Andrea D'Agnolo & Masaki Kashiwara

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ON A TOPOLOGICAL COUNTERPART OF
REGULARIZATION FOR HOLONOMIC $\mathscr{D}$-MODULES

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Abstract. — On a complex manifold, the embedding of the category of regular holonomic $\mathscr{D}$-modules into that of holonomic $\mathscr{D}$-modules has a left quasi-inverse functor $\mathscr{M} \to \mathscr{M}_{\mathrm{reg}}$, called regularization. Recall that $\mathscr{M}_{\mathrm{reg}}$ is reconstructed from the de Rham complex of $\mathscr{M}$ by the regular Riemann-Hilbert correspondence. Similarly, on a topological space, the embedding of sheaves into enhanced ind-sheaves has a left quasi-inverse functor, called here sheafification. Regularization and sheafification are intertwined by the irregular Riemann-Hilbert correspondence. Here, we study some of the properties of the sheafification functor. In particular, we provide a stalk formula for the sheafification of enhanced specialization and microlocalization.

Résumé (Sur un analogue topologique de la régularisation pour les $\mathscr{D}$-modules holonomes)

Sur une variété complexe lisse, l'inclusion de la catégorie des $\mathscr{D}$-modules holonomes réguliers dans celle des $\mathscr{D}$-modules holonomes admet un foncteur quasi-inverse à gauche $\mathscr{M} \to \mathscr{M}_{\mathrm{reg}}$, appelé régularisation. Rappelons que $\mathscr{M}_{\mathrm{reg}}$ est reconstruit à partir du complexe de de Rham de $\mathscr{M}$ par la correspondance de Riemann-Hilbert régulière. De même, sur un espace topologique, l'inclusion des faisceaux dans les ind-faisceaux enrichis admet un foncteur quasi-inverse à gauche, qu'on appelle ici faisceautisation. La régularisation et la faisceautisation sont échangées par la correspondance de Riemann-Hilbert irrégulière. Dans ce travail, nous étudions certaines des propriétés du foncteur de faisceautisation. En particulier, nous fournissons une formule qui calcule la fibre du faisceautisé de la spécialisation et de la microlocalisation enrichies.

Contents

1. Introduction

Let X be a complex manifold. The regular Riemann-Hilbert correspondence (see [7]) states that the de Rham functor induces an equivalence between the triangulated category of regular holonomic $\mathscr{D}$-modules and that of $\mathbb{C}$-constructible sheaves. More precisely, one has a diagram

$$
\begin{array}{ccc}
\mathrm{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathscr{D}_X) & & \\
\iota \big\uparrow & \searrow{\mathscr{D}\mathscr{R}} & \\
\mathrm{D}_{\mathrm{rh}}^{\mathrm{b}}(\mathscr{D}_X) & \xrightarrow[\underset{\Phi}{\sim}]{\mathscr{D}\mathscr{R}} & \mathrm{D}_{\mathbb{C}\text{-c}}^{\mathrm{b}}(\mathbb{C}_X)
\end{array}
$$

where ι is the embedding (i.e. fully faithful functor) of regular holonomic $\mathscr{D}$-modules into holonomic $\mathscr{D}$-modules, the triangle quasi-commutes, $\mathscr{D}\mathscr{R}$ is the de Rham functor, and Φ is an (explicit) quasi-inverse to $\mathscr{D}\mathscr{R}$.

The *regularization* functor $\mathrm{reg}\colon \mathrm{D}_{\mathrm{hol}}^{\mathrm{b}}(\mathscr{D}_X) \to \mathrm{D}_{\mathrm{rh}}^{\mathrm{b}}(\mathscr{D}_X)$ is defined by $\mathscr{M}_{\mathrm{reg}} := \Phi(\mathscr{D}\mathscr{R}(\mathscr{M}))$. It is a left quasi-inverse to ι, of transcendental nature. Recall that (ι, reg) is *not* a pair of adjoint functors.[1] Recall also that reg is conservative.[2]

Let $\mathbf{k}$ be a field and M be a good topological space. Consider the natural embeddings $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M) \overset{\iota}{\rightarrowtail} \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \overset{e}{\rightarrowtail} \mathrm{E}_{\mathrm{st}}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ of sheaves into ind-sheaves into stable enhanced ind-sheaves. One has pairs of adjoint functors (α, ι) and (e, Ish), and we set $\mathsf{sh} := \alpha\,\mathsf{Ish}$:

$$
\mathsf{sh}\colon \mathrm{E}_{\mathrm{st}}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \xrightarrow{\mathsf{Ish}} \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \xrightarrow{\alpha} \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M).
$$

We call Ish and sh the *ind-sheafification* and *sheafification* functor, respectively. The functor sh is a left quasi-inverse of $e\,\iota$.

For $\mathbf{k} = \mathbb{C}$ and $M = X$, the irregular Riemann-Hilbert correspondence (see [1]) intertwines[3] the pair (ι, reg) with the pair $(e\,\iota, \mathsf{sh})$. In particular, the pair $(e\,\iota, \mathsf{sh})$ is *not* a pair of adjoint functors in general.

With the aim of better understanding the rather elusive regularization functor, in this paper we study some of the properties of the ind-sheafification and sheafification functors. More precisely, the contents of the paper are as follows.

In Section 2, besides recalling notations, we establish some complementary results on ind-sheaves on bordered spaces that we need in the following. Further complements are provided in Appendix A.

Some functorial properties of ind-sheafification and sheafification are obtained in Section 3. In Section 4, we obtain a stalk formula for the sheafification of a pull-back by an embedding. (At the level of $\mathscr{D}$-modules, the interest of such a formula is due to the lack of commutation between the de Rham functor and the restriction functor.) Then, these results are used in Section 5 to obtain a stalk formula for the sheafification

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[1]By saying that (ι, reg) is a pair of adjoint functors, we mean that ι is the left adjoint of reg.

[2]In fact, if $\mathscr{M}_{\mathrm{reg}} \simeq 0$ then $\mathscr{D}\mathscr{R}(\mathscr{M}) \simeq \mathscr{D}\mathscr{R}(\mathscr{M}_{\mathrm{reg}}) \simeq 0$, and hence $\mathscr{M} \simeq 0$.

[3]Using formula (3.1) below, this follows from [1, Cor. 9.6.7].

of enhanced specialization and microlocalization. In particular, the formula for the specialization puts in a more geometric perspective what we called *multiplicity test functor* in [2, §6.3].

Finally, we provide in Appendix B a formula for the sections of a weakly constructible sheaf on a locally closed subanalytic subset, which could be of independent interest.

2. Notations and complements

We recall here some notions and results, mainly to fix notations, referring to the literature for details. In particular, we refer to [9] for sheaves, to [13] (see also [5, 3]) for enhanced sheaves, to [10] for ind-sheaves, and to [1] (see also [12, 8, 3]) for bordered spaces and enhanced ind-sheaves. We also add some complements.

– In this paper, $\mathbf{k}$ denotes a base field.
– A good space is a topological space which is Hausdorff, locally compact, countable at infinity, and with finite soft dimension.
– By subanalytic space we mean a subanalytic space which is also a good space.

2.1. **Bordered spaces.** — The category of bordered spaces has for objects the pairs $\mathsf{M} = (M, C)$ with M an open subset of a good space C. Set $\overset{\circ}{\mathsf{M}} := M$ and $\overset{\vee}{\mathsf{M}} := C$. A morphism $f \colon \mathsf{M} \to \mathsf{N}$ is a morphism $\overset{\circ}{f} \colon \overset{\circ}{\mathsf{M}} \to \overset{\circ}{\mathsf{N}}$ of good spaces such that the projection $\overline{\Gamma}_f \to \overset{\vee}{\mathsf{M}}$ is proper. Here, $\overline{\Gamma}_f$ denotes the closure in $\overset{\vee}{\mathsf{M}} \times \overset{\vee}{\mathsf{N}}$ of the graph Γ_f of $\overset{\circ}{f}$.

Note that $\mathsf{M} \mapsto \overset{\vee}{\mathsf{M}}$ is not a functor. The functor $\mathsf{M} \mapsto \overset{\circ}{\mathsf{M}}$ is right adjoint to the embedding $M \mapsto (M, M)$ of good spaces into bordered spaces. We will write for short $M = (M, M)$.

Note that the inclusion $k_{\mathsf{M}} \colon \overset{\circ}{\mathsf{M}} \to \overset{\vee}{\mathsf{M}}$ factors into

$$(2.1) \qquad k_{\mathsf{M}} \colon \overset{\circ}{\mathsf{M}} \xrightarrow{\ i_{\mathsf{M}}\ } \mathsf{M} \xrightarrow{\ j_{\mathsf{M}}\ } \overset{\vee}{\mathsf{M}}.$$

By definition, a subset Z of M is a subset of $\overset{\circ}{\mathsf{M}}$. We say that $Z \subset \mathsf{M}$ is open (resp. closed, locally closed) if it is so in $\overset{\circ}{\mathsf{M}}$. For a locally closed subset Z of M, we set $Z_\infty = (Z, \overline{Z})$ where $\overline{Z}$ is the closure of Z in $\overset{\vee}{\mathsf{M}}$. Note that $U_\infty \simeq (U, \overset{\vee}{\mathsf{M}})$ for $U \subset \mathsf{M}$ open.

We say that Z is a relatively compact subset of M if it is contained in a compact subset of $\overset{\vee}{\mathsf{M}}$. Note that this notion does not depend on the choice of $\overset{\vee}{\mathsf{M}}$. This means that if N is a bordered space with $\mathsf{N} \simeq \mathsf{M}$ and $\overset{\circ}{\mathsf{N}} = \overset{\circ}{\mathsf{M}}$, then Z is relatively compact in M if and only if it is so in N.

An *open covering* $\{U_i\}_{i \in I}$ of a bordered space M is an open covering of $\overset{\circ}{\mathsf{M}}$ which satisfies the condition: for any relatively compact subset Z of M there exists a finite subset I' of I such that $Z \subset \bigcup_{i \in I'} U_i$.

We say that a morphism $f \colon \mathsf{M} \to \mathsf{N}$ is

(i) *an open embedding* if $\overset{\circ}{f}$ is a homeomorphism from $\overset{\circ}{\mathsf{M}}$ onto an open subset of $\overset{\circ}{\mathsf{N}}$,

(ii) *borderly submersive* if there exists an open covering $\{U_i\}_{i \in I}$ of M such that for any $i \in I$ there exist a subanalytic space S_i and an open embedding $g_i \colon (U_i)_\infty \to S_i \times \mathsf{N}$ with a commutative diagram of bordered spaces

$$
\begin{array}{ccc}
(U_i)_\infty & \longrightarrow & \mathsf{M} \\
{\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle f} \\
S_i \times \mathsf{N} & \xrightarrow{\ p_i\ } & \mathsf{N}
\end{array}
$$

where p_i is the projection,

(iii) *semiproper* if $\overline{\Gamma}_f \to \overset{\vee}{\mathsf{N}}$ is proper,

(iv) *proper* if it is semiproper and $\overset{\circ}{f} \colon \overset{\circ}{\mathsf{M}} \to \overset{\circ}{\mathsf{N}}$ is proper,

(v) *self-cartesian* if the diagram

$$
\begin{array}{ccc}
\overset{\circ}{\mathsf{M}} & \xrightarrow{\ \overset{\circ}{f}\ } & \overset{\circ}{\mathsf{N}} \\
{\scriptstyle i_\mathsf{M}}\downarrow & & \downarrow{\scriptstyle i_\mathsf{N}} \\
\mathsf{M} & \xrightarrow{\ f\ } & \mathsf{N}
\end{array}
$$

is cartesian.

Recall that, by [1, Lem. 3.3.16], a morphism $f \colon \mathsf{M} \to \mathsf{N}$ is proper if and only if it is semiproper and self-cartesian.

2.2. Ind-sheaves on good spaces. — Let M be a good space.

We denote by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-vector spaces on M. For $S \subset M$ locally closed, we denote by $\mathbf{k}_S$ the extension by zero to M of the constant sheaf on S with stalk $\mathbf{k}$.

For $f \colon M \to N$ a morphism of good spaces, denote by $\otimes$, f^{-1}, $\mathrm{R}f_!$ and $\mathrm{R}\mathscr{H}om$, $\mathrm{R}f_*$, $f^!$ the six operations. Denote by $\boxtimes$ the exterior tensor product and by D_M the Verdier duality functor.

We denote by $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ the bounded derived category of ind-sheaves of $\mathbf{k}$-vector spaces on M, and by $\otimes$, f^{-1}, $\mathrm{R}f_{!!}$ and $\mathrm{R}\mathscr{I}hom$, $\mathrm{R}f_*$, $f^!$ the six operations. Denote by $\boxtimes$ the exterior tensor product and by D_M the Verdier duality functor.

There is a natural embedding $\iota_M \colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$. It has a left adjoint α_M, which in turn has a left adjoint β_M. The commutativity of these functors with the operations is as follows

(2.2)

	$\boxtimes$	f^{-1}	$\mathrm{R}f_*$	$f^!$	$\mathrm{R}f_{!!}$
ι	$\circ$	$\circ$	$\circ$	$\circ$	$\times$
α	$\circ$	$\circ$	$\circ$	$\times$	$\circ$
β	$\circ$	$\circ$	$\times$	$\times$	$\times$

where "$\circ$" means that the functors commute, and "$\times$" that they don't.

2.3. **Ind-sheaves on bordered spaces.** — Let M be a bordered space. Setting

$$\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{M}}) := \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{M}}^{\vee})/\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{M}\smallsetminus\overset{\circ}{\mathsf{M}}}^{\vee}),$$

one has $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{M}}) \simeq \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}})$.

The bounded derived category of ind-sheaves of $\mathbf{k}$-vector spaces on M is defined by

$$\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) := \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}^{\vee})/\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}\smallsetminus\overset{\circ}{\mathsf{M}}}^{\vee}).$$

There is a natural embedding

$$\iota_{\mathsf{M}} : \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}}) \simeq \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{M}}) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$$

induced by $\iota_{\mathsf{M}}^{\vee}$. It has a left adjoint

$$\alpha_{\mathsf{M}} : \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}}),$$

which in turn has a left adjoint β_{M}. One sets $\mathrm{R}\mathscr{H}om := \alpha_{\mathsf{M}}\mathrm{R}\mathscr{I}hom$, a functor with values in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}})$.

For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}})$, we often simply write F instead of $\iota_{\mathsf{M}}F$ in order to make notations less heavy.

For operations, we use the same notations as in the case of good spaces. Recall (see [1, Prop. 3.3.19][4]) that

$$\mathrm{R}f_{!!} \simeq \mathrm{R}f_{*} \quad \text{if } f \text{ is proper,}$$

$$(2.3) \qquad f^{!} \simeq f^{!}\mathbf{k}_{\overset{\circ}{\mathsf{N}}} \otimes f^{-1} \quad \text{if } f : \mathsf{M} \longrightarrow \mathsf{N} \text{ is borderly submersive.}$$

The last statement implies

$$(2.4) \qquad\qquad f^{!} \text{ commutes with } \alpha \text{ if } f \text{ is borderly submersive.}$$

With notations (2.1), (2.3) implies that

$$(2.5) \qquad\qquad i_{\mathsf{M}}^{-1} \simeq i_{\mathsf{M}}^{!}, \quad j_{\mathsf{M}}^{-1} \simeq j_{\mathsf{M}}^{!}.$$

The quotient functor $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}^{\vee}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ is isomorphic to $j_{\mathsf{M}}^{-1} \simeq j_{\mathsf{M}}^{!}$ and has a left adjoint $\mathrm{R}j_{\mathsf{M}!!}$ and a right adjoint $\mathrm{R}j_{\mathsf{M}*}$, both fully faithful.

