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DESENSITIZING CONTROL FOR THE HEAT EQUATION WITH RESPECT TO DOMAIN VARIATIONS

by Sylvain Ervedoza, Pierre Lissy & Yannick Privat

Abstract. — This article is dedicated to desensitizing issues for a quadratic functional involving the solution of the linear heat equation with respect to domain variations. This work can be seen as a continuation of [28], insofar as we generalize several of the results it contains and investigate new related properties. In our framework, we consider variations of the spatial domain on which the solution of the PDE is defined at each time, and investigate three main issues: (i) approximate desensitizing, (ii) approximate desensitizing combined with an exact desensitizing for a finite-dimensional subspace, and (iii) exact desensitizing. We provide positive answers to questions (i) and (ii) and partial results to question (iii).

Résumé (Contrôle désensibilisant pour l'équation de la chaleur par rapport à des variations du domaine)

Cet article est dédié à l'étude de problèmes de désensibilisation par rapport à des variations du domaine, pour des fonctionnelles quadratiques dépendant de la solution de l'équation de la chaleur linéaire. Ce travail peut être vu comme la suite du travail [28], dans la mesure où nous généralisons un certain nombre de résultats qu'il contient, et où nous nous intéressons à de nouvelles propriétés en lien avec ce travail. Nous considérons des variations du domaine spatial sur lequel la solution de l'EDP est définie, et nous nous intéressons à trois questions : (i) désensibilisation approchée, (ii) désensibilisation approchée couplée avec une propriété de désensibilisation exacte sur un sous-espace vectoriel de dimension finie, (iii) désensibilisation exacte. Nous donnons des réponses positives aux points (i) et (ii), et des résultats partiels au point (iii).

Contents

Mathematical subject classification (2020). — 35K05, 93C20, 49K20.
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1. Introduction

1.1. Desensitizing controls with respect to domain variations, framework

The goal of this article is to discuss a desensitizing control problem for the heat equation with respect to variations of the domains (note that this notion has been popularized by J.-L. Lions [23] under the wording "insensitizing" and can be found in the literature under both names "insentizing" or "desensitizing"). First results in this direction have already been obtained in [28]. Introducing this problem precisely requires some notations, which we choose similar to the ones in [28].

Let $T>0$ denote a horizon of time, ω and Θ be two open subsets of $\mathbb{R}^d$, $d\in\mathbb{N}^*$, and $\xi\in L^2(0,T;L^2(\mathbb{R}^d))$.

For a bounded, connected and open set Ω of $\mathbb{R}^d$ of class $\mathscr{C}^2$, we consider the shape functional J_h, indexed by $h\in L^2(0,T;L^2(\omega))$, defined by

$$J_h(\Omega)=\frac{1}{2}\int_0^T\int_\Theta y_{\Omega,h}(t,x)^2\,dx\,dt, \tag{1.1}$$

where $y_{\Omega,h}\in L^2((0,T)\times\mathbb{R}^d)$ is defined on $(0,T)\times\Omega$ as the unique weak solution of

$$\begin{cases} \dfrac{\partial y}{\partial t}-\Delta y=\xi+h\mathbb{1}_\omega & \text{in } (0,T)\times\Omega,\\ y=0 & \text{on } (0,T)\times\partial\Omega,\\ y(0,\cdot)=0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

($\mathbb{1}_\omega$ denotes the characteristic function of the set ω) extended by 0 outside $(0,T)\times\Omega$.

Originally, the desensitizing problem consists in finding a control function such that some functional depending on the solution of a partial differential equation is locally insensitive to perturbations of the initial condition. This issue was first raised by J.-L. Lions in [23]. We refer to Section 1.3 for bibliographical comments. Nevertheless, up to our knowledge, desensitizing properties with respect to shape variation issues have been first investigated in [28]. Let us recall here what we are talking about: given a bounded, connected and open set Ω_0 of $\mathbb{R}^d$ with $\mathscr{C}^2$ boundary, our goal is to determine, whenever it exists, a control function $h\in L^2(0,T;L^2(\omega))$ such that J_h does not depend on small variations of Ω in a neighbourhood of Ω_0 (which will be made precise in what follows) at first order. In other words, we want to choose the control function h in such a way that the functional J_h is desensitized with respect to small variations of the domain.

To give a precise meaning to this, we first remark that this problem is interesting only if the intersection of these two last sets with Ω_0 is nonempty, in which case the functional J_h only depends on $\Omega\cap\omega$ and $\Omega\cap\Theta$. Hence, in the following, we will assume that Ω_0 is a bounded, connected and open set of $\mathbb{R}^d$ with a $\mathscr{C}^2$ boundary and that ω and Θ are open subsets of Ω_0.

It is convenient to endow the set of domains with some differential structure. In what follows, we will use the notion of differentiation in the sense of Hadamard [13, 20], which is classically used in the framework of shape optimization. This means that perturbations of Ω_0 will be defined by means of well-chosen diffeomorphisms, which have the advantage of preserving some topological features such as connectedness, boundedness and regularity.

Accordingly, we introduce the class $W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ of admissible vector fields. Then, for each element $\boldsymbol{V}$ of $W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)$, there exists $\tau_{\boldsymbol{V}}>0$ such that for all $\tau\in[0,\tau_{\boldsymbol{V}})$, the mapping $\boldsymbol{T}_\tau:=\mathrm{Id}+\tau\boldsymbol{V}$ defines a diffeomorphism(1) in $\mathbb{R}^d$, i.e., the mapping $\boldsymbol{T}_\tau$ is invertible and $\boldsymbol{T}_\tau^{-1}\in W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)$. Furthermore, since $\boldsymbol{T}_\tau$ writes as a perturbation of the identity operator, one easily infers that $\boldsymbol{T}_\tau(\Omega_0)$ is a connected, bounded domain whose boundary $\boldsymbol{T}_\tau(\partial\Omega_0)$ is of class $\mathscr{C}^2$. It is notable that, in this framework, one has $\partial\boldsymbol{T}_\tau(\Omega_0)=\boldsymbol{T}_\tau(\partial\Omega_0)$. To sum up, the assumption that $\boldsymbol{V}$ belongs to $W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ is essentially technical and preserves the $\mathscr{C}^2$ regularity of the domain boundary once the domain deformation is applied [13, Chap. 7].

In the sequel, given $\boldsymbol{V}\in W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)$, we introduce the family $\{\Omega_{\tau\boldsymbol{V}}\}_{\tau\in[0,\tau_{\boldsymbol{V}})}$ of domains defined by
$$\Omega_{\tau\boldsymbol{V}}=(\mathrm{Id}+\tau\boldsymbol{V})(\Omega_0).$$
As a consequence of the above discussion, for $\tau\in[0,\tau_{\boldsymbol{V}})$, each domain $\Omega_{\tau\boldsymbol{V}}$ inherits the aforementioned properties.

It is then classical (see e.g. [20, Chap. 5]) that the map $\tau\mapsto J_h(\Omega_{\tau\boldsymbol{V}})$ is differentiable in a neighbourhood of $\tau=0$. In the following result, we provide a workable expression of the shape derivative $dJ_h(\Omega_{\tau\boldsymbol{V}})/d\tau|_{\tau=0}$.

Proposition 1.1 ([28, Proof of Prop. 1.1]). — *Let $\xi\in L^2(0,T;L^2(\mathbb{R}^d))$ and $h\in L^2(0,T;L^2(\omega))$. For all $\boldsymbol{V}\in W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)$, the mapping $\tau\mapsto J_h(\Omega_{\tau\boldsymbol{V}})$ is differentiable at $\tau=0$ and*
$$\frac{d}{d\tau}\left(J_h\left(\Omega_{\tau\boldsymbol{V}}\right)\right)\Big|_{\tau=0}=\int_{\partial\Omega_0}\boldsymbol{V}\cdot\boldsymbol{n}\left(\int_0^T\partial_n y_0\partial_n z_0\,dt\right)d\sigma, \tag{1.3}$$
where the pair (y_0,z_0) solves the coupled forward-backward system
$$\begin{cases}\dfrac{\partial y_0}{\partial t}-\Delta y_0=\xi+h\mathbb{1}_\omega & \text{in } (0,T)\times\Omega_0,\\ y_0=0 & \text{on } (0,T)\times\partial\Omega_0,\\ y_0(0,\cdot)=0 & \text{in } \Omega_0,\\ -\dfrac{\partial z_0}{\partial t}-\Delta z_0=\mathbb{1}_\Theta y_0 & \text{in } (0,T)\times\Omega_0,\\ z_0=0 & \text{on } (0,T)\times\partial\Omega_0,\\ z_0(T,\cdot)=0 & \text{in } \Omega_0.\end{cases} \tag{1.4}$$
($\mathbb{1}_\Theta$ denotes the characteristic function of the set Θ.)

We now recall the precise definitions of desensitizing controls that will be used next, introduced in [28, Def. 1.1](2) and much inspired of notions introduced in [25, 5].

(1)To be more precise, it is easy to see that the choice $\tau_{\boldsymbol{V}}=1/\|\boldsymbol{V}\|_{W^{3,\infty}}$ works, by applying the Banach fixed-point theorem.

(2)The authors restricted the properties below to diffeomorphisms $\boldsymbol{V}\in W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)$ of norm less than 1 in this reference, but an easy homogeneity argument enables to give an equivalent definition for any $\mathbf{V}\in W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)$.

Definition 1.2. — Let $\xi \in L^2(0,T;L^2(\mathbb{R}^d))$. We say that the control function $h \in L^2(0,T;L^2(\omega))$ desensitizes J_h exactly at Ω_0 at the first order with respect to domain variations if

$$\forall \boldsymbol{V} \in W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d), \quad \frac{d}{d\tau}\left(J_h\left(\Omega_{\tau\boldsymbol{V}}\right)\right)\Big|_{\tau=0} = 0. \tag{1.5}$$

Let $\mathcal{E}$ be a linear subspace of $W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)$. We say that the control function $h \in L^2(0,T;L^2(\omega))$ exactly desensitizes J_h for $\mathcal{E}$ at Ω_0 at the first order with respect to domain variations if

$$\forall \boldsymbol{V} \in \mathcal{E}, \quad \frac{d}{d\tau}\left(J_h\left(\Omega_{\tau\boldsymbol{V}}\right)\right)\Big|_{\tau=0} = 0. \tag{1.6}$$

Given $\varepsilon > 0$, we say that the control function $h \in L^2(0,T;L^2(\omega))$ ε-approximately desensitizes J_h at Ω_0 at the first order with respect to domain variations if

$$\forall \boldsymbol{V} \in W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d), \quad \left|\frac{d}{d\tau}\left(J_h\left(\Omega_{\tau\boldsymbol{V}}\right)\right)\Big|_{\tau=0}\right| \leqslant \varepsilon\|\boldsymbol{V}\|_{W^{3,\infty}(\mathbb{R}^d;\mathbb{R}^d)}. \tag{1.7}$$

Let us conclude this section by introducing interesting issues related to desensitizing of the solution of the heat equation with respect to domain variations, that will be tackled in what follows:

Q.1. (ε-approximate desensitizing). — Let $\xi \in L^2(0,T;L^2(\mathbb{R}^d))$ and $\varepsilon > 0$. Does there exist a control function $h \in L^2(0,T;L^2(\omega))$ that ε-approximately desensitizes J_h at Ω_0 in the sense of (1.7)?

Q.1′. (ε-approximate desensitizing and null/approximate controllability). — If the answer to Q.1 is yes, is it possible to choose $h \in L^2(0,T;L^2(\omega))$ in such a way that it is also a null control[(3)] or an ε-approximate control[(4)] for y_0 at time T?

Q.2. (ε-approximate desensitizing and exact desensitizing for a finite-dimensional subspace $\mathcal{E}$). — Let $\xi \in L^2(0,T;L^2(\mathbb{R}^d))$ and $\varepsilon > 0$. Does there exist a control $h \in L^2(0,T;L^2(\omega))$ that desensitizes J_h exactly for $\mathcal{E}$ in the sense of (1.6), and at the same time that ε-approximately desensitizes J_h in the sense of (1.7)?

Q.2′. (ε-approximate desensitizing, exact desensitizing for a finite-dimensional subspace $\mathcal{E}$, and null/approximate controllability). — If the answer to Q.2 is yes, is it possible to choose $h \in L^2(0,T;L^2(\omega))$ in such a way that it is also a null control or an ε-approximate control for y_0 at time T, in the sense given in Q.1′?

(3) The wording "null control" refers to a function h such that the solution (y_0, z_0) of (1.4) satisfies $y_0(T) = 0$ in Ω_0.

(4) Given $y_T \in L^2(\Omega_0)$ and $\varepsilon > 0$, the wording "ε-approximate control" refers to a control function $h \in L^2(0,T;L^2(\omega))$ such that the solution (y_0, z_0) of (1.4) satisfies $\|y_0(T) - y_T\|_{L^2(\Omega_0)} \leqslant \varepsilon$.

Q.3. (exact desensitizing). — Let $\xi \in L^2(0,T;L^2(\mathbb{R}^d))$. Is it possible to exactly desensitize the functional J_h in the sense of (1.5)?

If ω and Θ are strongly included in Ω_0 and $\omega\cap\Theta \neq \varnothing$, Q.1 has been solved in [28], whereas Q.2 has also been solved in [28] when $\mathcal{E}$ is of dimension 1 or 2. The goal of the present article is to extend the results of [28] to more general geometric settings and to the more general questions above-mentioned. To be more precise, in the next section, we will distinguish between the cases where ω and Θ intersect or not (see Figure 1 below), since approaches to deal with them and the results obtained are fairly different. In the case $\omega\cap\Theta = \varnothing$, Q.1 will be tackled in Theorem 1.6 and Q.2 will be tackled in Theorems 1.7. In the case $\omega\cap\Theta \neq \varnothing$, Q.1$'$ will be tackled in Theorem 1.10 and Q.2$'$ will be tackled in Theorem and 1.11. Finally, we will provide two partial answers to Q.3 in Theorems 1.14 and 1.15.

