}, x'_N)$. It is also immediate by definition of $\lim_\omega \mathscr{Y}_N$ that $\lim_\omega((X_N)^h_{\mathscr{Y}_N}, x'_N)$ is the horoballification of the asymptotic cone of the sequence (X_N, x'_N) over the family $\lim_\omega(\mathscr{Y}_N, x'_N)$ defined above.

This family $\lim_\omega(\mathscr{Y}_N, x'_N)$ consists of *convex* subsets (hence subtrees), such that any two share at most one point. This horoballification is therefore a tree-graded space in the sense of [DS05], with pieces being the combinatorial horoballs over the subtrees constituting $\lim_\omega \mathscr{Y}_N$. As a tree of 10-hyperbolic spaces, this space is 10-hyperbolic, contradicting the inequalities satisfied by the limits $\lim_\omega\{x_N, y_N, z_N, t_N\}$. Therefore, $X^h_{\mathscr{Y}}$ is ρ-locally 10-hyperbolic. As before, one may check that (under the same assumptions) $X^h_{\mathscr{Y}}$ is $(2 + 10C_0 + 10\delta_0)$-coarsely simply connected, and again this implies by

the Gromov-Cartan-Hadamard theorem that (under the same assumptions) $X^h_{\mathscr{Y}}$ is hyperbolic.

This implies the second point. Indeed, let us denote by $\frac{1}{\lambda}X$ the space X with metric rescaled by $1/\lambda$.

The previous claim shows that, under the assumption of the second point of the proposition, there exists $\lambda > 1$ such that $\lambda(\frac{1}{\lambda}X)^h_{\frac{1}{\lambda}\mathscr{Y}}$ is hyperbolic. Consider the map η between $X^h_{\mathscr{Y}} \to \lambda(\frac{1}{\lambda}X)^h_{\frac{1}{\lambda}\mathscr{Y}}$ that is identity on X and that sends $\{y\} \times \{n\}$ to $\{y\} \times \{\lambda \times (n + \lfloor \log_2 \lambda \rfloor)\}$ for all $y \in Y_i$ and all Y_i (and all n). All paths in $X^h_{\mathscr{Y}}$ that have only vertical segments in horoballs have their length expanded (under the map η) by a factor between 1 and $\lambda + \log_2 \lambda$. But the geodesics in $X^h_{\mathscr{Y}}$ and $\lambda(\frac{1}{\lambda}X)^h_{\frac{1}{\lambda}\mathscr{Y}}$ are paths whose components in horoballs consist of a vertical (descending) segment, followed by a single edge, followed by a vertical ascending segment (see [GM08]). Hence η is a quasi-isometry, and the space $X^h_{\mathscr{Y}}$ is hyperbolic. □

We continue with the persistence of quasi-convexity of Proposition 2.11.

Proof. — The strategy is similar to that in the previous proposition. The main claim is that for all ρ, there is $\delta_0 < 1, C_0 < 1$ such that if (X, d_X) is δ_0-hyperbolic, if $\mathscr{Y}$ is a collection of C_0-quasiconvex subsets and if Q is another C_0-quasiconvex subset of X, then Q is ρ-locally 10-quasiconvex in $X^{\mathrm{el}}_{\mathscr{Y}}$ (of course δ_0, C_0 will be very small).

To prove the claim, again, by contradiction, consider a sequence $X_N, \mathscr{Y}_N, Q_N$ of counter examples for $\delta_N = C_N = 1/N$ for $N = 1, 2, \dots$. There exist two points x_N, y_N in Q_N, at distance $\leqslant \rho$ from each other (for the electric metric), and a geodesic $[x_N, y_N]$ in $X^{\mathrm{el}}_{\mathscr{Y}}$ with a point z_N on it at distance > 10 from Q. We record a point z'_N in Q at minimal distance ($\leqslant \rho$ in any case) from z_N.

With a non principal ultrafilter ω, we may take the asymptotic cone of the family of pointed spaces (X^{el}_N, x_N). In $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}}, x_N)$, the sequences (y_N), $([x_N, y_N])$ and (z_N) have limits for which the distance inequalities persist, and we get that $\lim_\omega(Q_N, x_N)$ is not ρ-locally 10-quasiconvex in $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}}, x_N)$. But as we noticed in Corollary 2.21 $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}}, x_N)$ is a 2-quasi-tree of spaces of the form $(\lim_\omega(X_N, x'_N))^{\mathrm{el}}_{\mathscr{Y}\omega}$, which are the electrifications of $\mathbb{R}$-trees $\lim_\omega(X_N, x'_N)$ over a family of convex subsets (i.e., subtrees). In this space, $\lim_\omega(Q_N, x_N)$ is also a subforest of $\bigsqcup_{(u_N)\in\mathfrak{C}} \lim_\omega(X_N, u_N)$. Also observe that if (Q_N) is visible in two adjacent classes, then $\lim_\omega(Q_N, x_N)$ is adjacent to their common cone point over sequences $(Y^{(i)}_N), (Y^{(j)}_N)$. Hence $\lim_\omega(Q_N, x_N)$ is 2-quasiconvex in $\lim_\omega((X_N)^{\mathrm{el}}_{\mathscr{Y}\omega}, x_N)$, and this contradicts the inequalities satisfied by $\lim_\omega\{x_N, y_N, z_N, z'_N\}$. The claim is established for all ρ.

Now there exists ρ_0 such that, in any 1-hyperbolic space, any subset that is ρ_0-locally 10-quasiconvex is 10^{14}-globally quasiconvex (this classical fact, perhaps found elsewhere with other (better!) constants, follows also from the Gromov-Cartan-Hadamard theorem for instance). So, by choosing an appropriate ρ, we have proven that there is $\delta_0 < 1, C_0 < 1$ and C_1, such that if (X, d_X) is δ_0-hyperbolic, if $\mathscr{Y}$ is a

collection of C_0-quasiconvex subsets and if Q is another C_0-quasiconvex subset of X, then Q is C_1-quasiconvex in $X^{\mathrm{el}}_{\mathscr{Y}}$.

Coming back to the statement of the proposition, by rescaling our space, we have proven that if (X, d_X) is a δ-hyperbolic metric space with a collection $\mathscr{Y}$ of C-quasiconvex sets, and if Q is C-quasiconvex, then Q is λC_1-quasiconvex in $X^{el_\lambda}_{\mathscr{Y}}$ (as defined in the previous proof) for $\lambda = \max\{\delta/\delta_0, C/C_0\}$. Since $X^{el_\lambda}_{\mathscr{Y}}$ is quasi-isometric to $X^{\mathrm{el}}_{\mathscr{Y}}$, by a (λ, λ)-quasi-isometry, it follows that Q is C'-quasiconvex in $X^{\mathrm{el}}_{\mathscr{Y}}$ for C' depending only on δ, C. □

Finally, we consider the proposed converse in Proposition 2.12. We use the same strategy again. The claim is now the lemma below. To state it, we need to define the m-coarse path metric on an m-coarse path connected subspace of a metric space. A subset $Y \subset X$ of a metric space is m-coarse path connected if for any two points x, y in it there is a sequence $x_0 = x, x_1, \dots, x_r = y$ for some r such that $x_i \in Y$ and $d_X(x_i, x_{i+1}) \leqslant m$ for all i. We call such a sequence an m-coarse path or a path with mesh $\leqslant m$. The length of the coarse path $(x_0, \dots, x_r)$ is $\sum d_X(x_i, x_{i+1})$. The m-coarse path metric on Y is the distance obtained by taking the infimum of lengths of coarse paths between its points. An m-coarse geodesic is a coarse path realizing the coarse path metric between two points.

Lemma 2.22. — *Fix C^{el}_H, $R > 0$, $Q > 1$, $\varepsilon > 0$, and $\Delta > 10\varepsilon$. Then there exists $\delta_0, C_0, m_0 > 0$, such that the following holds:*

Assume that (X, d) is geodesic, δ_0-hyperbolic, with a collection $\mathscr{Y}$ of C_0-quasiconvex subsets. Further suppose that H is an m_0-coarsely connected subset of X which is C^{el}_H-quasiconvex in the electrification $X^{\mathrm{el}}_{\mathscr{Y}}$. Equip $H^{+\varepsilon\Delta}$ with its m_0-coarse path metric d_H.

Assume also that whenever H (Δ, ε)-meets a set $Y \in \mathscr{Y}$, there is a (Q, C_0)-quasi-geodesic path, which is an $(m_0/10)$-coarse path, in $H^{+\varepsilon\Delta}$ joining the meeting points in H.

Then for all $a, b \in H^{+\varepsilon\Delta}$ at d_H-distance at most R from each other, any m_0-coarse δ_0-quasi-geodesic of $H^{+\varepsilon\Delta}$ (for its coarse path metric) between a, b is $(\Delta \times C^{\mathrm{el}}_H)$-close to a geodesic of X.

Proof. — Suppose that the claim is false: For all choice of δ, C, m there is a counterexample. Set $\delta_N = C_N = 1/N$.

For each ε, there exists N such that, in a $(1/N)$-hyperbolic space, for any two points x, y, and any $(Q, 1/N)$-quasigeodesic p which is a $(1/N)$-coarse path between these two points, the ε-neighborhood of p contains the geodesics $[x, y]$. (This, for instance, is visible on an asymptotic cone).

Thus, it is possible to choose a sequence $m_N > 10/N$ decreasing to zero, such that pairs of $(9\Delta/10)$-long $(Q, 1/N)$-quasigeodesics with mesh $\leqslant 1/N$ in a $(1/N)$-hyperbolic spaces, with starting points at distance $\leqslant \Delta/10$ from each other, and ending points at distance $\leqslant \Delta/10$ from each other, necessarily lie at distance $(m_N/10)$ from one another.

Let then $X_N, H_N, \mathscr{Y}_N$ be a counterexample to our claim for these values: for each N there is a_N, b_N in $H_N^{+\Delta}$, R-close to each other for d_{H_N}, and a point $c_N \in H_N^{+\Delta}$ in a coarse δ_N-quasi-geodesic in $[a_N, b_N]_{d_{H_n}}$ at distance at least $(\Delta \times C_H^{\mathrm{el}})$ from a geodesic $[a_N, b_N]$ in X_N. However c_N is C_H^{el}-close to a geodesic $[a_N, b_N]_{\mathrm{el}}$ in X^{el}. Passing to an asymptotic cone, we find a map p_ω from an interval $[0, R]$ to a continuous path in $\lim_\omega(H_N, a_N)$ from a^ω to b^ω (which can be equal to a^ω) that passes through a point c^ω at distance $\geqslant (\Delta \times C_H^{\mathrm{el}})$ from the arc $[a^\omega, b^\omega]$ in $\lim_\omega(X_N, a_N)$.