The functors ι_{M}, α_{M} and β_{M} are exact. Moreover, ι_{M} and β_{M} are fully faithful. This was shown in [10] in the case of good spaces. The general case reduces to the former by the

Lemma 2.1. — *One has*

(i) $\iota_{\mathsf{M}} := j_{\mathsf{M}}^{-1} \iota_{\mathsf{M}}^{\vee} \mathrm{R}k_{\mathsf{M}*} \simeq \mathrm{R}i_{\mathsf{M}*} \iota_{\overset{\circ}{\mathsf{M}}}$,
(ii) $\alpha_{\mathsf{M}} \simeq k_{\mathsf{M}}^{-1} \alpha_{\mathsf{M}}^{\vee} \mathrm{R}j_{\mathsf{M}!!} \simeq \alpha_{\overset{\circ}{\mathsf{M}}} i_{\mathsf{M}}^{-1}$,
(iii) $\beta_{\mathsf{M}} \simeq \mathrm{R}i_{\mathsf{M}!!} \beta_{\overset{\circ}{\mathsf{M}}}$.

---

[4]The statement of this proposition is erroneous. The first isomorphism in loc. cit. may not hold under the condition that $\overset{\circ}{f}$ is topologically submersive. However, it holds if f is borderly submersive. The second isomorphism, i.e. (2.3), holds under the condition that $\overset{\circ}{f}$ is topologically submersive.

Proof. — One has

$$j_{\mathsf{M}}^{-1}\, \iota_{\mathsf{M}}^{\vee}\, \mathrm{R}k_{\mathsf{M}*} \underset{(*)}{\simeq} j_{\mathsf{M}}^{-1}\, \mathrm{R}k_{\mathsf{M}*}\, \iota_{\mathsf{M}}^{\circ} \simeq j_{\mathsf{M}}^{-1}\, \mathrm{R}j_{\mathsf{M}*}\, \mathrm{R}i_{\mathsf{M}*}\, \iota_{\mathsf{M}}^{\circ} \simeq \mathrm{R}i_{\mathsf{M}*}\, \iota_{\mathsf{M}}^{\circ},$$

where $(*)$ follows from (2.2).

This proves (i). Then (ii) and (iii) follow by adjunction. □

For bordered spaces, the commutativity of the functor α with the operations is as follows.

Lemma 2.2. — *Let $f\colon \mathsf{M} \to \mathsf{N}$ be a morphism of bordered spaces.*

(i) *There are a natural isomorphism and a natural morphism of functors*

$$\mathring{f}^{-1}\, \alpha_{\mathsf{N}} \simeq \alpha_{\mathsf{M}}\, f^{-1}, \qquad \alpha_{\mathsf{M}}\, f^{!} \longrightarrow \mathring{f}^{!}\, \alpha_{\mathsf{N}},$$

and the above morphism is an isomorphism if f is borderly submersive.

(ii) *There are natural morphisms of functors*

$$\mathrm{R}\mathring{f}_{!}\, \alpha_{\mathsf{M}} \longrightarrow \alpha_{\mathsf{N}}\, \mathrm{R}f_{!!}, \qquad \alpha_{\mathsf{N}}\, \mathrm{R}f_{*} \longrightarrow \mathrm{R}\mathring{f}_{*}\, \alpha_{\mathsf{M}},$$

which are isomorphisms if f is self-cartesian.

(iii) *For $K \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ and $L \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{N}})$ one has*

$$\alpha_{\mathsf{M}\times\mathsf{N}}(K \boxtimes L) \simeq (\alpha_{\mathsf{M}}K) \boxtimes (\alpha_{\mathsf{N}}L).$$

Proof

(i)(a) By Lemma 2.1(ii) and (2.2), one has

$$\mathring{f}^{-1}\, \alpha_{\mathsf{N}} \simeq \mathring{f}^{-1}\, \alpha_{\mathsf{N}}^{\circ}\, i_{\mathsf{N}}^{-1} \simeq \alpha_{\mathsf{M}}^{\circ}\, \mathring{f}^{-1}\, i_{\mathsf{N}}^{-1} \simeq \alpha_{\mathsf{M}}^{\circ}\, i_{\mathsf{M}}^{-1}\, f^{-1} \simeq \alpha_{\mathsf{M}}\, f^{-1}.$$

(i)(b) By Lemma 2.1(ii), the morphism is given by the composition

$$\alpha_{\mathsf{M}}^{\circ}\, i_{\mathsf{M}}^{-1}\, f^{!} \xrightarrow[\sim]{(*)} \alpha_{\mathsf{M}}^{\circ}\, \mathring{f}^{!}\, i_{\mathsf{N}}^{-1} \xrightarrow{(**)} \mathring{f}^{!}\, \alpha_{\mathsf{N}}^{\circ}\, i_{\mathsf{N}}^{-1}.$$

Here, $(*)$ follows from (2.5), and $(**)$ follows by adjunction from

$$\mathring{f}^{!} \longrightarrow \mathring{f}^{!}\, \iota_{\mathsf{N}}^{\circ}\, \alpha_{\mathsf{N}}^{\circ} \simeq \iota_{\mathsf{M}}^{\circ}\, \mathring{f}^{!}\, \alpha_{\mathsf{N}}^{\circ},$$

with the isomorphism due to (2.2). If f is borderly submersive, $(**)$ is an isomorphism by (2.4).

(ii)(a) By Lemma 2.1(ii), the morphism is given by

$$\mathrm{R}\mathring{f}_{!}\, \alpha_{\mathsf{M}}^{\circ}\, i_{\mathsf{M}}^{-1} \simeq \alpha_{\mathsf{N}}^{\circ}\, \mathrm{R}\mathring{f}_{!!}\, i_{\mathsf{M}}^{-1} \xrightarrow{(*)} \alpha_{\mathsf{N}}^{\circ}\, i_{\mathsf{N}}^{-1}\, \mathrm{R}f_{!!}.$$

Here $(*)$ follows by adjunction from $\mathrm{R}i_{\mathsf{N}!!}\, \mathrm{R}\mathring{f}_{!!}\, i_{\mathsf{M}}^{!} \simeq \mathrm{R}f_{!!}\, \mathrm{R}i_{\mathsf{M}!!}i_{\mathsf{M}}^{!} \to \mathrm{R}f_{!!}$, recalling (2.5). If f is self-cartesian, this is an isomorphism by cartesianity.

(ii)(b) By Lemma 2.1(ii) and (2.2), the morphism is given by the composition

$$\alpha_{\mathsf{N}}^{\circ}\, i_{\mathsf{N}}^{-1}\, \mathrm{R}f_{*} \xrightarrow{(*)} \alpha_{\mathsf{N}}^{\circ}\, \mathrm{R}\mathring{f}_{*}\, i_{\mathsf{M}}^{-1} \simeq \mathrm{R}\mathring{f}_{*}\, \alpha_{\mathsf{M}}^{\circ}\, i_{\mathsf{M}}^{-1}.$$

Here $(*)$ follows from Lemma A.3.

Recall (2.5). If f is self-cartesian, then $(*)$ is an isomorphism by cartesianity.

(iii) follows from $\alpha_{\mathsf{M}} \simeq \alpha_{\mathsf{M}}^{\circ}\, i_{\mathsf{M}}^{-1}$ and (2.2). □

2.4. ENHANCED IND-SHEAVES. — Denote by $t \in \mathbb{R}$ the coordinate on the affine line, consider the two-point compactification $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, and set $\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}})$. For M a bordered space, consider the projection

$$\pi_\mathsf{M} \colon \mathsf{M} \times \mathbb{R}_\infty \longrightarrow \mathsf{M}.$$

Denote by $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_\mathsf{M}) := \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M} \times \mathbb{R}_\infty})/\pi_\mathsf{M}^{-1}\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_\mathsf{M})$ the bounded derived category of enhanced ind-sheaves of $\mathbf{k}$-vector spaces on M. Denote by $\mathrm{Q} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M} \times \mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_\mathsf{M})$ the quotient functor, and by L^{E} and R^{E} its left and right adjoint, respectively. They are both fully faithful.

For $f \colon \mathsf{M} \to \mathsf{N}$ a morphism of bordered spaces, set

$$f_\mathbb{R} := f \times \mathrm{id}_{\mathbb{R}_\infty} \colon \mathsf{M} \times \mathbb{R}_\infty \longrightarrow \mathsf{N} \times \mathbb{R}_\infty.$$

Denote by $\overset{+}{\otimes}$, $\mathrm{E}f^{-1}$, $\mathrm{E}f_{!!}$ and $\mathrm{R}\mathscr{I}hom^+$, $\mathrm{E}f_*$, $\mathrm{E}f^!$ the six operations for enhanced ind-sheaves. Recall that $\overset{+}{\otimes}$ is the additive convolution in the t variable, and that the external operations are induced via Q by the corresponding operations for ind-sheaves, with respect to the morphism $f_\mathbb{R}$. Denote by $\overset{+}{\boxtimes}$ the exterior tensor product and by D^{E} the Verdier duality functor.

We have

$$\mathrm{L}^{\mathrm{E}}\mathrm{Q}(F) \simeq (\mathbf{k}_{\{t \geqslant 0\}} \oplus \mathbf{k}_{\{t \leqslant 0\}}) \overset{+}{\otimes} F$$
$$\text{and} \qquad \mathrm{R}^{\mathrm{E}}\mathrm{Q}(F) \simeq \mathrm{R}\mathscr{I}hom^+(\mathbf{k}_{\{t \geqslant 0\}} \oplus \mathbf{k}_{\{t \leqslant 0\}}, F).$$

The functors $\mathrm{R}\mathscr{I}hom^{\mathrm{E}}$ and $\mathrm{R}\mathscr{H}om^{\mathrm{E}}$, taking values in $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_\mathsf{M})$ and $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}})$, respectively, are defined by

$$(2.6) \qquad \mathrm{R}\mathscr{I}hom^{\mathrm{E}}(K_1, K_2) := \mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\mathscr{I}hom(F_1, \mathrm{R}^{\mathrm{E}}K_2)$$
$$\simeq \mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\mathscr{I}hom(\mathrm{L}^{\mathrm{E}}K_1, F_2),$$
$$(2.7) \qquad \mathrm{R}\mathscr{H}om^{\mathrm{E}}(K_1, K_2) := \alpha_\mathsf{M}\mathrm{R}\mathscr{I}hom^{\mathrm{E}}(K_1, K_2),$$

for $K_i \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_\mathsf{M})$ and $F_i \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M} \times \mathbb{R}_\infty})$ such that $K_i = \mathrm{Q}F_i$ $(i = 1, 2)$.

There is a natural decomposition

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_\mathsf{M}) \simeq \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_\mathsf{M}) \oplus \mathrm{E}^{\mathrm{b}}_-(\mathrm{I}\mathbf{k}_\mathsf{M}),$$

given by

$$K \longmapsto (\mathrm{Q}\,\mathbf{k}_{\{t \geqslant 0\}} \overset{+}{\otimes} K) \oplus (\mathrm{Q}\,\mathbf{k}_{\{t \leqslant 0\}} \overset{+}{\otimes} K).$$

Denote by $\mathrm{L}^{\mathrm{E}}_\pm$ and $\mathrm{R}^{\mathrm{E}}_\pm$ the left and right adjoint, respectively, of the quotient functor $\mathrm{Q} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M} \times \mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}_\pm(\mathrm{I}\mathbf{k}_\mathsf{M})$.

There are embeddings

$$\epsilon^\pm_\mathsf{M} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_\mathsf{M}) \rightarrowtail \mathrm{E}^{\mathrm{b}}_\pm(\mathrm{I}\mathbf{k}_\mathsf{M}) , \quad F \longmapsto \mathrm{Q}(\mathbf{k}_{\{\pm t \geqslant 0\}} \otimes \pi_\mathsf{M}^{-1}F),$$

and one sets $\epsilon_\mathsf{M}(F) := \epsilon^+_\mathsf{M}(F) \oplus \epsilon^-_\mathsf{M}(F) \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_\mathsf{M})$. Note that $\epsilon_\mathsf{M}(F) \simeq \mathrm{Q}(\mathbf{k}_{\{t=0\}} \otimes \pi_\mathsf{M}^{-1}F)$.

2.5. **Stable objects.** — Let M be a bordered space. Set

$$\mathbf{k}_{\{t \gg 0\}} := \text{``}\varinjlim_{a \to +\infty}\text{''} \mathbf{k}_{\{t \geqslant a\}} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M} \times \mathbb{R}_\infty}),$$

$$\mathbf{k}_{\mathsf{M}}^{\mathrm{E}} := \mathrm{Q}\,\mathbf{k}_{\{t \gg 0\}} \in \mathrm{E}_+^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}).$$

An object $K \in \mathrm{E}_+^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ is called *stable* if $\mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} K \overset{\sim}{\longrightarrow} K$. We denote by $\mathrm{E}_{\mathrm{st}}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ the full subcategory of $\mathrm{E}_+^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ of stable objects. The embedding $\mathrm{E}_{\mathrm{st}}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \rightarrowtail \mathrm{E}_+^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ has a left adjoint $\mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} *$, as well as a right adjoint $\mathrm{R}\mathscr{I}hom^+(\mathbf{k}_{\mathsf{M}}^{\mathrm{E}}, *)$.

There is an embedding

$$e_{\mathsf{M}} \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \rightarrowtail \mathrm{E}_{\mathrm{st}}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}), \quad F \longmapsto \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} \epsilon_{\mathsf{M}}(F) \simeq \mathrm{Q}(\mathbf{k}_{\{t \gg 0\}} \otimes \pi_{\mathsf{M}}^{-1} F).$$

Notation 2.3. — Let $S \subset T$ be locally closed subsets of M.

(i) For continuous maps $\varphi_\pm \colon T \to \overline{\mathbb{R}}$ such that $-\infty \leqslant \varphi_- \leqslant \varphi_+ < +\infty$, set

$$\mathsf{E}_{S|\mathsf{M}}^{\varphi_+ \triangleright \varphi_-} := \mathrm{Q}\,\mathbf{k}_{\{x \in S,\ -\varphi_+(x) \leqslant t < -\varphi_-(x)\}} \quad \in \mathrm{E}_+^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}),$$

$$\mathbb{E}_{S|\mathsf{M}}^{\varphi_+ \triangleright \varphi_-} := \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{E}_{S|\mathsf{M}}^{\varphi_+ \triangleright \varphi_-} \quad \in \mathrm{E}_{\mathrm{st}}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}),$$

where we write for short

$$\{x \in S,\ -\varphi_+(x) \leqslant t < -\varphi_-(x)\} := \{(x,t) \in \overset{\circ}{\mathsf{M}} \times \mathbb{R};\ x \in S,\ -\varphi_+(x) \leqslant t < -\varphi_-(x)\},$$

with $<$ the total order on $\overline{\overline{\mathbb{R}}}$. If $S = T$, we also write for short

$$\{-\varphi_+(x) \leqslant t < -\varphi_-(x)\} := \{x \in T,\ -\varphi_+(x) \leqslant t < -\varphi_-(x)\}.$$

(ii) For a continuous map $\varphi \colon T \to \mathbb{R}$, consider the object of $\mathrm{E}_+^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$

$$\mathsf{E}_{S|\mathsf{M}}^{\varphi} := \mathrm{Q}\,\mathbf{k}_{\{x \in S,\ t+\varphi(x) \geqslant 0\}} \quad \in \mathrm{E}_+^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}),$$

$$\mathbb{E}_{S|\mathsf{M}}^{\varphi} := \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} \mathsf{E}_{S|\mathsf{M}}^{\varphi} \quad \in \mathrm{E}_{\mathrm{st}}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}),$$

where we write for short

$$\{x \in S,\ t + \varphi(x) \geqslant 0\} = \{(x,t) \in \overset{\circ}{\mathsf{M}} \times \mathbb{R};\ x \in S,\ t + \varphi(x) \geqslant 0\}.$$

If $S = T$, we also write for short

$$\{t + \varphi(x) \geqslant 0\} := \{x \in T,\ t + \varphi(x) \geqslant 0\}.$$

Note that one has $\mathsf{E}_{S|\mathsf{M}}^{\varphi} \simeq \mathsf{E}_{S|\mathsf{M}}^{\varphi \triangleright -\infty}$, and that there is a short exact sequence

$$0 \longrightarrow \mathsf{E}_{S|\mathsf{M}}^{\varphi_+ \triangleright \varphi_-} \longrightarrow \mathsf{E}_{S|\mathsf{M}}^{\varphi_+} \longrightarrow \mathsf{E}_{S|\mathsf{M}}^{\varphi_-} \longrightarrow 0$$

in the heart of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ for the natural t-structure.