Remark 1.3. — According to Proposition 1.1, although all the above questions a priori depend on $\xi \in L^2(0,T;L^2(\mathbb{R}^d))$, in reality, they only depend on the restriction on ξ to Ω_0. Thus, in the following, we shall simply take $\xi \in L^2(0,T;L^2(\Omega_0))$ (extended by 0 on $\mathbb{R}^d$) without loss of generality.

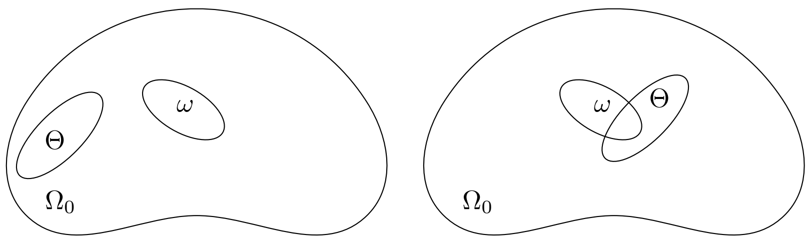

Figure 1. The two main situations investigated: (left) the intersection set of ω and Θ is empty; (right) the intersection set of ω and Θ is nonempty.

1.2. Main results. — As we said, the aforementioned desensitizing problems will be strongly dependent on the relative geometry of the various sets ω, Θ and Ω_0, and in particular depending if the set $\omega\cap\Theta$ is empty or not, but we will always make the following minimal assumption on both sets ω and Θ:

$$(\mathrm{H}_{\omega,\Theta}) \qquad \omega \text{ and } \Theta \text{ are two nonempty open subsets of } \Omega_0.$$

Case $\omega\cap\Theta = \varnothing$ and $\Theta \Subset \Omega_0$. — To be more precise, the first geometric setting we consider is the following:

$$\omega \text{ and } \Theta \text{ satisfy } (\mathrm{H}_{\omega,\Theta}), \quad \omega\cap\Theta = \varnothing, \quad \Theta \Subset \Omega_0, \text{ and } \Omega_0 \smallsetminus \overline{\Theta} \text{ is connected.} \tag{1.8}$$

In this setting, our first result is dedicated to an approximate controllability property, whose relation to desensitizing issue will become clear later on. We will then use it in a crucial way to derive one of the main results of this paper, allowing to partially answer the question Q.1.

Proposition 1.4. — *Assume the geometric setting* (1.8). *Then, given any* $(f_1, f_2) \in L^2(0,T;L^2(\partial\Omega_0))^2$, *for any* $\varepsilon > 0$ *and* $\xi \in L^2(0,T;L^2(\Omega_0))$, *there exists a control function* $h \in L^2(0,T;L^2(\omega))$ *such that the solution* (y,z) *of*

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = \xi + h\mathbb{1}_\omega & \text{in } (0,T)\times\Omega_0, \\ y = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ y(0,\cdot) = 0 & \text{in } \Omega_0, \\ -\dfrac{\partial z}{\partial t} - \Delta z = \mathbb{1}_\Theta y & \text{in } (0,T)\times\Omega_0, \\ z = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ z(T,\cdot) = 0 & \text{in } \Omega_0, \end{cases} \tag{1.9}$$

satisfies

$$\|\partial_n y - f_1\|_{L^2(0,T;L^2(\partial\Omega_0))} + \|\partial_n z - f_2\|_{L^2(0,T;L^2(\partial\Omega_0))} \leqslant \varepsilon. \tag{1.10}$$

Remark 1.5

– Looking carefully at the proof of Proposition 1.4, given in Section 2.2, it is easy to figure out that the last condition in (1.8) can be relaxed into the following one: ω intersects every connected component $\mathcal{O}$ of $\Omega_0 \smallsetminus \overline{\Theta}$ verifying that $\overline{\mathcal{O}} \cap \partial\Omega \neq \varnothing$.

– As it is classical for approximate controllability results (see [35, 16, 14]), one can reinforce the above results as follows: if F is a finite-dimensional subspace of $\big(L^2(0,T;L^2(\partial\Omega_0))\big)^2$ and $\mathbb{P}_F$ denotes the orthogonal projection on F in $\big(L^2(0,T;L^2(\partial\Omega_0))\big)^2$, then, for any $(f_1,f_2) \in L^2(0,T;L^2(\partial\Omega_0))^2$, for any $\varepsilon > 0$ and $\xi \in L^2(0,T;L^2(\Omega_0))$, there exists a control function $h \in L^2(0,T;L^2(\omega))$ such that the solution (y,z) of (1.9) satisfies (1.10) and

$$\mathbb{P}_F(\partial_n y, \partial_n z) = \mathbb{P}_F(f_1, f_2).$$

As we will see, the proof of Proposition 1.4 given in Section 2.2 mainly relies on a unique continuation property for the adjoint operator, which consists of coupled parabolic equations where the coupling coefficients are disjoint from the observation set. This kind of issues is known to be particularly difficult in the case of coupled parabolic systems (see e.g. [2] for partial results in one space dimension), and comes naturally when dealing with desensitizing problems. However, to our knowledge, the only works dealing with control and observation sets which do not intersect in this context are [29] in a 1d case and [22]. Though, our result is different, since the unique continuation property we need to prove Proposition 1.4 is not the one in [29, 22].

A straightforward application of Proposition 1.4, proved in Section 2.1, is the following one:

Theorem 1.6. — *Assume the geometric setting* (1.8). *Then, for all* $\xi \in L^2(0,T;L^2(\Omega_0))$ *and* $\varepsilon > 0$, *there exists a control function* $h \in L^2(0,T;L^2(\omega))$ *such that the solution*

(y_0, z_0) of (1.4) satisfies

$$\left\| \int_0^T \partial_n y_0 \partial_n z_0 \, dt \right\|_{L^1(\partial\Omega_0)} \leqslant \varepsilon. \tag{1.11}$$

In other words, according to (1.3), the functional J_h is ε-approximately desensitized by $h \in L^2(0,T;L^2(\omega))$ at Ω_0 in the sense of (1.7).

One can actually prove that the functional J_h can be made exactly desensitized to any finite-dimensional vector space of $W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$:

Theorem 1.7. — *Assume the geometric setting (1.8). Let $\mathcal{E}$ be a finite-dimensional linear subspace of $W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Then, for all $\xi \in L^2(0,T;L^2(\Omega_0))$ and for all $\varepsilon > 0$, there exists a control $h \in L^2(0,T;L^2(\omega))$ that desensitizes J_h exactly for $\mathcal{E}$ in the sense of (1.6) and that ε-approximately desensitizes J_h in the sense of (1.7).*

In fact, the main difficulty in Theorem 1.7 is the construction of h such that J_h is exactly desensitized for $\mathcal{E}$ in the sense of (1.6), since the map

$$h \in L^2(0,T;L^2(\omega)) \longmapsto \left(\int_0^T \partial_n y_0 \partial_n z_0 \, dt \right) \in L^1(\partial\Omega_0), \tag{1.12}$$

where the pair (y_0, z_0) solves (1.4), is not linear in h even in the case $\xi = 0$, but quadratic. Therefore, we use techniques specifically designed to deal with such kind of non-linearities, which consists in choosing the control functions in a vector space of much larger dimension than the number of constraints. Similarly to what has been done in another context for the stabilizability of the Navier-Stokes equation, see [11], if there are N constraints imposed by the exact desensitizing for $\mathcal{E}$, we look for control functions in a vector space of size (at most) $2N$ which is suitably designed. In particular, even if there is one constraint (i.e., if $\mathcal{E}$ is a vector space of dimension 1), we look for the control function in a vector space of dimension (at most) 2, thus preventing possible obstructions that may appear due to the quadratic nature of the map in (1.12) (see e.g. [3]). Details of the proof are given in Section 2.3.

Case $\omega \cap \Theta \neq \varnothing$. — The second geometric setting we consider is the case

$$\omega \text{ and } \Theta \text{ satisfy } (\mathrm{H}_{\omega,\Theta}), \quad \omega \cap \Theta \neq \varnothing. \tag{1.13}$$

This case is more favorable since the control set ω meets the observation set Θ, and as we shall see afterward, not only do all previously established results remain true, but this also allows to prove the existence of even better desensitizing controls.

On the ε-approximate desensitizing problem. — To start with, we first claim that Proposition 1.4 can be reinforced under the geometric setting (1.13).

Proposition 1.8. — *Assume the geometric setting (1.13). Then, given any $(f_1, f_2) \in (L^2(0,T;L^2(\partial\Omega_0)))^2$, $y_T \in L^2(\Omega)$, any $\varepsilon > 0$ and $\xi \in L^2(0,T;L^2(\Omega_0))$, there exists a control function $h \in L^2(0,T;L^2(\omega))$ such that the solution (y,z) of (1.9) satisfies*

$$\|\partial_n y - f_1\|_{L^2(0,T;L^2(\partial\Omega_0))} + \|\partial_n z - f_2\|_{L^2(0,T;L^2(\partial\Omega_0))} + \|y(T) - y_T\|_{L^2(\Omega_0)} \leqslant \varepsilon. \tag{1.14}$$

Besides, if the source term $\xi \in L^2(0,T;L^2(\Omega_0))$ is null-controllable in the sense that there exists $h_{\rm nc} \in L^2(0,T;L^2(\omega))$ such that the solution $y_{\rm nc}$ of

$$\begin{cases} \dfrac{\partial y_{\rm nc}}{\partial t} - \Delta y_{\rm nc} = \xi + h_{\rm nc}\mathbb{1}_\omega & \text{in } (0,T)\times\Omega_0, \\ y_{\rm nc} = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ y_{\rm nc}(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases} \tag{1.15}$$

satisfies

$$y_{\rm nc}(T) = 0 \text{ in } \Omega_0, \tag{1.16}$$

then, given any $(f_1,f_2) \in L^2(0,T;L^2(\partial\Omega_0))^2$, for any $\varepsilon > 0$, there exists a control function $h \in L^2(0,T;L^2(\omega))$ such that the solution (y,z) of (1.9) satisfies

$$\|\partial_n y - f_1\|_{L^2(0,T;L^2(\partial\Omega_0))} + \|\partial_n z - f_2\|_{L^2(0,T;L^2(\partial\Omega_0))} \leqslant \varepsilon. \tag{1.17}$$

and

$$y(T) = 0 \text{ in } \Omega_0. \tag{1.18}$$

Remark 1.9. — Determining a control function h such that the solution $y_{\rm nc}$ of (1.15) satisfies (1.16) is the well-known null-controllability problem with source term for the heat equation. This issue has been much investigated. By using duality arguments, this issue can be recast in terms of a so-called "observability inequality". More precisely, one can deduce from [17, Lem. 2.1] that there exists $C > 0$ such that

$$\Big\|\exp\Big(-\frac{C}{T-t}\Big)\varphi(t)\Big\|_{L^2(0,T;L^2(\Omega_0))} \leqslant C\|\varphi\|_{L^2(0,T;L^2(\omega))}, \tag{1.19}$$

where φ denotes the solution of the backward adjoint system

$$\begin{cases} -\dfrac{\partial \varphi}{\partial t} - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega_0, \\ \varphi = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ \varphi(T,\cdot) = \varphi^T & \text{in } \Omega_0. \end{cases}$$

From the observability inequality (1.19), we can deduce that for any $\xi \in L^2(0,T;L^2(\Omega_0))$ with $e^{C/(T-t)}\xi \in L^2(0,T;L^2(\Omega_0))$, one can find a null control $h_{\rm nc}$ to (1.15), i.e., a function $h_{\rm nc} \in L^2(0,T;L^2(\omega))$ such that the solution $y_{\rm nc}$ of (1.15) satisfies (1.16), as explained in [17, Proof of Th. 2.1].

Again, Proposition 1.8 is based on suitable unique continuation properties for the adjoint equation. However, here, since $\omega\cap\Theta \neq \varnothing$, the arguments are more standard for the proof of (1.14) than for the proof of Proposition 1.4. The possibility of further imposing (1.18) when ξ is a source term that can be null-controlled is much more subtle, and amounts to a suitable use of duality arguments, inspired by [24], and of observability estimates for the heat equation given in [17]. Details of the proof are given in Section 3.1.

As before, a straightforward application of Proposition 1.8 is the following result, whose proof is postponed to Section 3.2.

Theorem 1.10. — *Assume the geometric setting* (1.13). *Then, for all*

$$\xi \in L^2(0,T;L^2(\Omega_0)), \quad y_T \in L^2(\Omega_0), \quad \varepsilon > 0,$$

there exists a control function $h\in L^2(0,T;L^2(\omega))$ *such that the solution* (y_0,z_0) *of* (1.4) *satisfies* (1.11) *and*

$$\|y_0(T) - y_T\|_{L^2(\Omega_0)} \leqslant \varepsilon. \tag{1.20}$$

In other words, according to (1.3)*, the functional* J_h *is* ε*-approximately desensitized by a control* $h \in L^2(0,T;L^2(\omega))$ *at* Ω_0 *in the sense of* (1.7)*. Furthermore, the function* h *also* ε*-approximately controls the state* y_0 *of* (1.4) *at time* T*, in the sense that* (1.20) *is verified.*

Similarly, if the source term $\xi \in L^2(0,T;L^2(\Omega_0))$ *is null-controllable in the sense that there exists* $h_{\mathrm{nc}} \in L^2(0,T;L^2(\omega))$ *such that the solution* y_{nc} *of* (1.15) *satisfies* (1.16)*, then, there exists a control function* h *such that solution* (y_0,z_0) *of* (1.4) *satisfies* (1.11) *and*

$$y_0(T) = 0 \quad \textit{in} \ \ \Omega_0. \tag{1.21}$$

In other words, if the source term ξ *is null-controllable at time* $T > 0$*, the functional* J_h *is* ε*-approximately desensitized by a control* $h \in L^2(0,T;L^2(\omega))$ *at* Ω_0 *in the sense of* (1.7)*, which also steers the state* y_0 *of* (1.4) *exactly to* 0 *at time* T*.*

One can also improve Theorem 1.7 in the case of the geometric setting (1.13):

Theorem 1.11. — *Assume the geometric setting* (1.13)*, and let* $\mathcal{E}$ *be a finite-dimensional subspace of* $W^{3,\infty}(\mathbb{R}^d,\mathbb{R}^d)$*.*

Then, for all $\xi \in L^2(0,T;L^2(\Omega_0))$ *and* $y_T \in L^2(\Omega_0)$*, for all* $\varepsilon > 0$*, there exists a control* $h \in L^2(0,T;L^2(\omega))$ *that desensitizes* J_h *exactly for* $\mathcal{E}$ *in the sense of* (1.6)*,* ε*-approximately desensitizes* J_h *in the sense of* (1.7)*, and which approximately controls* y_0 *at time* T *in the sense of* (1.20)*.*

Besides, if the source term $\xi \in L^2(0,T;L^2(\Omega_0))$ *is null-controllable, then, there exists a control* $h \in L^2(0,T;L^2(\omega))$ *that desensitizes* J_h *exactly for* $\mathcal{E}$ *in the sense of* (1.6)*,* ε*-approximately desensitizes* J_h *in the sense of* (1.7)*, and which steers* y_0 *to* 0 *at time* T *in the sense of* (1.21)*.*

Here again, the proof of Theorem 1.11 is a rather simple adaptation of the one of Theorem 1.7, based on the stronger results given by Proposition 1.8.