However, it is at distance $\leqslant C_H^{\mathrm{el}}$ in the electrification of $\lim_\omega(X_N, a_N)$ by $\mathscr{Y}^\omega$. It follows that on the path in $\lim_\omega(X_N, a_N)$ from $[a^\omega, b^\omega]$ to c^ω, there must exist a segment of length $\geqslant \Delta$ belonging to the same $Y^\omega \in \mathscr{Y}^\omega$. Let us say that Y^ω is the limit of a sequence Y_N. Note that the limit path p_ω crosses this segment at least twice (once in either direction).

Thus, for N large enough, H_N (Δ, ε)-meets Y_N, with two pairs of meeting points $(r_1, r_2), (s_1, s_2)$ in H_N, where $d(r_1, r_2) \geqslant 9\Delta/10$ (and $(s_1, s_2) \geqslant 9\Delta/10$) and $d(r_1, s_1) \leqslant 3\Delta/10$ and $d(r_2, s_2) \leqslant 3\Delta/10$. By assumption, there is a $(Q, 1/N)$-quasi-geodesic path in $H^{+\varepsilon\Delta}$ from s_1 to s_2 and another from r_1 to r_2, with mesh $< m_N/10$. They have to fellow travel on a large subpath, and pass at distance $\leqslant m_N/10$ from each other, by choice of m_N. One can therefore find a shortcut that is still a path in $H^{+\Delta}$ of mesh $\leqslant m_N$, a contradiction. □

From the claim, we can prove the statement of the proposition. Consider a situation as in the statement. We may choose the coarse path connectivity constant of H to be more than 10 times the quasigeodesic constant of the last assumption there. Take ε, given by the assumption of the proposition, and $\Delta > \max\{100\varepsilon, \Delta_0\}$. Let Q be the quasi-geodesic constant given by the the assumption of the proposition on (Δ, ε)-meetings, and C_H^{el} be as given by the assumption. Take R larger (how large will be made clear in the proof).

Rescale the space X so that the hyperbolicity constant, the quasiconvexity constant of items of $\mathscr{Y}$, and the constant of coarse path connection of H are respectively smaller than δ_0, C_0, m_0 of the lemma above.

Note that the assumption of the proposition on (Δ, ε)-meetings is invariant under rescaling (except for the value of Δ_0). Thus, this assumption still holds, with the same ε, and for the specified Δ chosen above. The lemma applies, and $H^{+\varepsilon\Delta}$ is R-locally quasiconvex for the rescaled metric. By the local to global principle (in δ_0 hyperbolic spaces), with a suitable preliminary (large enough) choice of R, $H^{+\varepsilon\Delta}$ is then globally quasiconvex. After rescaling back to the original metric of X, $H^{+\lambda}$ is still quasiconvex for some λ (depending on $\varepsilon\Delta$, and the coefficient of rescaling); hence H is quasiconvex.

By construction, we also have the statement on the dependence of the quasiconvexity constant. □

2.5.4. *Coarse hyperbolic embeddedness and strong relative hyperbolicity.* — The following proposition establishes the equivalence of coarse hyperbolic embeddedness and strong relative hyperbolicity in the context of this paper.

Proposition 2.23. — *Assume that (X,d) is a metric space, and that $\mathscr{Y}$ is a collection of subspaces.*

If the horoballification $X^h_{\mathscr{Y}}$ of X over $\mathscr{Y}$ is hyperbolic, then $\mathscr{Y}$ is coarsely hyperbolically embedded in the sense of spaces.

If X is hyperbolic and if $\mathscr{Y}$ is coarsely hyperbolically embedded in the sense of spaces, then $X^h_{\mathscr{Y}}$ is hyperbolic.

We remark here parenthetically that the converse should be true without the assumption of hyperbolicity of X. However, this is not necessary for this paper.

Proof. — Assume that the horoballification $X^h_{\mathscr{Y}}$ of X over $\mathscr{Y}$ is δ-hyperbolic. The horoballs Y^h (corresponding to Y) are thus 10δ-quasiconvex. Therefore, by Proposition 2.10, the electrified space obtained by electrifying (coning off) the horoballs Y^h is hyperbolic.

Since by Proposition 2.8 this space is quasi-isometric to $(X^{\mathrm{el}}_{\mathscr{Y}}, d^{\mathrm{el}}_{\mathscr{Y}})$, it follows that the later is hyperbolic. This proves the first condition of Definition 2.2.

We want to prove the existence of a proper increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, such that the angular metric at each cone point v_Y (for $Y \in \mathscr{Y}$) of $(X^{\mathrm{el}}_{\mathscr{Y}}, d^{\mathrm{el}}_{\mathscr{Y}})$ is bounded below by $\psi \circ d|_Y$. Define

$$\psi(r) = \inf_{Y\in\mathscr{Y}} \inf_{\substack{y_1,y_2\in Y\\ d(y_1,y_2)\leqslant r}} \widehat{d}(y_1,y_2).$$

Of course, the angular metric at v_Y is bounded below by $\psi \circ d|_Y$. The function ψ is obviously increasing. We need to show that it is proper, i.e., that it goes to $+\infty$.

If ψ is not proper, then there exists $\theta_0 > 0$ such that for all D, there exist $Y \in \mathscr{Y}$ and $y, y' \in Y$ at d-distance greater than D but $\widehat{d}(y,y') \leqslant \theta_0$ (where $\widehat{d}$ is the angular metric on Y). We choose $D \gg \theta_0\delta$ (for instance $D = \exp(100(\theta_0+1)(\delta+1))$).

Consider a path in $(X^h_{\mathscr{Y}})^{\mathrm{el}}$ of length less than θ_0 from y to y' avoiding the cone point of Y. Because $D \gg \theta_0$, this path has to pass through other cone points. It can thus be chosen as a concatenation of $N+1$ geodesics whose vertices are y, y' and some cone points $v_1, \dots, v_N$ (corresponding to $Y_1, \dots, Y_N$ with $N < \theta_0$). Adjoining the (geodesic) path $[y, v_Y] \cup [v_Y, y']$ (where v_Y corresponds to the cone point for Y), we thus have a geodesic $(N+2)$-gon σ. Next replace each passage of σ through a cone point (v_i or v_Y) in $X^{\mathrm{el}}_{\mathscr{Y}}$ by a geodesic (μ_i or μ_Y respectively) in the corresponding horoball (Y_i^h or Y^h respectively) in $X^h_{\mathscr{Y}}$ to obtain a geodesic $(2N+2)$-gon P in $X^h_{\mathscr{Y}}$. The geodesic segments μ_i or μ_Y comprise $(n+1)$ alternate sides of this geodesic $(2N+2)$-gon.

Since $X^h_{\mathscr{Y}}$ is δ-hyperbolic, it follows that the mid-point m of μ_Y is at distance $\leqslant (2N+2)\delta$ from another edge of P. Note that m is in the horoball of Y, and because the distance in Y between y and y' is larger than D, we have that $d^h(m, Y)$ is at least $\log(D)/2$.

Since no other edge of P enters the horoball Y^h, this forces $\log(D)$ (and hence D) to be bounded in terms of θ_0 and δ: $D \leqslant \exp(4(N+1)\delta)$. Since $N \leqslant \theta_0$, this is a

contradiction with the choice of D. We can conclude that ψ is proper, and we have the first statement.

Let us consider the second statement. If X is hyperbolic and if $\mathscr{Y}$ is coarsely hyperbolically embedded in the sense of spaces, then elements of $\mathscr{Y}$ are uniformly quasiconvex in (X,d) by 2.7, and, by the property of the angular distance on any $Y \in \mathscr{Y}$, they are mutually cobounded. The statement then follows by Proposition 2.10. □

3. Algebraic height and intersection properties

3.1. Algebraic height. —

We recall here the general definition for height of finitely many subgroups.

Definition 3.1. — Let G be a group and $\{H_1,\dots,H_m\}$ be a finite collection of subgroups. Then the *algebraic height* of this collection is n if $(n+1)$ is the smallest number with the property that for any $(n+1)$ distinct left cosets $g_1H_{\alpha_1},\dots,g_{n+1}H_{\alpha_{n+1}}$, the intersection $\bigcap_{1\leqslant i\leqslant n+1} g_iH_{\alpha_i}g_i^{-1}$ is finite.

We shall describe this briefly by saying that algebraic height is the largest n for which the intersection of n *essentially distinct* conjugates of $H_1,\dots,H_m$ is infinite. Here 'essentially distinct' refers to the cosets of $H_1,\dots,H_m$ and not to the conjugates themselves.

For hyperbolic groups, one of the main theorems of [GMRS98] is the following:

Theorem 3.2 ([GMRS98]). — *Let G be a hyperbolic group and H a quasiconvex subgroup. Then the algebraic height of H is finite. Further, there exists R_0 such that if $H\cap gHg^{-1}$ is infinite, then g has a double coset representative with length at most R_0.*

The same conclusions hold for finitely many quasiconvex subgroups $\{H_1,\dots,H_n\}$ of G.

We quickly recall a proof of Theorem 3.2 for one subgroup H in order to generalize it to the context of mapping class groups and $\mathrm{Out}(F_n)$.

Proof. — Let G be hyperbolic, $X(=\Gamma_G)$ a Cayley graph of G with respect to a finite generating set (assumed to be δ-hyperbolic), and H a C_0-quasiconvex subgroup of G. Suppose that there exist N essentially distinct conjugates $\{H^{g_i}\}$, $i=1,\dots,N$, of H that intersect in an infinite subgroup. The N left-cosets g_iH are disjoint and share an accumulation point p in the boundary of G (in the limit set of $\bigcap_i H^{g_i}$). Since all g_iH are C_0-quasiconvex, there exist N disjoint quasi-geodesics $\sigma_1,\dots,\sigma_N$ (with same constants λ,μ depending only on C_0,δ) converging to p such that σ_i is in g_iH. Since X is δ-hyperbolic, there exists $R(=R(\lambda,\mu,\delta)=R(C_0,\delta))$ and a point p_0 sufficiently far along σ_1 such that all the quasi-geodesics $\sigma_1,\dots,\sigma_N$ pass through $B_R(p_0)$. Hence $N\leqslant \#(B_R(p_0))$ giving us finiteness of height.