2.6. CONSTRUCTIBLE OBJECTS. — A subanalytic bordered space is a bordered space M such that $\overset{\circ}{\mathsf{M}}$ is an open subanalytic subset of the subanalytic space $\overset{\vee}{\mathsf{M}}$. A morphism $f\colon \mathsf{M} \to \mathsf{N}$ of subanalytic bordered spaces is a morphism of bordered spaces such that Γ_f is subanalytic in $\overset{\vee}{\mathsf{M}} \times \overset{\vee}{\mathsf{N}}$. By definition, a subset Z of M is subanalytic if it is subanalytic in $\overset{\vee}{\mathsf{M}}$.

Let M be a subanalytic bordered space. Denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-}\mathrm{c}}(\mathbf{k}_{\mathsf{M}})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}})$ whose objects F are such that $\mathrm{R}k_{\mathsf{M}*}F$ (or equivalently, $\mathrm{R}k_{\mathsf{M}!}F$) is weakly $\mathbb{R}$-constructible, for $k_{\mathsf{M}}\colon \overset{\circ}{\mathsf{M}} \to \overset{\vee}{\mathsf{M}}$ the embedding. We similarly define the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}\mathrm{c}}(\mathbf{k}_{\mathsf{M}})$ of $\mathbb{R}$-constructible sheaves. Denote by $\mathrm{E}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-}\mathrm{c}}(\mathbf{I}\mathbf{k}_{\mathsf{M}})$ the strictly full subcategory of $\mathrm{E}^{\mathrm{b}}(\mathbf{I}\mathbf{k}_{\mathsf{M}})$ whose objects K are such that for any relatively compact open subanalytic subset U of M, one has

$$\pi_{\mathsf{M}}^{-1}\mathbf{k}_U \otimes K \simeq \mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} \mathrm{Q}\, F$$

for some $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-}\mathrm{c}}(\mathbf{k}_{\mathsf{M}\times\mathbb{R}_\infty})$. In particular, K belongs to $\mathrm{E}^{\mathrm{b}}_{\mathrm{st}}(\mathbf{I}\mathbf{k}_{\mathsf{M}})$. We similarly define the category $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}\mathrm{c}}(\mathbf{I}\mathbf{k}_{\mathsf{M}})$ of $\mathbb{R}$-constructible enhanced ind-sheaves.

3. SHEAFIFICATION

In this section, we discuss what we call here ind-sheafification and sheafification functor, and prove some of their functorial properties. Concerning constructibility, we use a fundamental result from [12, §6].

3.1. ASSOCIATED IND-SHEAF. — Let M be a bordered space. Let $i_0\colon \mathsf{M} \to \mathsf{M} \times \mathbb{R}_\infty$ be the embedding $x \mapsto (x, 0)$.

DEFINITION 3.1. — Let $K \in \mathrm{E}^{\mathrm{b}}(\mathbf{I}\mathbf{k}_{\mathsf{M}})$ and take $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{I}\mathbf{k}_{\mathsf{M}\times\mathbb{R}_\infty})$ such that $K \simeq \mathrm{Q}\, F$. We set

$$\begin{aligned}
\mathsf{Ish}_{\mathsf{M}}(K) &:= \mathrm{R}\mathscr{I}hom^{\mathrm{E}}(\mathrm{Q}\, \mathbf{k}_{\{t=0\}}, K) \\
&\simeq \mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t\geqslant 0\}} \oplus \mathbf{k}_{\{t\leqslant 0\}}, F) \\
&\simeq \mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}^{\mathrm{E}}\, K) \\
&\simeq \mathrm{R}\pi_{\mathsf{M}!!}\mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}^{\mathrm{E}}\, K) \\
&\simeq i_0^!\, \mathrm{R}^{\mathrm{E}}\, K \quad \in \mathrm{D}^{\mathrm{b}}(\mathbf{I}\mathbf{k}_{\mathsf{M}})
\end{aligned}$$

(see [1, Lem. 4.5.16]), and call it the *associated ind-sheaf* (in the derived sense) to K on M. We will write for short $\mathsf{Ish} = \mathsf{Ish}_{\mathsf{M}}$, if there is no fear of confusion.

Note that one has

$$\begin{aligned}
\mathsf{Ish}(K) &\simeq \mathrm{R}\mathscr{I}hom^{\mathrm{E}}(\mathrm{Q}\, \mathbf{k}_{\{t\geqslant 0\}}, K) &&\text{for } K \in \mathrm{E}^{\mathrm{b}}_+(\mathbf{I}\mathbf{k}_{\mathsf{M}}), \\
\mathsf{Ish}(K) &\simeq \mathrm{R}\mathscr{I}hom^{\mathrm{E}}(\mathbf{k}_{\mathsf{M}}^{\mathrm{E}}, K) &&\text{for } K \in \mathrm{E}^{\mathrm{b}}_{\mathrm{st}}(\mathbf{I}\mathbf{k}_{\mathsf{M}}).
\end{aligned}$$

LEMMA 3.2. — *The following are pairs of adjoint functors*

(i) $(\epsilon, \mathsf{Ish})\colon \mathrm{D}^{\mathrm{b}}(\mathbf{I}\mathbf{k}_{\mathsf{M}}) \underset{\mathsf{Ish}}{\overset{\epsilon}{\rightleftarrows}} \mathrm{E}^{\mathrm{b}}(\mathbf{I}\mathbf{k}_{\mathsf{M}})$,

(ii) $(\epsilon^+, \mathsf{Ish}) \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \underset{\mathsf{Ish}}{\overset{\epsilon^+}{\rightleftarrows}} \mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_{\mathsf{M}}),$

(iii) $(e, \mathsf{Ish}) \colon \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \underset{\mathsf{Ish}}{\overset{e}{\rightleftarrows}} \mathrm{E}^{\mathrm{b}}_{\mathrm{st}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}).$

Proof

(i) For $F \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ and $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ one has

$$\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})}(\epsilon(F), K) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M} \times \mathbb{R}_{\infty}})}(\pi^{-1} F \otimes \mathbf{k}_{\{t=0\}}, \mathrm{R}^{\mathrm{E}} K)$$
$$\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})}(F, \mathrm{R}\pi_{*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}^{\mathrm{E}} K))$$
$$\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})}(F, \mathsf{Ish}(K)).$$

(ii) and (iii) follow from (i), noticing that there are pairs of adjoint functors $(* \overset{+}{\otimes} \mathrm{Q}\,\mathbf{k}_{\{t \geqslant 0\}}, \iota)$ and $(* \overset{+}{\otimes} \mathbf{k}^{\mathrm{E}}_{\mathsf{M}}, \iota)$:

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \underset{\iota}{\overset{* \overset{+}{\otimes} \mathrm{Q}\,\mathbf{k}_{\{t \geqslant 0\}}}{\rightleftarrows}} \mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_{\mathsf{M}}) \underset{\iota}{\overset{* \overset{+}{\otimes} \mathbf{k}^{\mathrm{E}}_{\mathsf{M}}}{\rightleftarrows}} \mathrm{E}^{\mathrm{b}}_{\mathrm{st}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}).$$

Here we denote by ι the natural embeddings. □

Lemma 3.3. — *Let $f \colon \mathsf{M} \to \mathsf{N}$ be a morphism of bordered spaces.*

(i) *There are a natural morphism and a natural isomorphism of functors*

$$f^{-1}\,\mathsf{Ish}_{\mathsf{N}} \longrightarrow \mathsf{Ish}_{\mathsf{M}}\,\mathrm{E}f^{-1}, \quad f^{!}\,\mathsf{Ish}_{\mathsf{N}} \simeq \mathsf{Ish}_{\mathsf{M}}\,\mathrm{E}f^{!},$$

and the above morphism is an isomorphism if f is borderly submersive.

(ii) *There are a natural morphism and a natural isomorphism of functors*

$$\mathrm{R}f_{!!}\,\mathsf{Ish}_{\mathsf{M}} \longrightarrow \mathsf{Ish}_{\mathsf{N}}\,\mathrm{E}f_{!!}, \quad \mathrm{R}f_{*}\,\mathsf{Ish}_{\mathsf{M}} \simeq \mathsf{Ish}_{\mathsf{N}}\,\mathrm{E}f_{*},$$

and the above morphism is an isomorphism if f is proper.

(iii) *For $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ and $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{N}})$, there is a natural morphism*

$$\mathsf{Ish}(K) \boxtimes \mathsf{Ish}(L) \longrightarrow \mathsf{Ish}(K \overset{+}{\boxtimes} L).$$

Proof. — Recall that one sets $f_{\mathbb{R}} := f \times \mathrm{id}_{\mathbb{R}_{\infty}} \colon \mathsf{M} \times \mathbb{R}_{\infty} \to \mathsf{N} \times \mathbb{R}_{\infty}$.

(i) Let $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{N}})$ and set $G := \mathrm{R}^{\mathrm{E}} L \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{N} \times \mathbb{R}_{\infty}})$.

(i)(a) One has

$$f^{-1}\mathsf{Ish}_{\mathsf{N}}(L) \simeq f^{-1}\,\mathrm{R}\pi_{\mathsf{N}!!}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, G)$$
$$\simeq \mathrm{R}\pi_{\mathsf{M}!!}\,f_{\mathbb{R}}^{-1}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, G)$$
$$\underset{(*)}{\longrightarrow} \mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, f_{\mathbb{R}}^{-1}G)$$
$$\underset{(**)}{\longrightarrow} \mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}^{\mathrm{E}}\,\mathrm{E}f^{-1}L)$$
$$\simeq \mathsf{Ish}_{\mathsf{M}}(\mathrm{E}f^{-1}L).$$

Here, $(*)$ follows from [1, Prop. 3.3.13], and $(**)$ from Lemma A.4.

If f is borderly submersive, then $(*)$ is an isomorphism by [1, Prop. 3.3.19] and $(**)$ is an isomorphism by Lemma A.4.

(i)(b) Recall that $f_{\mathbb{R}}^{!}G \simeq \mathrm{R}^{\mathrm{E}}(\mathrm{E}f^{!}L)$. One has

$$
\begin{aligned}
f^{!}\,\mathsf{Ish}_{\mathsf{N}}(L) = f^{!}\,\mathrm{R}\pi_{\mathsf{N}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, G) \\
\simeq \mathrm{R}\pi_{\mathsf{M}*}\, f_{\mathbb{R}}^{!}\,\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, G) \\
\simeq \mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, f_{\mathbb{R}}^{!}\,G) \\
\simeq \mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}^{\mathrm{E}}(\mathrm{E}f^{!}L)) \\
\simeq \mathsf{Ish}_{\mathsf{M}}(\mathrm{E}f^{!}L).
\end{aligned}
$$

(ii) Let $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ and set $F := \mathrm{R}^{\mathrm{E}}\,K \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}\times\mathbb{R}_{\infty}})$.

(ii)(a) One has

$$
\begin{aligned}
\mathsf{Ish}_{\mathsf{N}}(\mathrm{E}f_{!!}K) = \mathrm{R}\pi_{\mathsf{N}!!}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}^{\mathrm{E}}\,\mathrm{E}f_{!!}K) \\
\longleftarrow \mathrm{R}\pi_{\mathsf{N}!!}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}f_{\mathbb{R}!!}F) \\
\underset{(*)}{\simeq} \mathrm{R}\pi_{\mathsf{N}!!}\,\mathrm{R}f_{\mathbb{R}!!}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, F) \\
\overset{\sim}{\longleftarrow} \mathrm{R}f_{!!}\,\mathrm{R}\pi_{\mathsf{M}!!}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, F) \\
= \mathrm{R}f_{!!}(\mathsf{Ish}_{\mathsf{M}}(K)).
\end{aligned}
$$

Here $(*)$ follows from [10, Lem. 5.2.8].

(ii)(b) Since $\mathrm{R}^{\mathrm{E}}(\mathrm{E}f_{*}K) \simeq \mathrm{R}f_{\mathbb{R}*}F$, one has

$$
\begin{aligned}
\mathsf{Ish}_{\mathsf{N}}(\mathrm{E}f_{*}K) \simeq \mathrm{R}\pi_{\mathsf{N}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}f_{\mathbb{R}*}F) \\
\simeq \mathrm{R}\pi_{\mathsf{M}*}\,\mathrm{R}f_{\mathbb{R}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, F) \\
\simeq \mathrm{R}f_{*}\,\mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, F).
\end{aligned}
$$

If f is proper, $f_{!} \simeq f_{*}$.

(iii) Set $F := \mathrm{R}^{\mathrm{E}}\,K \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}\times\mathbb{R}_{\infty}})$ and $G := \mathrm{R}^{\mathrm{E}}\,L \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{N}\times\mathbb{R}_{\infty}})$. Recall that $F \overset{+}{\boxtimes} G := \mathrm{R}m_{!!}(F \boxtimes G)$, where

$$
m\colon \mathsf{M} \times \mathbb{R}_{\infty} \times \mathsf{N} \times \mathbb{R}_{\infty} \longrightarrow \mathsf{M} \times \mathsf{N} \times \mathbb{R}_{\infty} \quad (x, t_1, y, t_2) \longmapsto (x, y, t_1 + t_2).
$$

Then, one has

$$
\begin{aligned}
\mathsf{Ish}(K) \boxtimes \mathsf{Ish}(L) \simeq \mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t_1=0\}}, F) \boxtimes \mathrm{R}\pi_{\mathsf{N}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t_2=0\}}, G) \\
\longrightarrow \mathrm{R}(\pi_{\mathsf{M}} \times \pi_{\mathsf{N}})_{*}\big(\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t_1=0\}}, F) \boxtimes \mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t_2=0\}}, G)\big) \\
\longrightarrow \mathrm{R}\pi_{\mathsf{M}\times\mathsf{N}*}\mathrm{R}m_{*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t_1=0\}} \boxtimes \mathbf{k}_{\{t_2=0\}}, F \boxtimes G) \\
\longrightarrow \mathrm{R}\pi_{\mathsf{M}\times\mathsf{N}*}\mathrm{R}\,\mathscr{I}hom\big(\mathrm{R}m_{!!}(\mathbf{k}_{\{t_1=0\}} \boxtimes \mathbf{k}_{\{t_2=0\}}), \mathrm{R}m_{!!}(F \boxtimes G)\big) \\
\simeq \mathrm{R}\pi_{\mathsf{M}\times\mathsf{N}*}\mathrm{R}\,\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, F \overset{+}{\boxtimes} G).
\end{aligned}
$$

One concludes using the natural morphism $F \overset{+}{\boxtimes} G \to \mathrm{R}^{\mathrm{E}}(K \overset{+}{\boxtimes} L)$. $\qquad\square$

3.2. Associated sheaf. — Let M be a bordered space.

Definition 3.4. — Let $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$.

(i) We set

$$\mathsf{sh}_{\mathsf{M}}(K) := \mathrm{R}\mathscr{H}om^{\mathrm{E}}(\mathrm{Q}\,\mathbf{k}_{\{t=0\}}, K)$$
$$= \alpha_{\mathsf{M}}\,\mathsf{Ish}_{\mathsf{M}}(K) \quad \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}}),$$

and call it the *associated sheaf* (in the derived sense) to K on $\overset{\circ}{\mathsf{M}}$. We will write for short $\mathsf{sh} = \mathsf{sh}_{\mathsf{M}}$, if there is no fear of confusion.

