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SOME SURFACES WITH MAXIMAL PICARD NUMBER

BY ARNAUD BEAUVILLE

ABSTRACT. — For a smooth complex projective variety, the rank ρ of the Néron-Severi group is bounded by the Hodge number $h^{1,1}$. Varieties with $\rho = h^{1,1}$ have interesting properties, but are rather sparse, particularly in dimension 2. We discuss in this note a number of examples, in particular those constructed from curves with special Jacobians.

RÉSUMÉ (Quelques surfaces dont le nombre de Picard est maximal). — Le rang ρ du groupe de Néron-Severi d'une variété projective lisse complexe est borné par le nombre de Hodge $h^{1,1}$. Les variétés satisfaisant à $\rho = h^{1,1}$ ont des propriétés intéressantes, mais sont assez rares, particulièrement en dimension 2. Dans cette note nous analysons un certain nombre d'exemples, notamment ceux construits à partir de courbes à jacobienne spéciale.

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1. INTRODUCTION

The *Picard number* of a smooth projective variety X is the rank ρ of the Néron-Severi group – that is, the group of classes of divisors in $H^2(X, \mathbb{Z})$. It is bounded by the Hodge number $h^{1,1} := \dim H^1(X, \Omega_X^1)$. We are interested here in varieties with maximal Picard number $\rho = h^{1,1}$. As we will see in §2, there are many examples of such varieties in dimension ≥ 3 , so we will focus on the case of surfaces.

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Apart from the well understood case of K3 and abelian surfaces, the quantity of known examples is remarkably small. In [Per82] Persson showed that some families of double coverings of rational surfaces contain surfaces with maximal Picard number (see Section 6.4 below); some scattered examples have appeared since then. We will review them in this note and examine in particular when the product of two curves has maximal Picard number – this provides some examples, unfortunately also quite sparse.

2. GENERALITIES

Let X be a smooth projective variety over \mathbb{C} . The Néron-Severi group $\text{NS}(X)$ is the subgroup of algebraic classes in $H^2(X, \mathbb{Z})$; its rank ρ is the *Picard number* of X . The natural map $\text{NS}(X) \otimes \mathbb{C} \rightarrow H^2(X, \mathbb{C})$ is injective and its image is contained in $H^{1,1}$, hence $\rho \leq h^{1,1}$.

PROPOSITION 1. — *The following conditions are equivalent:*

- (i) $\rho = h^{1,1}$;
- (ii) *The map $\text{NS}(X) \otimes \mathbb{C} \rightarrow H^{1,1}$ is bijective;*
- (iii) *The subspace $H^{1,1}$ of $H^2(X, \mathbb{C})$ is defined over \mathbb{Q} .*
- (iv) *The subspace $H^{2,0} \oplus H^{0,2}$ of $H^2(X, \mathbb{C})$ is defined over \mathbb{Q} .*

Proof. — The equivalence of (iii) and (iv) follows from the fact that $H^{2,0} \oplus H^{0,2}$ is the orthogonal of $H^{1,1}$ for the scalar product on $H^2(X, \mathbb{C})$ associated to an ample class. The rest is clear. \square

When X satisfies these equivalent properties we will say for short that X is ρ -*maximal* (one finds the terms singular, exceptional or extremal in the literature).

REMARKS

(1) A variety with $H^{2,0} = 0$ is ρ -maximal. We will implicitly exclude this trivial case in the discussion below.

(2) Let X, Y be two ρ -maximal varieties, with $H^1(Y, \mathbb{C}) = 0$. Then $X \times Y$ is ρ -maximal. For instance $X \times \mathbb{P}^n$ is ρ -maximal, and $Y \times C$ is ρ -maximal for any curve C .

(3) Let Y be a submanifold of X ; if X is ρ -maximal and the restriction map $H^2(X, \mathbb{C}) \rightarrow H^2(Y, \mathbb{C})$ is bijective, Y is ρ -maximal. By the Lefschetz theorem, the latter condition is realized if Y is a complete intersection of smooth ample divisors in X , of dimension ≥ 3 . Together with Remark 2, this gives many examples of ρ -maximal varieties of dimension ≥ 3 ; thus we will focus on finding ρ -maximal *surfaces*.

PROPOSITION 2. — *Let $\pi : X \dashrightarrow Y$ be a rational map of smooth projective varieties.*

- (a) *If $\pi^* : H^{2,0}(Y) \rightarrow H^{2,0}(X)$ is injective (in particular if π is dominant), and X is ρ -maximal, so is Y .*
- (b) *If $\pi^* : H^{2,0}(Y) \rightarrow H^{2,0}(X)$ is surjective and Y is ρ -maximal, so is X .*

Note that since π is defined on an open subset $U \subset X$ with $\text{codim}(X \setminus U) \geq 2$, the pull back map $\pi^* : H^2(Y, \mathbb{C}) \rightarrow H^2(U, \mathbb{C}) \cong H^2(X, \mathbb{C})$ is well defined.

Proof. — Hironaka’s theorem provides a diagram

$$\begin{array}{ccc} & \widehat{X} & \\ b \swarrow & & \searrow \widehat{\pi} \\ X & \overset{\pi}{\dashrightarrow} & Y \end{array}$$

where $\widehat{\pi}$ is a morphism, and b is a composition of blowing-ups with smooth centers. Then $b^* : H^{2,0}(X) \rightarrow H^{2,0}(\widehat{X})$ is bijective, and \widehat{X} is ρ -maximal if and only if X is ρ -maximal; so replacing π by $\widehat{\pi}$ we may assume that π is a morphism.

(a) Let $V := (\pi^*)^{-1}(\text{NS}(X) \otimes \mathbb{Q})$. We have

$$V \otimes_{\mathbb{Q}} \mathbb{C} = (\pi^*)^{-1}(\text{NS}(X) \otimes \mathbb{C}) = (\pi^*)^{-1}(H^{1,1}(X)) = H^{1,1}(Y)$$

(the last equality holds because π^* is injective on $H^{2,0}(Y)$ and $H^{0,2}(Y)$), hence Y is ρ -maximal.