Further, any such σ_i furnishes a double coset representative g_i' of g_i (say by taking a word that gives the shortest distance between H and the coset g_iH) of length bounded in terms of R. This furnishes the second conclusion. □

Remark 3.3. — A word about generalizing the above argument to a family $\mathscr{H}$ of finitely many subgroups is necessary. The place in the above argument where $\mathscr{H}$ consists of a singleton is used essentially is in declaring that the N left-cosets g_iH are disjoint. This might not be true in general (e.g. $H_1 < H_2$ for a family having two elements). However, by the pigeon-hole principle, choosing N_1 large enough, any N_1 distinct conjugates $\{H_i^{g_i}\}$, $i = 1, \dots, N$, $H_i \in \mathscr{H}$ must contain N essentially distinct conjugates $\{H^{g_i}\}$, $i = 1, \dots, N$ of some $H \in \mathscr{H}$ and then the above argument for a single $H \in \mathscr{H}$ goes through.

Remark 3.4. — A number of other examples of finite algebraic height may be obtained from certain special subgroups of relatively hyperbolic groups, mapping class groups and $\mathrm{Out}(F_n)$. These will be discussed after we introduce geometric height later in the paper.

3.2. Geometric i-fold intersections. — Given a finite family of subgroups of a group we define collections of geometric i-fold intersections.

Definition 3.5. — Let G be endowed with a left invariant word metric d. Let $\mathscr{H}$ be a finite family of subgroups of G.

For $i \in \mathbb{N}, i \geqslant 2$, define the *geometric i-fold intersection*, or simply the i-fold intersection of cosets of $\mathscr{H}$, $\mathscr{H}_i$, to be the set of subsets J of G for which there exist $H_1, \dots, H_i \in \mathscr{H}$ and $g_1, \dots, g_i \in G$, and $\Delta \in \mathbb{N}$ satisfying:

$$J = \left(\bigcap_j (g_j H_j)^{+\Delta} \right)$$

and $\bigcap_j (g_j H_j)^{+\Delta}$ is not in the 20δ-neighborhood of $\bigcap (g_j H_j)^{+\Delta - 2\delta}$, and the diameter of J is at least 10Δ.

Geometric i-fold intersections are thus, by definition, intersections of thickenings of cosets. The condition that the diameter of the intersection is larger than 10 times the thickening is merely to avoid counting myriads of too small intersections.

The next proposition establishes that the collection of such intersections is again closed under intersection.

Proposition 3.6. — *Consider $J \in \mathscr{H}_j$, and $K \in \mathscr{H}_k$ for $j < k$ and let Δ_J, Δ_K be constants as in Definition 3.5 for defining J and K respectively. Write $J = \left(\bigcap_i \left(g_i H_i\right)^{+\Delta_J}\right)$ and $K = \left(\bigcap_i (g_i' H_i')^{+\Delta_K}\right)$, and let $\Delta_0 > \max(\Delta_J, \Delta_K)$.*

Assume that J and K (Δ, ε)-meet, for some $\Delta > 20\Delta_0$, and for $\varepsilon < 1/50$.

Then either $K \subset J$, or for any pair of (Δ, ε)-meeting points of J and K, there is $L \in \mathscr{H}_{j+1}$ contained in J, that contains it.

Proof. — Let x, y be (Δ, ε)-meeting points of J and K. If $K \not\subset J$, we can assume that x, y are in $(g_1' H_1')^{+\Delta_K + \varepsilon\Delta}$ for some $g_1' H_1'$ not contained in the collection $\{g_i H_i\}$.

Notice that x, y are in $\bigcap_i (g_i H_i)^{+\Delta_J + \varepsilon\Delta} \cap (g_1' H_1')^{+\Delta_K + \varepsilon\Delta}$, hence in $\bigcap_i (g_i H_i)^{+\Delta'} \cap (g_1' H_1')^{+\Delta'}$ for Δ' the greater of $\Delta_K + \varepsilon\Delta$ and $\Delta_J + \varepsilon\Delta$.

We argue by contradiction. Suppose that x, y are contained in the 20δ-thickening of a 2δ-lesser intersection. It follows that there are x', y' such that $d(x, x') \leqslant 20\delta$, $d(y, y') < 20\delta$ and still $d(x', J) \leqslant \varepsilon\Delta - 2\delta$, and $d(y', J) \leqslant \varepsilon\Delta - 2\delta$. But by definition of (Δ, ε)-meeting, this is a contradiction.

Finally, the diameter of the intersection of Δ'-thickenings of our cosets, is larger than Δ. Since the thickening constant is $\Delta' \leqslant \Delta_0 + \varepsilon\Delta$, the ratio of the thickening constant by the diameter is at most $(\Delta_0 + \varepsilon\Delta)/\Delta$ which is less than $1/10$, hence the result. □

Let (G, d) be a group with a word metric and $H < G$ a subgroup. The restriction of d on H will be called the *induced metric* on H from G.

Proposition 3.7. — *Let (G, d) be a group with a δ-hyperbolic word metric (not necessarily locally finite).*

Assume that $A_1, \dots, A_n$ are C-quasiconvex subsets of G. Then for all $\Delta > C + 20\delta$, the intersection $\bigcap A_i^{+\Delta}$ is (4δ)-quasiconvex in (G, d).

Moreover, if A and B are C-quasiconvex subsets of G, and if $\Pi_B(A)$ denotes the set of nearest points projections of A on B, then, either $\Pi_B(A) \subset A^{+3C+10\delta}$ or $\operatorname{diam} \Pi_B(A) \leqslant 4C + 20\delta$.

Proof. — Consider $x, y \in \bigcap A_i^{+\Delta}$ and take a_i, b_i some nearest point projection on A_i. On a geodesic $[x, y]$ take p at distance greater than 4δ from x and y. Hyperbolicity applied to the quadrilateral (x, a_i, b_i, y) tells us that x is 4δ-close to $[x_i', a_i] \cup [a_i, b_i] \cup [b_i, y_i']$, where x_i' and y_i' are the points of, respectively $[x, a_i]$ and $[y, b_i]$, at distance 4δ from, respectively, x and y.

Let us call $[x_i', a_i], [b_i, y_i']$ the approaching segment, and $[a_i, b_i]$ the traveling segments. Hence for each i, p is closed to either an approaching segment, or the traveling segment, with subscript i.

– If p is close to an approaching segment of index i, then it is in $A_i^{+\Delta}$.

– If x is close to the traveling segment of index i, then it is at distance at most $C + 10\delta$ from A_i, hence in $A_i^{+\Delta}$ because $\Delta > C + 10\delta$.

We thus obtain that $[x, y]$ remains at distance 4δ from $\bigcap A_i^{+\Delta}$.

To prove the second statement, take a_0, b_0 in A and B respectively realizing the distance (up to δ if necessary). Let $b \in \Pi_B(A)$, and assume that it is the projection of a. In the quadrilateral a, a_0, b, b_0, the geodesic $[a, a_0]$ stays in A^{+C} and $[b, b_0]$ is in B^{+C}. Since b is a projection, $[b, a]$ fellow-travels $[b, b_0]$ for less than $2C + 10\delta$, and similarly for $[b_0, a_0]$ with $[b_0, b]$. By hyperbolicity $[b, b_0]$ thus stays 10δ close to $[a, a_0]$ except for the part $(2C + 10\delta)$-close to either b or b_0. It follows that either $b \in A^{+(3C+10\delta)}$ or b is at distance $\leqslant 4C + 20\delta$ from b_0. Thus $\Pi_B(A) \subset A^{+3C+10\delta}$ or $\operatorname{diam} \Pi_B(A) \leqslant 4C + 20\delta$. □

3.3. Algebraic i-fold intersections. — We provide now a more algebraic (group theoretic) treatment of the preceding discussion. This is in keeping with the more well-known setup of intersections of subgroups and their conjugates, cf. [GMRS98].

Given a finite family of subgroups of a group we first define collections of i-fold conjugates or algebraic i-fold intersections.

Definition 3.8. — Let G be endowed with a left invariant word metric d. Let $\mathscr{H}$ be a family of subgroup of G. For $i \in \mathbb{N}, i \geqslant 2$, define $\mathscr{H}_i$ to be the set of subgroups J of G for which there exists $H_1, \ldots, H_i \in \mathscr{H}$ and $g_1, \ldots, g_i \in G$ satisfying:

– the cosets $g_j H_j$ are pairwise distinct (and hence as in [GMRS98] we use the terminology that the conjugates $\{g_j H_j g_j^{-1}, j = 1, \ldots, i\}$ are *essentially distinct*)
– J is the intersection $\bigcap_j g_j H_j g_j^{-1}$.
– J is unbounded in (G, d).

We shall call $\mathscr{H}_i$ the family of *algebraic i-fold intersections* or simply, *i-fold conjugates*.

The second point in the following definition is motivated by the behavior of nearest point projections of cosets of quasiconvex subgroups of hyperbolic groups on each other. Let (G, d) be hyperbolic and H_1, H_2 be quasiconvex. Let aH_1, bH_2 be cosets and $c = a^{-1}b$. Then the nearest point projection of bH_2 onto aH_1 is the (left) a-translate of the nearest point projection of cH_2 onto H_1. Let $\Pi_B(A)$ denote the (nearest-point) projection of A onto B. Then $\Pi_{H_1}(cH_2)$ lies in a bounded neighborhood (say D_0-neighborhood) of $H_2^c \cap H_1$ and so $\Pi_{aH_1}(bH_2)$ lies in a D_0-neighborhood of $bH_2b^{-1}a \cap aH_1$. The latter does lie in a bounded neighborhood of $(H_2)^b \cap (H_1)^a$, but this bound depends on a, b and is not uniform. Hence the somewhat convoluted way of stating the second property below. The language of nearest-point projections below is in the spirit of [Mj10, Mj14] while the notion of geometric i-fold intersections discussed earlier is in the spirit of [DGO17].

Definition 3.9. — Let G be a group and d a word metric on G.