Remark 1.12. — Remark that Theorems 1.10 and 1.11 can be reinterpreted in terms of *robustness* of null and approximate controllability properties: they notably enable us to find a null or approximate control h for the heat equation

$$\begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = \xi + h\mathbb{1}_\omega & \text{in } (0,T)\times\Omega_0, \\ \qquad\quad y = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ \ \ y\,(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

so that the functional J_h is robust to small variations of the domain, in the sense that this control makes J_h insensitive at the first order to small perturbations of the domain.

On the exact desensitizing problem. — Note that in both geometric settings discussed so far, the question of exact desensitizing control has not been addressed. We now propose to study some cases in which we can solve the desensitizing problem. Let us start with the rather straightforward case $\Theta \Subset \omega$.

Proposition 1.13. — *Let ω and Θ be non-empty open subsets of Ω_0 such that*

$$\Theta \Subset \omega. \tag{1.22}$$

Then, for all $\xi \in L^2(0,T;L^2(\Omega_0))$, there exists $h \in L^2(0,T;L^2(\omega))$ such that the functional J_h in (1.1) *is exactly desensitized in the sense of* (1.5).

Proposition 1.13, proved in Section 4.1, in fact considers an easy case, in which we can ensure that with a suitable choice of a control function, the solution y_0 of (1.4) vanishes in $(0,T)\times\Theta$, so that the associated function z_0 satisfying (1.4) vanishes in $(0,T)\times\Omega_0$ and the result easily follows from (1.3).

Let us now consider a more subtle case, in which the outer boundary of Θ is included in ω.

Theorem 1.14. — *Let ω and Θ be smooth non-empty open subsets of Ω_0 such that*

$$\partial\Theta \text{ has only one connected component,} \quad \Theta \Subset \Omega_0, \quad \text{and} \quad \partial\Theta \subset \omega. \tag{1.23}$$

Then, for all $\xi \in L^2(0,T;L^2(\Omega_0))$, there exists $h \in L^2(0,T;L^2(\omega))$ such that the functional J_h in (1.1) *is exactly desensitized in the sense of* (1.5).

The strategy to prove this theorem is to choose the control function h such that the solution z_0 of (1.4) vanishes close to the boundary $\partial\Omega_0$. Thus, using (1.3), the exact desensitizing of J_h will immediately follow. In order to do that, we will interpret the function y_0 in (1.4) as a control function for z_0 whose goal is to impose the condition $z_0 = 0$ outside $(0,T)\times\Theta$, and we then define h in terms of y by (1.4). We refer to Section 4.2 for the proof of Theorem 1.14.

These two positive results should very likely not be considered as the usual case. In fact, we can discuss the case $\Theta = \Omega_0$ with more details:

Theorem 1.15. — *Assume that Ω_0 is smooth (of class $\mathscr{C}^\infty$), that $\Theta = \Omega_0$ and that ω is a non-empty open subset of Ω_0 such that $\omega \Subset \Omega_0$. Then, there exists a function $\xi \in L^2(0,T;L^2(\Omega_0))$ such that there exists no $h \in L^2(0,T;L^2(\omega))$ such that J_h satisfies* (1.5). *In other words, there are some $\xi \in L^2(0,T;L^2(\Omega_0))$ such that the exact desensitizing problem cannot be solved.*

The proof of this result is given in Section 4.3, and in fact only involves regularity issues.

1.3. Bibliographical comments. — We comment briefly on the bibliography, emphasizing particularly the works related to the heat equation (linear or non-linear) and dedicated approaches to solving problems related to functional desensitizing.

The question of functional desensitizing has been first introduced in [25, 5]. However, in [25, 5], the functionals under consideration were desensitized with respect to perturbation of the initial datum or of the source term, while we are discussing a new kind of desensitizing, with respect to perturbation of the domain.

Still, our approach is of course strongly inspired by the one developed in [25, 5], in which it was shown how unique continuation properties can be used to solve approximate desensitizing problems. It was then further developed to many settings, in particular when the control set and the observation set intersect.

Regarding the standard issue of desensitizing a given functional (often the L^2 norm of the state in some observation subset) involving the solution of the heat equation with respect to initial data, the general approach consists in recasting the (exact or approximated) desensitizing property in terms of an adjoint state, leading to consider a coupled system of forward-backward heat equations. Hence, exact desensitizing comes to investigate a null-controllability property which can in general be recast through an observability inequality (see [32, 4, 6] where Carleman based approaches are considered and [33], in which a Fourier approach is used). The question of ε-approximated desensitizing comes in general to solve an approximate controllability problem, leading to derive a unique continuation property (see [29, 22], in which spectral methods are employed). We also mention [18, 19, 10, 9, 8] where a functional involving the solution of another equations arising in Fluid Mechanics is considered.

1.4. Further comments and open problems. — In this article, we investigate and discuss three desensitizing properties with respect to domain variations. To conclude this introduction, we outline three open issues and hints that complement the study presented in this article and that we plan to address in the future.

Open problem #1. — Note that we were not able to answer questions Q.1′ and Q.2′ when $\omega \cap \Theta = \varnothing$. The main difference with the case $\omega \cap \Theta \neq \varnothing$ is that the approximate controllability results we are able to prove in the case $\omega \cap \Theta = \varnothing$ is weaker than in the case $\omega \cap \Theta \neq \varnothing$, compare Proposition 1.4 and Proposition 1.8. As one can check from the proofs, the stronger statement in Proposition 1.8 comes by duality from unique continuation properties for a coupled parabolic system, namely the unique continuation property (3.2) for the solutions of (3.1). Whether this unique continuation property holds when $\omega \cap \Theta = \varnothing$ is an open problem.

Open problem #2. — Can we answer the three questions Q.1 – Q.2 – Q.3 posed in this article, analyzing desensitizing issues with respect to domain variations, when the statement of problem is modified as follows:

– the heat equation (1.2) is replaced by more general controlled parabolic equations, e.g. semi-linear problems, Stokes or Navier-Stokes systems or a controlled wave equation (the analysis of desensitizing issues with respect to initial data in these two cases have been notably investigated in [18] for Stokes equations, and in [31, 30, 1] for waves). Note that this is very likely difficult to handle with the arguments developed here, since they are based on approximate controllability statements (recall Propositions 1.4 and 1.8);

– the shape functional with respect to which desensitizing is performed is replaced by a more general one of the kind

$$\int_0^T \int_\Theta j(y(t,x), \nabla y(t,x))\, dx\, dt,$$

where $j : \mathbb{R}^{1+d} \to \mathbb{R}$ is a given function, and y denotes the solution of the considered controlled system.

Open problem #3. — Can the answers provided in this article related to exact desensitizing be completed? In particular, what can be expected in the case where $\Theta \subsetneq \Omega_0$? Is it possible to answer positively or negatively to questions Q.1′ and Q.2′ when $\omega \cap \Theta = \varnothing$? Can one identify the set of functions ξ in $L^2(0,T;L^2(\mathbb{R}^d))$ for which Q.3 holds true?

1.5. Outline. — This article is organized as follows. Section 2 studies the case (1.8), i.e., $\omega \cap \Theta = \varnothing$ and $\Theta \Subset \Omega_0$, and gives the proofs of Proposition 1.4, Theorem 1.6, and Theorem 1.7. Section 3 then focuses on the case (1.13), i.e., $\omega \cap \Theta \neq \varnothing$, and provides the proofs of Proposition 1.8, Theorem 1.10 and Theorem 1.11. Then, Section 4 presents the proofs of the results on exact desensitizing control, namely Proposition 1.13, Theorem 1.14 and Theorem 1.15.

2. The case $\omega \cap \Theta = \varnothing$ and $\Theta \Subset \Omega_0$

In this whole section, we assume the geometric setting described in (1.8).

2.1. Proof of Proposition 1.4. — Proposition 1.4 can be recast in an abstract form into the problem: show that

$$\overline{\operatorname{Ran}\mathcal{L}} = (L^2(0,T;L^2(\partial\Omega_0)))^2,$$

where $\mathcal{L}$ is the operator defined for $h \in L^2(0,T;L^2(\omega))$ by

$$\mathcal{L}h = (\partial_n y_h, \partial_n z_h) \in L^2((0,T)\times\partial\Omega_0)^2,$$

where (y_h, z_h) solves

$$\begin{cases} \dfrac{\partial y_h}{\partial t} - \Delta y_h = h\mathbb{1}_\omega & \text{in } (0,T)\times\Omega_0, \\ y_h = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ y_h(0,\cdot) = 0 & \text{in } \Omega_0, \\ -\dfrac{\partial z_h}{\partial t} - \Delta z_h = \mathbb{1}_\Theta y_h & \text{in } (0,T)\times\Omega_0, \\ z_h = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ z_h(T,\cdot) = 0 & \text{in } \Omega_0. \end{cases} \tag{2.1}$$

Notice that $\mathcal{L}$ is bounded, thanks to [15, Th. 5, p. 382] and the continuity of the operator $f \in H^2(\Omega_0) \mapsto \partial_n f \in L^2((0,T) \times \partial\Omega_0)$. Therefore, by standard arguments from functional analysis, Proposition 1.4 is equivalent to showing the dual property $\operatorname{Ker} \mathcal{L}^* = \{0\}$ (we refer for instance to [7, Cor. 2.18]). Using the arguments developed in [28, Prop. 2.1], one can compute explicitly $\mathcal{L}^*$ and deduce that $\operatorname{Ker} \mathcal{L}^* = \{0\}$ is equivalent to the following unique continuation problem: If $(g_1, g_2) \in (L^2(0,T; L^2(\partial\Omega_0)))^2$, and (ψ, φ) solves

$$
\begin{cases}
-\dfrac{\partial \psi}{\partial t} - \Delta \psi = \mathbb{1}_\Theta \varphi & \text{in } (0,T) \times \Omega_0, \\
\psi = g_1 & \text{on } (0,T) \times \partial\Omega_0, \\
\psi(T, \cdot) = 0 & \text{in } \Omega_0, \\
\dfrac{\partial \varphi}{\partial t} - \Delta \varphi = 0 & \text{in } (0,T) \times \Omega_0, \\
\varphi = g_2 & \text{on } (0,T) \times \partial\Omega_0, \\
\varphi(0, \cdot) = 0 & \text{in } \Omega_0,
\end{cases}
\tag{2.2}
$$

then, we have the following unique continuation property:

$$
\psi = 0 \text{ in } \ (0,T) \times \omega \implies g_1 = g_2 = 0. \tag{2.3}
$$

We now prove this unique continuation property. Let $(g_1, g_2) \in (L^2(0,T; L^2(\partial\Omega_0)))^2$ be such that the solution (ψ, φ) of (2.2) satisfies $\psi = 0$ in $(0,T) \times \omega$. From [27, (15.17–18), p. 86], we notably infer that $(\psi, \varphi) \in L^2((0,T) \times \Omega_0)$.