(b) Let W be the \mathbb{Q} -vector subspace of $H^2(Y, \mathbb{Q})$ such that

$$W \otimes_{\mathbb{Q}} \mathbb{C} = H^{2,0}(Y) \oplus H^{0,2}(Y).$$

Then π^*W is a \mathbb{Q} -vector subspace of $H^2(X, \mathbb{Q})$, and

$$(\pi^*W) \otimes \mathbb{C} = \pi^*(W \otimes \mathbb{C}) = \pi^*(H^{2,0}(Y) \oplus H^{0,2}(Y)) = H^{2,0}(X) \oplus H^{0,2}(X),$$

so X is ρ -maximal. □

3. ABELIAN VARIETIES

There is a nice characterization of ρ -maximal abelian varieties ([Kat75], [Lan75]):

PROPOSITION 3. — *Let A be an abelian variety of dimension g . We have*

$$\text{rk}_{\mathbb{Z}} \text{End}(A) \leq 2g^2.$$

The following conditions are equivalent:

- (i) A is ρ -maximal;
- (ii) $\text{rk}_{\mathbb{Z}} \text{End}(A) = 2g^2$;
- (iii) A is isogenous to E^g , where E is an elliptic curve with complex multiplication.
- (iv) A is isomorphic to a product of mutually isogenous elliptic curves with complex multiplication.

(The equivalence of (i), (ii) and (iii) follows easily from Lemma 1 below; the only delicate point is (iii) \Rightarrow (iv), which we will not use.)

Coming back to the surface case, suppose that our abelian variety A contains a surface S such that the restriction map $H^{2,0}(A) \rightarrow H^{2,0}(S)$ is surjective. Then S is ρ -maximal if A is ρ -maximal (Proposition 2(b)). Unfortunately this situation seems to be rather rare. We will discuss below (Proposition 6) the case of $\text{Sym}^2 C$ for a curve C . Another interesting example is the *Fano surface* F_X parametrizing the lines

contained in a smooth cubic threefold X , embedded in the intermediate Jacobian JX [CG72]. There are some cases in which JX is known to be ρ -maximal:

PROPOSITION 4

(a) For $\lambda \in \mathbb{C}$, $\lambda^3 \neq 1$, let X_λ (resp. E_λ) be the cubic in \mathbb{P}^4 (resp. \mathbb{P}^2) defined by $X_\lambda: X^3 + Y^3 + Z^3 - 3\lambda XYZ + T^3 + U^3 = 0$, $E_\lambda: X^3 + Y^3 + Z^3 - 3\lambda XYZ = 0$.

If E_λ is isogenous to E_0 , JX_λ and F_{X_λ} are ρ -maximal. The set of $\lambda \in \mathbb{C}$ for which this happens is countably infinite.

(b) Let $X \subset \mathbb{P}^4$ be the Klein cubic threefold $\sum_{i \in \mathbb{Z}/5} X_i^2 X_{i+1} = 0$. Then JX and F_X are ρ -maximal.

Proof. — Part (a) is due to Rouleau [Rou11], who proves that JX_λ (for any λ) is isogenous to $E_0^3 \times E_\lambda^2$. Since the family $(E_\lambda)_{\lambda \in \mathbb{C}}$ is not constant, there is a countably infinite set of $\lambda \in \mathbb{C}$ for which E_λ is isogenous to E_0 , hence JX_λ and therefore F_{X_λ} are ρ -maximal.

Part (b) follows from a result of Adler [Adl81], who proves that JX is isogenous (actually isomorphic) to E^5 , where E is the elliptic curve whose endomorphism ring is the ring of integers of $\mathbb{Q}(\sqrt{-11})$ (see also [Rou09] for a precise description of the group $\text{NS}(X)$). \square

4. PRODUCTS OF CURVES

PROPOSITION 5. — Let C, C' be two smooth projective curves, of genus g and g' respectively. The following conditions are equivalent:

- (i) The surface $C \times C'$ is ρ -maximal;
- (ii) There exists an elliptic curve E with complex multiplication such that JC is isogenous to E^g and JC' to $E^{g'}$.

Proof. — Let p, p' be the projections from $C \times C'$ to C and C' . We have

$$H^{1,1}(C \times C') = p^* H^2(C, \mathbb{C}) \oplus p'^* H^2(C', \mathbb{C}) \oplus (p^* H^{1,0}(C) \otimes p'^* H^{0,1}(C')) \oplus (p^* H^{0,1}(C) \otimes p'^* H^{1,0}(C')),$$

hence $h^{1,1}(C \times C') = 2gg' + 2$. On the other hand we have

$$\text{NS}(C \times C') = p^* \text{NS}(C) \oplus p'^* \text{NS}(C') \oplus \text{Hom}(JC, JC')$$

([LB92], Th. 11.5.1), hence $C \times C'$ is ρ -maximal if and only if $\text{rk Hom}(JC, JC') = 2gg'$. Thus the Proposition follows from the following (well-known) lemma:

LEMMA 1. — Let A and B be two abelian varieties, of dimension a and b respectively. The \mathbb{Z} -module $\text{Hom}(A, B)$ has rank $\leq 2ab$; equality holds if and only if there exists an elliptic curve E with complex multiplication such that A is isogenous to E^a and B to E^b .

Proof. — There exist simple abelian varieties A_1, \dots, A_s , with distinct isogeny classes, and nonnegative integers $p_1, \dots, p_s, q_1, \dots, q_s$ such that A is isogenous to $A_1^{p_1} \times \dots \times A_s^{p_s}$ and B to $A_1^{q_1} \times \dots \times A_s^{q_s}$. Then

$$\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M_{p_1, q_1}(K_1) \times \dots \times M_{p_s, q_s}(K_s),$$

where K_i is the (possibly skew) field $\mathrm{End}(A_i) \otimes_{\mathbb{Z}} \mathbb{Q}$. Put $a_i := \dim A_i$. Since K_i acts on $H^1(A_i, \mathbb{Q})$ we have $\dim_{\mathbb{Q}} K_i \leq b_1(A_i) = 2a_i$, hence

$$\mathrm{rk} \mathrm{Hom}(A, B) \leq \sum_i 2p_i q_i a_i \leq 2 \left(\sum_i p_i a_i \right) \left(\sum_i q_i a_i \right) = 2ab.$$