A finite family $\mathscr{H} = \{H_1, \ldots, H_m\}$ of subgroups of G, each equipped with a word-metric d_i is said to have the *uniform qi-intersection property* if there exist $C_1, \ldots, C_n, \ldots$ such that

(1) for all n, and all $H \in \mathscr{H}_n$, H has a conjugate H' such that if d' is any *induced metric* on H' from some $H_i \in \mathscr{H}$, then (H', d') is (C_1, C_1)-qi-embedded in (G, d),

(2) for all n, let $(\mathscr{H}_n)_0$ be a choice of conjugacy representatives of elements of $\mathscr{H}_n$ that are C_1-quasiconvex in (G, d); Let $\mathscr{C}\mathscr{H}_n$ denote the collection of left cosets of elements of $(\mathscr{H}_n)_0$; for all $A, B \in \mathscr{C}\mathscr{H}_n$ with $A = aA_0, B = bB_0$, and $A_0, B_0 \in (\mathscr{H}_n)_0$, $\Pi_B(A)$ either has diameter bounded by C_n for the metric d, or $\Pi_B(A)$ lies in a (left) a-translate translate of a C_n-neighborhood of $A_0 \cap B_0^c$, where $c = a^{-1}b$.

In keeping with the spirit of the previous subsection, we provide a geometric version of the above definition below.

Definition 3.10. — Let G be a group and d a word metric on G.

A finite family $\mathscr{H} = \{H_1, \ldots, H_m\}$ of subgroups of G, each equipped with a word-metric d_i is said to have the *uniform geometric qi-intersection property* if there exist $C_1, \ldots, C_n, \ldots$ such that

(1) for all n, and all $H \in \mathscr{H}_n$, (H, d) is C_n-coarsely path connected, and (C_1, C_1)-qi-embedded in (G, d) (for its coarse path metric),

(2) for all $A, B \in \mathscr{H}_n$ either $\mathrm{diam}_{G,d}(\Pi_B(A)) \leqslant C_n$, or $\Pi_B(A) \subset A^{+C_n}$ for d.

Remark 3.11. — The second condition of Definition 3.10 follows from the first condition if d is hyperbolic by Proposition 3.7. Further, the first condition holds for such (G, d) so long as Δ is taken of the order of the quasiconvexity constants (again by Proposition 3.7).

Note further that if G is hyperbolic (with respect to a not necessarily locally finite word metric) and H is C-quasiconvex, then by Proposition 3.6 the collection of geometric n-fold intersections $\mathscr{H}_n$ is mutually cobounded for the metric of $(G, d)^{\mathrm{el}}_{\mathscr{H}_{n+1}}$ (as in Definition 2.9).

3.4. Existing results on algebraic intersection properties. — We start with the following result due to Short.

Theorem 3.12 ([Sho91, Prop. 3]). — *Let G be a group generated by the finite set S. Suppose G acts properly on a uniformly proper geodesic metric space (X, d), with a base point x_0. Given C_0, there exists C_1 such that if H_1, H_2 are subgroups of G for which the orbits $H_i x_0$ are C_0-quasiconvex in (X, d) (for $i = 1, 2$) then the orbit $(H_1 \cap H_2)x_0$ is C_1-quasiconvex in (X, d).*

We remark here that in the original statement of [Sho91, Prop. 3], X is itself the Cayley graph of G with respect to S, but the proof there goes through without change to the general context of Proposition 3.12.

In particular, for G (Gromov) hyperbolic, or $G = \mathrm{Mod}(S)$ acting on Teichmüller space $\mathrm{Teich}(S)$ (equipped with the Teichmüller metric) and $\mathrm{Out}(F_n)$ acting on Outer space cv_N (with the symmetrized Lipschitz metric), the notions of (respectively) quasiconvex subgroups or convex cocompact subgroups of $\mathrm{Mod}(S)$ or $\mathrm{Out}(F_n)$ (see Sections 4.3 and 4.4 below for the lemmas) are independent of the finite generating sets chosen. Hence we have the following.

Theorem 3.13. — *Let G be either $\mathrm{Mod}(S)$ or $\mathrm{Out}(F_n)$ equipped with some finite generating set. Given C_0, there exists C_1 such that if H_1, H_2 are C_0-convex cocompact subgroups of G, then $H_1 \cap H_2$ is C_1-convex cocompact in G.*

The corresponding statement for relatively hyperbolic groups and relatively quasiconvex groups is due to Hruska. For completeness we recall it.

Definition 3.14 ([Osi06b, Hru10]). — Let G be finitely generated hyperbolic relative to a finite collection $\mathscr{P}$ of parabolic subgroups. A subgroup $H \leqslant G$ is *relatively quasiconvex* if the following holds.

Let S be some (any) finite relative generating set for $(G, \mathscr{P})$, and let P be the union of all $P_i \in \mathscr{P}$. Let $\overline{\Gamma}$ denote the Cayley graph of G with respect to the generating set $S \cup P$ and d the word metric on G. Then there is a constant $C_0 = C_0(S, d)$ such that

for each geodesic $\gamma \subset \overline{\Gamma}$ joining two points of H, every vertex of γ lies within C_0 of H (measured with respect to d).

Theorem 3.15 ([Hru10]). — *Let G be finitely generated hyperbolic relative to $\mathscr{P}$. Given C_0, there exists C_1 such that if H_1, H_2 are C_0-relatively quasiconvex subgroups of G, then $H_1 \cap H_2$ is C_1-relatively quasiconvex in G.*

4. Geometric height and graded geometric relative hyperbolicity

We are now in a position to define the geometric analog of height. There are two closely related notions possible, one corresponding to the geometric notion of i-fold intersections and one corresponding to the algebraic notion of i-fold conjugates. The former is relevant when one deals with subsets and the latter when one deals with subgroups.

Definition 4.1. — Let G be a group, with a left invariant word metric $d(= d_G)$ with respect to some (not necessarily finite) generating set. Let $\mathscr{H}$ be a family of subgroups of G.

The *geometric height*, of $\mathscr{H}$ in (G, d) (for d) is the minimal number $i \geqslant 0$ so that the collection $\mathscr{H}_{i+1}$ of $(i+1)$-fold intersections consists of uniformly bounded sets.

If H is a single subgroup, its geometric height is that of the family $\{H\}$.

Remark 4.2 (Comparing notions of height)

– Geometric height is related to algebraic height, but is more flexible, since in the former, we allow the group G to have an infinite generating set. We are then free to apply the operations of electrification, horoballification in the context of non-proper graphs.

– In the case of a locally finite word metric, algebraic height is less than or equal to geometric height. Equality holds if all bounded intersections are uniformly bounded.

– For a locally finite word metric, finiteness of algebraic height implies that i-fold conjugates are finite (and hence bounded in any metric) for all sufficiently large i. Hence finiteness of geometric height follows from finiteness of algebraic height and of a uniform bound on the diameter of the finite intersections.

– When the metric on a Cayley graph is not locally finite, we do not know of any general statement that allows us to go directly from finiteness of diameter of an intersection of thickenings of cosets (geometric condition) to finiteness of diameter of intersections of conjugates (algebraic condition). Some of the technical complications below are due to this difficulty in going from geometric intersections to algebraic intersections.

We generalize Definition 1.3 of graded relative hyperbolicity to the context of geometric height as follows.

Definition 4.3. — Let G be a group, d the word metric with respect to some (not necessarily finite) generating set and $\mathscr{H}$ a finite collection of subgroups. Let $\mathscr{H}_i$ be the collection of all i-fold conjugates of $\mathscr{H}$. Let $(\mathscr{H}_i)_0$ be a choice of conjugacy

representatives, and $\mathscr{C}\mathscr{H}_i$ the set of left cosets of elements of $(\mathscr{H}_i)_0$ Let d_i be the metric on (G,d) obtained by electrifying the elements of $\mathscr{C}\mathscr{H}_i$. Let $\mathscr{C}\mathscr{H}_{\mathbb{N}}$ be the graded family $(\mathscr{C}\mathscr{H}_i)_{i\in\mathbb{N}}$.

We say that G is *graded geometric relatively hyperbolic* with respect to $\mathscr{C}\mathscr{H}_{\mathbb{N}}$ if

(1) $\mathscr{H}$ has geometric height n for some $n\in\mathbb{N}$, and for each i there are finitely many orbits of i-fold intersections,

(2) for all $i\leqslant n+1$, $\mathscr{C}\mathscr{H}_{i-1}$ is coarsely hyperbolically embedded in (G,d_i),

(3) there is D_i such that all items of $\mathscr{C}\mathscr{H}_i$ are D_i-coarsely path connected in (G,d).

Remark 4.4 (Comparing geometric and algebraic graded relative hyperbolicity)

Note that the second condition of Definition 4.3 is equivalent, by Proposition 2.23, to saying that (G,d_i) is strongly hyperbolic relative to the collection $\mathscr{H}_{i-1}$. This is exactly the third (more algebraic) condition in Definition 1.3. Also, the third condition of Definition 4.3 is the analog of (and follows from) the second (more algebraic) condition in Definition 1.3.

Thus finite geometric height along with (algebraic) graded relative hyperbolicity implies graded geometric relative hyperbolicity.

The rest of this section furnishes examples of finite height in both its geometric and algebraic incarnations.

4.1. Hyperbolic groups

Proposition 4.5. — *Let (G,d) be a hyperbolic group with a locally finite word metric, and let H be a quasiconvex subgroup of G. Then H has finite geometric height.*

More precisely, if C is the quasi-convexity constant of H in (G,d), and if δ be the hyperbolicity constant in (G,d), and if N is the cardinality of a ball of (G,d) of radius $2C+10\delta$, and if $g_0H,\dots,g_kH$ are distinct cosets of H for which there exists Δ such that the total intersection $\bigcap_{i=0}^k (g_iH)^{+\Delta}$ has diameter more than 10Δ, and more that 100δ, then there exists $x\in G$ such that each g_iH intersects the ball of radius N around x.

First note that the second statement implies the first in the (by the third point of Remark 4.2). We will directly prove the second. The proof is similar to the finiteness of the algebraic height. Also note that the second statement can be rephrased in terms of double cosets representatives of the g_i: under the assumption on the total intersection, and if $g_0=1$, there are double coset representatives of the g_i of length at most $2(2C+10\delta)$.

Proof. — Assume that there exists $\Delta>0$, and elements $1=g_0,g_1,\dots,g_k$ for which the cosets g_iH are distinct, and $\bigcap_{i=0}^k (g_iH)^{+\Delta}$ has diameter larger than 10Δ and than 100δ.

First we treat the case $\Delta>5\delta$. Pick $y_1,y_2\in\bigcap_{i=0}^k (g_iH)^{+\Delta}$ at distance 10Δ from each other, and pick $x\in[y_1,y_2]$ at distance larger than $\Delta+10\delta$ from both y_i. For each i an application of hyperbolicity and quasi-convexity tells us that x is at distance

at most $2C + 10\delta$ from each of g_iH. The ball of radius $2C + 10\delta$ around x thus meets each coset g_iH.