We first work in the set $(0,T) \times (\Omega_0 \setminus \overline{\Theta})$. There, ψ satisfies the usual backward heat equation

$$
-\frac{\partial \psi}{\partial t} - \Delta \psi = 0 \quad \text{in } (0,T) \times (\Omega_0 \setminus \overline{\Theta}),
$$

where we do not specify any initial or boundary conditions. Therefore, since $\omega \subset \Omega_0 \setminus \overline{\Theta}$ and $\Omega_0 \setminus \overline{\Theta}$ is connected thanks to Assumption (1.8), using the Holmgren theorem for the heat equation (see e.g. [21, §8.6 & Th. 8.6.5]), we thus infer that $\psi = 0$ in $(0,T) \times (\Omega_0 \setminus \overline{\Theta})$. In particular, using [34, Prop. 7.1.3], since ψ the unique weak solution of (2.2) in the sense of [34, Def. 7.1.2, p. 342], for any $\zeta \in H^2(\Omega_0) \cap H^1_0(\Omega_0)$ such that $\Delta\zeta \in H^1_0(\Omega_0)$, for any $t \in [0,T]$, we have

$$
\int_0^t \int_{(0,T) \times \partial\Omega_0} g_1 \partial_n \zeta \, d\sigma dt = 0.
$$

Differentiating this inequality with respect to t (which is possible because

$$
g_1(\cdot, x) \partial_n \zeta(x) \in L^2(0,T) \quad \text{for almost all } x \in (0,T) \times \partial\Omega_0,
$$

since the Neumann trace operator $f \in H^2(\Omega_0) \mapsto \partial_n f \in L^2(\partial\Omega_0)$ is well-defined), we obtain that for any $t \in [0,T]$, we have

$$
\int_{\partial\Omega_0} g_1(t, \cdot) \partial_n \zeta \, d\sigma = 0.
$$

Since $\{\zeta \in H^2(\Omega_0) \cap H_0^1(\Omega_0) \mid \Delta\zeta \in H_0^1(\Omega_0)\}$ is dense in $H^2(\Omega_0) \cap H_0^1(\Omega_0)$ for the H^2-norm, using the continuity of the Neumann trace operator

$$f \in H^2(\Omega_0) \longmapsto \partial_n f \in L^2((0,T) \times \partial\Omega_0),$$

we deduce that for any $\zeta \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$, we have, for any $t \in [0,T]$,

$$\int_{\partial\Omega_0} g_1(t,\cdot)\partial_n\zeta d\sigma = 0.$$

Since the extended trace operator

$$f \in H^2(\Omega) \longmapsto (f_{|\partial\Omega_0}, \partial_n f_{|\partial\Omega_0}) \in H^{3/2}(\partial\Omega_0) \times H^{1/2}(\partial\Omega_0)$$

is surjective, taking into account that $0 \in H^{3/2}(\partial\Omega_0)$, we deduce that for any $h \in H^{1/2}(\partial\Omega_0)$, we have, for any $t \in [0,T]$,

$$\int_{\partial\Omega_0} g_1(t,\cdot)h d\sigma = 0.$$

Since $H^{1/2}(\partial\Omega_0)$ is dense in $L^2(\partial\Omega_0)$ for the $L^2(\partial\Omega_0)$-norm, we deduce that $g_1 = 0$ on $(0,T) \times \partial\Omega_0$, so that $\psi \equiv 0$ on $\Omega_0 \smallsetminus \overline{\Theta}$ in the classical sense. Notably, $\partial_n\psi = 0$ on the whole boundary $\partial\Omega_0$ since $\Theta \Subset \Omega_0$.

We next remark that, by standard local regularity results for solutions of the heat equation, the function φ is smooth away from the boundary $\{0,T\} \times \partial\Omega_0$ (of class $\mathscr{C}^\infty((0,T) \times \Omega_0)$). Accordingly, $\mathbb{1}_\Theta\varphi$ belongs to $L^2(0,T;L^2(\Omega))$, and by standard maximal regularity results, the solution ψ belongs to

$$L^2(0,T;H^2_{\mathrm{loc}}(\Omega_0)) \cap H^1(0,T;L^2_{\mathrm{loc}}(\Omega_0)).$$

In view of this regularity, and due to the fact that $\psi = 0$ in a neighbourhood of $(0,T) \times \partial\Omega_0$, we can multiply equation $(2.2)_{(1)}$ by φ and perform several integration by parts, using $g_1 = \partial_n\psi = 0$ on $\partial\Omega_0$, and $\psi = 0$ in $(0,T) \times (\Omega_0 \smallsetminus \overline{\Theta})$:

$$\begin{aligned}
\int_0^T \int_\Theta |\varphi|^2 \, dx \, dt &= \int_0^T \int_{\Omega_0} \varphi\Big(-\frac{\partial\psi}{\partial t} - \Delta\psi\Big) \, dx \, dt \\
&= -\int_{\Omega_0} \varphi(\cdot,x)\psi(\cdot,x) \, dx\Big|_0^T - \int_0^T \int_{\partial\Omega_0} \varphi\partial_n\psi d\sigma dt \\
&\qquad + \int_0^T \int_{\partial\Omega_0} \partial_n\varphi\psi d\sigma dt + \int_0^T \int_{\Omega_0} \Big(\frac{\partial\varphi}{\partial t} - \Delta\varphi\Big)\psi \, dx \, dt \\
&= 0.
\end{aligned}$$

Therefore, $\varphi = 0$ in $(0,T) \times \Theta$, and by the classical unique continuation properties for the heat equation, $\varphi = 0$ in $(0,T) \times \Omega_0$, and in particular $g_2 = 0$. This concludes the proof of (2.3) for the solutions of (2.2), which proves that $\operatorname{Ker} \mathcal{L}^* = \{0\}$. Hence, Proposition 1.4 follows.

2.2. Proof of Theorem 1.6. — The proof of Theorem 1.6 mainly reduces to Proposition 1.4. Indeed, from Proposition 1.4 with $f_1 = f_2 = 0$, for any $\varepsilon > 0$, there exists a control function $h \in L^2(0,T;L^2(\omega))$ such that the solution (y_0, z_0) of (1.4) satisfies

$$\|\partial_n y_0\|_{L^2(0,T;L^2(\partial\Omega_0))} + \|\partial_n z_0\|_{L^2(0,T;L^2(\partial\Omega_0))} \leqslant \sqrt{\varepsilon}. \tag{2.4}$$

Accordingly, using (1.3), we infer (1.11) and

$$\forall\, \boldsymbol{V} \in \mathcal{E}, \quad \left| \frac{d}{d\tau} \left(J\left(\Omega_{\tau \boldsymbol{V}}\right)\right)\Big|_{\tau=0} \right| \leqslant \varepsilon \|\boldsymbol{V}\cdot\boldsymbol{n}\|_{L^\infty(\partial\Omega_0)},$$

which concludes the proof of Theorem 1.6, by using the fact that $W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ is included into the space of continuous vector fields, so that

$$\|\boldsymbol{V}\|_{L^\infty(\partial\Omega_0)} \leqslant \|\boldsymbol{V}\|_{\mathscr{C}^0(\overline{\Omega_0})} \leqslant \|\boldsymbol{V}\|_{W^{3,\infty}(\Omega_0)}.$$

2.3. Proof of Theorem 1.7. — Before proving Theorem 1.7, let us give an insight of the strategy of our proof. It will be divided into three steps.

– First step: we will treat the case of the exact desensitizing for J_h with respect to a finite dimensional space $\mathcal{E}$ of dimension 1, in the sense of (1.6), in order to explain the main idea behind our proof. This case was already studied and analyzed with different techniques in [28].

– Second step: we will explain how to modify our first step to the case of the exact desensitizing for J_h with respect to any finite dimensional space $\mathcal{E}$, in the sense of (1.6), by using the Brouwer fixed point Theorem.

– Last step: we will explain how the construction made in the previous step together with the use of Proposition 1.4 ensures that one can simultaneously solve the ε-approximate desensitizing of J_h and its exact desensitizing with respect to a finite dimensional space $\mathcal{E}$.

Recall that we assume the geometric setting (1.8).

First step: exact desensitizing in the case $\mathcal{E} = \operatorname{Span}\{\boldsymbol{V}\}$. — Let us fix some $\boldsymbol{V} \in W^{3,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ supposed to be non-zero, and let us consider the case $\mathcal{E} = \operatorname{Span}\{\boldsymbol{V}\}$, i.e., the case of a one-dimensional vector space $\mathcal{E}$, and only focus on the proof of exact desensitizing of J_h with respect to $\mathcal{E} = \operatorname{Span}\{\boldsymbol{V}\}$.

Recall that, according to (1.3), the exact desensitizing problem for $\mathcal{E}$ is equivalent to determining a control function $h \in L^2(0,T;L^2(\omega_0))$ such that

$$\int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n})\left(\int_0^T \partial_n y_0 \partial_n z_0 \, dt\right) d\sigma = 0, \tag{2.5}$$

where (y_0, z_0) solves (1.4).

Of course, if $\boldsymbol{V}\cdot\boldsymbol{n} = 0$ on $\partial\Omega_0$ (which may happen since we only assumed that $\boldsymbol{V}$ is non-zero as a function defined in $\mathbb{R}^d$), then (2.5) is automatically verified and the problem is trivial. Hence, from now on, we assume that $\boldsymbol{V}\cdot\boldsymbol{n}$ does not vanish identically on $\partial\Omega_0$.

To study condition (2.5), we introduce the pairs (y_ξ, z_ξ) and (y_h, z_h) as the solutions of the linear systems

$$\begin{cases} \dfrac{\partial y_\xi}{\partial t} - \Delta y_\xi = \xi & \text{in } (0,T)\times\Omega_0, \\ y_\xi = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ y_\xi(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases} \tag{2.6}$$

$$\begin{cases} -\dfrac{\partial z_\xi}{\partial t} - \Delta z_\xi = y_\xi\chi_\Theta & \text{in } (0,T)\times\Omega_0, \\ z_\xi = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ z_\xi(T,\cdot) = 0 & \text{in } \Omega_0, \end{cases} \tag{2.7}$$

$$\begin{cases} \dfrac{\partial y_h}{\partial t} - \Delta y_h = h\mathbb{1}_\omega & \text{in } (0,T)\times\Omega_0, \\ y_h = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ y_h(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases} \tag{2.8}$$

$$\begin{cases} -\dfrac{\partial z_h}{\partial t} - \Delta z_h = y_h\chi_\Theta & \text{in } (0,T)\times\Omega_0, \\ z_h = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ z_h(T,\cdot) = 0 & \text{in } \Omega_0. \end{cases} \tag{2.9}$$

This allows to decompose the solution (y_0, z_0) of (1.4) as

$$y_0 = y_\xi + y_h, \quad \text{and} \quad z_0 = z_\xi + z_h.$$

Now, we introduce the function

$$\mathcal{U} : L^2(0,T; L^2(\omega)) \longrightarrow \mathbb{R}$$

defined for $h \in L^2(0,T;L^2(\omega))$ by

$$\mathcal{U}(h) = \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n}) \bigg(\int_0^T (\partial_n y_\xi + \partial_n y_h)(\partial_n z_\xi + \partial_n z_h)\, dt \bigg)\, d\sigma,$$

so that condition (2.5) can be simply reformulated as $\mathcal{U}(h) = 0$.

Our goal is to find $h \in L^2(0,T;L^2(\omega))$ such that $\mathcal{U}(h) = 0$. To construct such a function h, we will look for a two-dimensional vector space spanned by two elements h_1 and h_2 in $L^2(0,T;L^2(\omega))$ such that the function $\mathcal{U}$ vanishes for at least one element of $\operatorname{Span}\{h_1, h_2\}$, i.e., we want to show that

$$\exists\, (h_1, h_2) \in L^2(0,T;L^2(\omega))^2,\ \exists\, (\lambda_1,\lambda_2)\in\mathbb{R}^2 \text{ such that } \mathcal{U}(\lambda_1 h_1 + \lambda_2 h_2) = 0. \tag{2.10}$$

To show that this can be done, we observe that the map

$$h \in L^2(0,T;L^2(\omega)) \longrightarrow \big(\partial_n y_h, \partial_n z_h\big) \in \big(L^2(0,T;L^2(\partial\Omega_0))\big)^2$$

is linear, hence it is obvious that the function $\mathcal{U}$ can be decomposed as follows

$$\mathcal{U}(h) = Q(h) + L(h) + C,$$

where Q is quadratic in h, L is linear in h and C does not depend on h:

$$\begin{aligned}
Q(h) &= \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n})\Big(\int_0^T \partial_n y_h \partial_n z_h \, dt\Big)\, d\sigma, \\
L(h) &= \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n})\,\Big(\int_0^T (\partial_n y_\xi \partial_n z_h + \partial_n y_h \partial_n z_\xi)\, dt\Big)\, d\sigma, \\
C &= \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n})\Big(\int_0^T \partial_n y_\xi \partial_n z_\xi \, dt\Big) d\sigma.
\end{aligned}\tag{2.11}$$

Accordingly, problem (2.10) amounts to finding h_1, h_2 in $L^2(0,T;L^2(\omega))$ and λ_1, λ_2 in $\mathbb{R}$ such that

$$\lambda_1^2 Q_{11}(h_1) + \lambda_1\lambda_2 Q_{12}(h_1,h_2) + \lambda_2^2 Q_{22}(h_2) + \lambda_1 L_1(h_1) + \lambda_2 L_2(h_2) + C = 0,\tag{2.12}$$

where

$$\begin{aligned}
Q_{11}(h_1) &= \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n})\Big(\int_0^T \partial_n y_{h_1} \partial_n z_{h_1} \, dt\Big) d\sigma, \\
Q_{12}(h_1,h_2) &= \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n})\Big(\int_0^T (\partial_n y_{h_1} \partial_n z_{h_2} + \partial_n y_{h_2} \partial_n z_{h_1})\, dt\Big) d\sigma, \\
Q_{22}(h_2) &= \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n})\Big(\int_0^T \partial_n y_{h_2} \partial_n z_{h_2} \, dt\Big) d\sigma, \\
L_1(h_1) &= \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n})\Big(\int_0^T (\partial_n y_{h_1} \partial_n z_\xi + \partial_n y_\xi \partial_n z_{h_1})\, dt\Big) d\sigma, \\
L_2(h_2) &= \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n})\Big(\int_0^T (\partial_n y_{h_2} \partial_n z_\xi + \partial_n y_\xi \partial_n z_{h_2})\, dt\Big) d\sigma.
\end{aligned}$$

Our strategy then reduces to choose h_1 and h_2 such that the Neumann traces $(\partial_n y_{h_i}, \partial_n z_{h_i})$, $i=1,2$ in $L^2(0,T;L^2(\partial\Omega_0))$ for the solutions of (2.8)–(2.9) with h_i, allows to guarantee the existence of a solution $(\lambda_1,\lambda_2)\in\mathbb{R}^2$ to (2.12).