(ii) We say that K is of *sheaf type* (in the derived sense) if it is in the essential image of

$$e_{\mathsf{M}}\,\iota_{\mathsf{M}}\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}}) \rightarrowtail \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}).$$

One has

$$\mathsf{sh}_{\mathsf{M}}(K) \simeq \mathrm{R}\mathscr{H}om^{\mathrm{E}}(\mathrm{Q}\,\mathbf{k}_{\{t\geqslant 0\}}, K), \qquad \text{for } K \in \mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_{\mathsf{M}}),$$

(3.1) $$\mathsf{sh}_{\mathsf{M}}(K) \simeq \mathrm{R}\mathscr{H}om^{\mathrm{E}}(\mathbf{k}_{\mathsf{M}}^{\mathrm{E}}, K), \qquad \text{for } K \in \mathrm{E}^{\mathrm{b}}_{\mathrm{st}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}).$$

Lemma 3.5. — *One has* $\mathsf{sh}_{\mathsf{M}} \simeq \mathsf{sh}_{\overset{\circ}{\mathsf{M}}}\,\mathrm{E}i_{\mathsf{M}}^{-1}$.

Proof. — Recall that $i_{\mathsf{M}}^{-1} \simeq i_{\mathsf{M}}^{!}$. Using Lemma 2.1(ii), one has

$$\alpha_{\mathsf{M}}\,\mathsf{Ish}_{\mathsf{M}} \simeq \alpha_{\overset{\circ}{\mathsf{M}}}\,i_{\mathsf{M}}^{!}\,\mathrm{R}\mathscr{I}hom^{\mathrm{E}}(\mathrm{Q}\,\mathbf{k}_{\{t=0\}}, K)$$
$$= \alpha_{\overset{\circ}{\mathsf{M}}}\,i_{\mathsf{M}}^{!}\,\mathrm{R}\pi_{\mathsf{M}*}\mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}^{\mathrm{E}}\,K)$$
$$\simeq \alpha_{\overset{\circ}{\mathsf{M}}}\,\mathrm{R}\pi_{\overset{\circ}{\mathsf{M}}*}\,i_{\mathsf{M}\times\mathbb{R}_{\infty}}^{!}\,\mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}^{\mathrm{E}}\,K)$$
$$\simeq \alpha_{\overset{\circ}{\mathsf{M}}}\,\mathrm{R}\pi_{\overset{\circ}{\mathsf{M}}*}\,\mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, i_{\mathsf{M}\times\mathbb{R}_{\infty}}^{!}\,\mathrm{R}^{\mathrm{E}}\,K)$$
$$\simeq \alpha_{\overset{\circ}{\mathsf{M}}}\,\mathrm{R}\pi_{\overset{\circ}{\mathsf{M}}*}\,\mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, \mathrm{R}^{\mathrm{E}}\,\mathrm{E}i_{\mathsf{M}}^{!}K)$$
$$\simeq \alpha_{\overset{\circ}{\mathsf{M}}}\,\mathrm{R}\mathscr{I}hom^{\mathrm{E}}(\mathrm{Q}\,\mathbf{k}_{\{t=0\}}, \mathrm{E}i_{\mathsf{M}}^{!}K). \qquad \square$$

Let M be a bordered space, and consider the natural morphisms of good spaces

$$\overset{\circ}{\mathsf{M}} \times \mathbb{R} \xrightarrow{\;k\;} \overset{\circ}{\mathsf{M}} \times \overline{\mathbb{R}} \xrightarrow{\;\overline{\pi}\;} \overset{\circ}{\mathsf{M}}.$$

We write $\overline{t}$ for points of $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

An important tool in this framework is given by

Proposition 3.6 ([12, Cor. 6.6.6])

Let M *be a bordered space. Then, for* $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\overset{\circ}{\mathsf{M}}\times\mathbb{R}})$ *one has*

$$\mathsf{sh}_{\mathsf{M}}(\mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} \mathrm{Q}\,F) \simeq \mathrm{R}\overline{\pi}_{*}(\mathbf{k}_{\{-\infty < \overline{t} \leqslant +\infty\}} \otimes \mathrm{R}k_{*}F).$$

Consider the natural morphism $j\colon M \times \mathbb{R}_{\infty} \to M \times \overline{\mathbb{R}}$, and the embeddings $i_{\pm\infty}\colon M \to M \times \overline{\mathbb{R}}$, $x \mapsto (x, \pm\infty)$. Using the above proposition and [1, Prop. 4.3.10, Lem. 4.3.13], we get

Corollary 3.7. — *Let* M *be a bordered space. Then, for* $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{M} \times \mathbb{R}}^{\circ})$ *one has*

$$\mathsf{sh}_{\mathsf{M}}(\mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} \mathrm{Q}\, F) \simeq i_{+\infty}^{-1} \mathrm{R} j_* \, \mathrm{L}_+^{\mathrm{E}} \, \mathrm{Q}\, F$$
$$\simeq i_{-\infty}^{-1} \mathrm{R} j_* \, \mathrm{R}_+^{\mathrm{E}} \, \mathrm{Q}\, F[-1]$$
$$\simeq \mathrm{R} \pi_* \, \mathrm{L}_+^{\mathrm{E}} \, \mathrm{Q}\, F$$
$$\simeq \mathrm{R} \pi_! \, \mathrm{R}_+^{\mathrm{E}} \, \mathrm{Q}\, F.$$

Consider the functors

(3.2) $$\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{M}}^{\circ}) \underset{\mathsf{sh}_{\mathsf{M}}}{\overset{e_{\mathsf{M}}\, \iota_{\mathsf{M}}}{\rightleftarrows}} \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}}).$$

As explained in the Introduction, $(e_{\mathsf{M}}\, \iota_{\mathsf{M}},\ \mathsf{sh}_{\mathsf{M}})$ is *not* an adjoint pair of functors in general.

Proposition 3.8. — *Consider the functors* (3.2).

   (i) sh_{M} *is a left quasi-inverse to* $e_{\mathsf{M}}\, \iota_{\mathsf{M}}$.
   (ii) *The property of being of sheaf type is local*[5] *on* M, *and* $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ *is of sheaf type if and only if* $K \simeq e_{\mathsf{M}}\, \iota_{\mathsf{M}}\big(\mathsf{sh}_{\mathsf{M}}(K)\big)$.

Proof
   (i) By Proposition 3.6, for $L \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathsf{M}}^{\circ})$, one has

$$\mathsf{sh}_{\mathsf{M}}\, e_{\mathsf{M}}\, \iota_{\mathsf{M}}(L) \simeq \mathsf{sh}_{\mathsf{M}}\big(\mathbf{k}_{\mathsf{M}}^{\mathrm{E}} \overset{+}{\otimes} \mathrm{Q}(\mathbf{k}_{\{t=0\}} \otimes \pi^{-1}\iota_{\mathsf{M}}L)\big)$$
$$\simeq \mathrm{R}\overline{\pi}_*\big(\mathbf{k}_{\{-\infty < \overline{t} \leqslant +\infty\}} \otimes \mathbf{k}_{\{\overline{t}=0\}} \otimes \overline{\pi}^{-1}L\big)$$
$$\simeq \mathrm{R}\overline{\pi}_!\big(\mathbf{k}_{\{\overline{t}=0\}} \otimes \overline{\pi}^{-1}L\big)$$
$$\simeq \big(\mathrm{R}\overline{\pi}_!\, \mathbf{k}_{\{\overline{t}=0\}}\big) \otimes L \simeq L.$$

   (ii) follows from (i).   □

By Lemmas 2.2 and 3.3, one gets

Lemma 3.9. — *Let* $f\colon \mathsf{M} \to \mathsf{N}$ *be a morphism of bordered spaces.*
   (i) *There are natural morphisms of functors*

$$\mathring{f}^{-1}\, \mathsf{sh}_{\mathsf{N}} \longrightarrow \mathsf{sh}_{\mathsf{M}}\, \mathrm{E}f^{-1}, \quad \mathsf{sh}_{\mathsf{M}}\, \mathrm{E}f^! \longrightarrow \mathring{f}^!\, \mathsf{sh}_{\mathsf{N}},$$

which are isomorphisms if f *is borderly submersive.*
   (ii) *There are natural morphisms of functors*

$$\mathrm{R}\mathring{f}_!\, \mathsf{sh}_{\mathsf{M}} \longrightarrow \mathsf{sh}_{\mathsf{N}}\, \mathrm{E}f_{!!}, \quad \mathsf{sh}_{\mathsf{N}}\, \mathrm{E}f_* \longrightarrow \mathrm{R}\mathring{f}_*\, \mathsf{sh}_{\mathsf{M}}.$$

The first morphism is an isomorphism if f *is proper. The second morphism is an isomorphism if* f *is self-cartesian, and in particular if* f *is proper.*

---

[5] Saying that a property $\mathscr{P}(\mathsf{M})$ is local on M means the following. For any open covering $\{U_i\}_{i \in I}$ of M, $\mathscr{P}(\mathsf{M})$ is true if and only if $\mathscr{P}\big((U_i)_\infty\big)$ is true for any $i \in I$.

(iii) *For $K \in \mathrm{E}^\mathrm{b}(\mathrm{I}\mathbf{k}_\mathsf{M})$ and $L \in \mathrm{E}^\mathrm{b}(\mathrm{I}\mathbf{k}_\mathsf{N})$, there is a natural morphism*

$$\mathsf{sh}(K) \boxtimes \mathsf{sh}(L) \longrightarrow \mathsf{sh}(K \overset{+}{\boxtimes} L).$$

EXAMPLE 3.10. — Let $M = \mathbb{R}_x$, $U = \{x > 0\}$. By Corollary 3.7 one has

$$\mathrm{R}^\mathrm{E}\,\mathsf{E}_{U|M}^{1/x} \simeq \mathbf{k}_{\{x>0,\ xt<-1\}}[1], \qquad\qquad \mathrm{R}^\mathrm{E}\,\mathsf{E}_{U|M}^{-1/x} \simeq \mathbf{k}_{\{x\geqslant 0,\ xt<1\}}[1],$$

$$\mathsf{sh}(\mathbb{E}_{U|M}^{1/x}) \simeq \mathbf{k}_{\{x>0\}}, \qquad\qquad \mathsf{sh}(\mathbb{E}_{U|M}^{-1/x}) \simeq \mathbf{k}_{\{x\geqslant 0\}}.$$

Note that, denoting by $i\colon \{0\} \to M$ the embedding, one has

$$i^!(\mathsf{sh}(\mathbb{E}_{U|M}^{1/x})) \not\simeq \mathsf{sh}(\mathrm{E}i^!(\mathbb{E}_{U|M}^{1/x})), \quad i^{-1}(\mathsf{sh}(\mathbb{E}_{U|M}^{-1/x})) \not\simeq \mathsf{sh}(\mathrm{E}i^{-1}(\mathbb{E}_{U|M}^{-1/x})).$$

In fact, on one hand one has $i^!(\mathsf{sh}(\mathbb{E}_{U|M}^{1/x})) \simeq \mathbf{k}[-1]$ and $\mathrm{E}i^!(\mathbb{E}_{U|M}^{1/x}) \simeq 0$, and on the other hand one has $i^{-1}(\mathsf{sh}(\mathbb{E}_{U|M}^{-1/x})) \simeq \mathbf{k}$ and $\mathrm{E}i^{-1}(\mathbb{E}_{U|M}^{-1/x}) \simeq 0$.

Note also that sh is *not* conservative, since $\mathsf{sh}(\mathbb{E}_{U|X}^{2/x \triangleright 1/x}) \simeq 0$.

EXAMPLE 3.11. — Let $X \subset \mathbb{C}_z$ be an open neighborhood of the origin, and set $\dot{X} = X \smallsetminus \{0\}$. The real oriented blow-up $p\colon X_0^\mathsf{rb} \to X$ with center the origin is defined by $X_0^\mathsf{rb} := \{(r, w) \in \mathbb{R}_{\geqslant 0} \times \mathbb{C}; \ |w| = 1, \ rw \in X\}$, $p(r, w) = rw$. Denote by $S_0 X = \{r = 0\}$ the exceptional divisor.

Let $f \in \mathscr{O}_X(*0)$ be a meromorphic function with pole order $d > 0$ at the origin. With the identification $\dot{X} \simeq \{r > 0\} \subset X_0^\mathsf{rb}$, the set $I := S_0 X \smallsetminus \overline{\{z \in \dot{X} \ ; \ \mathrm{Re}\, f(z) \geqslant 0\}}$ is the disjoint union of d open non-empty intervals. Here $\overline{\{\cdot\}}$ is the closure in X_0^rb. Then,[6] recalling Notation 2.3,

$$\mathsf{sh}(\mathbb{E}_{\dot{X}|X}^{\mathrm{Re}\,f}) \simeq \mathsf{sh}(\mathrm{E}p_*\mathbb{E}_{\dot{X}|X_0^\mathsf{rb}}^{\mathrm{Re}\,f \circ p}) \simeq \mathrm{R}p_*\mathsf{sh}(\mathbb{E}_{\dot{X}|X_0^\mathsf{rb}}^{\mathrm{Re}\,f \circ p}) \simeq \mathrm{R}p_!\,\mathbf{k}_{I \sqcup \dot{X}}.$$

Recall that, for $\mathbf{k} = \mathbb{C}$, the Riemann-Hilbert correspondence of [1] associates the meromorphic connection $d - df$ with $\mathbb{E}_{\dot{X}|X}^{\mathrm{Re}\,f}$ by the functor $\mathscr{D}\mathscr{R}_X^\mathrm{E}$.

3.3. (Weak-) constructibility. — An important consequence of Proposition 3.6 is

PROPOSITION 3.12 ([12, Th. 6.6.4]). — *Let M be a subanalytic bordered space. The functor sh_M induces functors*

$$\mathsf{sh}_\mathsf{M}\colon \mathrm{E}_{\mathrm{w}\text{-}\mathbb{R}\text{-}\mathrm{c}}^\mathrm{b}(\mathrm{I}\mathbf{k}_\mathsf{M}) \longrightarrow \mathrm{D}_{\mathrm{w}\text{-}\mathbb{R}\text{-}\mathrm{c}}^\mathrm{b}(\mathbf{k}_\mathsf{M}),$$

$$\mathsf{sh}_\mathsf{M}\colon \mathrm{E}_{\mathbb{R}\text{-}\mathrm{c}}^\mathrm{b}(\mathrm{I}\mathbf{k}_\mathsf{M}) \longrightarrow \mathrm{D}_{\mathbb{R}\text{-}\mathrm{c}}^\mathrm{b}(\mathbf{k}_\mathsf{M}).$$

PROPOSITION 3.13. — *Let M be a subanalytic bordered space. For $K \in \mathrm{E}_{\mathbb{R}\text{-}\mathrm{c}}^\mathrm{b}(\mathrm{I}\mathbf{k}_\mathsf{M})$ there is a natural isomorphism*

$$\mathsf{sh}_\mathsf{M}(\mathrm{D}_\mathsf{M}^\mathrm{E}K) \xrightarrow{\sim} \mathrm{D}_{\mathring{\mathsf{M}}}(\mathsf{sh}_\mathsf{M}K).$$

---

[6]The analogue result for ind-sheaves was obtained in [11, Prop. 7.3] and [6, Prop. 3.14], at the level of cohomology groups.

Proof. — Recall that $\mathsf{sh}_\mathsf{M} \simeq \mathsf{sh}_{\overset{\circ}{\mathsf{M}}} \mathrm{E}i_\mathsf{M}^{-1}$ and $\mathrm{E}i_\mathsf{M}^{-1} \simeq \mathrm{E}i_\mathsf{M}^!$. Since $\mathrm{E}i_\mathsf{M}^{-1} \mathrm{D}_\mathsf{M}^\mathrm{E} \simeq \mathrm{D}_{\overset{\circ}{\mathsf{M}}}^\mathrm{E} \mathrm{E}i_\mathsf{M}^{-1}$, we may assume that $\mathsf{M} = \overset{\circ}{\mathsf{M}} = M$ is a subanalytic space.