If $\Delta \leqslant 5\delta$, we pick $y_1, y_2 \in \bigcap_{i=0}^{k}(g_iH)^{+\Delta}$ at distance 100δ from each other, and take x at distance greater than 10δ from both ends. The end of the proof is the same. □

4.2. Relatively hyperbolic groups. — If G is hyperbolic relative to a collection of subgroups $\mathscr{P}$, then Hruska and Wise defined in [HW09] the relative height of a subgroup H of G as n if $(n+1)$ is the smallest number with the property that for any $(n+1)$ elements $g_0, \dots, g_n$ such that the g_iH are $(n+1)$-distinct cosets, the intersections of conjugates $\bigcap_i g_iHg_i^{-1}$ is finite or parabolic.

The notion of relative algebraic height is actually the geometric height for the relative distance, which is given by a word metric over a generating set that is the union of a finite set and a set of conjugacy representatives of the elements of $\mathscr{P}$. Indeed, in a relatively hyperbolic group, the subgroups that are bounded in the relative metric are precisely those that are finite or parabolic. We give a quick argument. It follows from the lemma of relative quasiconvexity that a subgroup having finite diameter in the electric metric on G (rel. $\mathscr{P}$) is relatively quasiconvex. It is also true [DGO17] that the normalizer of any $P \in \mathscr{P}$ is itself and that the subgroup generated by any P and any infinite order element $g \in G \smallsetminus P$ contains the free product of conjugates of P by $g^{kn}, k \in \mathbb{Z}$. Since any proper supergroup of P necessarily contains such a g, it follows that no proper supergroup of P can be of finite diameter in the electric metric on G (rel. $\mathscr{P}$). It follows that bounded subgroups are precisely the finite subgroups or those contained inside parabolic subgroups.

The notion of relative height can actually be extended to define the height of a collection of subgroups $H_1, \dots, H_k$, as in the case for the algebraic height.

Hruska and Wise proved that relatively quasiconvex subgroups have finite relative height. More precisely:

Theorem 4.6 ([HW09, Th. 1.4, Cors. 8.5-8.7]). — *Let $(G, \mathscr{P})$ be relatively hyperbolic, let S be a finite relative generating set for G and Γ be the Cayley graph of G with respect to S. Then for $\sigma \geqslant 0$, there exists $C \geqslant 0$ such that the following holds.*

Let $H_1, \dots, H_n$ be a finite collection of σ-relatively quasiconvex subgroups of $(G, \mathscr{P})$. Suppose that there exist distinct cosets $\{g_mH_{\alpha_m}\}$ with $\alpha_m \in \{1, \dots, n\}$, $m = 1, \dots, n$, such that $\bigcap_m g_mH_{\alpha_m}g_m^{-1}$ is not contained in a parabolic $P \in \mathscr{P}$. Then there exists a vertex $z \in G$ such that the ball of radius C in Γ intersects every coset $g_mH_{\alpha_m}$.

Further, for any $i \in \{1, \dots, n\}$, there are only finitely many double cosets of the form $H_ig_iH_{\alpha_i}$ such that $H_i \cap \bigcap_i g_iH_{\alpha_i}g_i^{-1}$ is not contained in a parabolic $P \in \mathscr{P}$.

Let G be a relatively hyperbolic group, and let H be a relatively quasiconvex subgroup. Then H has finite relative algebraic height.

This allows us to give an example of geometric height in our setting.

Proposition 4.7. — *Let $(G, \mathscr{P})$ be a relatively hyperbolic group, and (G, d) a relative word metric d (i.e., a word metric over a generating set that is the union of a finite set and of a set of conjugacy representatives of the elements of $\mathscr{P}$, and hence, in general, not a finite generating set). Let H be a relatively quasiconvex subgroup. Then, H has finite geometric height for d.*

This just a rephrasing of Hruska and Wise's result Theorem 4.6. The proof is similar to that in the hyperbolic groups case, using for instance cones instead of balls.

4.3. Mapping class groups. — Another source of examples arise from convex-cocompact subgroups of mapping class groups, and of $\mathrm{Out}(F_n)$ for a free group F_n. We establish finiteness of both algebraic and geometric height of convex cocompact subgroups of mapping class groups in this subsubsection. In the following S will be a closed oriented surface of genus greater than 2, and $\mathrm{Teich}(S)$ and $\mathrm{CC}(S)$ will denote respectively the Teichmüller space and Curve Complex of S.

Definition 4.8 ([FM02]). — A finitely generated subgroup H of the mapping class group $\mathrm{Mod}(S)$ for a surface S (with or without punctures) is σ-convex cocompact if for some (any) $x \in \mathrm{Teich}(S)$, the Teichmüller space of S, the orbit $Hx \subset \mathrm{Teich}(S)$ is σ-quasiconvex with respect to the Teichmüller metric.

Kent-Leininger [KL08] and Hamenstädt [Ham08] prove the following:

Theorem 4.9. — *A finitely generated subgroup H of the mapping class group* $\mathrm{Mod}(S)$ *is convex cocompact if and only if for some (any) $x \in \mathrm{CC}(S)$, the curve complex of S, the orbit $Hx \subset \mathrm{CC}(S)$ is qi-embedded in* $\mathrm{CC}(S)$.

One important ingredient in Kent-Leininger's proof of Theorem 4.9 is a lifting of the limit set of H in $\partial\,\mathrm{CC}(S)$ (the boundary of the curve complex) to $\partial\,\mathrm{Teich}(S)$ (the boundary of Teichmüller space). What is important here is that $\mathrm{Teich}(S)$ is a proper metric space unlike $\mathrm{CC}(S)$. Further, they show using a theorem of Masur [Mas80], that any two Teichmüller geodesics converging to a point on the limit set Λ_H (in $\partial\,\mathrm{Teich}(S)$) of a convex cocompact subgroup H are asymptotic. An alternate proof is given by Hamenstädt in [Ham10]. With these ingredients in place, the proof of Theorem 4.10 below is an exact replica of the proof of Theorem 3.2 above:

Theorem 4.10 (Height from the Teichmüller metric). — *Let G be the mapping class group of a surface S, and* $\mathrm{Teich}(S)$ *the corresponding Teichmüller space with the Teichmüller metric, and with a base point z_0. Then for $\sigma \geqslant 0$, there exists $C \geqslant 0$, and $D \geqslant 0$ such that the following holds.*

Let $H_1, \dots, H_n$ be a finite collection of σ-convex cocompact subgroups of G. Suppose that there exist distinct cosets $\{g_m H_{\alpha_m}\}$ with $\alpha_m \in \{1, \dots, n\}$, $m = 1, \dots, n$, such that, for some Δ, $\bigcap_m (g_m H_{\alpha_m})^{+\Delta}$ is larger than $\max\{10\Delta, D\}$. *Then there exists a point $z \in \mathrm{Teich}(S)$ such that the ball of radius C in* $\mathrm{Teich}(S)$ *intersects every image of z_0 by a coset $g_m H_{\alpha_m} z_0$.*

Further, for any $i \in \{1, \dots, n\}$, there are only finitely many double cosets of the form $H_i g_i H_{\alpha_i}$ such that $H_i \cap \bigcap_i g_i H_{\alpha_i} g_i^{-1}$ is infinite.

The collection $\{H_1, \dots, H_n\}$ has finite algebraic height.

A more geometric strengthening of Theorem 4.10 can be obtained as follows using recent work of Durham and Taylor [DT15], who have given an intrinsic quasi-convexity interpretation of convex cocompactness, by proving that convex cocompact subgroups of mapping class groups are stable: in a word metric, they are undistorted, and quasi geodesics with end points in the subgroup remain close to each other [DT15].

Theorem 4.11 (Height from a word metric). — *Let G be the mapping class group of a surface S and d the word metric with respect to a finite generating set. Then for $\sigma \geqslant 0$, and any subgroup H that is σ-convex cocompact, the group H has finite geometric height in (G, d).*

Moreover, any σ-convex cocompact subgroup H has finite geometric height in (G, d_1), where d_1 is the word metric with respect to any (not necessarily finite) generating set.

Proof. — Assume that the theorem is false: there exists σ such that for all k, and all D, there exists a σ-convex cocompact subgroup H, with a collection of distinct cosets $\{g_m H, m = 0, \dots, k\}$ (with $g_0 = 1$), satisfying the property that $\bigcap_m (g_m H)^{+\Delta}$ has diameter larger than $\max\{10\Delta, D\}$.

Let a, b be two points in $\bigcap_m (g_m H)^{+\Delta}$ such that $d(a, b) \geqslant \max\{10\Delta, D\}$. For each i, let a_i, b_i in $g_i H$ be at distance at most Δ from a and b respectively. Consider γ_i geodesics in H from $g_i^{-1} a_i$ to $g_i^{-1} b_i$. Consider also a_i' and b_i'-nearest point projections of a_0 and b_0 on $g_i \gamma_i$. Finally, denote by $g_i \gamma_i'$ the subpath of $g_i \gamma_i$ between a_i' and b_i'

By [DT15, Prop. 5.7], H is quasiconvex in G (for a fixed chosen word metric), and for each i, $g_i \gamma_i$ is a $f(\sigma)$-quasi-geodesic (for some function f).

We thus obtain from a_0 to b_0 a family of $k+1$ paths, namely γ_0 and (one for all i), the concatenation $\eta_i = [a_0, a_i'] \cdot g_i \gamma_i' \cdot [b_i', b_0]$. For D large enough, the paths η_i are $2f(\sigma)$-quasigeodesics in G.

Stability of H ([DT15, Th. 1.1]) implies that, there exists $R(\sigma)$ such that in G, the paths remain at mutual Hausdorff distance at most $R(\sigma)$. This is thus also true in the Teichmüller space by the orbit map. Hence it follows that all the subpaths $g_i \gamma_i'$ are at distance at most $2R(\sigma)$ from each other, but are disjoint, and all lie in a thick part of the Teichmüller space, where the action is uniformly proper. This leads to a contradiction.

Since the diameter of intersections can only go down if the generating set is increased, the last statement follows. □

4.4. Out(F_n). — Following Dowdall-Taylor [DT14], we say that a finitely generated subgroup H of Out(F_n) is *σ-convex cocompact* if

(1) all non-trivial elements of H are atoroidal and fully irreducible,

(2) for some (any) $x \in cv_n$, the (projectivized) Outer space for F_n, the orbit $Hx \subset cv_n$ is σ-quasiconvex with respect to the Lipschitz metric.