The last inequality is strict unless $s = a_1 = 1$, in which case the first one is strict unless $\dim_{\mathbb{Q}} K_1 = 2$. The lemma, and therefore the Proposition, follow. \square

The most interesting case occurs when $C = C'$. Then:

PROPOSITION 6. — *Let C be a smooth projective curve. The following conditions are equivalent:*

- (i) *The Jacobian JC is ρ -maximal;*
- (ii) *The surface $C \times C$ is ρ -maximal;*
- (iii) *The symmetric square $\mathrm{Sym}^2 C$ is ρ -maximal.*

Proof. — The equivalence of (i) and (ii) follows from Proposition 5. The Abel-Jacobi map $\mathrm{Sym}^2 C \rightarrow JC$ induces an isomorphism

$$H^{2,0}(JC) \cong \wedge^2 H^0(C, K_C) \xrightarrow{\sim} H^{2,0}(\mathrm{Sym}^2 C),$$

thus (i) and (iii) are equivalent by Proposition 2. \square

When the equivalent conditions of Proposition 6 hold, we will say that C has *maximal correspondences* (the group $\mathrm{End}(JC)$ is often called the group of divisorial correspondences of C).

By Proposition 3 the Jacobian JC is then isomorphic to a product of isogenous elliptic curves with complex multiplication. Though we know very few examples of such curves, we will give below some examples with $g = 4$ or 10.

For $g = 2$ or 3, there is a countably infinite set of curves with maximal correspondences ([HN65], [Hof91]). The point is that any indecomposable principally polarized abelian variety of dimension 2 or 3 is a Jacobian; thus it suffices to construct an indecomposable principal polarization on E^g , where E is an elliptic curve with complex multiplication, and this is easily translated into a problem about hermitian forms of rank g on certain rings of quadratic integers.

This approach works only for $g = 2$ or 3; moreover it does not give an explicit description of the curves. Another method is by using automorphism groups, with the help of the following easy lemma:

LEMMA 2. — Let G be a finite group of automorphisms of C , and let $H^0(C, K_C) = \bigoplus_{i \in I} V_i$ be a decomposition of the G -module $H^0(C, K_C)$ into irreducible representations. Assume that there exists an elliptic curve E and for each $i \in I$, a nontrivial map $\pi_i : C \rightarrow E$ such that $\pi_i^* H^0(E, K_E) \subset V_i$. Then JC is isogenous to E^g .

In particular if $H^0(C, K_C)$ is an irreducible G -module and C admits a map onto an elliptic curve E , then JC is isogenous to E^g .

Proof. — Let η be a generator of $H^0(E, K_E)$. Let $i \in I$; the forms $g^* \pi_i^* \eta$ for $g \in G$ generate V_i , hence there exists a subset A_i of G such that the forms $g^* \pi_i^* \eta$ for $g \in A_i$ form a basis of V_i .

Put $\Pi_i = (g \circ \pi_i)_{g \in A_i} : C \rightarrow E^{A_i}$, and $\Pi = (\Pi_i)_{i \in I} : C \rightarrow E^g$. By construction $\Pi^* : H^0(E^g, \Omega_{E^g}^1) \rightarrow H^0(C, K_C)$ is an isomorphism. Therefore the map $JC \rightarrow E^g$ deduced from Π is an isogeny. \square

In the examples which follow, and in the rest of the paper, we put $\omega := e^{2\pi i/3}$.

EXAMPLE 1. — We consider the family (C_t) of genus 2 curves given by $y^2 = x^6 + tx^3 + 1$, for $t \in \mathbb{C} \setminus \{\pm 2\}$. It admits the automorphisms

$$\tau : (x, y) \mapsto \left(\frac{1}{x}, \frac{y}{x^3} \right) \quad \text{and} \quad \psi : (x, y) \mapsto (\omega x, y).$$

The forms dx/y and $x dx/y$ are eigenvectors for ψ and are exchanged (up to sign) by τ ; it follows that the action of the group generated by ψ and τ on $H^0(C_t, K_{C_t})$ is irreducible.

Let E_t be the elliptic curve defined by $v^2 = (u+2)(u^3 - 3u + t)$; the curve C_t maps onto E_t by

$$(x, y) \mapsto \left(x + \frac{1}{x}, \frac{y(x+1)}{x^2} \right).$$

By Lemma 2 JC_t is isogenous to E_t^2 . Since the j -invariant of E_t is a non-constant function of t , there is a countably infinite set of $t \in \mathbb{C}$ for which E_t has complex multiplication, hence C_t has maximal correspondences.

EXAMPLE 2. — Let C be the genus 2 curve $y^2 = x(x^4 - 1)$; its automorphism group is a central extension of \mathfrak{S}_4 by the hyperelliptic involution σ ([LB92], 11.7); its action on $H^0(C, K_C)$ is irreducible.

Let E be the elliptic curve $E: v^2 = u(u+1)(u-2\alpha)$, with $\alpha = 1 - \sqrt{2}$. The curve C maps to E by

$$(x, y) \mapsto \left(\frac{x^2 + 1}{x - 1}, \frac{y(x - \alpha)}{(x - 1)^2} \right).$$

The j -invariant of E is 8000, so E is the elliptic curve $\mathbb{C}/\mathbb{Z}[\sqrt{-2}]$ ([Sil94], Prop. 2.3.1).

EXAMPLE 3 (The \mathfrak{S}_4 -invariant quartic curves). — Consider the standard representation of \mathfrak{S}_4 on \mathbb{C}^3 . It is convenient to view \mathfrak{S}_4 as the semi-direct product $(\mathbb{Z}/2)^2 \rtimes \mathfrak{S}_3$,

with \mathfrak{S}_3 (resp. $(\mathbb{Z}/2)^2$) acting on \mathbb{C}^3 by permutation (resp. change of sign) of the basis vectors. The quartic forms invariant under this representation form the pencil

$$(C_t)_{t \in \mathbb{P}^1} : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0.$$

According to [DK93], this pencil was known to Ciani. It contains the Fermat quartic ($t = 0$) and the Klein quartic ($t = \frac{3}{2}(1 \pm i\sqrt{7})$).