(i) Let us construct a natural morphism

$$\mathsf{sh}(\mathrm{D}^\mathrm{E}K) \longrightarrow \mathrm{D}(\mathsf{sh}K).$$

By adjunction, it is enough to construct a natural morphism

$$\mathsf{sh}(\mathrm{D}^\mathrm{E}K) \otimes \mathsf{sh}(K) \longrightarrow \omega_M.$$

Note that we have a morphism

$$\mathrm{D}^\mathrm{E}K \overset{+}{\otimes} K \longrightarrow \omega_M^\mathrm{E}.$$

Let $\delta\colon M \to M \times M$ be the diagonal embedding, so that $\mathrm{D}^\mathrm{E}K \overset{+}{\otimes} K \simeq \mathrm{E}\delta^{-1}(\mathrm{D}^\mathrm{E}K \overset{+}{\boxtimes} K)$. There are natural morphisms

$$\mathsf{sh}(\mathrm{D}^\mathrm{E}K) \otimes \mathsf{sh}(K) \simeq \delta^{-1}\big(\mathsf{sh}(\mathrm{D}^\mathrm{E}K) \boxtimes \mathsf{sh}(K)\big)$$

$$\underset{(*)}{\longrightarrow} \delta^{-1}\big(\mathsf{sh}(\mathrm{D}^\mathrm{E}K \overset{+}{\boxtimes} K)\big)$$

$$\underset{(**)}{\longrightarrow} \mathsf{sh}\big(\mathrm{E}\delta^{-1}(\mathrm{D}^\mathrm{E}K \overset{+}{\boxtimes} K)\big)$$

$$\longrightarrow \mathsf{sh}(\omega_M^\mathrm{E}) \simeq \omega_M,$$

where $(*)$ is due to Lemma 3.9(iii), and $(**)$ is due to Lemma 3.9(i).

(ii) By (i), the problem is local on M. Hence, we may assume that $K \simeq \mathbf{k}_M^\mathrm{E} \overset{+}{\otimes} \mathrm{Q}\, F$ for $F \in \mathrm{D}_{\mathbb{R}\text{-c}}^\mathrm{b}(\mathbf{k}_{M \times \mathbb{R}_\infty})$. Consider the morphisms

$$k\colon M \times \mathbb{R}_\infty \xrightarrow{i^\pm} M \times (\mathbb{R} \cup \{\pm\infty\}, \overline{\mathbb{R}}) \xrightarrow{j^\pm} M \times \overline{\mathbb{R}}.$$

Since

$$\mathbf{k}_{\{-\infty < \overline{t} \leqslant +\infty\}} \otimes \mathrm{R}k_* F \simeq \mathrm{R}j_!^+ \, \mathrm{R}i_*^+ F \simeq \mathrm{R}j_*^- \, \mathrm{R}i_!^- F,$$

Proposition 3.6 gives

$$\mathsf{sh}_M(K) \simeq \mathrm{R}\overline{\pi}_* \, \mathrm{R}j_!^+ \, \mathrm{R}i_*^+ F$$
$$\simeq \mathrm{R}\overline{\pi}_* \, \mathrm{R}j_*^- \, \mathrm{R}i_!^- F.$$

By [1, Prop. 4.8.3] one has

$$\mathrm{D}_M^\mathrm{E}(\mathbf{k}_M^\mathrm{E} \overset{+}{\otimes} \mathrm{Q}\, F) \simeq \mathbf{k}_M^\mathrm{E} \overset{+}{\otimes} \mathrm{Q}\, a^{-1}\mathrm{D}_{M \times \mathbb{R}_\infty}F,$$

where $a\colon M \times \mathbb{R}_\infty \to M \times \mathbb{R}_\infty$ is given by $a(x,t) = (x,-t)$. Then, one has

$$\mathsf{sh}_M(\mathrm{D}_M^\mathrm{E}K) \simeq \mathsf{sh}_M(\mathbf{k}_M^\mathrm{E} \overset{+}{\otimes} \mathrm{Q}\, a^{-1} \mathrm{D}_{M \times \mathbb{R}_\infty} F)$$
$$\simeq \mathrm{R}\overline{\pi}_* \, \mathrm{R}j_!^+ \, \mathrm{R}i_*^+ a^{-1} \mathrm{D}_{M \times \mathbb{R}_\infty} F$$
$$\simeq \mathrm{R}\overline{\pi}_* \, \mathrm{R}j_!^- \, \mathrm{R}i_*^- \mathrm{D}_{M \times \mathbb{R}_\infty} F$$
$$\simeq \mathrm{D}_M(\mathrm{R}\overline{\pi}_* \, \mathrm{R}j_*^- \, \mathrm{R}i_!^- F)$$
$$\simeq \mathrm{D}_M(\mathsf{sh}_M(K)). \qquad \square$$

Lemma 3.14. — *Let* M *and* N *be bordered spaces. Let* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{\mathsf{M}})$ *and* $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{N}})$. *Then*

$$\mathsf{sh}(\epsilon(F) \overset{+}{\boxtimes} L) \simeq F \boxtimes \mathsf{sh}(L).$$

Proof. — For $G := \mathrm{R}^{\mathrm{E}} L \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{N} \times \mathbb{R}_\infty})$, one has

$$\mathsf{sh}(\epsilon(F) \overset{+}{\boxtimes} L) \simeq \alpha_{\mathsf{M} \times \mathsf{N}} \, \mathrm{R}\pi_{\mathsf{M} \times \mathsf{N}*} \, \mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, F \boxtimes G)$$

$$\underset{(\mathrm{a})}{\simeq} \alpha_{\mathsf{M} \times \mathsf{N}} \, \mathrm{R}\pi_{\mathsf{M} \times \mathsf{N}*} \big(F \boxtimes \mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, G) \big)$$

$$\underset{(\mathrm{b})}{\simeq} \alpha_{\mathsf{M} \times \mathsf{N}} \big(F \boxtimes \mathrm{R}\pi_{\mathsf{N}*} \mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, G) \big)$$

$$\simeq F \boxtimes \alpha_{\mathsf{N}} \, \mathrm{R}\pi_{\mathsf{N}*} \mathrm{R}\mathscr{I}hom(\mathbf{k}_{\{t=0\}}, G),$$

where (a) follows from [1, Cor. 2.3.5] and (b) follows from Proposition A.2 in Appendix A. $\qquad\square$

4. Stalk formula

As we saw in Example 3.10, sheafification does not commute with the pull-back by a closed embedding, in general. We provide here a stalk formula for the sheafification of such a pull-back, using results from Appendix B.

4.1. Restriction and stalk formula. — Let M be a subanalytic bordered space. Recall Notation 2.3.

Let $N \subset \mathsf{M}$ be a closed subanalytic subset, denote by $i \colon N_\infty \to \mathsf{M}$ the embedding. To illustrate the difference between $\mathsf{sh}\, \mathrm{E}i^{-1}$ and $i^{-1}\mathsf{sh}$ note that on one hand, by [2, Lem. 2.4.1], for $K \in \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ and $y_0 \in N$ one has[7]

$$\big(i^{-1}\mathsf{sh}(K)\big)_{y_0} \simeq \mathsf{sh}(K)_{y_0}$$

$$\simeq \varinjlim_{U \ni y_0} \mathrm{RHom}^{\mathrm{E}}(\mathsf{E}^0_{U|\mathsf{M}}, K),$$

where U runs over the open neighborhoods of y_0 in $\mathring{\mathsf{M}}$. On the other hand,

Proposition 4.1. — *Let* $\varphi \colon \mathsf{M} \to \mathbb{R}_\infty$ *be a morphism of subanalytic bordered spaces, set* $N := \mathring{\varphi}^{-1}(0) \subset \mathsf{M}$, *and denote by* $i \colon N_\infty \to \mathsf{M}$ *the embedding. For* $y_0 \in N$ *and* $K \in \mathrm{E}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$ *one has*

$$\mathsf{sh}(\mathrm{E}i^{-1}K)_{y_0} \simeq \varinjlim_{\substack{U \ni y_0 \\ \delta,\varepsilon \to 0+}} \mathrm{RHom}^{\mathrm{E}}(\mathsf{E}^{0 \rhd -\delta|\mathring{\varphi}(x)|^{-\varepsilon}}_{U|\mathsf{M}}, K),$$

where U *runs over the open neighborhoods of* y_0 *in* $\mathring{\mathsf{M}}$. *Here, we set* $-\delta|\mathring{\varphi}(x)|^{-\varepsilon} = -\infty$ *for* $\mathring{\varphi}(x) = 0$.

---

[7]Recall from [2, §2.1] that, for any $c, d \in \mathbb{Z}$, small filtrant inductive limits exist in $\mathrm{D}^{[c,d]}(\mathbf{k})$, the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$ whose objects V satisfy $H^j(V) = 0$ for $j < c$ or $j > d$. That is, uniformly bounded small filtrant inductive limits exist in $\mathrm{D}^{\mathrm{b}}(\mathbf{k})$.

More generally, for $T \subset N$ a compact subset one has

$$(4.1) \qquad \mathrm{R}\Gamma(T; \mathsf{sh}(\mathrm{E}i^{-1}K)) \simeq \varinjlim_{\substack{U \supset T \\ \delta,\varepsilon \to 0+}} \mathrm{RHom}^{\mathrm{E}}(\mathsf{E}_{U|\mathsf{M}}^{0\rhd -\delta|\overset{\circ}{\varphi}(x)|^{-\varepsilon}}, K),$$

where U runs over the open neighborhoods of T in $\overset{\circ}{\mathsf{M}}$.

Proof. — Let us prove the isomorphism (4.1). Since $T \subset N \subset \overset{\circ}{\mathsf{M}}$ is compact, we may assume that $\mathsf{M} = \overset{\circ}{\mathsf{M}} =: M$ is a subanalytic space.

(i) On the right hand side of (4.1), we may assume that U runs over the open *subanalytic* neighborhoods of T in M. Up to shrinking M around T, we can assume that there exists $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-}\mathrm{c}}(\mathbf{k}_{M\times\mathbb{R}_\infty})$ such that $K \simeq \mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} \mathrm{Q}\, F$. For $c \in \mathbb{R}$, and U an open relatively compact subanalytic subset of M containing T, set

$$U_{c,\delta,\varepsilon} := \{(x,t) \in U \times \mathbb{R}; \, t + c < \delta|\varphi(x)|^{-\varepsilon}\}.$$

Note that $\mathrm{L}^{\mathrm{E}}\, \mathsf{E}_{U|M}^{c\rhd c-\delta|\varphi(x)|^{-\varepsilon}} \simeq \mathbf{k}_{U_{c,\delta,\varepsilon}} \otimes \mathbf{k}_{\{t\geqslant -c\}}$. Then, one has

$$\mathrm{RHom}^{\mathrm{E}}(\mathsf{E}_{U|M}^{0\rhd -\delta|\varphi(x)|^{-\varepsilon}}, K) \simeq \varinjlim_{c\to+\infty} \mathrm{RHom}^{\mathrm{E}}(\mathrm{Q}\,\mathbf{k}_{\{t\geqslant -c\}} \overset{+}{\otimes} \mathsf{E}_{U|M}^{0\rhd -\delta|\varphi(x)|^{-\varepsilon}}, \mathrm{Q}\,F)$$

$$\simeq \varinjlim_{c\to+\infty} \mathrm{Hom}\,(\mathrm{L}^{\mathrm{E}}\, \mathsf{E}_{U|M}^{c\rhd c-\delta|\varphi(x)|^{-\varepsilon}}, F)$$

$$\simeq \varinjlim_{c\to+\infty} \mathrm{Hom}\,(\mathbf{k}_{U_{c,\delta,\varepsilon}} \otimes \mathbf{k}_{\{t\geqslant -c\}}, F)$$

$$\underset{(*)}{\simeq} \varinjlim_{c\to+\infty} \mathrm{Hom}\,(\mathbf{k}_{U_{c,\delta,\varepsilon}}, \mathbf{k}_{\{t>-c\}} \otimes F)$$

$$\simeq \varinjlim_{c\to+\infty} \mathrm{R}\Gamma\big(U_{c,\delta,\varepsilon};\, \mathbf{k}_{\{t>-c\}} \otimes F\big)$$

$$\simeq \varinjlim_{c\to+\infty} \mathrm{R}\Gamma\big(U_{c,\delta,\varepsilon} \cap \{t \geqslant -c\};\, \mathbf{k}_{\{t>-c\}} \otimes F\big).$$

Here $(*)$ follows from the same argument used in the proof of the second isomorphism in [12, (6.6.2)].

(ii) Let us deal with the left hand side of (4.1). Consider the natural maps

$$\begin{array}{ccc}
N \times \mathbb{R}_\infty & \xrightarrow{\ i_{\mathbb{R}}\ } & M \times \mathbb{R}_\infty \\
\pi_N \downarrow & & \downarrow \pi_M \\
N & \xrightarrow{\ i\ } & M
\end{array}$$

and set, for $S = M, N$,

$$\mathbf{k}_{S\times\{t>*\}} := \text{``}\varinjlim_{c\to+\infty}\text{''}\, \mathbf{k}_{S\times\{t>-c\}} \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{S\times\mathbb{R}_\infty}).$$

Noticing that $\mathrm{E}i^{-1}K \simeq \mathbf{k}_N^{\mathrm{E}} \overset{+}{\otimes} \mathrm{Q}\, i_{\mathbb{R}}^{-1}F$, by [12, Prop. 6.6.5] one has

$$\mathsf{sh}(\mathrm{E}i^{-1}K) \simeq \alpha_N \mathrm{R}\pi_{N*}\big(\mathbf{k}_{N\times\{t>*\}} \otimes i_{\mathbb{R}}^{-1}F\big)$$

$$\simeq \alpha_N \mathrm{R}\pi_{N*} i_{\mathbb{R}}^{-1}\big(\mathbf{k}_{M\times\{t>*\}} \otimes F\big).$$

Hence

$$
\begin{aligned}
\mathrm{R}\varGamma(T; \mathsf{sh}(\mathrm{E}i^{-1}K)) &\simeq \varinjlim_{V} \mathrm{R}\varGamma(V; \mathsf{sh}(\mathrm{E}i^{-1}K)) \\
&\simeq \varinjlim_{c,V} \mathrm{R}\varGamma(V; \mathrm{R}\pi_{N*}i_{\mathbb{R}}^{-1}(\mathbf{k}_{M\times\{t>-c\}} \otimes F)) \\
&\simeq \varinjlim_{c,V} \mathrm{R}\varGamma(V \times \mathbb{R}; i_{\mathbb{R}}^{-1}(\mathbf{k}_{M\times\{t>-c\}} \otimes F)) \\
&\simeq \varinjlim_{c,V} \mathrm{R}\varGamma(V \times \{t \geqslant -c\}; i_{\mathbb{R}}^{-1}(\mathbf{k}_{M\times\{t>-c\}} \otimes F)) \\
&\simeq \varinjlim_{c,V,W} \mathrm{R}\varGamma(W; \mathbf{k}_{M\times\{t>-c\}} \otimes F),
\end{aligned}
$$

where $c \to +\infty$, V runs over the system of open relatively compact subanalytic neighborhoods of T in N, and $W = W_{c,V}$ runs over the system of open subanalytic subsets of $M \times \{\bar{t} \in \overline{\mathbb{R}}; +\infty \geqslant \bar{t} \geqslant -c\}$, such that $W \supset V \times \{t \in \mathbb{R}; t \geqslant -c\}$. Here, the last isomorphism follows from Corollary B.3.

(iii) For $c \in \mathbb{R}$ consider the following inductive systems: I_c is the set of tuples (U, δ, ε) as in (i); J_c is the set of tuples (V, W) as in (ii). We are left to show the cofinality of the functor $\phi\colon I_c \to J_c$, $(U, \delta, \varepsilon) \mapsto (U \cap N, U_{c,\delta,\varepsilon} \cap \{t \geqslant -c\})$.