Remark 4.12. — The above lemma, while not explicit in [DT14], is implicit in Section 1.2 of that paper.

Also, a word about the metric on cv_n is in order. The statements in [DT14] are made with respect to the *unsymmetrized* metric on outer space. However, convex cocompact subgroups have orbits lying in the thick part; and hence the unsymmetrized and symmetrized metrics are quasi-isometric to each other. We assume henceforth, therefore, that we are working with the symmetrized metric, to which the conclusions of [DT14] apply via this quasi-isometry.

The following theorem gives a characterization of convex cocompact subgroups in this context and is the analog of Theorem 4.9.

Theorem 4.13 ([DT14]). — *Let H be a finitely generated subgroup of $\mathrm{Out}(F_n)$ all whose non-trivial elements are atoroidal and fully irreducible. Then H is convex cocompact if and only if for some (any) $x \in \mathscr{F}_n$ (the free factor complex of F_n), the orbit $Hx \subset \mathscr{F}_n$ is qi-embedded in $\mathscr{F}_n$.*

Dowdall and Taylor also show [DT14, Th. 4.1] that any two quasi-geodesics in cv_n converging to the same point p on the limit set Λ_H (in ∂cv_n) of a convex cocompact subgroup H are asymptotic. More precisely, given λ, μ and $p \in \Lambda_H$ there exists $C_0 (= C_0(\lambda, \mu, p))$ such that any two (λ, μ)-quasi-geodesics in cv_n converging to p are asymptotically C_0-close. As observed before in the context of Theorem 4.10, this is adequate for the proof of Theorem 4.10 to go through:

Theorem 4.14. — *Let $G = \mathrm{Out}(F_n)$, and cv_n the Outer space for G with a base point z_0. Then for $\sigma \geqslant 0$, there exists $C \geqslant 0$ such that the following holds.*

Let $H_1, \ldots, H_n$ be a finite collection of σ-convex cocompact subgroups of G. Suppose that there exist distinct cosets $\{g_m H_{\alpha_m}\}$ with $\alpha_m \in \{1, \ldots, n\}$, $m = 1, \ldots, n$, such that $\bigcap_m g_m H_{\alpha_m} g_m^{-1}$ is infinite. Then there exists a point $z \in cv_n$ such that the ball of radius C in cv_n intersects every image of z_0 by a coset $g_m H_{\alpha_m} z_0$.

Further, for any $i \in \{1, \ldots, n\}$, there are only finitely many double cosets of the form $H_i g_i H_{\alpha_i}$ such that $H_i \cap \bigcap_i g_i H_{\alpha_i} g_i^{-1}$ is infinite.

The collection $\{H_1, \ldots, H_n\}$ has finite algebraic height.

Since an analog of the stability result of [DT15] in the context of $\mathrm{Out}(F_n)$ is missing at the moment, we cannot quite get an analog of Theorem 4.11.

4.5. Algebraic and geometric qi-intersection property: Examples

In the proposition below we shall put parentheses around (geometric) to indicate that the statement holds for both the qi-intersection property as well as the geometric qi-intersection property.

Proposition 4.15

(1) *Let H be a quasiconvex subgroup of a hyperbolic group G, endowed with a locally finite word metric. Then, $\{H\}$ satisfies the uniform (geometric) qi-intersection property.*

(2) *Let H be a relatively quasiconvex subgroup of a relatively hyperbolic group $(G, \mathscr{P})$. Let $\mathscr{P}_0$ be a set of conjugacy representatives of groups in $\mathscr{P}$, and d a word metric on G over a generating set $S = S_0 \cup \mathscr{P}_0$, where S_0 is finite. Then $\{H\}$ satisfies the uniform (geometric) qi-intersection property with respect to d.*

(3) *Let H be a convex-cocompact subgroup of the Mapping Class Group* $\mathrm{Mod}(\Sigma)$ *of an oriented closed surface Σ of genus $\geqslant 2$. If d is a word metric on* $\mathrm{Mod}(\Sigma)$ *that makes it quasi-isometric to the curve complex of Σ, then H satisfies the uniform (geometric) qi-intersection property with respect to d.*

(4) *Let H be a convex-cocompact subgroup of* $\mathrm{Out}(F_n)$ *for some $n \geqslant 2$. If d is a word metric on* $\mathrm{Out}(F_n)$ *that makes it quasi-isometric to the free factor complex of F_n, then H satisfies the uniform (geometric) qi-intersection property with respect to d.*

Proof. — All four cases have similar proofs. Consider the first point.

Case 1: G hyperbolic, H quasiconvex. — Let h be the height of H (which is finite by Theorem 3.2): every $h+1$-fold intersection of conjugates of H is finite, but some h-fold intersection is infinite.

– The first conditions of Definition 3.9 and Definition 3.10 follow from this finiteness and Proposition 4.5 and Theorem 3.12.

– The second condition of Definition 3.10 follows from Proposition 3.7.

– We prove the second condition of Definition 3.9 (on mutual coboundedness of elements of $\mathscr{C}\mathscr{H}_i$) iteratively.

By Theorem 3.12, there exists C_h such that two elements of $\mathscr{C}\mathscr{H}_h$ are C_h-quasiconvex in (G, d). Let $D > 0$. If A and B are two distinct such elements such that the projection of A on B has diameter greater than D, then there are D/C_h pairs of elements (a_i, b_i) in $A \times B$, such that $a_i^{-1} b_i$ are elements of length at most $20\delta C_h$. Choose N_0 larger than the cardinality of finite subgroups of G. By a standard pigeon hole argument, if D is large enough, there are N_0 such pairs for which $a_i^{-1} b_i$ take the same value. It follows that there are two essentially distinct conjugates of elements of $\mathscr{H}_h$ that intersect on a subset of at least N_0 elements, hence on an infinite subgroup. This contradicts the definition of height, and it follows that D is bounded, and elements of $\mathscr{C}\mathscr{H}_h$ are mutually cobounded.

We continue by descending induction. Assume that the second property of Definition 3.9 is established for $\mathscr{C}\mathscr{H}_{i+1}$. By Proposition 2.10 it follows that (G, d_{i+1}) is hyperbolic. Let δ_{i+1} be its hyperbolicity constant. By Proposition 2.11, there exists C_i such that two elements of $\mathscr{C}\mathscr{H}_i$ are C_i-quasiconvex in (G, d_{i+1}).

Again take A and B two distinct elements of $\mathscr{C}\mathscr{H}_i$ such that the projection of A on B has diameter greater than $D > 1000\delta_{i+1}$ for d_{i+1}. Then there are at least D/C_i pairs of elements (a_i, b_i) in $A \times B$, such that $a_i^{-1} b_i$ is an element of length at most $20\delta_{i+1} C_h$ for the metric d_{i+1}, and for all i there exists i' such that the segments

$[a_i, b_i], [a_{i'}, b_{i'}]$ are $(200\delta_{i+1})$-far from one another. Apply the Proposition 2.12 to each geodesic $[a_i, b_i]$ to find quasi-geodesics q_i from a_i to b_i in (G, d) (this can also be done by Lemma 2.15). We know that in (G, d), A and B are quasiconvex (for the constant C_h). By their definition, and by hyperbolicity, the paths q_i end at bounded distance of a shortest-point projection of a_i to B (for d). Therefore, since (G, d) is hyperbolic, and since the q_i are far from one another for d, it follows that the q_i are actually short for the metric d (shorter than $(200\delta C_h)$). Since there are D/C_i pairs of elements (a_i, b_i), by the pigeon hole argument, there is an element g_0 (of length at most $(200\delta C_h)$ in the metric d) such that for $D/(C_i \times B_{G,d}(200\delta C_h))$ such pairs, the difference $a^{-1}b$ equals g_0. If D is large enough, $D/(C_i \times B_{G,d}(200\delta C_h))$ is larger than the cardinality of the finite order elements of G. It follows that the two essentially distinct conjugates of elements of $\mathscr{H}_i$, corresponding to the cosets A and B, intersect on a set of size larger than any finite subgroup of G (and of diameter larger than 3 in d_{i+1}). Thus the intersection is an infinite subgroup of G. This subgroup is necessarily among the conjugates of some $\mathscr{H}_j$ for $j \geqslant i+1$, but therefore must have diameter 2 in the metric d_{i+1}.

Case 2: G relatively hyperbolic, H relatively quasiconvex. — The geometric height of H for the relative metric is finite, by Proposition 4.7. Let h be its value. The first points of Definition 3.9 follows from this finiteness and Theorem 3.15.

The second point has a similar proof as the first case, except that the pigeon hole argument needs to be made precise because the relative metric (G, d) is not locally finite.

Let $D > 0$. If A and B are two distinct elements of $\mathscr{C}\mathscr{H}_h$ such that the projection of A on B has diameter greater than D, then there are D/C_H pairs of elements (a_i, b_i) in $A \times B$, such that $a_i^{-1}b_i$ are elements of length at most $20\delta C_h$. Moreover, if $D > 100\delta C_h$, for each $[a_i, b_i]$, there is $[a_j, b_j]$ such that both segments are short (for d) and are at distance at least $(50\delta C_h)$ from each other. It follows that, in the Cone-off Cayley graph of G, the maximal angle of $[a_i, b_i]$ at the cone vertices is uniformly bounded by $(100\delta C_h) + 2(2C_h + 5\delta)$. Indeed, consider α and β quasi-geodesic paths in A and B respectively, from a_i to a_j and from b_i to b_j. By hyperbolicity and quasi-geodesy, at distance $30\delta C_h$ from a_i and b_i, there is a path of length $2(2C_h + 5\delta)$ joining α to β. Being too short, this path cannot possibly intersect $[a_i, b_i]$. There is thus a path from a_i to b_i of length at most $2 \times (30\delta C_h) + 2(2C_h + 5\delta)$ that does not intersect $[a_i, b_i]$ outside its end points. It follows indeed that the maximal angle of $[a_i, b_i]$ is at most $2 \times (30\delta C_h) + 2(2C_h + 5\delta) + 20\delta C_h$.

From this bound on angles, we may use the fact that the angular metric at each cone point is locally finite (by definition of relative hyperbolicity) and the bound on the length in the metric d, to get that all the elements $a_i^{-1}b_i$ are in a finite set, independent of D. We can now use the pigeon hole argument, as in the hyperbolic case, and conclude similarly that D is bounded.