Let us take $t \notin \{2, -1, -2, \infty\}$; then C_t is smooth. The action of \mathfrak{S}_4 on $H^0(C_t, K)$, given by the standard representation, is irreducible. Moreover the involution $x \mapsto -x$ has 4 fixed points, hence the quotient curve E_t has genus 1. It is given by the degree 4 equation

$$u^2 + tu(y^2 + z^2) + y^4 + z^4 + ty^2z^2 = 0$$

in the weighted projective space $\mathbb{P}(2, 1, 1)$. Thus E_t is a double covering of \mathbb{P}^1 branched along the zeroes of the polynomial $(t + 2)(y^4 + z^4) + 2ty^2z^2$. The cross-ratio of these zeroes is $-(t + 1)$, so E_t is the elliptic curve $y^2 = x(x - 1)(x + t + 1)$. By Lemma 2 JC_t is isogenous to E_t^3 . For a countably infinite set of t the curve E_t has complex multiplication, thus C_t has maximal correspondences. For $t = 0$ we recover the well known fact that the Jacobian of the Fermat quartic curve is isogenous to $(\mathbb{C}/\mathbb{Z}[i])^3$.

EXAMPLE 4. — Consider the genus 3 hyperelliptic curve $H: y^2 = x(x^6 + 1)$. The space $H^0(H, K_H)$ is spanned by $dx/y, xdx/y, x^2dx/y$. This is a basis of eigenvectors for the automorphism $\tau : (x, y) \mapsto (\omega x, \omega^2 y)$. On the other hand the involution $\sigma : (x, y) \mapsto (1/x, -y/x^4)$ exchanges dx/y and x^2dx/y , hence the summands of the decomposition

$$H^0(H, K_H) = \left\langle \frac{dx}{y}, x^2 \frac{dx}{y} \right\rangle \oplus \left\langle x \frac{dx}{y} \right\rangle$$

are irreducible under the group \mathfrak{S}_3 generated by σ and τ .

Let E_i be the elliptic curve $v^2 = u^3 + u$, with endomorphism ring $\mathbb{Z}[i]$. Consider the maps f and g from H to E_i given by

$$f(x, y) = (x^2, xy) \quad g(x, y) = \left(\lambda^2 \left(x + \frac{1}{x} \right), \frac{\lambda^3 y}{x^2} \right) \quad \text{with } \lambda^{-4} = -3.$$

We have

$$f^* \frac{du}{v} = \frac{2x dx}{y} \quad \text{and} \quad g^* \frac{du}{v} = \lambda^{-1} (x^2 - 1) \frac{dx}{y}.$$

Thus we can apply Lemma 2, and we find that JH is isogenous to E_i^3 .

Thus JH is isogenous to the Jacobian of the Fermat quartic F_4 (Example 3). In particular we see that the surface $H \times F_4$ is ρ -maximal.

We now arrive to our main example in higher genus. Recall that we put $\omega = e^{2\pi i/3}$.

PROPOSITION 7. — *The Fermat sextic curve $C_6: X^6 + Y^6 + Z^6 = 0$ has maximal correspondences. Its Jacobian JC_6 is isogenous to E_ω^{10} , where E_ω is the elliptic curve $\mathbb{C}/\mathbb{Z}[\omega]$.*

The first part can be deduced from the general recipe given by Shioda to compute the Picard number of $C_d \times C_d$ for any d [Shi81]. Let us give an elementary proof. Let $G := T \rtimes \mathfrak{S}_3$, where \mathfrak{S}_3 acts on \mathbb{C}^3 by permutation of the coordinates and T is the group of diagonal matrices t with $t^6 = 1$.

$$\text{Let } \Omega = \frac{XdY - YdX}{Z^5} = \frac{YdZ - ZdY}{X^5} = \frac{ZdX - XdZ}{Y^5} \in H^0(C, K_C(-3)).$$

A basis of eigenvectors for the action of T on $H^0(C_6, K)$ is given by the forms $X^a Y^b Z^c \Omega$, with $a + b + c = 3$; using the action of \mathfrak{S}_3 we get a decomposition into irreducible components:

$$H^0(C_6, K) = V_{3,0,0} \oplus V_{2,1,0} \oplus V_{1,1,1},$$

where $V_{\alpha,\beta,\gamma}$ is spanned by the forms $X^a Y^b Z^c \Omega$ with $\{a, b, c\} = \{\alpha, \beta, \gamma\}$.

Let us use affine coordinates $x = X/Z$, $y = Y/Z$ on C_6 . We consider the following maps from C_6 onto E_ω : $v^2 = u^3 - 1$:

$$f(x, y) = (-x^2, y^3), \quad g(x, y) = \left(2^{-2/3}x^{-2}y^4, \frac{1}{2}(x^3 - x^{-3})\right);$$

and, using for E_ω the equation $\xi^3 + \eta^3 + 1 = 0$, $h(x, y) = (x^2, y^2)$.

We have

$$\begin{aligned} f^* \frac{du}{v} &= -\frac{2xdx}{y^3} = -2XY^2 \Omega \in V_{2,1,0}, \\ g^* \frac{du}{v} &= -2^{4/3}Y^3 \Omega \in V_{3,0,0}, \\ h^* \frac{d\xi}{\eta^2} &= 2XYZ \Omega \in V_{1,1,1}, \end{aligned}$$

so the Proposition follows from Lemma 2. \square

By Proposition 2 every quotient of C_6 has again maximal correspondences. There are four such quotient which have genus 4:

- The quotient by an involution $\alpha \in T$, which we may take to be $\alpha : (X, Y, Z) \mapsto (X, Y, -Z)$. The canonical model of C_6/α is the image of C_6 by the map

$$(X, Y, Z) \mapsto (X^2, XY, Y^2, Z^2);$$

its equations in \mathbb{P}^3 are $xz - y^2 = x^3 + z^3 + t^3 = 0$. Projecting onto the conic $xz - y^2 = 0$ realizes C_6/α as the cyclic triple covering $v^3 = u^6 + 1$ of \mathbb{P}^1 .