Given $(V, W) \in J_c$, we look for $(U, \delta, \varepsilon) \in I_c$ such that

$$
U \cap N \subset V \quad \text{and} \quad U_{c,\delta,\varepsilon} \cap \{t \geqslant -c\} \subset W.
$$

Let U be a subanalytic relatively compact open neighborhood of T in M such that $\overline{U} \cap N \subset V$. With notations as in Lemma B.1, set

$$
X = M \times \{\bar{t} \in \overline{\mathbb{R}}; \bar{t} \geqslant -c\}, \quad W = W, \quad T = \overline{U} \times \{\bar{t} \in \overline{\mathbb{R}}; \bar{t} \geqslant -c\}, \quad f(x,\bar{t}) = \varphi(x)
$$

and
$$
g(x,\bar{t}) = (\bar{t} + c + 1)^{-1}.
$$

Note that $g(x, +\infty) = 0$. Since (B.1) is satisfied, Lemma B.1(ii) provides $C > 0$ and $n \in \mathbb{Z}_{>0}$ such that

$$
\{(x,\bar{t}) \in \overline{U} \times \overline{\mathbb{R}}; \bar{t} \geqslant -c, \ Cg(x,\bar{t})^n > |\varphi(x)|\} \subset W.
$$

Then

$$
\{(x,t) \in U \times \mathbb{R}; t \geqslant -c, \ C(t + c + 1)^{-n} > |\varphi(x)|\} \subset W.
$$

One concludes by noticing that the set on the left hand side contains $U_{c,\delta,\varepsilon} \cap \{t \geqslant -c\}$ for $\delta = C^{1/n}$ and $\varepsilon = 1/n$. $\qquad\square$

5. Specialization and microlocalization

Using results from the previous section, we establish here a stalk formula for the natural enhancement of Sato's specialization and microlocalization functors, as introduced in [4].

5.1. **Real oriented blow-up transforms.** — Let M be a real analytic manifold and $N \subset M$ a closed submanifold. Denote by $S_N M$ the sphere normal bundle. Consider the real oriented blow-up M_N^{rb} of M with center N, which enters the commutative diagram with cartesian square

$$
\begin{array}{ccc}
S_N M & \overset{i}{\lhook\joinrel\longrightarrow} M_N^{\mathsf{rb}} \overset{j}{\longleftarrow\joinrel\rhook} & (M \smallsetminus N)_\infty \\
\sigma \downarrow & \square \quad p \downarrow \quad \swarrow j_N & \\
N & \overset{}{\underset{i_N}{\lhook\joinrel\longrightarrow}} M &
\end{array}
$$

Note that $(M \smallsetminus N)_\infty := (M \smallsetminus N, M) \simeq (M \smallsetminus N, M_N^{\mathsf{rb}})$.

Recall the blow-up transform of [4, §4.4]

$$
\mathrm{E}\nu^{\mathsf{rb}} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \longrightarrow \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{S_N M}), \quad K \longmapsto \mathrm{E}i^{-1}\mathrm{E}j_* \mathrm{E}j_N^{-1}K.
$$

A *sectorial neighborhood* of $\theta \in S_N M$ is an open subset $U \subset M \smallsetminus N$ such that $S_N M \cup j(U)$ is a neighborhood of θ in M_N^{rb}. We write $U \overset{\cdot}{\ni} \theta$ to indicate that U is a sectorial neighborhood of θ. We say that $U \subset M \smallsetminus N$ is a sectorial neighborhood of $Z \subset S_N M$, and we write $U \overset{\cdot}{\supset} Z$, if U is a sectorial neighborhood of each $\theta \in Z$.

Lemma 5.1. — *Let* $\varphi \colon M \to \mathbb{R}$ *be a subanalytic continuous map such that* $N = \varphi^{-1}(0)$. *Let* $K \in \mathrm{E}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-}\mathrm{c}}(\mathrm{I}\mathbf{k}_M)$. *For* $\theta_0 \in S_N M$, *one has*

$$
\mathsf{sh}\big(\mathrm{E}\nu_N^{\mathsf{rb}}(K)\big)_{\theta_0} \simeq \varinjlim_{\delta,\varepsilon,U} \mathrm{RHom}^{\mathrm{E}}(\mathsf{E}_{U|M}^{0\rhd -\delta|\varphi(x)|^{-\varepsilon}}, K),
$$

where $\delta, \varepsilon \to 0+$ *and* $U \overset{\cdot}{\ni} \theta_0$. *More generally, if* $Z \subset S_N M$ *is a* closed *subset one has*

$$
\mathrm{R}\Gamma\big(Z; \mathsf{sh}(\mathrm{E}\nu_N^{\mathsf{rb}}(K))\big) \simeq \varinjlim_{\delta,\varepsilon,U} \mathrm{RHom}^{\mathrm{E}}(\mathsf{E}_{U|M}^{0\rhd -\delta|\varphi(x)|^{-\varepsilon}}, K),
$$

where $\delta, \varepsilon \to 0+$ *and* $U \overset{\cdot}{\supset} Z$.

Proof. — Let us prove the last statement. Note that in M_N^{rb} one has $S_N M = (\varphi \circ p)^{-1}(0)$. Hence, by Proposition 4.1,

$$
\mathrm{R}\Gamma\big(Z; \mathsf{sh}(\mathrm{E}\nu_N^{\mathsf{rb}}(K))\big) \simeq \varinjlim_{\delta,\varepsilon,\widetilde{U}} \mathrm{RHom}^{\mathrm{E}}(\mathsf{E}_{\widetilde{U}|M_N^{\mathsf{rb}}}^{0\rhd -\delta|\varphi(p(\widetilde{x}))|^{-\varepsilon}}, \mathrm{E}j_* \mathrm{E}j_N^{-1}K),
$$

where $\widetilde{U} \subset M_N^{\mathsf{rb}}$ runs over the neighborhoods of $i(Z)$. Then

$$
\begin{aligned}
\mathrm{R}\Gamma\big(Z; \mathsf{sh}(\mathrm{E}\nu_N^{\mathsf{rb}}(K))\big) &\simeq \varinjlim_{\delta,\varepsilon,\widetilde{U}} \mathrm{RHom}^{\mathrm{E}}(\mathrm{E}j_{N!!}\mathrm{E}j^{-1}\mathsf{E}_{\widetilde{U}|M_N^{\mathsf{rb}}}^{0\rhd -\delta|\varphi(p(\widetilde{x}))|^{-\varepsilon}}, K) \\
&\simeq \varinjlim_{\delta,\varepsilon,\widetilde{U}} \mathrm{RHom}^{\mathrm{E}}(\mathrm{E}j_{N!!}\mathsf{E}_{j^{-1}(\widetilde{U})|(M\smallsetminus N)_\infty}^{0\rhd -\delta|\varphi(x)|^{-\varepsilon}}, K) \\
&\simeq \varinjlim_{\delta,\varepsilon,\widetilde{U}} \mathrm{RHom}^{\mathrm{E}}(\mathsf{E}_{j_N(j^{-1}(\widetilde{U}))|M}^{0\rhd -\delta|\varphi(x)|^{-\varepsilon}}, K).
\end{aligned}
$$

One concludes by noticing that $U \overset{\cdot}{\supset} Z$ if and only if $U = j_N(j^{-1}(\widetilde{U}))$ for some neighborhood $\widetilde{U}$ of $i(Z)$ in M_N^{rb}. $\qquad\square$

5.2. SHEAFIFICATION ON VECTOR BUNDLES. — Recall from [4, §2.2] that any morphism $p\colon M \to \mathsf{S}$, from a good space to a bordered space, admits a bordered compactification $p_\infty\colon M_\infty \to \mathsf{S}$ such that $(M_\infty)^\circ = M$ and p_∞ is semiproper. Moreover, such a bordered compactification is unique up to isomorphism.
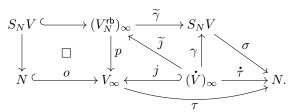
Let $\tau\colon V \to N$ be a vector bundle. Denote by V_∞ its bordered compactification, and by $o\colon N \to V$ the zero section.

The natural action of $\mathbb{R}_{>0}$ on V extends to an action of the bordered group[8] $(\mathbb{R}_{>0}^\times)_\infty := (\mathbb{R}_{>0}, \overline{\mathbb{R}})$ on V_∞. Denote by $\mathrm{E}^{\mathrm{b}}_{(\mathbb{R}_{>0}^\times)_\infty}(\mathrm{I}\mathbf{k}_{V_\infty})$ the category of conic enhanced ind-sheaves on V_∞ (see [4, §4.1]).

LEMMA 5.2. — *For $K \in \mathrm{E}^{\mathrm{b}}_{(\mathbb{R}_{>0}^\times)_\infty}(\mathrm{I}\mathbf{k}_{V_\infty})$, one has*

$$o^{-1}\mathsf{sh}(K) \simeq \mathsf{sh}(\mathrm{E}o^{-1}K), \quad o^!\mathsf{sh}(K) \simeq \mathsf{sh}(\mathrm{E}o^!K).$$

Proof. — We shall prove only the first isomorphism since the proof of the second is similar.

With the identification $N \simeq o(N) \subset V$, set $\dot{V} = V \smallsetminus N$. Consider the commutative diagram, associated with the real oriented blow-up of V with center N,

Here $(\dot{V})_\infty$ denotes the bordered compactification of $\dot{V} \to V_\infty$.

Consider the distinguished triangle

$$\mathrm{E}j_{!!}\mathrm{E}j^{-1}K \longrightarrow K \longrightarrow \mathrm{E}o_*\mathrm{E}o^{-1}K \xrightarrow{+1} .$$

One has

$$o^{-1}\mathsf{sh}(\mathrm{E}o_*\mathrm{E}o^{-1}K) \underset{(*)}{\simeq} o^{-1}o_*\mathsf{sh}(\mathrm{E}o^{-1}K) \simeq \mathsf{sh}(\mathrm{E}o^{-1}K),$$

where $(*)$ holds since o is proper. Hence, we can assume

$$K \simeq \mathrm{E}j_{!!}\mathrm{E}j^{-1}K$$

and, since $\mathrm{E}o^{-1}K \simeq 0$, we have to show

$$o^{-1}\mathsf{sh}(K) \simeq 0.$$

Recall that $\mathrm{E}j^{-1}K \simeq \mathrm{E}\gamma^{-1}K^{\mathsf{sph}}$ for $K^{\mathsf{sph}} := \mathrm{E}\gamma_*\mathrm{E}j^{-1}K$. Then one has

$$\begin{aligned}
K &\simeq \mathrm{E}j_{!!}\mathrm{E}\gamma^{-1}K^{\mathsf{sph}} \\
&\simeq \mathrm{E}p_*\mathrm{E}\widetilde{\jmath}_{!!}\mathrm{E}\widetilde{\jmath}^{-1}\mathrm{E}\widetilde{\gamma}^{-1}K^{\mathsf{sph}} \\
&\simeq \mathrm{E}p_*\big(\mathbf{k}_{V_N^{\mathsf{rb}} \smallsetminus S_N V} \otimes \mathrm{E}\widetilde{\gamma}^{-1}K^{\mathsf{sph}}\big).
\end{aligned}$$

_____

[8]a group object in the category of bordered spaces

Thus, recalling that $o^{-1}\mathsf{sh}(K) \simeq \mathrm{R}\tau_*\mathsf{sh}(K)$ since $\mathsf{sh}(K)$ is conic,

$$o^{-1}\mathsf{sh}(K) \simeq \mathrm{R}\tau_*\mathsf{sh}\big(\mathrm{E}p_*(\mathbf{k}_{V_N^{\mathrm{rb}} \smallsetminus S_N V} \otimes \mathrm{E}\widetilde{\gamma}^{-1}K^{\mathsf{sph}})\big)$$
$$\underset{(*)}{\simeq} \mathrm{R}\tau_*\mathrm{R}p_*\mathsf{sh}\big(\mathbf{k}_{V_N^{\mathrm{rb}} \smallsetminus S_N V} \otimes \mathrm{E}\widetilde{\gamma}^{-1}K^{\mathsf{sph}}\big)$$
$$\simeq \mathrm{R}\sigma_*\mathrm{R}\widetilde{\gamma}_*\mathsf{sh}\big(\mathbf{k}_{V_N^{\mathrm{rb}} \smallsetminus S_N V} \otimes \mathrm{E}\widetilde{\gamma}^{-1}K^{\mathsf{sph}}\big),$$

where $(*)$ holds since p is proper. It is then enough to show

$$\mathrm{R}\widetilde{\gamma}_*\mathsf{sh}(\mathbf{k}_{V_N^{\mathrm{rb}} \smallsetminus S_N V} \otimes \mathrm{E}\widetilde{\gamma}^{-1}K^{\mathsf{sph}}) \simeq 0.$$

Since $\widetilde{\gamma}$ is borderly submersive and $\widetilde{\gamma}^! \mathbf{k}_{S_N V} \simeq \mathbf{k}_{V_N^{\mathrm{rb}} \smallsetminus S_N V}[1]$, one has by (2.3)

$$\mathbf{k}_{V_N^{\mathrm{rb}} \smallsetminus S_N V} \otimes \mathrm{E}\widetilde{\gamma}^{-1}K^{\mathsf{sph}} \simeq \mathrm{E}\widetilde{\gamma}^! K^{\mathsf{sph}}[-1].$$

Hence one obtain

$$\mathrm{R}\widetilde{\gamma}_*\mathsf{sh}(\mathbf{k}_{V_N^{\mathrm{rb}} \smallsetminus S_N V} \otimes \mathrm{E}\widetilde{\gamma}^{-1}K^{\mathsf{sph}}) \simeq \mathrm{R}\widetilde{\gamma}_*\mathsf{sh}(\mathrm{E}\widetilde{\gamma}^! K^{\mathsf{sph}}[-1])$$
$$\underset{(*)}{\simeq} \mathrm{R}\widetilde{\gamma}_*\widetilde{\gamma}^!\mathsf{sh}(K^{\mathsf{sph}}[-1])$$
$$\simeq \mathrm{R}\widetilde{\gamma}_*\mathrm{R}\mathscr{H}om\big(\mathbf{k}_{V_N^{\mathrm{rb}}},\ \widetilde{\gamma}^!\mathsf{sh}(K^{\mathsf{sph}}[-1])\big)$$
$$\simeq \mathrm{R}\mathscr{H}om\big(\mathrm{R}\widetilde{\gamma}_!\,\mathbf{k}_{V_N^{\mathrm{rb}}},\ \mathsf{sh}(K^{\mathsf{sph}}[-1])\big),$$

where $(*)$ follows from Lemma 3.9(i). Then the desired result follows from

$$\mathrm{R}\widetilde{\gamma}_!\,\mathbf{k}_{V_N^{\mathrm{rb}}} \simeq 0. \qquad \square$$

5.3. **SPECIALIZATION AND MICROLOCALIZATION.** — Let us recall from [4] the natural enhancement of Sato's specialization and microlocalization functors.

Let M be a real analytic manifold and $N \subset M$ a closed submanifold. Consider the normal and conormal bundles

$$T_N M \xrightarrow{\ \tau\ } N \xleftarrow{\ \varpi\ } T_N^* M,$$

and denote by $(T_N M)_\infty$ and $(T_N^* M)_\infty$ the bordered compactification of τ and ϖ, respectively.

Denote by $(p, s)\colon M_N^{\mathsf{nd}} \to M \times \mathbb{R}$ the normal deformation of M along N (see [9, §4.1]). Setting $\Omega := s^{-1}(\mathbb{R}_{>0})$, one has morphisms

$$(5.1) \qquad (T_N M)_\infty \xhookrightarrow{\ i\ } (M_N^{\mathsf{nd}})_\infty \xhookleftarrow{\ j\ } \Omega_\infty \xrightarrow{\ p_\Omega\ } M,$$

where $(M_N^{\mathsf{nd}})_\infty$ is the bordered compactification of p, and $p_\Omega = p|_\Omega$. The enhanced Sato's specialization functor is defined by

$$\mathrm{E}\nu_N\colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \longrightarrow \mathrm{E}^{\mathrm{b}}_{(\mathbb{R}_{>0}^\times)_\infty}(\mathrm{I}\mathbf{k}_{(T_N M)_\infty}), \quad K \longmapsto \mathrm{E}i^{-1}\mathrm{E}j_*\mathrm{E}p_\Omega^{-1}K.$$

Sato's Fourier transform have natural enhancements (see e.g. [4, §5.2])

$$(\cdot)^\wedge\colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{(T_N M)_\infty}) \longrightarrow \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{(T_N^* M)_\infty}),$$
$$^{\mathsf{L}}(\cdot)\colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{(T_N M)_\infty}) \longrightarrow \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{(T_N^* M)_\infty}),$$

and we denote by $(\cdot)^\vee$ and $^\lrcorner(\cdot)$ their respective quasi-inverses. Recall that $(\cdot)^\wedge$ and $(\cdot)^\vee$ take values in conic objects, and that $^\llcorner(\cdot)$ and $^\lrcorner(\cdot)$ send conic objects to conic objects.