Let us choose $(\gamma_{i,y},\gamma_{i,z})$ in $(L^2(0,T;L^2(\partial\Omega_0)))^2$ for $i=1,2$ as follows:

$$\begin{aligned}
\gamma_{1,y} &= \frac{(\boldsymbol{V}\cdot\boldsymbol{n})}{\|\boldsymbol{V}\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega_0)}}\,\frac{\mathbb{1}_{(0,T/2)}}{T}, &\qquad \gamma_{1,z} &= \mathbb{1}_{(T/2,T)}, \\
\gamma_{2,y} &= \frac{(\boldsymbol{V}\cdot\boldsymbol{n})}{\|\boldsymbol{V}\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega_0)}}\,\frac{\mathbb{1}_{(T/2,T)}}{T}, &\qquad \gamma_{2,z} &= \mathbb{1}_{(0,T/2)}.
\end{aligned}$$

We easily have that

$$\begin{aligned}
&\int_0^T \gamma_{1,y}(t,x)\gamma_{1,z}(t,x)\, dt = 0, \quad \text{for all } x\in\partial\Omega_0, \\
&\int_0^T \gamma_{2,y}(t,x)\gamma_{2,z}(t,x)\, dt = 0, \quad \text{for all } x\in\partial\Omega_0, \\
&\int_0^T (\gamma_{1,y}(t,x)\gamma_{2,z}(t,x) + \gamma_{2,y}(t,x)\gamma_{1,z}(t,x))\, dt = \frac{(\boldsymbol{V}\cdot\boldsymbol{n})(x)}{\|\boldsymbol{V}\cdot\boldsymbol{n}\|^2_{L^2(\partial\Omega_0)}}, \quad \text{for all } x\in\partial\Omega_0.
\end{aligned}$$

If it were possible to find some $h_1 \in L^2(0,T;L^2(\omega))$ and $h_2 \in L^2(0,T;L^2(\omega))$ such that

$$(\partial_n y_{h_1}, \partial_n z_{h_1}, \partial_n y_{h_2}, \partial_n z_{h_2}) = (\gamma_{1,y}, \gamma_{1,z}, \gamma_{2,y}, \gamma_{2,z}), \tag{2.13}$$

then, we would have $Q_{11} = Q_{22} = 0$ and $Q_{12} = 1$, so that equation (2.12) with $\lambda_2 = |\lambda_1|$ would become

$$\lambda_1|\lambda_1| + \lambda_1 L_1(h_1) + |\lambda_1| L_2(h_2) + C = 0,$$

which can obviously be solved for some $\lambda_1 \in \mathbb{R}$ according to the intermediate value theorem, since the left hand-side goes to $-\infty$ when $\lambda_1 \to -\infty$ and to $+\infty$ when $\lambda_1 \to +\infty$ while being continuous on $\mathbb{R}$.

Unfortunately, we cannot a priori find $h_1 \in L^2(0,T;L^2(\omega))$ and $h_2 \in L^2(0,T;L^2(\omega))$ such that (2.13) *exactly* holds, but Proposition 1.4 ensures that (2.13) *approximately* holds, in the following sense: for any $\alpha > 0$, there exists $h_1^\alpha \in L^2(0,T;L^2(\omega))$ and $h_2^\alpha \in L^2(0,T;L^2(\omega))$ such that

$$\begin{aligned}
&\|\partial_n y_{h_1^\alpha} - \gamma_{1,y}\|_{L^2(0,T;L^2(\partial\Omega_0))} \leqslant \alpha,\\
&\|\partial_n q_{h_1^\alpha} - \gamma_{1,q}\|_{L^2(0,T;L^2(\partial\Omega_0))} \leqslant \alpha,\\
&\|\partial_n y_{h_2^\alpha} - \gamma_{2,y}\|_{L^2(0,T;L^2(\partial\Omega_0))} \leqslant \alpha,\\
&\|\partial_n q_{h_2^\alpha} - \gamma_{2,q}\|_{L^2(0,T;L^2(\partial\Omega_0))} \leqslant \alpha.
\end{aligned}$$

Accordingly, with this choice of h_1^α and h_2^α, the quadratic part Q that is given in (2.11) is only slightly perturbed in the sense that

$$|Q_{11}(h_1^\alpha)| + |Q_{12}(h_1^\alpha, h_2^\alpha) - 1| + |Q_{22}(h_2^\alpha)| \leqslant C\alpha,$$

where C only depends on the norm of $(\gamma_{i,y}, \gamma_{i,z})_{i\in\{1,2\}}$ in $L^2(0,T;L^2(\partial\Omega_0))$. Therefore, taking $\alpha > 0$ such that $C\alpha \leqslant 1/2$, and choosing $(h_1,h_2) = (h_1^\alpha, h_2^\alpha)$, we get that

$$|Q(\lambda_1 h_1 + |\lambda_1| h_2) - \lambda_1|\lambda_1|| \leqslant \frac{|\lambda_1|^2}{2}.$$

Accordingly, the continuous function $\lambda_1 \in \mathbb{R} \mapsto Q(\lambda_1 h_1 + |\lambda_1| h_2)$ goes to $-\infty$ as $\lambda_1 \to -\infty$ and to $+\infty$ as $\lambda_1 \to \infty$, and hence, the function

$$\lambda_1 \in \mathbb{R} \longmapsto \mathcal{U}(\lambda_1 h_1 + |\lambda_1| h_2)$$

inherits the same property. Hence, it vanishes for some $\lambda_1 \in \mathbb{R}$.

This concludes the proof of exact desensitizing of J_h for a vector space $\mathcal{E}$ is of dimension 1.

Second step: exact desensitizing for a finite-dimensional vector space $\mathcal{E}$. — Now, we assume that $\mathcal{E}$ is of finite dimension $N \geqslant 2$. Our goal is to mimic the method developed when $\mathcal{E}$ was a one-dimensional vector space, replacing the intermediate value theorem by a Brouwer fixed point argument.

Let $\mathscr{E} = \{\boldsymbol{V}\cdot\boldsymbol{n},\, \boldsymbol{V} \in \mathcal{E}\}$, which is itself a finite dimensional subspace of $L^2(\partial\Omega_0)$ of dimension $M \leqslant N$, and choose an orthonormal basis $(\boldsymbol{V}_k \cdot \boldsymbol{n})_{k\in[\![1,M]\!]}$ of $\mathscr{E}$ for the canonical inner product on $L^2(\partial\Omega_0)$.

Following the previous case, for all $k \in [\![1, M]\!]$, we introduce

$$\mathcal{U}_k(h) := \int_{\partial\Omega_0} (\boldsymbol{V}_k \cdot \boldsymbol{n}) \bigg(\int_0^T (\partial_n y_\xi + \partial_n y_h)(\partial_n z_\xi + \partial_n z_h)\, dt \bigg)\, d\sigma, \tag{2.14}$$

where y_h, z_h, y_ξ and z_ξ are defined in (2.6), (2.7), (2.8) and (2.9).

According to (1.6), the desensitizing problem for J_h for the family $\mathcal{E}$ amounts to finding a function $h \in L^2(0,T;L^2(\omega))$ such that for all $k \in [\![1, M]\!]$, $\mathcal{U}_k(h) = 0$.

As in the first step, for all $k \in [\![1, M]\!]$, the function $\mathcal{U}_k$ can be decomposed as

$$\mathcal{U}_k(h) = Q_k(h) + L_k(h) + C_k, \tag{2.15}$$

where Q_k is quadratic in h, L_k is linear in h and C_k does not depend on h:

$$Q_k(h) = \int_{\partial\Omega_0} (\boldsymbol{V}_k \cdot \boldsymbol{n}) \bigg(\int_0^T \partial_n y_h \partial_n z_h \, dt \bigg) d\sigma, \tag{2.16}$$

$$L_k(h) = \int_{\partial\Omega_0} (\boldsymbol{V}_k \cdot \boldsymbol{n}) \,\bigg(\int_0^T (\partial_n y_\xi \partial_n z_h + \partial_n y_h \partial_n z_\xi)\, dt \bigg) d\sigma, \tag{2.17}$$

$$C_k = \int_{\partial\Omega_0} (\boldsymbol{V}_k \cdot \boldsymbol{n}) \bigg(\int_0^T \partial_n y_\xi \partial_n z_\xi \, dt \bigg) d\sigma. \tag{2.18}$$

For each $k \in [\![1, M]\!]$, we introduce the following elements of $L^2(0;T;L^2(\partial\Omega_0))$:

$$\begin{aligned}
\gamma_{k,1,y} &= \frac{(\boldsymbol{V}_k \cdot \boldsymbol{n})}{\|\boldsymbol{V}_k \cdot n\|^2_{L^2(\partial\Omega_0)}} \frac{M}{T} \mathbb{1}_{((k-1)T/M,(2k-1)T/(2M))},\\
\gamma_{k,1,z} &= \mathbb{1}_{((2k-1)T/(2M),kT/M)},\\
\gamma_{k,2,y} &= \frac{(\boldsymbol{V}_k \cdot \boldsymbol{n})}{\|\boldsymbol{V}_k \cdot n\|^2_{L^2(\partial\Omega_0)}} \frac{M}{T} \mathbb{1}_{((2k-1)T/(2M),kT/M)},\\
\gamma_{k,2,z} &= \mathbb{1}_{((k-1)T/M,(2k-1)T/(2M))}.
\end{aligned}$$

It is then easy to check that

$$\begin{gathered}\forall\, (i,j,k) \in [\![1, M]\!]^3,\ (a,b) \in \{1,2\}^2,\\
\int_{\partial\Omega_0} \boldsymbol{V}_k \cdot \boldsymbol{n} \bigg(\int_0^T (\gamma_{i,a,y}\gamma_{j,b,z} + \gamma_{j,b,y}\gamma_{i,a,z})\, dt \bigg) d\sigma = \delta_{i,j,k} \mathbb{1}_{a\neq b},\end{gathered} \tag{2.19}$$

where $\delta_{i,j,k}$ denotes the Kronecker symbol ($\delta_{i,j,k} = 1$ if and only if $i = j = k$, and $= 0$ otherwise).

Now, for $k \in [\![1, M]\!]$ and $a \in \{1,2\}$, using Proposition 1.4, for any $\alpha > 0$, there exists $h^\alpha_{k,a} \in L^2(0,T;L^2(\omega))$ such that the solution $(y_{h^\alpha_{k,a}}, z_{h^\alpha_{k,a}})$ of (2.8)–(2.9) satisfies:

$$\|\partial_n y_{h^\alpha_{k,a}} - \gamma_{k,a,y}\|_{L^2(0,T;L^2(\partial\ \Omega_0))} + \|\partial_n z_{h^\alpha_{k,a}} - \gamma_{k,a,y}\|_{L^2(0,T;L^2(\partial\ \Omega_0))} \leqslant \alpha. \tag{2.20}$$

Using (2.19) and (2.20), we easily show that, for any $\lambda = (\lambda_k)_{k\in[\![1,M]\!]} \in \mathbb{R}^M$, for any $k \in [\![1, M]\!]$,

$$\bigg| Q_k\bigg(\sum_{j=1}^M (\lambda_j h^\alpha_{j,1} + |\lambda_j| h^\alpha_{j,2}) \bigg) - \lambda_k |\lambda_k| \bigg| \leqslant C\alpha \|\lambda\|^2_{\mathbb{R}^M},$$

for some $C>0$ independent of α, where Q_k is defined in (2.16). Therefore, choosing $\alpha>0$ small enough such that $C\alpha\leqslant 1/(2M)$ and dropping the superscript α from now on, we have, for all $\lambda=(\lambda_k)_{k\in[\![1,M]\!]}\in\mathbb{R}^M$, for any $k\in[\![1,M]\!]$,

$$\left|Q_k\Big(\sum_{j=1}^M(\lambda_j h_{j,1}+|\lambda_j|h_{j,2})\Big)-\lambda_k|\lambda_k|\right|\leqslant\frac{1}{2M}\|\lambda\|^2_{\mathbb{R}^M}. \tag{2.21}$$

Our next goal is to check that

$$\exists\,(\lambda_k)_{k\in[\![1,M]\!]}\in\mathbb{R}^M,\ \forall\,k\in[\![1,M]\!],\quad \mathcal{U}_k\Big(\sum_{j=1}^M(\lambda_j h_{j,1}+|\lambda_j|h_{j,2})\Big)=0, \tag{2.22}$$

where $\mathcal{U}_k$ is defined in (2.14). Based on the decomposition (2.15), in order to do that, we will use a fixed point argument. We introduce the continuous function $s:\mathbb{R}\to\mathbb{R}$ given by

$$s(y)=\begin{cases}\sqrt{y} & \text{if } y\geqslant 0,\\ -\sqrt{-y} & \text{if } y<0.\end{cases}$$

Now, let us define the mapping

$$F:\lambda=(\lambda_k)_{k\in[\![1,M]\!]}\in\mathbb{R}^M\longmapsto\widehat{\lambda}=(\widehat{\lambda_k})_{k\in[\![1,M]\!]}\in\mathbb{R}^M,$$

where, for all $k\in[\![1,M]\!]$, $\widehat{\lambda_k}$ is defined as

$$\widehat{\lambda_k}=s\bigg(-\Big(Q_k\Big(\sum_{j=1}^M(\lambda_j h_{j,1}+|\lambda_j|h_{j,2})\Big)-\lambda_k|\lambda_k|\Big)\\ -L_k\Big(\sum_{j=1}^M(\lambda_j h_{j,1}+|\lambda_j|h_{j,2})\Big)-C_k\bigg),$$

for Q_k, L_k and C_k given in (2.16)–(2.17)–(2.18). Notably, by definition of s, we have

$$\widehat{\lambda_k}\,|\widehat{\lambda_k}|=-\Big(Q_k\Big(\sum_{j=1}^M(\lambda_j h_{j,1}+|\lambda_j|h_{j,2})\Big)-\lambda_k|\lambda_k|\Big)\\ -L_k\Big(\sum_{j=1}^M(\lambda_j h_{j,1}+|\lambda_j|h_{j,2})\Big)-C_k. \tag{2.23}$$

Using the identity (2.23), one easily checks that if $(\lambda_k)_{k\in[\![1,M]\!]}\in\mathbb{R}^M$ is a fixed point of F, then, it solves the problem (2.22).

It is clear that the function F is continuous in $\mathbb{R}^M$. We will simply show that it maps a ball into itself and conclude using Brouwer fixed point theorem. In order to prove that, we use the bound (2.21) and the following straightforward estimates: for all $k\in[\![1,M]\!]$,

$$\left|L_k\Big(\sum_{j=1}^M(\lambda_j h_{j,1}+|\lambda_j|h_{j,2})\Big)\right|\leqslant C\,\|(\partial_n y_\xi,\partial_n z_\xi)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2}\|\lambda\|_{\mathbb{R}^M},$$

$$|C_k|\leqslant\widetilde{C}\,\|(\partial_n y_\xi,\partial_n z_\xi)\|^2_{(L^2(0,T;L^2(\partial\Omega_0)))^2},$$

where $\widetilde{C}$ is a positive constant (independent of $k\in[\![1,M]\!]$ and of ξ).