- The quotient by an involution $\beta \in \mathfrak{S}_3$, say $\beta : (X, Y, Z) \mapsto (Y, X, Z)$. The canonical model of C_6/β is the image of C_6 by the map

$$(X, Y, Z) \mapsto ((X + Y)^2, Z(X + Y), Z^2, XY);$$

its equations are $xz - y^2 = x(x - 3t)^2 + z^3 - 2t^3 = 0$.

Since the quadric containing their canonical model is singular, the two genus 4 curves C_6/α and C_6/β have a unique g_3^1 . The associated triple covering $C_6/\alpha \rightarrow \mathbb{P}^1$ is cyclic, while the corresponding covering $C_6/\beta \rightarrow \mathbb{P}^1$ is not. Therefore the two curves are not isomorphic.

- The quotient by an element of order 3 of T acting freely, say $\gamma : (X, Y, Z) \mapsto (X, \omega Y, \omega^2 Z)$. The canonical model of C_6/γ is the image of C_6 by the map

$$(X, Y, Z) \mapsto (X^3, Y^3, Z^3, XYZ);$$

its equations are $x^2 + y^2 + z^2 = t^3 - xyz = 0$. Projecting onto the conic $x^2 + y^2 + z^2 = 0$ realizes C_6/γ as the cyclic triple covering $v^3 = u(u^4 - 1)$ of \mathbb{P}^1 ; thus C_6/γ is not isomorphic to C_6/α or C_6/β .

- The quotient by an element of order 3 of \mathfrak{S}_3 acting freely, say $\delta : (X, Y, Z) \mapsto (Y, Z, X)$. The canonical model of C_6/δ is the image of C_6 by the map

$$(X, Y, Z) \mapsto (X^3 + Y^3 + Z^3, XYZ, X^2Y + Y^2Z + Z^2X, XY^2 + YZ^2 + ZX^2).$$

It is contained in the smooth quadric $(x+y)^2 + 5y^2 - 2zt = 0$, so C_6/δ is not isomorphic to any of the 3 previous curves.

Thus we have found four non-isomorphic curves of genus 4 with Jacobian isogenous to E_ω^4 . The product of any two of these curves is a ρ -maximal surface.

COROLLARY 1. — *The Fermat sextic surface $S_6 : X^6 + Y^6 + Z^6 + T^6 = 0$ is ρ -maximal.*

Proof. — This follows from Propositions 7, 2 and Shioda’s trick: there exists a rational dominant map $\pi : C_6 \times C_6 \dashrightarrow S_6$, given by

$$\pi((X, Y, Z), (X', Y', Z')) = (XZ', YZ', iX'Z, iY'Z). \quad \square$$

REMARK 4. — Since the Fermat plane quartic has maximal correspondences (Example 2), the same argument gives the classical fact that the Fermat quartic surface is ρ -maximal. It follows from the explicit formula for $\rho(S_d)$ given in [Aok83] that S_d is ρ -maximal (for $d \geq 4$) only for $d = 4$ and 6.

Again every quotient of the Fermat sextic is ρ -maximal. For instance, the quotient of S_6 by the automorphism $(X, Y, Z, T) \mapsto (X, Y, Z, \omega T)$ is the double covering of \mathbb{P}^2 branched along C_6 : it is a ρ -maximal K3 surface. The quotient of S_6 by the involution $(X, Y, Z, T) \mapsto (X, Y, -Z, -T)$ is given in \mathbb{P}^5 by the equations

$$y^2 - xz = v^2 - uw = x^3 + z^3 + u^3 + w^3 = 0;$$

it is a complete intersection of degrees $(2, 2, 3)$, with 12 ordinary nodes. Other quotients have p_g equal to 2, 3, 4 or 6.

5. QUOTIENTS OF SELF-PRODUCTS OF CURVES

The method of the previous section may sometimes allow to prove that certain quotients of a product $C \times C$ have maximal Picard number. Since we have very few examples we will refrain from giving a general statement and contend ourselves with one significant example.

Let C be the curve in \mathbb{P}^4 defined by

$$u^2 = xy, \quad v^2 = x^2 - y^2, \quad w^2 = x^2 + y^2.$$

It is isomorphic to the modular curve $X(8)$ [FSM13]. Let $\Gamma \subset \mathrm{PGL}(5, \mathbb{C})$ be the subgroup of diagonal elements changing an even number of signs of u, v, w ; Γ is isomorphic to $(\mathbb{Z}/2)^2$ and acts freely on C .

PROPOSITION 8

- (a) JC is isogenous to $E_i^3 \times E_{\sqrt{-2}}^2$, where $E_\alpha = \mathbb{C}/\mathbb{Z}[\alpha]$ for $\alpha = i$ or $\sqrt{-2}$.
- (b) The surface $(C \times C)/\Gamma$ is ρ -maximal.

Proof

(a) The form $\Omega := (xdy - ydx)/uvw$ generates $H^0(C, K_C(-1))$, and is Γ -invariant; thus multiplication by Ω induces a Γ -equivariant isomorphism

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1)) \xrightarrow{\sim} H^0(C, K_C).$$

Let V and L be the subspaces of $H^0(C, K_C)$ corresponding to $\langle u, v, w \rangle$ and $\langle x, y \rangle$. The projection $(u, v, w, x, y) \mapsto (u, v, w)$ maps C onto the quartic curve $F: 4u^4 + v^4 - w^4 = 0$; the induced map $f: C \rightarrow F$ identifies F with the quotient of C by the involution $(u, v, w, x, y) \mapsto (u, v, w, -x, -y)$, and we have $f^*H^0(F, K_F) = V$.

The quotient curve $H := C/\Gamma$ is the genus 2 curve $z^2 = t(t^4 - 1)$ [Bea13]. The pull-back of $H^0(H, K_H)$ is the subspace invariant under Γ , that is L . Thus JC is isogenous to $JF \times JH$. From examples 1 and 2 of §4 we conclude that JC is isogenous to $E_i^3 \times E_{\sqrt{-2}}^2$.