Finally, Sato's microlocalization functor have a natural enhancement

$$\mathrm{E}\mu_N \colon \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_M) \longrightarrow \mathrm{E}^{\mathrm{b}}_+(\mathrm{I}\mathbf{k}_{(T_N^*M)_\infty}) \cap \mathrm{E}^{\mathrm{b}}_{(\mathbb{R}^\times_{>0})_\infty}(\mathrm{I}\mathbf{k}_{(T_N^*M)_\infty}),$$

defined by $\mathrm{E}\mu_N(K) := {}^\llcorner \mathrm{E}\nu_N(K)$. Recall that $\mathrm{E}\mu_N(K) \simeq \mathrm{E}\nu_N(K)^\wedge$ by [4, Prop. 5.3].

Consider the natural morphisms

$$S_N M \xleftarrow{\gamma} (\dot{T}_N M)_\infty \xrightarrow{u} (T_N M)_\infty \xleftarrow{o} N,$$

where $\dot{T}_N M$ is the complement of the zero-section, and o is the embedding of the zero-section. Here $(\dot{T}_N M)_\infty$ denotes the bordered compactification of $\dot{T}_N M \to (T_N M)_\infty$. Recall that one has

$$\mathrm{E}\gamma^{-1} \circ \mathrm{E}\nu_N^{\mathrm{rb}} \simeq \mathrm{E}u^{-1} \circ \mathrm{E}\nu_N.$$

Recall from [9, §4.1] that the normal cone $C_N(S) \subset T_N M$ to $S \subset M$ along N is defined by $C_N(S) := T_N M \cap \overline{p_\Omega^{-1}(S)}$, where $\overline{(\cdot)}$ denotes the closure in M_N^{nd}.

PROPOSITION 5.3. — *Let $\varphi \colon M \to \mathbb{R}$ be a continuous subanalytic function such that $N = \varphi^{-1}(0)$. For $v_0 \in T_N M$, $\xi_0 \in T_N^* M$, and $K \in \mathrm{E}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-}\mathrm{c}}(\mathrm{I}\mathbf{k}_M)$, one has*

(i) $\mathsf{sh}\big(\mathrm{E}\nu_N(K)\big)_{v_0} \simeq \varinjlim_{\delta,\varepsilon,U} \mathrm{RHom}^{\mathrm{E}}(\mathsf{E}^{0\triangleright -\delta|\varphi(x)|^{-\varepsilon}}_{U|M}, K)$,

(ii) $\mathsf{sh}\big(\mathrm{E}\mu_N(K)\big)_{\xi_0} \simeq \varinjlim_{\delta,\varepsilon,W,Z} \mathrm{RHom}^{\mathrm{E}}(\mathsf{E}^{0\triangleright -\delta|\varphi(x)|^{-\varepsilon}}_{W\cap Z|M}, K)$,

where $\delta, \varepsilon \to 0+$, U runs over the open subsets of M such that $v_0 \notin C_N(M \smallsetminus U)$, W runs over the open neighborhoods of $\varpi(\xi_0)$ in M, and Z runs over the closed subsets of M such that

$$C_N(Z)_{\varpi(\xi_0)} \subset \{v \in (T_N M)_{\varpi(\xi_0)}\,;\, \langle v, \xi_0\rangle > 0\} \cup \{0\}.$$

Proof

(i)(a) Assume that $v_0 \in \dot{T}_N M$, and set $\theta_0 = \gamma(v_0)$. Then, one has

$$\begin{aligned}
\mathsf{sh}\big(\mathrm{E}\nu_N(K)\big)_{v_0} &\underset{(*)}{\simeq} \mathsf{sh}\big(\mathrm{E}u^{-1}\mathrm{E}\nu_N(K)\big)_{v_0} \\
&\simeq \mathsf{sh}\big(\mathrm{E}\gamma^{-1}\mathrm{E}\nu_N^{\mathrm{rb}}(K)\big)_{v_0} \\
&\underset{(**)}{\simeq} \mathsf{sh}\big(\mathrm{E}\nu_N^{\mathrm{rb}}(K)\big)_{\theta_0},
\end{aligned}$$

where $(*)$ and $(**)$ follow from Lemma 3.9(i). Then, the statement follows from Lemma 5.1, by noticing that $U \overset{\bullet}{\ni} \theta_0$ if and only if $v_0 \notin C_N(M \smallsetminus U)$.

(i)(b) Assume that $v_0 = o(y_0)$ for $y_0 \in N$, where $o \colon N \to T_N M$ is the embedding of the zero section. Then, Lemma 5.2 gives

$$\begin{aligned}
\mathsf{sh}\big(\mathrm{E}\nu_N(K)\big)_{o(y_0)} &\simeq \big(o^{-1}\mathsf{sh}(\mathrm{E}\nu_N(K))\big)_{y_0} \\
&\simeq \big(\mathsf{sh}(\mathrm{E}o^{-1}\mathrm{E}\nu_N(K))\big)_{y_0} \\
&\underset{(*)}{\simeq} \big(\mathsf{sh}(\mathrm{E}i_N^{-1}K)\big)_{y_0},
\end{aligned}$$

where $(*)$ follows from [4, Lem. 4.8(i)], with $i_N \colon N \to M$ denoting the embedding. Then the statement follows from Proposition 4.1.

(ii) For $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}^{\times}_{>0}}(\mathbf{k}_{T^*_N M})$ one has

$$\mathrm{RHom}^{\mathrm{E}}(\epsilon(F), \mathrm{E}\mu_N(K)) = \mathrm{RHom}^{\mathrm{E}}(\epsilon(F), {}^{\mathsf{L}}\mathrm{E}\nu_N(K))$$
$$\simeq \mathrm{RHom}^{\mathrm{E}}({}^{\mathsf{J}}\epsilon(F), \mathrm{E}\nu_N(K))$$
$$\simeq \mathrm{RHom}^{\mathrm{E}}(\epsilon(F^\vee), \mathrm{E}\nu_N(K)).$$

Hence

$$\mathsf{sh}\bigl(\mathrm{E}\mu_N(K)\bigr)_{\xi_0} \simeq \varinjlim_{V \ni \xi_0} \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_V), \mathrm{E}\mu_N(K))$$
$$\simeq \varinjlim_{V \ni \xi_0} \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_V^\vee), \mathrm{E}\nu_N(K))$$
$$\underset{(*)}{\simeq} \varinjlim_{V \ni \xi_0} \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_{V^\circ}), \mathrm{E}\nu_N(K)),$$

where V runs over the conic open neighborhoods of ξ_0 in $T^*_N M$, and

$$V^\circ := \{v \in T_N M \,;\, \langle v, \xi \rangle \geqslant 0, \ \forall \xi \in V\}$$

denotes the polar cone. Here $(*)$ follows by noticing that ξ_0 has a fundamental system of open conic neighborhoods $V \subset T^*_N M$ such that $\varpi|_V$ has convex fibers.

We are left with computing

$$\varinjlim_{V \ni \xi_0} \mathrm{RHom}^{\mathrm{E}}(\epsilon(F), \mathrm{E}\nu_N(K))$$

for $F = \mathbf{k}_{V^\circ}$. For this, setting $D = \varpi(V)$, and considering the distinguished triangle

$$\mathbf{k}_{\tau^{-1}(D) \smallsetminus V^\circ} \longrightarrow \mathbf{k}_{\tau^{-1}(D)} \longrightarrow \mathbf{k}_{V^\circ} \xrightarrow{\;+1\;},$$

we will instead compute the cases where $F = \mathbf{k}_{\tau^{-1}(D)}$ or $F = \mathbf{k}_{\tau^{-1}(D) \smallsetminus V^\circ}$.

(ii)(a) On one hand, one has

$$\mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_{\tau^{-1}(D)}), \mathrm{E}\nu_N(K)) \simeq \mathrm{RHom}^{\mathrm{E}}(\mathrm{E}\tau^{-1}\epsilon(\mathbf{k}_D), \mathrm{E}\nu_N(K))$$
$$\simeq \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_D), \mathrm{E}\tau_* \mathrm{E}\nu_N(K))$$
$$\simeq \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_D), \mathrm{E}i_N^{-1}K),$$

where we recall that $i_N \colon N \to M$ denotes the embedding. Thus, noticing that $D = \varpi(V)$ is a system of neighborhoods of $\varpi(\xi_0)$,

$$\varinjlim_{V \ni \xi_0} \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_{\tau^{-1}(D)}), \mathrm{E}\nu_N(K)) \simeq \varinjlim_{D \ni \varpi(\xi_0)} \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_D), \mathrm{E}i_N^{-1}K)$$
$$\simeq \mathsf{sh}\bigl(\mathrm{E}i_N^{-1}K\bigr)_{\varpi(\xi_0)}$$
$$\underset{(*)}{\simeq} \varinjlim_{\delta, \varepsilon, W} \mathrm{RHom}^{\mathrm{E}}(\mathrm{E}^{0 \rhd -\delta|\varphi(x)|^{-\varepsilon}}_{W|M}, K),$$

where $(*)$ follows from Proposition 4.1, and δ, ε, W are as in the statement of the present proposition.

(ii)(b) On the other hand, setting $\widetilde{V} = \gamma(\tau^{-1}(D) \smallsetminus V^\circ) \subset S_N M$, one has

$$\mathbf{k}_{\tau^{-1}(D) \smallsetminus V^\circ} \simeq \mathrm{R} u_! \, \gamma^{-1} \mathbf{k}_{\widetilde{V}}.$$

Hence, since $u \colon (\dot{T}_N M)_\infty \to (T_N M)_\infty$ is semiproper, one has

$$\begin{aligned}
\mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_{\tau^{-1}(D) \smallsetminus V^\circ}), \mathrm{E}\nu_N(K)) &\simeq \mathrm{RHom}^{\mathrm{E}}(\mathrm{E}u_{!!}\mathrm{E}\gamma^{-1}\epsilon(\mathbf{k}_{\widetilde{V}}), \mathrm{E}\nu_N(K)) \\
&\simeq \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_{\widetilde{V}}), \mathrm{E}\gamma_* \mathrm{E}u^{-1}\mathrm{E}\nu_N(K)) \\
&\simeq \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_{\widetilde{V}}), \mathrm{E}\nu_N^{\mathsf{rb}}(K)).
\end{aligned}$$

Note that when V runs over the neighborhoods of ξ_0, $\widetilde{V}$ runs over the neighborhoods of $S = \gamma(\{\xi_0\}^{\circ a})$. Here a denotes the antipodal map. Thus

$$\begin{aligned}
\varinjlim_{V \ni \xi_0} \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_{\tau^{-1}(D) \smallsetminus V^\circ}), \mathrm{E}\nu_N(K)) &\simeq \varinjlim_{V \ni \xi_0} \mathrm{RHom}^{\mathrm{E}}(\epsilon(\mathbf{k}_{\widetilde{V}}), \mathrm{E}\nu_N^{\mathsf{rb}}(K)) \\
&\simeq \varinjlim_{V \ni \xi_0} \mathrm{RHom}\,(\mathbf{k}_{\widetilde{V}}, \mathsf{sh}(\mathrm{E}\nu_N^{\mathsf{rb}}(K))) \\
&\simeq \mathrm{R}\Gamma\,(S; \mathsf{sh}(\mathrm{E}\nu_N^{\mathsf{rb}}(K))) \\
&\underset{(*)}{\simeq} \varinjlim_{\delta, \varepsilon, \Omega} \mathrm{RHom}^{\mathrm{E}}(\mathrm{E}_{\Omega|M}^{0 \triangleright -\delta|\varphi(x)|^{-\varepsilon}}, K),
\end{aligned}$$

where $\delta, \varepsilon \to 0+$, and $\Omega \overset{\cdot}{\supset} S$. Here, $(*)$ follows from Lemma 5.1. $\qquad\square$

Appendix A. Complements on enhanced ind-sheaves

We provide here some complementary results on (enhanced ind-)sheaves that we need in this paper.

Proposition A.1. — *Let* M *be a subanalytic bordered space, and* N *a bordered space. Then, for any* $F \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}\mathrm{c}}(\mathbf{k}_\mathsf{M})$ *and* $K \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_\mathsf{N})$ *we have*

$$(\mathrm{A.1}) \qquad\qquad \mathrm{D}_\mathsf{M} F \boxtimes K \simeq \mathrm{R}\mathscr{I}hom(p^{-1}F, q^! K).$$

Here, $p \colon \mathsf{M} \times \mathsf{N} \to \mathsf{M}$ *and* $q \colon \mathsf{M} \times \mathsf{N} \to \mathsf{N}$ *are the projections.*

Proof. — By [1, Prop. 2.3.4], one has

$$\mathrm{D}_{\overset{\vee}{\mathsf{M}}} \mathrm{R} j_{\mathsf{M}!} F \boxtimes \mathrm{R} j_{\mathsf{N}!!} K \simeq \mathrm{R}\mathscr{I}hom(\overset{\vee}{p}^{-1} \mathrm{R} j_{\mathsf{M}!} F, \overset{\vee}{q}^! \mathrm{R} j_{\mathsf{N}!!} K),$$

where $\overset{\vee}{p}$ and $\overset{\vee}{q}$ are the projections from $\overset{\vee}{\mathsf{M}} \times \overset{\vee}{\mathsf{N}}$, and $j_\mathsf{M} \colon \mathsf{M} \to \overset{\vee}{\mathsf{M}}$ is the natural morphism.

Applying $j_{\mathsf{M} \times \mathsf{N}}^{-1}$, (A.1) follows. $\qquad\square$

Proposition A.2. — *Let* $\mathsf{M}, \mathsf{N}, F, K$ *be as in the preceding proposition. Let* $f \colon \mathsf{N} \to \mathsf{S}$ *be a morphism of bordered spaces, and let* $f' = \mathrm{id}_\mathsf{M} \times f \colon \mathsf{M} \times \mathsf{N} \to \mathsf{M} \times \mathsf{S}$. *Then, we have*

$$\mathrm{R} f'_*(F \boxtimes K) \simeq F \boxtimes \mathrm{R} f_* K.$$

Proof. — Let $p_{\mathsf{N}}\colon \mathsf{M} \times \mathsf{N} \to \mathsf{M}$ and $q_{\mathsf{N}}\colon \mathsf{M} \times \mathsf{N} \to \mathsf{N}$ be the projections. We define similarly p_{S} and q_{S}. Then, the preceding proposition implies

$$\begin{aligned}
\mathrm{R}f'_* (F \boxtimes K) &\simeq \mathrm{R}f'_* \, \mathrm{R}\mathscr{I}hom\big(p_{\mathsf{N}}^{-1}\mathrm{D}_{\mathsf{M}}F, q_{\mathsf{N}}^! K\big) \\
&\simeq \mathrm{R}f'_* \, \mathrm{R}\mathscr{I}hom\big(f'^{-1}p_{\mathsf{S}}^{-1}\mathrm{D}_{\mathsf{M}}F, q_{\mathsf{N}}^! K\big) \\
&\simeq \mathrm{R}\,\mathscr{I}hom\big(p_{\mathsf{S}}^{-1}\mathrm{D}_{\mathsf{M}}F, \mathrm{R}f'_* q_{\mathsf{N}}^! K\big) \\
&\simeq \mathrm{R}\,\mathscr{I}hom\big(p_{\mathsf{S}}^{-1}\mathrm{D}_{\mathsf{M}}F, q_{\mathsf{S}}^! \mathrm{R}f_* K\big) \\
&\simeq F \boxtimes \mathrm{R}f_* K. \hspace{4cm}\square
\end{aligned}$$

Lemma A.3. — *Let us consider a commutative square of bordered spaces*

$$\begin{array}{ccc}
\mathsf{M}' & \xrightarrow{\ g'\ } & \mathsf{M} \\
f' \downarrow & & \downarrow f \\
\mathsf{N}' & \xrightarrow{\ g\ } & \mathsf{N}.
\end{array}$$

For any $F \in \mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{M}})$, one has a canonical morphism in $\mathrm{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{N}'})$

$$g^{-1}\mathrm{R}f_* F \longrightarrow \mathrm{R}f'_* g'^{-1}F.$$

If the square is cartesian and g is borderly submersive, then the above morphism is an isomorphism.