Accordingly, by denoting $\|\cdot\|_{\mathbb{R}^M}$ the euclidean norm, one has for all $\lambda \in \mathbb{R}^M$,

$$\begin{aligned}\|\widehat{\lambda}\|^2_{\mathbb{R}^M} &\leqslant \frac{1}{2}\|\lambda\|^2_{\mathbb{R}^M} + \widetilde{C}\,\|(\partial_n y_\xi, \partial_n z_\xi)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2}\|\lambda\|_{\mathbb{R}^M} \\ &\qquad + \widetilde{C}\,\|(\partial_n y_\xi, \partial_n z_\xi)\|^2_{(L^2(0,T;L^2(\partial\Omega_0)))^2} \\ &\leqslant \frac{3}{4}\|\lambda\|^2_{\mathbb{R}^M} + 2\widetilde{C}\,\|(\partial_n y_\xi, \partial_n z_\xi)\|^2_{(L^2(0,T;L^2(\partial\Omega_0)))^2}.\end{aligned}$$

In particular, setting

$$R = 2\sqrt{2\widetilde{C}}\,\|(\partial_n y_\xi, \partial_n z_\xi)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2},$$

where $\widetilde{C}$ denotes the constant in the previous estimate, the closed ball of $\mathbb{R}^M$ of radius R (for the euclidean norm) is stable by F. Therefore, by Brouwer fixed point theorem, there exists $\lambda \in \mathbb{R}^M$ in the closed ball of radius R such that $F(\lambda) = \lambda$.

This proves the existence of $\lambda = (\lambda_k)_{k\in[\![1,M]\!]} \in \mathbb{R}^M$ satisfying (2.22) and with the bound

$$\|\lambda\|_{\mathbb{R}^M} \leqslant \widehat{C}\,\|(\partial_n y_\xi, \partial_n z_\xi)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2}, \tag{2.24}$$

for some positive constant $\widehat{C}$. In particular, this bound implies that the corresponding control

$$h = \sum_{j=1}^{M}(\lambda_j h_{j,1} + |\lambda_j| h_{j,2}), \tag{2.25}$$

and the corresponding controlled trajectory (y_h, z_h) of (2.8)–(2.9) satisfies

$$\begin{aligned}\|h\|_{L^2(0,T;L^2(\omega))} &+ \|(y_h, z_h)\|_{(L^2(0,T;H^2(\Omega_0)))^2} \\ &+ \|(\partial_n y_h, \partial_n z_h)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2} \\ &\qquad\leqslant C\|(\partial_n y_\xi, \partial_n z_\xi)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2},\end{aligned} \tag{2.26}$$

for some positive constant C.

Last step: adding the approximate desensitizing property. — The idea there is to decompose the approximate and exact desensitizing problems. To be more precise, we will choose the control h in two steps, under the form $h = h_0 + h_1$, where h_0 is used to get the approximate desensitizing property, and h_1 is then chosen afterward to get the exact desensitizing property in the directions of $\mathcal{E}$.

Given $\xi \in L^2(0,T;L^2(\Omega_0))$ and $\varepsilon > 0$, we set

$$\varepsilon_0 = \frac{\sqrt{\varepsilon}}{C+1}, \quad \text{where } C \text{ is the constant in (2.26).}$$

According to Proposition 1.4, there exists $h_0 \in L^2(0,T;L^2(\omega))$ such that the solution (y,z) of (1.9) satisfies

$$\|(\partial_n y, \partial_n z)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2} \leqslant \varepsilon_0. \tag{2.27}$$

Now, we set $\xi_1 = \xi + h_0 \mathbb{1}_\omega$, which belongs to $L^2(0,T;L^2(\Omega_0))$. According to the previous paragraph applied for the source term ξ_1, there exists $h_1 \in L^2(0,T;L^2(\omega))$ such that

$$\forall \boldsymbol{V} \in \mathcal{E}, \quad \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n}) \bigg(\int_0^T \partial_n y_0 \partial_n z_0 \, dt \bigg) \, d\sigma = 0, \tag{2.28}$$

where (y_0, z_0) is the solution of

$$\begin{cases} \dfrac{\partial y_0}{\partial t} - \Delta y_0 = \xi_1 + h_1 \mathbb{1}_\omega = \xi + (h_0 + h_1)\mathbb{1}_\omega & \text{in } (0,T)\times\Omega_0, \\ y_0 = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ y_0(0,\cdot) = 0 & \text{in } \Omega_0, \\ -\dfrac{\partial z_0}{\partial t} - \Delta z_0 = \mathbb{1}_\Theta y_0 & \text{in } (0,T)\times\Omega_0, \\ z_0 = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ z_0(T,\cdot) = 0 & \text{in } \Omega_0. \end{cases} \tag{2.29}$$

Besides,

$$\partial_n y_0 = \partial_n y + \partial_n y_{h_1}, \quad \partial_n z_0 = \partial_n z + \partial_n z_{h_1},$$

so that the bounds (2.26) and (2.27) imply

$$\|(\partial_n y_0, \partial_n z_0)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2} \leqslant (C+1)\varepsilon_0,$$

where C is the constant in (2.26). We then easily get that, for all $\boldsymbol{V} \in W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$,

$$\begin{aligned} \bigg| \int_{\partial\Omega_0} (\boldsymbol{V}\cdot\boldsymbol{n}) \bigg(\int_0^T \partial_n y_0 \partial_n z_0 \, dt \bigg) \, d\sigma \bigg| &\leqslant (C+1)^2 \varepsilon_0^2 \|\boldsymbol{V}\cdot\boldsymbol{n}\|_{L^\infty(\partial\Omega_0)} \\ &\leqslant \varepsilon \|\boldsymbol{V}\cdot\boldsymbol{n}\|_{L^\infty(\partial\Omega_0)} \\ &\leqslant \varepsilon \|\boldsymbol{V}\cdot\boldsymbol{n}\|_{W^{3,\infty}(\Omega_0)}. \end{aligned}$$

In other words, $h = h_0 + h_1$ exactly desensitizes J_h for $\mathcal{E}$ and ε-approximately desensitizes J_h.

3. The case $\omega \cap \Theta \neq \varnothing$

In this whole section, we assume the geometric setting (1.13).

3.1. Proof of Proposition 1.8. — Similarly to the proof of Proposition 1.4, we reformulate the first part of Proposition 1.8, i.e., the approximate controllability property (1.14), as the density of the range of the operator $\mathcal{L}$ defined for $h \in L^2(0,T;L^2(\omega))$ with values in $(L^2(0,T;L^2(\partial\Omega_0)))^2 \times L^2(\Omega_0)$ by

$$\mathcal{L}(h) = (\partial_n y, \partial_n z, y(T)),$$

where (y,z) is the solution of (2.1). Arguments similar to the one developed in the proof of Proposition 1.4 imply that $\mathcal{L}$ is a bounded operator. By using similar classical arguments resting upon duality, as in the proof of Proposition 1.4, one has $\overline{\operatorname{Ran}\mathcal{L}} = (L^2(0,T;L^2(\partial\Omega_0)))^2 \times L^2(\Omega_0)$ if and only if $\operatorname{Ker}\mathcal{L}^* = \{0\}$ (we refer for instance to [7, Cor. 2.18]). Using the arguments developed in [28, Prop. 2.1] together with [12, Th. 2.43, p. 56] (in order to take into account the term $y(T)$ in the definition of $\mathcal{L}$), one can compute explicitly $\mathcal{L}^*$ and deduce that $\operatorname{Ker}\mathcal{L}^* = \{0\}$ is equivalent to the following unique continuation problem: if $(g_1, g_2, \psi_T) \in (L^2(0,T; L^2(\partial\Omega_0)))^2 \times L^2(\Omega_0)$, and (ψ, φ) solves

$$\tag{3.1} \begin{cases} -\dfrac{\partial \psi}{\partial t} - \Delta\psi = \mathbb{1}_\Theta \varphi & \text{in } (0,T)\times\Omega_0, \\ \psi = g_1 & \text{on } (0,T)\times\partial\Omega_0, \\ \psi(T,\cdot) = \psi_T & \text{in } \Omega_0, \\ \dfrac{\partial \varphi}{\partial t} - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega_0, \\ \varphi = g_2 & \text{on } (0,T)\times\partial\Omega_0, \\ \varphi(0,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

then, we have the following unique continuation property:

$$\tag{3.2} \psi = 0 \text{ in } (0,T)\times\omega \implies g_1 = g_2 = 0 \text{ and } \psi_T = 0.$$

Let us then take $(g_1, g_2, \psi_T) \in (L^2(0,T; L^2(\partial\Omega_0)))^2 \times L^2(\Omega_0)$, (ψ, φ) solving (3.1) with $\psi = 0$ in $(0,T)\times\omega$. Then the equation (3.1) on ψ implies that $\varphi = 0$ in $(0,T)\times(\omega\cap\Theta)$. The classical unique continuation property for the heat equation (for instance, by the Holmgren uniqueness theorem, see e.g. [21, §8.6 & Th. 8.6.5]) then applies to φ and $\varphi = 0$ in $(0,T)\times\Omega_0$. Following arguments similar to the proof of Proposition Proposition 1.4, we deduce that $g_2 = 0$, and that ψ solves the heat equation

$$-\frac{\partial \psi}{\partial t} - \Delta\psi = 0, \quad \text{in } (0,T)\times\Omega_0.$$

Since $\psi = 0$ in $(0,T)\times\omega$, we immediately deduce again, from the Holmgren uniqueness theorem (see e.g. [21, §8.6 & Th. 8.6.5]), that $\psi = 0$ in $(0,T)\times\Omega_0$. Since $\psi \in C^0([0,T], H^{-1}(\Omega_0))$ (see e.g. [34, Prop. 7.1.3]), the initial datum ψ_T vanishes as well. Following arguments similar to the proof of Proposition Proposition 1.4, we deduce that $g_1 = 0$.

This finishes the proof of the unique continuation property (3.2) for solutions of (3.1), hence the proof of the first part of Proposition 1.8, i.e., of the approximate controllability property (1.14).

Let us now focus on the proof of the null-controllability property when $\xi \in L^2(0,T; L^2(\Omega_0))$ can be steered to 0 using a control $h_{\mathrm{nc}} \in L^2(0,T; L^2(\omega))$, in the sense that the solution y_{nc} of (1.15) satisfies (1.16).

Then, for any $\varepsilon > 0$ and f_1, f_2 in $L^2(0,T; L^2(\partial\Omega_0))$, we look for y, z and h respectively as $y = y_{\mathrm{nc}} + y_1$, $z = z_1$ and $h = h_{\mathrm{nc}} + h_1$, where (y_1, z_1) satisfies

$$\tag{3.3} \begin{cases} \dfrac{\partial y_1}{\partial t} - \Delta y_1 = h_1 \mathbb{1}_\omega & \text{in } (0,T)\times\Omega_0, \\ y_1 = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ y_1(0,\cdot) = 0 & \text{in } \Omega_0, \\ -\dfrac{\partial z_1}{\partial t} - \Delta z_1 = \mathbb{1}_\Theta (y_{\mathrm{nc}} + y_1) & \text{in } (0,T)\times\Omega_0, \\ z_1 = 0 & \text{on } (0,T)\times\partial\Omega_0, \\ z_1(T,\cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

and

$$\begin{cases} \|\partial_n y_1 - \widetilde{f}_1\|_{L^2(0,T;L^2(\partial\Omega_0))} + \|\partial_n z_1 - f_2\|_{L^2(0,T;L^2(\partial\Omega_0))} \leqslant \varepsilon, \\ y_1(T) = 0 \text{ in } \Omega_0, \end{cases} \tag{3.4}$$

where

$$\widetilde{f}_1 = f_1 - \partial_n y_{\mathrm{nc}}. \tag{3.5}$$

In order to do that, we consider the functional K_ε defined for

$$(g_1, g_2, \psi_T) \in (L^2(0,T; L^2(\partial\Omega_0)))^2 \ \times L^2(\Omega_0)$$

by

$$\begin{aligned} K_\varepsilon(g_1, g_2, \psi_T) = \frac{1}{2} \int_0^T \int_\omega |\psi(t,x)|^2 \, dx \, dt \\ + \int_0^T \int_{\partial\Omega_0} \big(\widetilde{f}_1 g_1 + f_2 g_2\big) \, d\sigma dt + \int_0^T \int_{\Omega_0} \mathbb{1}_\Theta y_0 \varphi + \varepsilon \|(g_1, g_2)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2}, \end{aligned}$$

where (ψ, φ) is the solution of (3.1) corresponding to (g_1, g_2, ψ_T).