(b) We have Γ -equivariant isomorphisms

$$\begin{aligned} H^{1,1}(C \times C) &= H^2(C, \mathbb{C}) \oplus H^2(C, \mathbb{C}) \oplus (H^{1,0} \boxtimes H^{0,1}) \oplus (H^{0,1} \boxtimes H^{1,0}) \\ &= \mathbb{C}^2 \oplus \mathrm{End}(H^0(C, K_C))^{\oplus 2} \end{aligned}$$

(where Γ acts trivially on \mathbb{C}^2), hence

$$H^{1,1}((C \times C)/\Gamma) = \mathbb{C}^2 \oplus \mathrm{End}_\Gamma(H^0(C, K_C))^{\oplus 2}.$$

As a Γ -module we have $H^0(C, K_C) = L \oplus V$, where Γ acts trivially on L and V is the sum of the 3 nontrivial one-dimensional representations of Γ . Thus

$$\mathrm{End}_\Gamma(H^0(C, K_C)) = \mathbb{M}_2(\mathbb{C}) \times \mathbb{C}^3.$$

Similarly we have $\mathrm{NS}((C \times C)/\Gamma) \otimes \mathbb{Q} = \mathbb{Q}^2 \oplus (\mathrm{End}_\Gamma(JC) \otimes \mathbb{Q})$ and

$$\mathrm{End}_\Gamma(JC) \otimes \mathbb{Q} = (\mathrm{End}(JH) \otimes \mathbb{Q}) \times (\mathrm{End}_\Gamma(JF) \otimes \mathbb{Q})^3 = \mathbb{M}_2(\mathbb{Q}(\sqrt{-2})) \times \mathbb{Q}(i)^3,$$

hence the result. □

COROLLARY 2 ([ST10]). — *Let $\Sigma \subset \mathbb{P}^6$ be the surface of cuboids, defined by*

$$t^2 = x^2 + y^2 + z^2, \quad u^2 = y^2 + z^2, \quad v^2 = x^2 + z^2, \quad w^2 = x^2 + y^2.$$

Σ has 48 ordinary nodes; its minimal desingularization S is ρ -maximal.

Indeed Σ is a quotient of $(C \times C)/\Gamma$ [Bea13]. □

(The result has been obtained first in [ST10] with a very different method.)

6. OTHER EXAMPLES

6.1. ELLIPTIC MODULAR SURFACES. — Let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ such that $-I \notin \Gamma$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the Poincaré upper half-plane \mathbb{H} ; let Δ_Γ be the compactification of the Riemann surface \mathbb{H}/Γ . The universal elliptic curve over \mathbb{H} descends to \mathbb{H}/Γ , and extends to a smooth projective surface B_Γ over Δ_Γ , the *elliptic modular surface* attached to Γ . In [Shi69] Shioda proves that B_Γ is ρ -maximal.⁽¹⁾

Now take $\Gamma = \Gamma(5)$, the kernel of the reduction map $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/5)$. In [Liv81] Livne constructed a $\mathbb{Z}/5$ -covering $X \rightarrow B_{\Gamma(5)}$, branched along the sum of the 25 5-torsion sections of $B_{\Gamma(5)}$. The surface X satisfies $c_1^2 = 3c_2 (= 225)$, hence it is a ball quotient and therefore rigid. By analyzing the action of $\mathbb{Z}/5$ on $H^{1,1}(X)$ Livne shows that $H^{1,1}(X)$ is not defined over \mathbb{Q} , hence X is not ρ -maximal. This seems to be the only known example of a surface which cannot be deformed to a ρ -maximal surface.

6.2. SURFACES WITH $p_g = K^2 = 1$. — The minimal surfaces with $p_g = K^2 = 1$ have been studied by Catanese [Cat79] and Todorov [Tod80]. Their canonical model is a complete intersection of type $(6, 6)$ in the weighted projective space $\mathbb{P}(1, 2, 2, 3, 3)$. The moduli space \mathcal{M} is smooth of dimension 18.

PROPOSITION 9. — *The ρ -maximal surfaces are dense in \mathcal{M} .*

Proof. — We can replace \mathcal{M} by the Zariski open subset \mathcal{M}_a parametrizing surfaces with ample canonical bundle. Let $S \in \mathcal{M}_a$, and let $f : \mathcal{S} \rightarrow (B, \mathfrak{o})$ be a local versal deformation of S , so that $S \cong \mathcal{S}_\mathfrak{o}$. Let L be the lattice $H^2(S, \mathbb{Z})$, and $k \in L$ the class of K_S . We may assume that B is simply connected and fix an isomorphism of local systems $R^2 f_* (\mathbb{Z}) \xrightarrow{\sim} L_B$, compatible with the cup-product and mapping the canonical class $[K_{\mathcal{S}/B}]$ onto k . This induces for each $b \in B$ an isometry $\varphi_b : H^2(\mathcal{S}_b, \mathbb{C}) \xrightarrow{\sim} L_{\mathbb{C}}$, which maps $H^{2,0}(\mathcal{S}_b)$ onto a line in $L_{\mathbb{C}}$; the corresponding point $\wp(b)$ of $\mathbb{P}(L_{\mathbb{C}})$ is the period of \mathcal{S}_b . It belongs to the complex manifold

$$\Omega := \{[x] \in \mathbb{P}(L_{\mathbb{C}}) \mid x^2 = 0, x \cdot k = 0, x \cdot \bar{x} > 0\}.$$

Associating to $x \in \Omega$ the real 2-plane $P_x := \langle \mathrm{Re}(x), \mathrm{Im}(x) \rangle \subset L_{\mathbb{R}}$ defines an isomorphism of Ω onto the Grassmannian of positive oriented 2-planes in $L_{\mathbb{R}}$.

The key point is that the image of the *period map* $\wp : B \rightarrow \Omega$ is open [Cat79]. Thus we can find b arbitrarily close to \mathfrak{o} such that the 2-plane P_b is defined over \mathbb{Q} , hence $H^{2,0}(\mathcal{S}_b) \oplus H^{0,2}(\mathcal{S}_b) = P_b \otimes_{\mathbb{R}} \mathbb{C}$ is defined over \mathbb{Q} . \square

REMARK 5. — The proof applies to all surfaces with $p_g = 1$ for which the image of the period map is open (for instance to K3 surfaces); unfortunately this seems to be a rather exceptional situation.

⁽¹⁾I am indebted to I. Dolgachev and B. Totaro for pointing out this reference.