The rest of the argument is also by descending induction. Assume that the second property of Definition 3.9 is established for $\mathscr{C}\mathscr{H}_{i+1}$. We proceed in a very similar way as in the hyperbolic case, with the difference is that, after establishing that the

paths q_i are small for the metric d, one needs to check that their angles at cone points are bounded, which is done by the argument we just used. We provide the details now.

By Proposition 2.10 it follows that (G, d_{i+1}) is hyperbolic. Let δ_{i+1} be its hyperbolicity constant. By Proposition 2.11, there exists C_i such that two elements of $\mathscr{C}\mathscr{H}_i$ are C_i-quasiconvex in (G, d_{i+1}).

Take A and B two distinct elements of $\mathscr{C}\mathscr{H}_i$ such that the projection of A on B has diameter greater than some constant D for d_{i+1}. Take a quasigeodesic in the projection of A on B, of length D. Then there are at least D/C_i pairs of elements (a_i, b_i) in $A \times B$, with b_i on that quasigeodesic, and such that $a_i^{-1} b_i$ is an element of length at most $20\delta_{i+1}C_h$ for the metric d_{i+1}, and for all i there exists i' such that the segments $[a_i, b_i], [a_{i'}, b_{i'}]$ are $(200\delta_{i+1})$-far from one another. Apply the Proposition 2.12 (or alternatively 2.15) to each geodesic $[a_i, b_i]$ to find quasi-geodesics q_i from a_i to b_i in (G, d). We know that in (G, d), A and B are quasiconvex (for the constant C_h). By their definition, and by hyperbolicity, the paths q_i end at bounded distance of a shortest-point projection of a_i to B (for d). Therefore, since (G, d) is hyperbolic, and since the q_i are far from one another for d, it follows that the q_i are actually short for the metric d (shorter than $(200\delta C_h)$). By the argument used at the initial step of the descending induction, we also have an uniform upper bound on the maximal angle of these paths, and therefore on the number of elements of G that label one of the paths q_i.

Since there are D/C_i pairs of elements (a_i, b_i), if D is large enough, by the pigeon hole argument, there is an element g_0 (of length at most $(200\delta C_h)$ in the metric d), and a pair (a_{i_0}, b_{i_0}), such that $a_i^{-1} b_i = g_0$ and such that for $1000\delta_{i+1}C_i$ other such pairs (a_j, b_j), the difference $a_j^{-1} b_j$ is also equal to g_0. The intersection of two essentially distinct conjugates of elements of $\mathscr{H}_i$, corresponding to the cosets A and B, thus contains $a_{i_0}^{-1} a_j$ for all those indices j. There are indices j for which $a_{i_0}^{-1} a_j$ labels a quasi-geodesic paths in A of length at least $1000\delta_{i+1}C_i$. Such an element is either loxodromic, or elliptic with fixed point at the midpoint $[a_{i_0}, a_j]$. But if all of them are elliptic, for two indices j_1, j_2, we get two different fixed points, hence the product of the elements $a_{i_0}^{-1} a_{j_1} a_{i_0}^{-1} a_{j_2}$ is loxodromic.

This element is in the intersection of conjugates of elements of $\mathscr{H}_i$, thus is in a subgroup among the conjugates of some $\mathscr{H}_j$ for $j \geqslant i+1$, but therefore must have diameter 2 in the metric d_{i+1}, and cannot contain loxodromic elements. This is thus a contradiction.

Cases 3 and 4: $G = \mathrm{Mod}(\Sigma)$ or $\mathrm{Out}(F_n)$, H convex cocompact. — Consider the Teichmüller metric on Teichmüller space $(\mathrm{Teich}(\Sigma), d_T)$ and the (symmetrization of the) Lipschitz metric on Outer space (cv_n, d_S) respectively for $\mathrm{Mod}(\Sigma)$ and $\mathrm{Out}(F_n)$. Though $\mathrm{Teich}(\Sigma)$ and cv_n are non-hyperbolic, they are proper metric spaces.

For the mapping class group $\mathrm{Mod}(\Sigma)$, the curve complex $(\mathrm{CC}\,\Sigma), d)$ is hyperbolic and quasi-isometric to $(\mathrm{Mod}(\Sigma), d)$, where d is the word-metric on $\mathrm{Mod}(\Sigma)$ obtained by taking as generating set a finite generating set of $\mathrm{Mod}(\Sigma)$ along with *all* elements of certain sub-mapping class groups (see [MM99]).

Similarly for $\mathrm{Out}(F_n)$, the free factor complex $(\mathscr{F}_n, d)$ is hyperbolic, and is quasi-isometric to $(\mathrm{Out}(F_n), d)$ for a certain word metric over an infinite generating set ([BF14]). This establishes that the hypotheses in the statements of Cases 3 and 4 are not vacuous.

Recall that if a subgroup H of $\mathrm{Mod}(\Sigma)$ or $\mathrm{Out}(F_n)$ is C-convex co-compact, then by Theorem 4.9 (and 4.13) the orbit of a base point in $\mathrm{CC}\,\Sigma)$ (or $\mathscr{F}_n$) is a quasi-isometric image of the orbit of a base point in Teichmüller space.

Finiteness of height of convex cocompact subgroups follows from Theorems 4.11 and 4.14 for $G = \mathrm{Mod}(\Sigma)$ and $\mathrm{Out}(F_n)$ respectively. The first condition of Definition 3.9 now follows from Theorems 3.13.

We now proceed with proving the second condition of Definition 3.9. We first remark that, given C, there exists Δ, C' such that if A, B are cosets of C-convex co-compact subgroups, and if $a_1, a_2 \in A$, $b_1, b_2 \in B$ are such that, in $\mathrm{CC}\,\Sigma)$, $d(a_1, b_1)$ and $d(a_2, b_2)$ are at most $10C\delta$ and that $d(a_1, a_2)$ and $d(b_1, b_2)$ are larger than Δ then, $d_T(a_i, b_i) \leqslant C'$ for both $i = 1, 2$. Indeed, by definition of convex cocompactness, the segment $[a_1, a_2]$ in Teichmüller space maps on a parametrized quasi-geodesic in the curve complex. A result of Dowdall Duchin and Masur ensures that Teichmüller geodesics that make progress in the curve complex, are contracting in Teichmüller space [DDM14, Th. A] (see the formulation done and proved in [DH15, Prop. 3.6]). Thus the segment $[a_1, a_2]$ is contracting in Teichmüller space: any Teichmüller geodesic whose projection in the curve complex fellow-travels that of $[a_1, a_2]$ has to be uniformly close to $[a_1, a_2]$. Applying that to the segment $[b_1, b_2]$, it follows that it must remain at bounded distance (for Teichmüller distance) from $[a_1, a_2]$, as demanded.

A similar statement is valid for $\mathrm{Out}(F_n)$ with the objects that we introduced, it suffice to use [DH15, Prop. 4.17], an arrangement of Dowdall-Taylor's result [DT14], in place of the Dowdall-Duchin-Masur criterion.

With this estimate, one can easily adapt the proof of the first case to get the result. □

5. From quasiconvexity to graded relative hyperbolicity

Recall that we defined graded geometric relative hyperbolicity in Definition 4.3.

5.1. Ensuring geometric graded relative hyperbolicity

Proposition 5.1. — *Let G be a group, d a word metric on G with respect to some (not necessarily finite) generating set, such that (G, d) is hyperbolic. Let H be a subgroup of G. If $\{H\}$ has finite geometric height for d and has the uniform qi-intersection property, then $(G, \{H\}, d)$ has graded geometric relative hyperbolicity.*

Proof. — As in Definition 3.9, $\mathscr{H}_n$ denotes the collection of intersections of n essentially distinct conjugates of H. Let $(\mathscr{H}_n)_0$ denote a set of conjugacy representatives of $(\mathscr{H}_n)$ that are C_1-quasiconvex, and let $\mathscr{C}\mathscr{H}_n$ denote the collection of cosets of elements of $(\mathscr{H}_n)_0$. Let d_n be the metric on $X = (G, d)$ after electrifying the elements of $\mathscr{C}\mathscr{H}_n$.

By Definition 3.9 and Remark 3.11, for all n, all elements of $\mathscr{C}\mathscr{H}_n$ and of $\mathscr{C}\mathscr{H}_{n+1}$ are C_1-quasiconvex in (G,d). Therefore, by Proposition 2.11, all elements of $\mathscr{C}\mathscr{H}_n$ are C_1'-quasiconvex in (G,d_{n+1}) for some C_1' depending on the hyperbolicity of d, and on C_1.

By Definition 3.9, $\mathscr{C}\mathscr{H}_n$ is mutually cobounded in the metric d_{n+1}. Proposition 2.10 now shows that the horoballification of (G,d_{n+1}) over $\mathscr{C}\mathscr{H}_n$ is hyperbolic, for all n. Proposition 2.23 then guarantees that $\mathscr{C}\mathscr{H}_n$ is coarsely hyperbolically embedded in (G,d_{n+1}). Since H is assumed to have finite geometric height, $(G,\{H\},d)$ has graded geometric relative hyperbolicity. □

5.2. Graded relative hyperbolicity for quasiconvex subgroups

Proposition 5.2. — *Let H be a quasiconvex subgroup of a hyperbolic group G, with a word metric d (with respect to a finite generating set). Then the pair $(G,\{H\})$ has graded geometric relative hyperbolicity, and graded relative hyperbolicity.*

Proof. — For the word metric d with respect to a finite generating set, graded geometric relative hyperbolicity agrees with the notion of graded relative hyperbolicity (Definition 1.3).

By Theorem 3.2, H has finite height. By Proposition 4.15 it satisfies the uniform qi-intersection property 3.9. Therefore, by Proposition 5.1, the pair $(G,\{H\})$ has graded relative hyperbolicity.

Finally, note that since the word metric we use is locally finite, and all i-fold intersections are quasiconvex, graded relative hyperbolicity follows. □

Proposition 5.3. — *Let $(G,\mathscr{P})$ be a finitely generated relatively hyperbolic group. Let H be a relatively quasiconvex subgroup. Let S be a finite relative generating set of G (relative to $\mathscr{P}$) and let d be the word metric with respect to $S\cup\mathscr{P}$. Then $(G,\{H\},d)$ has graded relative hyperbolicity as well as graded geometric relative hyperbolicity.*

Proof. — The proof is similar to that of Proposition 5.2. By Theorem 4.6, H has finite relative height, hence it has finite geometric height for the relative metric (see Example 4.7).