Then, we endow the set

$$X_0 = (L^2(0,T; L^2(\partial\Omega_0)))^2 \ \times L^2(\Omega_0)$$

with the norm

$$\|(g_1, g_2, \psi_T)\|_{\mathrm{obs}}^2 = \int_0^T \int_\omega |\psi(t,x)|^2 \, dx \, dt + \int_0^T \int_{\partial\Omega_0} \big(|g_1|^2 + |g_2|^2\big) \, d\sigma dt,$$

where ψ is the solution of (3.1). The fact that this defines a norm comes from the unique continuation property (3.2). Then, we define

$$X_{\mathrm{obs}} = \overline{X_0}^{\|\cdot\|_{\mathrm{obs}}}.$$

We claim that K_ε can be extended continuously to X_{obs}. Indeed, let us emphasize that for $(g_1, g_2, \psi_T) \in X_{\mathrm{obs}}$, $\psi|_{(0,T)\times\omega}$ is well defined by density as an element of $L^2(0,T; L^2(\omega))$ since the function $(g_1, g_2, \psi_T) \in X_0 \mapsto \psi|_{(0,T)\times\omega}$ is well-defined on X_0 and continuous (by construction) for the topology of X_{obs}, and that we have the following straightforward estimate of the solution φ of (3.1): there exists $C > 0$ such that for any $(g_1, g_2, \psi_T) \in X_0$,

$$\|\varphi\|_{L^2(0,T;L^2(\Omega_0))} \leqslant C \|g_2\|_{L^2(0,T;L^2(\partial\Omega_0))} \leqslant C \|(g_1, g_2, \psi_T)\|_{\mathrm{obs}}. \tag{3.6}$$

Now, given $(g_1, g_2, \psi_T) \in X_0$, ψ satisfies the backward heat equation (3.1). We then introduce the solution ψ_1 of

$$\begin{cases} -\dfrac{\partial \psi_1}{\partial t} - \Delta \psi_1 = 0 & \text{in } (0,T) \times \Omega_0, \\ \psi_1 = g_1 & \text{on } (0,T) \times \partial\Omega_0, \\ \psi_1(T, \cdot) = 0 & \text{in } \Omega_0, \end{cases}$$

which belongs to $L^2(0,T; L^2(\Omega_0))$, with

$$\|\psi_1\|_{L^2(0,T;L^2(\Omega_0))} \leqslant C \|g_1\|_{L^2(0,T;L^2(\partial\Omega_0))}.$$

Next, $\psi - \psi_1$ satisfies the heat equation with homogeneous Dirichlet boundary conditions, so that classical Carleman estimates, see for instance [34, Th. 9.4.1] (after having bounded the weight function $\alpha(x)$ by some constant $C > 0$ from above and by 1 from below) implies

$$\begin{aligned}\|e^{-C/(t(T-t))}(\psi - \psi_1)\|_{L^2(0,T;L^2(\Omega_0))}\\ \leqslant C\|\psi - \psi_1\|_{L^2(0,T;L^2(\omega))} + C\| - \partial_t\psi - \Delta\psi\|_{L^2(0,T;L^2(\Omega_0))}.\end{aligned}$$

Using that the backward heat equation is well posed, standard energy estimates show that

$$\begin{aligned}\|e^{-C/(T-t)}(\psi - \psi_1)\|_{L^2(0,T;L^2(\Omega_0))}\\ \leqslant C\|\psi - \psi_1\|_{L^2(0,T;L^2(\omega))} + C\| - \partial_t\psi - \Delta\psi\|_{L^2(0,T;L^2(\Omega_0))}.\end{aligned}$$

In particular,

$$\begin{aligned}\|e^{-C/(T-t)}\psi\|_{L^2(0,T;L^2(\Omega_0))} &\leqslant C\|\psi\|_{L^2(0,T;L^2(\omega))} + C\|\varphi\|_{L^2(0,T;L^2(\Omega_0))}\\ &\qquad + C\|g_1\|_{L^2(0,T;L^2(\partial\Omega_0))}\\ &\leqslant C\|(g_1, g_2, \psi_T)\|_{\mathrm{obs}}.\end{aligned}$$

Hence, by density of X_0 in X_{obs}, to each $(g_1, g_2, \psi_T) \in X_{\mathrm{obs}}$, we can associate a solution (ψ, φ) to

$$\begin{cases} -\dfrac{\partial\psi}{\partial t} - \Delta\psi = \mathbb{1}_\Theta\varphi & \text{in } (0,T)\times\Omega_0,\\ \qquad\quad \psi = g_1 & \text{on } (0,T)\times\partial\Omega_0,\\ \dfrac{\partial\varphi}{\partial t} - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega_0,\\ \qquad\quad \varphi = g_2 & \text{on } (0,T)\times\partial\Omega_0,\\ \quad \varphi(0,\cdot) = 0 & \text{in } \Omega_0,\end{cases}$$

such that for all $T' < T$, $\psi \in L^2(0,T';L^2(\Omega_0))$. We can thus apply the unique continuation property (3.2) with T replaced by T', using the same strategy as before relying on Holmgren's uniqueness theorem, and we obtain that if, for $(g_1, g_2, \psi_T) \in X_{\mathrm{obs}}$ we have $\psi|_{(0,T')\times\omega} = 0$ then φ and ψ vanishes identically on $(0,T')$. Since $T' \in (0,T)$ is arbitrary, we deduce that the unique continuation property (3.2) extends to $(g_1, g_2, \psi_T) \in X_{\mathrm{obs}}$.

Classical contradiction arguments relying on (3.2) (see e.g. [14]) then give that

$$\liminf_{\|(g_1,g_2,\psi_T)\|_{\mathrm{obs}}\to\infty} \frac{K_\varepsilon(g_1, g_2, \psi_T)}{\|(g_1, g_2, \psi_T)\|_{\mathrm{obs}}} \geqslant \varepsilon. \tag{3.7}$$

Indeed, let us prove (3.7) by contradiction. Assume that there exists a sequence $(g_{1,n}, g_{2,n}, \psi_{T,n}) \in X_{\mathrm{obs}}$ such that

$$\begin{cases}\rho_n := \|(g_{1,n}, g_{2,n}, \psi_{T,n})\|_{\mathrm{obs}} \to \infty \text{ as } n\to\infty,\\ \alpha := \limsup_{n\to\infty} \dfrac{K_\varepsilon(g_{1,n}, g_{2,n}, \psi_{T,n})}{\|(g_{1,n}, g_{2,n}, \psi_{T,n})\|_{\mathrm{obs}}} < \varepsilon.\end{cases} \tag{3.8}$$

We first renormalize the corresponding functions, and set

$$(\widetilde{g}_{1,n},\widetilde{g}_{2,n},\widetilde{\psi}_{T,n})=\frac{(g_{1,n},g_{2,n},\psi_{T,n})}{\|(g_{1,n},g_{2,n},\psi_{T,n})\|_{\mathrm{obs}}},$$

so that

$$\|(\widetilde{g}_{1,n},\widetilde{g}_{2,n},\widetilde{\psi}_{T,n})\|_{\mathrm{obs}}=1. \tag{3.9}$$

Therefore, there exists $(\widetilde{g}_1,\widetilde{g}_2,\widetilde{\psi}_T)$ in X_{obs} such that

$$(\widetilde{g}_{1,n},\widetilde{g}_{2,n},\widetilde{\psi}_{T,n}) \text{ weakly converges to } (\widetilde{g}_1,\widetilde{g}_2,\widetilde{\psi}_T) \text{ in } X_{\mathrm{obs}} \text{ as } \ n\longrightarrow\infty. \tag{3.10}$$

According to (3.8), we have

$$\begin{multline}\frac{1}{2}(\rho_n)^2\int_0^T\int_\omega|\widetilde{\psi}_n|^2\,dx\,dt+\rho_n\Big(\int_0^T\int_{\partial\Omega_0}\big(\widetilde{f}_1\widetilde{g}_{1,n}+f_2\widetilde{g}_{2,n}\big)\,d\sigma dt\\ +\int_0^T\int_{\Omega_0}\chi_\Theta y_0\widetilde{\varphi}_n+\varepsilon\|(\widetilde{g}_{1,n},\widetilde{g}_{2,n})\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2}\Big)\leqslant\alpha\rho_n.\end{multline} \tag{3.11}$$

By (3.6) and (3.10), the sequence

$$\begin{multline}\Big(\int_0^T\int_{\partial\Omega_0}\big(\widetilde{f}_1\widetilde{g}_{1,n}+f_2\widetilde{g}_{2,n}\big)\,d\sigma dt+\int_0^T\int_{\Omega_0}\chi_\Theta y_0\widetilde{\varphi}_n\\ +\varepsilon\|(\widetilde{g}_{1,n},\widetilde{g}_{2,n})\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2}\Big)_{n\in\mathbb{N}}\end{multline}$$

is uniformly bounded in n. Therefore, dividing by ρ_n and using that $\rho_n\to+\infty$ when $n\to\infty$, we infer

$$\lim_{n\to\infty}\Big(\int_0^T\int_\omega|\widetilde{\psi}_n|^2\,dx\,dt\Big)=0,$$

hence

$$\widetilde{\psi}=0 \text{ in } (0,T)\times\omega.$$

We also deduce, according to (3.9), that

$$\lim_{n\to\infty}\|(\widetilde{g}_{1,n},\widetilde{g}_{2,n})\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2}=1.$$

Furthermore, using the unique continuation property (3.2), which has been shown to be also valid for elements of X_{obs}, we deduce that

$$(\widetilde{g}_1,\widetilde{g}_2,\widetilde{\psi}_T)=(0,0,0).$$

In particular, according to (3.10) and the equation satisfied by $\widetilde{\varphi}$,

$$\begin{multline}\lim_{n\to\infty}\Big(\int_0^T\int_{\partial\Omega_0}\big(\widetilde{f}_1\widetilde{g}_{1,n}+f_2\widetilde{g}_{2,n}\big)\,d\sigma dt+\int_0^T\int_{\Omega_0}\chi_\Theta y_0\widetilde{\varphi}_n\\ +\varepsilon\|(\widetilde{g}_{1,n},\widetilde{g}_{2,n})\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2}\Big)=\varepsilon,\end{multline}$$

which is in contradiction with (3.11), since α was assumed to be smaller than ε by (3.8). This concludes the proof of the coercivity estimate (3.7).

Therefore, since the functional K_ε is also obviously strictly convex on X_{obs}, it admits a unique minimizer $(g_1^*, g_2^*, \psi_T^*) \in X_{\text{obs}}$. Writing the Euler-Lagrange equation satisfied by the minimizer gives that, setting

$$h_1 = \psi^* \quad \text{in } (0,T)\times\omega,$$

the corresponding solution (y_1, z_1) of (3.3) satisfies (3.4). We refer for instance to [14, Th. 1.2 and its proof] for further details about the approach considered here. To conclude the proof of Proposition 1.8, we set $(y,z) = (y_1 + y_{\text{nc}}, z_1)$. Since (y_1, z_1) satisfies (3.3) and y_{nc} satisfies (1.15), (y,z) is a solution of (1.9), verifying moreover (1.17) and (1.18), thanks to (3.4), (3.5) and (1.16).

3.2. Proof of Theorem 1.10. — The proof of Theorem 1.10 mainly reduces to Proposition 1.8, similarly as for the proof of Theorem 1.6, that follows from Proposition 1.4.

When the goal is to get the approximate controllability for y_0 at time $T > 0$, i.e., (1.20), the only novelty is that for any $\varepsilon > 0$ and $y_T \in L^2(\Omega_0)$, we should take the control function $h \in L^2(0,T;L^2(\omega))$ such that the solution (y_0, z_0) of (1.4) satisfies (2.4) and (1.20), which can be done according to Proposition 1.8.

When the goal is to get null-controllability of y_0 at time T when ξ is null-controllable, the argument is basically the same, by using a control function $h \in L^2(0,T;L^2(\omega))$ such that the solution (y_0, z_0) of (1.4) satisfies the estimate (2.4), i.e.,

$$\|\partial_n y_0\|_{L^2(0,T;L^2(\partial\Omega_0))} + \|\partial_n z_0\|_{L^2(0,T;L^2(\partial\Omega_0))} \leqslant \sqrt{\varepsilon},$$

and (1.18).

Details are left to the reader.

3.3. Proof of Theorem 1.11. — Here again, the proof of Theorem 1.11 is very similar to the one of Theorem 1.7. We focus on the case in which we only want approximate controllability of y_0 at time T since the other case in which we want null-controllability at time T when ξ is null-controllable can be deduced similarly, following the proof of Theorem 1.7.

To be more precise, we will choose the control h in two steps, under the form $h = h_0 + h_1$, where h_0 is used to get the approximate desensitizing property and the approximate controllability of y_0 at time T, and h_1 is then chosen afterward to get the exact desensitizing property in the directions of $\mathcal{E}$.

We assume that $\mathcal{E}$ is of finite dimension $N \geqslant 2$. Let $\mathscr{E} = \{\boldsymbol{V}\cdot\boldsymbol{n} \mid \boldsymbol{V} \in \mathcal{E}\}$, which is itself a finite dimensional subspace of $L^2(\partial\Omega_0)$ of dimension $M \leqslant N$. Now, we proceed exactly as in the proof of Theorem 1.7 by choosing, for a parameter $\alpha > 0$ to be chosen later, for $k \in [\![1, M]\!]$, $a \in \{1,2\}$, control functions $h_{k,a}^\alpha$ such that (2.20) holds and

$$\|y_{h_{k,a}^\alpha}(T)\|_{L^2(\Omega_0)} \leqslant 1, \tag{3.12}$$

where $y_{h_{k,a}^\alpha}$ solves (2.8)–(2.9). Note that this can be done according to Proposition 1.8.

Then, the same arguments as before yields the following result: there exists h of the form (2.25) with $\lambda \in \mathbb{R}^M$ satisfying (2.24) such that for all $\boldsymbol{V} \in \mathcal{E}$,

$$\int_{\partial\Omega_0} \boldsymbol{V} \cdot \boldsymbol{n} \bigg(\int_0^T (\partial_n y_\xi + \partial_n y_h)(\partial_n z_\xi + \partial_n z_h)\, dt \bigg)\, d\sigma = 0,$$

where (y_ξ, z_ξ) solves (2.6)–(2.7) and (y_h, z_h) solves (2.8)–(2.9). Besides, combining (2.24), (2.25) and (3.12), the corresponding controlled trajectory (y_h, z_h) of (2.8)–(2.9) satisfies:

$$\begin{aligned} (3.13)\quad \|h\|_{L^2(0,T;L^2(\omega))} + \|(y_h, z_h)\|_{(L^2(0,T;H^2(\Omega_0)))^2} + \|y_h(T)\|_{L^2(\Omega_0)} \\ + \|(\partial_n y_h, \partial_n z_h)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2} \\ \leqslant C\|(\partial_n y_\xi, \partial_n z_\xi)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2}. \end{aligned}$$

Therefore, to solve the problem of approximate desensitizing of J_h, exact desensitizing of J_h for $\mathcal{E}$ and approximate controllability (1.20), we do as in the proof of Theorem 1.11: for any $\varepsilon > 0$, setting

$$\varepsilon_0 = \frac{\min\{\sqrt{\varepsilon}, \varepsilon\}}{C+1},$$

where C is the constant in (3.13), we start by taking $h_0 \in L^2(0,T;L^2(\omega))$ such that the solution (y,z) of (1.9) satisfies

$$\|(\partial_n y, \partial_n z)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2} + \|y(T) - y_T\|_{L^2(\Omega_0)} \leqslant \varepsilon_0,$$

which can be done according to Proposition 1.8.