6.3. **TODOROV SURFACES.** — In [Tod81] Todorov constructed a series of regular surfaces with $p_g = 1$, $2 \leq K^2 \leq 8$, which provide counter-examples to the Torelli theorem. The construction is as follows: let $K \subset \mathbb{P}^3$ be a Kummer surface. We choose k double points of K in general position (this can be done with $0 \leq k \leq 6$), and a general quadric $Q \subset \mathbb{P}^3$ passing through these k points. The *Todorov surface* S is the double covering of K branched along $K \cap Q$ and the remaining $16 - k$ double points. It is a minimal surface of general type with $p_g = 1$, $K^2 = 8 - k$, $q = 0$. If moreover we choose K ρ -maximal (that is, $K = E^2/\{\pm 1\}$, where E is an elliptic curve with complex multiplication), then S is ρ -maximal by Proposition 2(b).

Note that by varying the quadric Q we get a continuous, non-constant family of ρ -maximal surfaces.

6.4. **DOUBLE COVERS.** — In [Per82] Persson constructs ρ -maximal double covers of certain rational surfaces by allowing the branch curve to acquire some simple singularities (see also [BE87]). He applies this method to find ρ -maximal surfaces in the following families:

- Horikawa surfaces, that is, surfaces on the “Noether line” $K^2 = 2p_g - 4$, for $p_g \not\equiv -1 \pmod{6}$;
- Regular elliptic surfaces;
- Double coverings of \mathbb{P}^2 .

In the latter case the double plane admits (many) rational singularities; it is unknown whether there exists a ρ -maximal surface S which is a double covering of \mathbb{P}^2 branched along a smooth curve of even degree ≥ 8 .

6.5. **HYPERSURFACES AND COMPLETE INTERSECTIONS.** — Probably the most natural families to look at are smooth surfaces in \mathbb{P}^3 , or more generally complete intersections. Here we may ask for a smooth surface S , or for the minimal resolution of a surface with rational double points (or even any surface deformation equivalent to a complete intersection of given type). Here are the examples that we know of:

- The quintic surface $x^3yz + y^3zt + z^3tx + t^3xy = 0$ has four A_9 singularities; its minimal resolution is ρ -maximal [Sch11]. It is not yet known whether there exists a smooth ρ -maximal quintic surface.
- The Fermat sextic is ρ -maximal (§4, Corollary 1).
- The complete intersection $y^2 - xz = v^2 - uw = x^3 + z^3 + u^3 + w^3 = 0$ of type $(2, 2, 3)$ in \mathbb{P}^5 has 12 nodes; its minimal desingularization is ρ -maximal (end of §4).
- The surface of cuboids is a complete intersection of type $(2, 2, 2, 2)$ in \mathbb{P}^6 with 48 nodes; its minimal desingularization is ρ -maximal (§5, Corollary 2).

7. THE COMPLEX TORUS ASSOCIATED TO A ρ -MAXIMAL VARIETY

For a ρ -maximal variety X , let T_X be the \mathbb{Z} -module $H^2(X, \mathbb{Z})/\text{NS}(X)$. We have a decomposition

$$T_X \otimes \mathbb{C} = H^{2,0} \oplus H^{0,2}$$

defining a weight 1 Hodge structure on T_X , hence a complex torus $\mathcal{T} := H^{0,2}/p_2(T_X)$, where $p_2 : T_X \otimes \mathbb{C} \rightarrow H^{2,0}$ is the second projection. Via the isomorphism $H^{0,2} = H^2(X, \mathcal{O}_X)$, \mathcal{T}_X is identified with the cokernel of the natural map $H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X)$.

The exponential exact sequence gives rise to an exact sequence

$$0 \longrightarrow \text{NS}(X) \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X^*) \xrightarrow{\partial} H^3(X, \mathbb{Z}),$$

hence to a short exact sequence

$$0 \longrightarrow \mathcal{T}_X \longrightarrow H^2(X, \mathcal{O}_X^*) \xrightarrow{\partial} H^3(X, \mathbb{Z}),$$

so that \mathcal{T}_X appears as the “continuous part” of the group $H^2(X, \mathcal{O}_X^*)$.

EXAMPLE 5. — Consider the elliptic modular surface B_Γ of Section 6.1. The space $H^0(B_\Gamma, K_{B_\Gamma})$ can be identified with the space of cusp forms of weight 3 for Γ ; then the torus \mathcal{T}_{B_Γ} is the complex torus associated to this space by Shimura (see [Shi69]).

EXAMPLE 6. — Let $X = C \times C'$, with J_C isogenous to E^g and $J_{C'}$ to $E^{g'}$ (Proposition 5). The torus \mathcal{T}_X is the cokernel of the map

$$i \otimes i' : H^1(C, \mathbb{Z}) \otimes H^1(C', \mathbb{Z}) \longrightarrow H^1(C, \mathcal{O}_C) \otimes H^1(C', \mathcal{O}_{C'}),$$

where i and i' are the embeddings

$$H^1(C, \mathbb{Z}) \hookrightarrow H^1(C, \mathcal{O}_C) \quad \text{and} \quad H^1(C', \mathbb{Z}) \hookrightarrow H^1(C', \mathcal{O}_{C'}).$$

We want to compute \mathcal{T}_X up to isogeny, so we may replace the left hand side by a finite index sublattice. Thus, writing $E = \mathbb{C}/\Gamma$, we may identify i with the diagonal embedding $\Gamma^g \hookrightarrow \mathbb{C}^g$, and similarly for i' ; therefore $i \otimes i'$ is the diagonal embedding of $(\Gamma \otimes \Gamma)^{gg'}$ in $\mathbb{C}^{gg'}$. Put $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$; the image Γ' of $\Gamma \otimes \Gamma$ in \mathbb{C} is spanned by $1, \tau, \tau^2$; since E has complex multiplication, τ is a quadratic number, hence Γ has finite index in Γ' . Finally we obtain that \mathcal{T}_X is isogenous to $E^{gg'}$.