Next, by Proposition 4.15, H satisfies the uniform qi-intersection property for a relative metric, and graded geometric relative hyperbolicity follows from Proposition 5.1.

Again, since G has a word metric with respect to a finite relative generating set, and H and all i-fold intersections are relatively quasiconvex as well, the above argument furnishes graded relative hyperbolicity as well. □

Similarly, replacing the use of Theorem 3.2 by Theorems 4.10 and 4.14, one obtains the following.

Proposition 5.4. — *Let G be the mapping class group $\mathrm{Mod}(S)$ (respectively $\mathrm{Out}(F_n)$). Let d be a word metric on G making it quasi-isometric to the curve complex $\mathrm{CC}(S)$ (respectively the free factor complex $\mathscr{F}_n$). Let H be a convex cocompact subgroup of G. Then $(G,\{H\},d)$ has graded relative hyperbolicity.*

Again, replacing the use of Theorem 3.2 by Theorem 4.11, we obtain:

Proposition 5.5. — *Let G be the mapping class group* $\mathrm{Mod}(S)$*. Let d be a word metric on G making it quasi-isometric to the curve complex* $\mathrm{CC}(S)$*. Let H be a convex cocompact subgroup of G. Then $(G, \{H\}, d)$ has graded geometric relative hyperbolicity.*

Remark 5.6. — Since we do not have an exact (geometric) analog of Theorem 4.11 for $\mathrm{Out}(F_n)$ (more precisely an analog of the stability result of [DT15]) as of now, we have to content ourselves with the slightly weaker Proposition 5.4 for $\mathrm{Out}(F_n)$.

6. From graded relative hyperbolicity to quasiconvexity

6.1. A sufficient condition

Proposition 6.1. — *Let G be a group and d a hyperbolic word metric with respect to a (not necessarily finite) generating set. Let H be a subgroup such that $(G, \{H\}, d)$ has graded geometric relative hyperbolicity. Then H is quasiconvex in (G, d).*

Proof. — Assume $(G, \{H\}, d)$ has graded geometric relative hyperbolicity as in Definition 4.3. Then H has finite geometric height in (G, d). Let k be this height. Thus, $\mathscr{H}_{k+1}$ is a collection of uniformly bounded subsets, and d_{k+1} is quasi-isometric to d. It follows that (G, d_{k+1}) is hyperbolic.

Further, by Definition 4.3, $\mathscr{H}_k$ is hyperbolically embedded in (G, d_{k+1}). This means in particular that the electrification $(G, d_{k+1})^{\mathrm{el}}_{\mathscr{H}_k}$ is hyperbolic. Since (G, d_k) is quasi-isometric to $(G, d_{k+1})^{\mathrm{el}}_{\mathscr{H}_k}$ (being the restriction of the metric on G) it follows that (G, d_k) is hyperbolic as well. Further, by Corollary 2.7 the elements of $\mathscr{H}_k$, are uniformly quasiconvex in (G, d_{k+1}).

We now argue by descending induction on i.

The inductive hypothesis for $(i+1)$. — We assume that d_{i+1} is a hyperbolic metric on G, and that there is a constant c_{i+1} such that, for all $j \geqslant 1$ the elements of $\mathscr{H}_{i+j}$ are uniformly c_{i+1}-quasiconvex in (G, d).

We assume the inductive hypothesis for $i+1$ (i.e., as stated), and we now prove it for i.

Of course, we also assume, as in the statement of the proposition, that $\mathscr{H}_i$ is coarsely hyperbolically embedded in (G, d_{i+1}). Hence d_i is a hyperbolic metric on G.

We will now check that the assumptions of Proposition 2.12 are satisfied for $(X, d) = (G, d_{i+1})$, $\mathscr{Y} = \mathscr{H}_{i+1}$, and $H_{i,\ell}$ arbitrary in $\mathscr{H}_i$.

Elements of $\mathscr{H}_i$ in (G, d_{i+1}) are uniformly quasiconvex in (G, d_{i+1}): this follows from Corollary 2.7. We will write C_i for their quasiconvexity constant.

A second step is to check that, for some uniform Δ_0 and ε, for all $\Delta > \Delta_0$, when an element $H_{i,\ell}$ of $\mathscr{H}_i$ (Δ, ε)-meets an item of $\mathscr{H}_{i+1}$, then $H^{+\varepsilon\Delta}$ contains a quasigeodesic between the meeting points in H. Thus, fix $\varepsilon < 1/100$, and take Δ_0 larger than 20 times the thickening constants for the definition of elements in $\mathscr{H}_i$ (which is possible by finiteness of number of orbits of i-fold intersections). Assume $H_{i,\ell}$ (Δ, ε)-meets $Y \in \mathscr{H}_{i+1}$. Then, by definition of i-fold intersections 3.5, and Proposition 3.6, either

the pair of meeting points is in an item of $\mathscr{H}_{i+1}$ inside $H_{i,\ell}$, or $Y \subset H_{i,\ell}$. In both cases, by the inductive assumption, there is a path in $H_{i,\ell}^{+\varepsilon\Delta}$ between the meeting points in $H_{i,\ell}$ that is a quasigeodesic for d. Hence the second assumption of Proposition 2.12 is satisfied.

We can thus conclude by that proposition that $H_{i,\ell}$ is quasiconvex in (G,d) for a uniform constant, and therefore the inductive assumption holds for i.

By induction it is then true for $i=0$, hence the first statement of the proposition holds, i.e., quasiconvexity follows from graded geometric relative hyperbolicity. □

We shall deduce various consequences of Proposition 6.1 below. However, before we proceed, we need the following observation since we are dealing with spaces/graphs that are not necessarily proper.

Observation 6.2. — Let X be a (not necessarily proper) hyperbolic graph. For all $C_0 \geqslant 0$, there exists $C_1 \geqslant 0$ such that the following holds:

Let H be a hyperbolic group acting uniformly properly on X, i.e., for all D_0 there exists N such that for any $x \in X$, any D_0 ball in X contains at most N orbit points of Hx.

Then a C_0-quasiconvex orbit of H is (C_1, C_1)-quasi-isometrically embedded in X.

Combining Proposition 6.1 with Observation 6.2 we obtain the following:

Proposition 6.3. — *Let G be a group and d a hyperbolic word metric with respect to a (not necessarily finite) generating set. Let H be a subgroup such that*

(1) *$(G, \{H\}, d)$ has graded geometric relative hyperbolicity,*
(2) *The action of H on (G,d) is uniformly proper.*

Then H is hyperbolic and H is qi-embedded in (G,d).

Proof. — Quasi-convexity of H in (G,d) was established in Proposition 6.1. Qi-embeddedness of H follows from Observation 6.2. Hyperbolicity of H is an immediate consequence. □

6.2. The main theorem. — We assemble the pieces now to prove the following main theorem of the paper.

Theorem 6.4. — *Let (G,d) be one of the following:*

(1) *G a hyperbolic group and d the word metric with respect to a finite generating set S.*
(2) *G is finitely generated and hyperbolic relative to $\mathscr{P}$, S a finite relative generating set, and d the word metric with respect to $S \cup \mathscr{P}$.*
(3) *G is the mapping class group $\mathrm{Mod}(S)$ and d the metric obtained by electrifying the subgraphs corresponding to sub mapping class groups so that (G,d) is quasi-isometric to the curve complex $\mathrm{CC}(S)$.*

(4) *G is $\mathrm{Out}(F_n)$ and d the metric obtained by electrifying the subgroups corresponding to subgroups that stabilize proper free factors so that (G,d) is quasi-isometric to the free factor complex $\mathcal{F}_n$.*

Then (respectively)

(1) *H is quasiconvex if and only if $(G,\{H\})$ has graded geometric relative hyperbolicity,*

(2) *H is relatively quasiconvex if and only if $(G,\{H\},d)$ has graded geometric relative hyperbolicity,*

(3) *H is convex cocompact in $\mathrm{Mod}(S)$ if and only if $(G,\{H\},d)$ has graded geometric relative hyperbolicity and the action of H on the curve complex is uniformly proper,*

(4) *H is convex cocompact in $\mathrm{Out}(F_n)$ if and only if $(G,\{H\},d)$ has graded geometric relative hyperbolicity and the action of H on the free factor complex is uniformly proper.*

Proof. — The forward implications of quasiconvexity to graded geometric relative hyperbolicity in the first 3 cases are proved by Propositions 5.2, 5.3, 5.4 and 5.5 and case 4 by Proposition 5.4. In cases (3) and (4) properness of the action of H on the curve complex follows from convex cocompactness.

We now proceed with the reverse implications. Again, the reverse implications of (1) and (2) are direct consequences of Proposition 6.1.

The proofs of the reverse implications of (3) and (4) are similar. Proposition 6.3 proves that any orbit of H on either the curve complex $\mathrm{CC}(S)$ or the free factor complex $\mathcal{F}_n$ is qi-embedded. Convex cocompactness now follows from Theorems 4.9 and 4.13. □

6.3. Examples. — We give a couple of examples below to show that finiteness of geometric height does not necessarily follow from quasiconvexity.

Example 6.5. — Let $G_1 = \pi_1(S)$ and $H = \langle h \rangle$ be a cyclic subgroup corresponding to a simple closed curve. Let $G_2 = H_1 \oplus H_2$, where each H_i is isomorphic to $\mathbb{Z}$. Let $G = G_1 *_{H=H_1} G_2$. Let d be the metric obtained on G with respect to some finite generating set along with all elements of H_2. Then G_1 is quasiconvex in (G,d), but G_1 does not have finite geometric height.

Note however, that the action of G_1 on (G,d) is not acylindrical. We now furnish another example to show that graded geometric relative hyperbolicity does not necessarily follow from quasiconvexity even if we assume acylindricity.

Example 6.6. — Let $G = \langle a_i, b_i : i \in \mathbb{N}, a_{2i}^{b_i} = a_{2i-1} \rangle$ and let F be the (free) subgroup generated by $\{a_i\}$. Then $F^{b_i} \cap F = \langle a_{2i-1} \rangle$ for all i. Let d be the word metric on G with respect to the generators a_i, b_i. Then the action of F on (G,d) is acylindrical and F is quasiconvex. However there are infinitely many double coset representatives corresponding to b_i such that $F^{b_i} \cap F$ is infinite.

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