Setting $\xi_1 = \xi + h_0 \mathbb{1}_\omega$, which belongs to $L^2(0,T;L^2(\Omega_0))$, by the previous paragraph and the estimate (3.13) applied for the source term ξ_1, there exists $h_1 \in L^2(0,T;L^2(\omega))$ such that the identity (2.28) holds for all $\boldsymbol{V} \in \mathcal{E}$, where (y_0, z_0) denotes the solution of (2.29) and the solution (y_{h_1}, z_{h_1}) of (2.8)–(2.9) satisfies the bound

$$\|y_{h_1}(T)\|_{L^2(\Omega_0)} + \|(\partial_n y_{h_1}, \partial_n z_{h_1})\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2} \leqslant C\varepsilon_0.$$

Besides,

$$y_0(T) = y(T) + y_{h_1}(T), \quad \partial_n y_0 = \partial_n y + \partial_n y_{h_1}, \quad \partial_n z_0 = \partial_n z + \partial_n z_{h_1},$$

so that

$$\|y_0(T) - y_T\|_{L^2(\Omega_0)} + \|(\partial_n y_0, \partial_n z_0)\|_{(L^2(0,T;L^2(\partial\Omega_0)))^2} \leqslant (C+1)\varepsilon_0,$$

where C is the constant in (3.13). Then, we easily get that, for all $\boldsymbol{V} \in W^{3,\infty}(\mathbb{R}^d, \mathbb{R}^d)$,

$$\begin{aligned} \bigg| \int_{\partial\Omega_0} (\boldsymbol{V} \cdot \boldsymbol{n}) \bigg(\int_0^T \partial_n y_0 \partial_n z_0 \, dt \bigg)\, d\sigma \bigg| &\leqslant (C+1)^2 \varepsilon_0^2 \|\boldsymbol{V} \cdot \boldsymbol{n}\|_{L^\infty(\partial\Omega_0)} \\ &\leqslant \varepsilon \|\boldsymbol{V} \cdot \boldsymbol{n}\|_{L^\infty(\partial\Omega_0)} \\ &\leqslant \varepsilon \|\boldsymbol{V} \cdot \boldsymbol{n}\|_{W^{3,\infty}(\Omega_0)}, \end{aligned}$$

while

$$\|y_0(T) - y_T\|_{L^2(\Omega_0)} \leqslant (C+1)\varepsilon_0 \leqslant \varepsilon.$$

In other words, $h = h_0 + h_1$ exactly desensitizes J_h for $\mathcal{E}$, ε-approximately desensitizes J_h and ε-approximately controls y_0 at time T.

4. The exact desensitizing problem

4.1. Proof of Proposition 1.13. — We fix $\xi \in L^2(0,T;L^2(\mathbb{R}^d))$, and we introduce the solution y_ξ of

$$\begin{cases} \partial_t y_\xi - \Delta y_\xi = \xi, & (t,x) \in (0,T)\times\Omega_0, \\ y_\xi(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega_0, \\ y_\xi(0,x) = 0, & x\in\Omega_0, \end{cases}$$

According to (1.22), there exists a smooth function $\eta = \eta(x)$ such that $\eta = 1$ in $\Omega_0 \setminus \omega$ and $\eta = 0$ in Θ. Then, we set

$$y_0(t,x) = \eta(x) y_\xi(t,x), \quad \text{for } (t,x)\in(0,T)\times\Omega_0,$$

which solves (1.4) with control function

$$\mathbb{1}_\omega(x)h(t,x) = (\eta(x)-1)\xi(t,x) - [\Delta,\eta]y_\xi(t,x), \quad \text{for } (t,x)\in(0,T)\times\Omega_0,$$

where $[\Delta,\eta]y_\xi(t,x) := \Delta(\eta y_\xi) - \eta\Delta y_\xi$. Note that $\mathbb{1}_\omega h$ is localized in ω because of the support properties of η.

Therefore, y_0 vanishes in $(0,T)\times\Theta$, hence the associated function z_0 such that (y_0,z_0) satisfies (1.4) is identically zero. In particular, according to (1.3), we immediately have the exact desensitizing property (1.5).

4.2. Proof of Theorem 1.14. — We start by introducing open sets $\omega_0, \omega_1, \omega_2$, and ω_3 such that

$$\partial\Theta \subset \omega_0 \Subset \omega_1 \Subset \omega_2 \Subset \omega_3 \Subset \omega,$$

which is possible thanks to Assumption (1.23). We also introduce a smooth function $\eta_{23} = \eta_{23}(x)$ taking value 1 in $\Omega\setminus\overline{\omega_3}$ and vanishing in $\overline{\omega_2}$.

We fix $\xi\in L^2(0,T;L^2(\mathbb{R}^d))$, and we introduce the solution y_ξ to

$$\begin{cases} \partial_t y_\xi - \Delta y_\xi = \eta_{23}\xi, & (t,x)\in(0,T)\times\Omega_0, \\ y_\xi(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega_0, \\ y_\xi(0,x) = 0, & x\in\Omega, \end{cases}$$

and the solution z_ξ to

$$\begin{cases} -\partial_t z_\xi - \Delta z_\xi = \eta_{12}\; y_\xi \mathbb{1}_\Theta, & (t,x)\in(0,T)\times\Omega_0, \\ z_\xi(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega_0, \\ z_\xi(T,x) = 0, & x\in\Omega_0. \end{cases} \tag{4.1}$$

where $\eta_{12} = \eta_{12}(x)$ is a smooth function taking value 1 in $\Omega_0\setminus\overline{\omega_2}$ and vanishing in $\overline{\omega_1}$.

Then, we introduce a smooth function $\eta_{01} = \eta_{01}(x)$ such that η_{01} vanishes in ω_0 and equal to 1 in $\Omega_0\setminus\omega_1$, so that $\mathbb{1}_\Theta\eta_{01}$ is actually a smooth function taking value 1 in $\Theta\setminus\omega_1$ and vanishing in $\omega_0\cup(\Omega_0\setminus\Theta)$. Then $z_0(t,x) = \mathbb{1}_\Theta(x)\eta_{01}(x)z_\xi(t,x)$ satisfies

$$\begin{cases} -\partial_t z_0 - \Delta z_0 = \eta_{01}\eta_{12}\; y_\xi\mathbb{1}_\Theta - [\Delta, \mathbb{1}_\Theta\eta_{01}]\; z_\xi, & (t,x)\in(0,T)\times\Omega_0, \\ z_0(t,x) = 0, & (t,x)\in(0,T)\times\partial\Omega_0, \\ z_0(T,x) = 0, & x\in\Omega_0, \end{cases}$$

and, by construction,

$$z_0(t,x)=0 \quad \text{for all } (t,x)\in(0,T)\times(\Omega_0\setminus\Theta).$$

Now, we remark that by construction, $[\Delta,\mathbb{1}_\Theta\eta_{01}]z_\xi$ is localized in $\omega_1\cap\Theta$. Besides, local regularity results for (4.1) imply that z_ξ is $\mathscr{C}^2([0,T]\times\overline{\omega_1\cap\Theta})$. We then take

$$y_0(t,x)=\eta_{01}(x)\eta_{12}(x)y_\xi(t,x)-[\Delta,\mathbb{1}_\Theta\eta_{01}]z_\xi(t,x), \quad \text{for all } (t,x)\in(0,T)\times\Omega_0,$$

which satisfies (1.4) for h given for $(t,x)\in(0,T)\times\Omega_0$ by

$$\begin{aligned}h(t,x)=(\eta_{01}(x)\eta_{12}(x)\eta_{23}(x)-1)\xi(t,x)-[\Delta,\eta_{01}\eta_{12}]y_\xi(t,x)\\ -(\partial_t-\Delta)([\Delta,\mathbb{1}_\Theta\eta_{01}]z_\xi)(t,x).\end{aligned}$$

This control function h is localized in $(0,T)\times\omega$ due to the conditions on the support of η_{01}, η_{12}, η_{23}. This concludes the proof of Theorem 1.14.

4.3. Proof of Theorem 1.15. — Firstly, we consider a function $g\in\mathscr{C}^4(\partial\Omega_0)$ such that g is nowhere $\mathscr{C}^5(\partial\Omega_0)$ (such functions form a dense set in the sense of Baire of $\mathscr{C}^4(\partial\Omega_0)$ and thus exist).

Then, we introduce a function $q_*\in H^4(\Omega_0)$ such that

$$\begin{aligned} &q_*(x)=0 \text{ for } x\in\partial\Omega_0, &&\partial_n q_*(x)=g(x) \text{ for } x\in\partial\Omega_0,\\ &\Delta q_*(x)=0 \text{ for } x\in\partial\Omega_0, &&\partial_n\Delta q_*(x)=g(x) \text{ for } x\in\partial\Omega_0,\end{aligned}$$

whose existence is guaranteed by classical trace theorems, see e.g. [26, Chap. 1, Th. 8.3], and we choose a smooth non-negative function $\eta=\eta(t)$ such that

$$\eta(0)=\eta(T)=\eta'(0)=0 \quad\text{and}\quad \int_0^T\eta(t)^2\,dt=1.$$

Then, we set, for $(t,x)\in(0,T)\times\Omega_0$,

$$\begin{aligned} z_\xi(t,x)&=\eta(t)q_*(x),\\ y_\xi(t,x)&=-\partial_t z_\xi(t,x)-\Delta z_\xi(t,x)=-\eta'(t)q_*(x)-\eta(t)\Delta q_*(x),\\ \xi(t,x)&=\partial_t y_\xi(t,x)-\Delta y_\xi(t,x).\end{aligned}$$

Note that $\xi\in L^2(0,T;L^2(\Omega_0))$ since $q_*\in H^4(\Omega_0)$.

Assume that we can solve the exact desensitizing problem for this choice of ξ. Hence, from (1.5) and formula (1.3), there exists $h\in L^2(0,T;L^2(\omega))$ such that

$$\forall\, x\in\partial\Omega_0,\quad \int_0^T\partial_n y_0(t,x)\partial_n z_0(t,x)\,dt=0, \tag{4.2}$$

where (y_0,z_0) solves (1.4). Now, we decompose y_0 as

$$y(t,x)=y_\xi(t,x)+y_h(t,x),$$

where y_h is the solution of (2.8), and z_0 as

$$z_0(t,x)=z_\xi(t,x)+z_h(t,x),\quad (t,x)\in(0,T)\times\Omega_0,$$

where z_h solves (2.9).

From (4.2), for all $x \in \partial\Omega_0$,

$$\begin{aligned}0 &= \int_0^T (\partial_n y_\xi + \partial_n y_h)(\partial_n z_\xi + \partial_n z_h)\, dt\\ &= -(g(x))^2 + g(x) \int_0^T \left(\eta(t)\partial_n y_h(t,x) - (\eta'(t)+\eta(t))\partial_n z_h(t,x)\right)\, dt\\ &\qquad\qquad + \int_0^T \partial_n y_h(t,x)\, \partial_n z_h(t,x)\, dt,\end{aligned}$$

where we used that

$$\partial_n y_\xi = -(\eta+\eta')g, \quad \partial_n z_\xi = \eta g.$$

Since $\omega \Subset \Omega_0$, the regularizing properties of the heat equation imply that y_h is smooth close to the boundary $[0,T]\times\partial\Omega_0$, and thus so is z_h. Therefore, the quantities

$$\begin{aligned}a_0(x) &= \int_0^T \left(\eta(t)\partial_n y_h(t,x) - (\eta'(t)+\eta(t))\partial_n z_h(t,x)\right)\, dt,\\ a_1(x) &= \int_0^T \partial_n y_h(t,x)\, \partial_n z_h(t,x)\, dt,\end{aligned}$$

are smooth ($\mathscr{C}^\infty$) in $\partial\Omega_0$. Since for all $x \in \partial\Omega_0$, $g(x)$ is a real root to the polynomial $-X^2 + Xa_0(x) + a_1(x)$, we necessarily have that for all $x \in \partial\Omega_0$, $a_0(x)^2 + 4a_1(x) \geqslant 0$ and for all $x \in \partial\Omega_0$,

$$g(x) \in \Big\{\frac{1}{2}\big(a_0(x) + \sqrt{a_0(x)^2+4a_1(x)}\big),\ \frac{1}{2}\big(a_0(x) - \sqrt{a_0(x)^2+4a_1(x)}\big)\Big\}.$$

In particular, if there exists $x_0 \in \partial\Omega$ such that $a_0(x_0)^2 > 4a_1(x_0)$, since g is continuous, there is a sign $s \in \{-1,1\}$ such that in a neighbourhood of x_0 (in $\partial\Omega$) in which $a_0^2 + 4a_1$ stays positive,

$$g(x) = \frac{1}{2}\big(-a_0(x) + s\sqrt{a_0(x)^2+4a_1(x)}\big),$$

implying in particular that g is smooth ($\mathscr{C}^\infty$) in a neighbourhood of x_0, which contradicts the choice of g.

Thus, for all $x \in \partial\Omega_0$, we should have $a_0(x)^2+4a_1(x) = 0$, so that $g(x) = -a_0(x)/2$. But this would again imply that g is smooth in $\partial\Omega_0$, thus contradicting the choice of g.

We have thus obtained a contradiction. There cannot be any control $h \in L^2(0,T;L^2(\omega))$ such that the condition (4.2) holds.

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