For the surface $X = (C \times C)/\Gamma$ studied in §5 an analogous argument shows that \mathcal{T}_X is isogenous to $A = E_i^4 \times E_{\sqrt{-2}}^3$. This is still an abelian variety of type CM, in the sense that $\text{End}(A) \otimes \mathbb{Q}$ contains an étale \mathbb{Q} -algebra of maximal dimension $2 \dim(A)$. There seems to be no reason why this should hold in general. However it is true in the special case $h^{2,0} = 1$ (e.g. for holomorphic symplectic manifolds):

PROPOSITION 10. — *If $h^{2,0}(X) = 1$, the torus \mathcal{T}_X is an elliptic curve with complex multiplication.*

Proof. — Let T'_X be the pull back of $H^{2,0} + H^{0,2}$ in $H^2(X, \mathbb{Z})$; then $p_2(T'_X)$ is a sublattice of finite index in $p_2(T_X)$. Choosing an ample class $h \in H^2(X, \mathbb{Z})$ defines a quadratic form on $H^2(X, \mathbb{Z})$ which is positive definite on T'_X . Replacing again T'_X by a finite index sublattice we may assume that it admits an orthogonal basis (e, f) with $e^2 = a, f^2 = b$. Then $H^{2,0}$ and $H^{0,2}$ are the two isotropic lines of $T'_X \otimes \mathbb{C}$; they are spanned by the vectors $\omega = e + \tau f$ and $\bar{\omega} = e - \tau f$, with $\tau^2 = -a/b$. We have $e = \frac{1}{2}(\omega + \bar{\omega})$ and $f = \frac{1}{2\tau}(\omega - \bar{\omega})$; therefore multiplication by $\frac{1}{2\tau}\bar{\omega}$ induces an

isomorphism of $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ onto $H^{0,2}/p_2(T'_X)$, hence \mathcal{T}_X is isogenous to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ and

$$\text{End}(\mathcal{T}_X) \otimes \mathbb{Q} = \mathbb{Q}(\tau) = \mathbb{Q}\left(\sqrt{-\text{disc}(T'_X)}\right). \quad \square$$

8. HIGHER CODIMENSION CYCLES

A natural generalization of the question considered here is to look for varieties X for which the group $H^{2p}(X, \mathbb{Z})_{\text{alg}}$ of algebraic classes in $H^{2p}(X, \mathbb{Z})$ has maximal rank $h^{p,p}$. Very few nontrivial cases seem to be known. The following is essentially due to Shioda:

PROPOSITION 11. — *Let F_d^n be the Fermat hypersurface of degree d and even dimension $n = 2\nu$. For $d = 3, 4$, the group $H^n(F_d^n, \mathbb{Z})_{\text{alg}}$ has maximal rank $h^{\nu,\nu}$.*

Proof. — According to [Shi79] we have

$$\text{rk } H^n(F_3^n, \mathbb{Z})_{\text{alg}} = 1 + \frac{n!}{(\nu)!^2} \quad \text{and} \quad \text{rk } H^n(F_4^n, \mathbb{Z})_{\text{alg}} = \sum_{k=0}^{k=\nu+1} \frac{(n+2)!}{(k!)^2(n+2-2k)!}.$$

On the other hand, let $R_d^n := \mathbb{C}[X_0, \dots, X_{n+1}]/(X_0^{d-1}, \dots, X_{n+1}^{d-1})$ be the Jacobian ring of F_d^n ; Griffiths theory [Gri69] provides an isomorphism of the primitive cohomology $H^{\nu,\nu}(F_d^n)_o$ with the component of degree $(\nu+1)(d-2)$ of R_d^n . Since this ring is the tensor product of $(n+2)$ copies of $\mathbb{C}[T]/(T^{d-1})$, its Poincaré series $\sum_k \dim(R_d^n)_k T^k$ is $(1+T+\dots+T^{d-2})^{n+2}$. Then an elementary computation gives the result. \square

In the particular case of cubic fourfolds we have more examples:

PROPOSITION 12. — *Let F be a cubic form in 3 variables, such that the curve $F(x, y, z) = 0$ in \mathbb{P}^2 is an elliptic curve with complex multiplication; let X be the cubic fourfold defined by $F(x, y, z) + F(u, v, w) = 0$ in \mathbb{P}^5 . The group $H^4(X, \mathbb{Z})_{\text{alg}}$ has maximal rank $h^{2,2}(X)$.*

Proof. — Let u be the automorphism of X defined by

$$u(x, y, z; u, v, w) = (x, y, z; \omega u, \omega v, \omega w).$$

We observe that u acts trivially on the (one-dimensional) space $H^{3,1}(X)$. Indeed Griffiths theory [Gri69] provides a canonical isomorphism

$$\text{Res} : H^0(\mathbb{P}^5, K_{\mathbb{P}^5}(2X)) \xrightarrow{\sim} H^{3,1}(X);$$

the space $H^0(\mathbb{P}^5, K_{\mathbb{P}^5}(2X))$ is generated by the meromorphic form Ω/G^2 , with

$$\begin{aligned} \Omega &= xdy \wedge dz \wedge du \wedge dv \wedge dw - ydx \wedge dz \wedge du \wedge dv \wedge dw + \dots, \\ G &= F(x, y, z) + F(u, v, w). \end{aligned}$$

The automorphism u acts trivially on this form, and therefore on $H^{3,1}(X)$.

Let F be the variety of lines contained in X . We recall from [BD85] that F is a holomorphic symplectic fourfold, and that there is a natural isomorphism of Hodge structures $\alpha : H^4(X, \mathbb{Z}) \xrightarrow{\sim} H^2(F, \mathbb{Z})$. Therefore the automorphism u_F of F induced by u is symplectic. Let us describe its fixed locus.

The fixed locus of u in X is the union of the plane cubics E given by $x = y = z = 0$ and E' given by $u = v = w = 0$. A line in X preserved by u must have (at least) two fixed points, hence must meet both E and E' ; conversely, any line joining a point of E to a point of E' is contained in X , and preserved by u . This identifies the fixed locus A of u_F to the abelian surface $E \times E'$. Since u_F is symplectic A is a symplectic submanifold, that is, the restriction map $H^{2,0}(F) \rightarrow H^{2,0}(A)$ is an isomorphism. By our hypothesis A is ρ -maximal, so F is ρ -maximal by Proposition 2. Since α maps $H^4(X, \mathbb{Z})_{\text{alg}}$ onto $\text{NS}(F)$ this implies the Proposition. \square

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