Proof. — The morphism is induced by adjunction from

$$\mathrm{R}f_* F \longrightarrow \mathrm{R}f_* \mathrm{R}g'_* g'^{-1}F \xrightarrow{\ \sim\ } \mathrm{R}g_* \mathrm{R}f'_* g'^{-1}F.$$

Assume that the square is cartesian and g is borderly submersive. Then we may assume that $\mathsf{N}' = S \times \mathsf{N}$ and $\mathsf{M}' = S \times \mathsf{M}$ for a subanalytic space S, and that g and g' are the second projections. Hence the assertion follows from

$$\mathrm{R}f'_* g'^{-1}F \simeq \mathrm{R}f'_*(\mathbf{k}_S \boxtimes F) \simeq \mathbf{k}_S \boxtimes \mathrm{R}f_* F \simeq g^{-1}\mathrm{R}f_* F,$$

which is a consequence of Proposition A.2. $\hspace{3cm}\square$

Lemma A.4. — *For $f\colon \mathsf{M} \to \mathsf{N}$ a morphism of bordered spaces and $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathsf{N}})$ there is a natural morphism $f_{\mathbb{R}}^{-1}(\mathrm{R}^{\mathrm{E}} K) \to \mathrm{R}^{\mathrm{E}}(\mathrm{E}f^{-1}K)$. If f is borderly submersive, then the previous morphism is an isomorphism.*

Proof. — The morphism in the statement follows by adjunction from the isomorphism $\mathrm{Q}_{\mathsf{M}}(f_{\mathbb{R}}^{-1}\mathrm{R}^{\mathrm{E}} K) \simeq \mathrm{E}f^{-1}K$. If f is borderly submersive, we have

$$\mathrm{R}\pi_{\mathsf{M}*}f_{\mathbb{R}}^{-1}\mathrm{R}^{\mathrm{E}} K \underset{(*)}{\simeq} f^{-1}\mathrm{R}\pi_{\mathsf{N}*}\mathrm{R}^{\mathrm{E}} K \simeq 0,$$

where $(*)$ follows from Lemma A.3. Hence, the fact that the morphism in the statement is an isomorphism follows from [1, Prop. 4.4.4(ii-b)]. $\hspace{2cm}\square$

Appendix B. Complements on weak constructibility

In this appendix we obtain a formula for the sections, on a locally closed subanalytic subset, of a weakly constructible sheaf. This result might be of independent interest.

B.1. Łojasiewicz's inequalities. — Let M be a subanalytic space.

Lemma B.1. — *Let $T \subset M$ be a compact subanalytic subset, and let $f, g \colon M \to \mathbb{R}$ be continuous subanalytic functions.*

(i) *Assume that $T \cap f^{-1}(0) \subset g^{-1}(0)$. Then there exist $\varepsilon > 0$ and $n \in \mathbb{Z}_{>0}$ such that*

$$\varepsilon |g(x)|^n \leqslant |f(x)| \quad \text{for } x \in T.$$

(ii) *Let $W \subset M$ be an open subanalytic subset, and assume that*

(B.1) $\{x \in T \,;\, g(x) > 0, \ f(x) = 0\} \subset W.$

Then there exist $\varepsilon > 0$ and $n \in \mathbb{Z}_{>0}$ such that

$$\{x \in T \,;\, g(x) > 0, \ \varepsilon g(x)^n > |f(x)|\} \subset W.$$

Proof. — Consider the subanalytic map $(f, g) \colon M \to \mathbb{R}^2_{(t,u)}$.

(i) The set $Z = (f, g)(T)$ is a compact subanalytic subset of $\mathbb{R}^2$, and we have

$$Z \cap \{(t, u) \,;\, t = 0\} \subset \{(t, u) \,;\, u = 0\}.$$

Hence, there exist $\varepsilon > 0$ and $n \in \mathbb{Z}_{>0}$ such that

$$Z \subset \{(t, u) \in \mathbb{R}^2 \,;\, \varepsilon |u|^n \leqslant |t|\}.$$

This gives the statement.

(ii) Let $T' = T \cap g^{-1}(\mathbb{R}_{\geqslant 0}) \smallsetminus W$. Since $T' \cap f^{-1}(0) \subset g^{-1}(0)$, (i) gives

$$T' \subset \{x \in M \,;\, \varepsilon |g(x)|^n \leqslant |f(x)|\},$$

which implies the desired result.   □

Theorem B.2. — *Let M be a subanalytic space, and $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-}\mathrm{c}}(\mathbf{k}_M)$. Then, for any locally closed subanalytic subset Z of M, and any open subanalytic subset W of M such that $Z \subset W$, there exists $U \subset W$ open subanalytic in M, such that Z is a closed subset of U and*

$$\mathrm{R}\Gamma(U; F) \xrightarrow{\;\sim\;} \mathrm{R}\Gamma(Z; F).$$

The proof is given in Section B.3 after the preparation of the next subsection.

Corollary B.3. — *Let M be a subanalytic bordered space, Z a locally closed subanalytic subset of M, and let $F \in \mathrm{D}^{\mathrm{b}}_{\mathrm{w}\text{-}\mathbb{R}\text{-}\mathrm{c}}(\mathbf{k}_{\mathsf{M}})$. Then, there is an isomorphism*

$$\mathrm{R}\Gamma(Z; F) \xleftarrow{\;\sim\;} \text{``}\varinjlim_{U}\text{''}\, \mathrm{R}\Gamma(U; F),$$

where U runs over the open subanalytic subsets of M such that $Z \subset U$.

B.2. Barycentric decomposition. — We will use here the language of simplicial complexes, for which we refer to [9, §8.1].

Let $\Sigma = (S, \Delta)$ be a simplicial complex, with S the set of vertices, and Δ the set of simplexes (i.e., finite subsets of S). Recall that one sets $|\Sigma| := \bigcup_{\sigma \in \Delta} |\sigma|$, where

$$|\sigma| := \{x \in \mathbb{R}^S ;\ \textstyle\sum_p x(p) = 1,\ x(p) = 0 \text{ for } p \notin \sigma,\ x(p) > 0 \text{ for } p \in \sigma\}.$$

Here, $\mathbb{R}^S$ denote the set of maps $S \to \mathbb{R}$ equipped with the product topology.

For a subset Z of $|\Sigma|$, we set

$$\Delta_Z := \{\sigma \in \Delta ;\ |\sigma| \subset Z\}.$$

A subset Z of $|\Sigma|$ is called Σ-constructible if Z is a union of simplexes.

Lemma B.4. — *Let Z be a Σ-constructible subset of $|\Sigma|$.*

(i) *the following conditions are equivalent.*

    (a) *Z is closed,*

    (b) *if $\tau, \sigma \in \Delta$ satisfy $\sigma \in \Delta_Z$ and $\tau \subset \sigma$, then $\tau \in \Delta_Z$.*

(ii) *the following conditions are equivalent.*

    (a) *Z is open*

    (b) *if $\tau, \sigma \in \Delta$ satisfy $\sigma \in \Delta_Z$ and $\sigma \subset \tau$, then $\tau \in \Delta_Z$.*

(iii) *the following conditions are equivalent.*

    (a) *Z is locally closed,*

    (b) *if $\sigma_1, \sigma_2, \sigma_3 \in \Delta$ satisfy $\sigma_1, \sigma_3 \in \Delta_Z$ and $\sigma_1 \subset \sigma_2 \subset \sigma_3$, then $\sigma_2 \in \Delta_Z$.*

Proof. — (i) follows from $\overline{|\sigma|} = \bigcup_{\tau \in \Delta, \tau \subset \sigma} |\tau|$. (ii) and (iii) follow from (i). $\qquad \square$

For $\sigma \in \Delta$, we set

$$U(\sigma) = \bigcup_{\sigma \subset \tau \in \Delta} |\tau| = \{x \in |\Sigma| ;\ x(s) > 0 \text{ for any } s \in \sigma\}.$$

It is the smallest open Σ-constructible subset containing $|\sigma|$.

Let us denote by $\mathrm{D}^{\mathrm{b}}_{w\text{-}\Sigma\text{-}c}(\mathbf{k}_{|\Sigma|})$ the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_{|\Sigma|})$ whose objects are weakly $|\Sigma|$-constructible. By [9, Prop. 8.1.4], we have

Lemma B.5. — *Let $F \in \mathrm{D}^{\mathrm{b}}_{w\text{-}\Sigma\text{-}c}(\mathbf{k}_{|\Sigma|})$ and $\sigma \in \Delta$. Then, one has*

$$\mathrm{R}\Gamma(U(\sigma); F) \xrightarrow{\;\sim\;} \mathrm{R}\Gamma(|\sigma|; F).$$

Let $\mathsf{B}(\Sigma) = (S_{\mathsf{B}(\Sigma)}, \Delta_{\mathsf{B}(\Sigma)})$ be the barycentric decomposition of Σ defined as follows:

$$S_{\mathsf{B}(\Sigma)} = \Delta,$$

$$\Delta_{\mathsf{B}(\Sigma)} = \{\widetilde{\sigma} ;\ \widetilde{\sigma} \text{ is a finite totally ordered subset of } \Delta\}.$$

Here, $\Delta_{\mathsf{B}(\Sigma)}$ is ordered by the inclusion relation. Then there is a homeomorphism $f \colon |\mathsf{B}(\Sigma)| \xrightarrow{\;\sim\;} |\Sigma|$ defined as follows. For $\sigma \in \Delta = S_{\mathsf{B}(\Sigma)}$, let $e_\sigma \in |\Sigma|$ be given by

$$e_\sigma(s) = \begin{cases} \dfrac{1}{\#\sigma} & \text{if } s \in \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Then, we define

$$f(x) = \sum_{\sigma \in S_{\mathsf{B}(\Sigma)}} x(\sigma) e_\sigma \quad \text{for any } x \in |\mathsf{B}(\Sigma)| \subset \mathbb{R}^{S_{\mathsf{B}(\Sigma)}}.$$

That is, $f(x) \in \mathbb{R}^S$ is given by

$$\big(f(x)\big)(s) = \sum_{\sigma \ni s,\ \sigma \in S_{\mathsf{B}(\Sigma)}} \frac{x(\sigma)}{\#\sigma} \quad \text{for any } s \in S.$$

Note that we have

(B.2) $$f(|\widetilde{\sigma}|) \subset |\max(\widetilde{\sigma})| \quad \text{for any } \widetilde{\sigma} \in \Delta_{\mathsf{B}(\Sigma)},$$

where $\max(\widetilde{\sigma}) \in \Delta$ is the largest member of $\widetilde{\sigma}$ in Δ. Conversely, for $y \in |\Sigma|$ one has

$$y \in f(|\widetilde{\sigma}|),$$

where $\widetilde{\sigma} \in \Delta_{\mathsf{B}(\Sigma)}$ is given by

$$\widetilde{\sigma} := \{\sigma \in \Delta;\ \sigma = \{s \in S;\ y(s) \geqslant a\} \text{ for some } a \in \mathbb{R}_{>0}\}.$$

LEMMA B.6. — *Let $Z \subset |\Sigma|$ be a locally closed Σ-constructible subset. Then for any $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \Delta_{\mathsf{B}(\Sigma)}$ such that $\widetilde{\sigma}_1 \cup \widetilde{\sigma}_2 \in \Delta_{\mathsf{B}(\Sigma)}$ and $f(|\widetilde{\sigma}_1|), f(|\widetilde{\sigma}_2|) \subset Z$, we have $f(|\widetilde{\sigma}_1 \cup \widetilde{\sigma}_2|) \subset Z$.*

Proof. — Set $\widetilde{\tau} = \widetilde{\sigma}_1 \cup \widetilde{\sigma}_2$. We have $|\max(\widetilde{\sigma}_1)|, |\max(\widetilde{\sigma}_2)| \subset Z$. Then the desired result follows from the fact that $\max(\widetilde{\tau})$ is equal to either $\max(\widetilde{\sigma}_1)$ or $\max(\widetilde{\sigma}_2)$. Hence $|\widetilde{\tau}| \subset |\max(\widetilde{\tau})| \subset Z$. □

B.3. PROOF OF THEOREM B.2

LEMMA B.7. — *Let $\Sigma = (S, \Delta)$ be a simplicial complex. Let $Z \subset |\Sigma|$ be a Σ-constructible locally closed subset such that*

(B.3) $$\text{for any } \sigma_1, \sigma_2 \in \Delta_Z \text{ such that } \sigma_1 \cup \sigma_2 \in \Delta, \text{ one has } \sigma_1 \cup \sigma_2 \in \Delta_Z.$$

Set

$$U := \bigcup_{\sigma \in \Delta_Z} U(\sigma).$$

Then, for $F \in \mathrm{D}^{\mathrm{b}}_{w\text{-}\Sigma\text{-}c}(\mathbf{k}_{|\Sigma|})$ one has

$$\mathrm{R}\Gamma(U; F) \xrightarrow{\sim} \mathrm{R}\Gamma(Z; F).$$

Proof. — Let us remark that U is an open subset and Z is a closed subset of U. Hence it is enough to how that

$$\mathrm{R}\Gamma(U; F \otimes \mathbf{k}_{U \smallsetminus Z}) \simeq 0.$$

Thus, we reduce the problem to prove that $\mathrm{R}\Gamma(U; F) \simeq 0$ under the condition that $F \in \mathrm{D}^{\mathrm{b}}_{w\text{-}\Sigma\text{-}c}(\mathbf{k}_{|\Sigma|})$ satisfies $F|_Z \simeq 0$.

Let us take the open covering $\mathfrak{U} := \{U(\sigma)\}_{\sigma \in \Delta_Z}$ of U. For $\sigma_1, \ldots, \sigma_\ell \in \Delta_Z$, if $\bigcap_{1 \leqslant k \leqslant \ell} U(\sigma_k) \neq \varnothing$, then $\sigma := \bigcup_{1 \leqslant k \leqslant \ell} \sigma_k \in \Delta_Z$ by condition (B.3) and $\bigcap_{1 \leqslant k \leqslant \ell} U(\sigma_k) = U(\sigma)$.

Hence, one has by Lemma B.5

$$\mathrm{R}\Gamma(\textstyle\bigcap_{1 \leqslant k \leqslant \ell} U(\sigma_k); F) \xrightarrow{\sim} \mathrm{R}\Gamma(|\sigma|; F) \simeq 0.$$

Thus, we have $R\varGamma(\bigcap_{1 \leqslant k \leqslant \ell} U(\sigma_k); F) \simeq 0$ for any $\sigma_1, \ldots, \sigma_\ell \in \Delta_Z$. We conclude that $R\varGamma(U; F) \simeq R\varGamma(\mathfrak{U}; F) \simeq 0$. $\square$

Proof of Theorem B.2. — There exists a simplicial complex $\Sigma = (S, \Delta)$ and a subanalytic isomorphism $M \simeq |\Sigma|$ such that Z and W are Σ-constructible and F is weakly Σ-constructible (after identifying M and $|\Sigma|$). Let $\widetilde{\Sigma} = (\widetilde{S}, \widetilde{\Delta})$ be the barycentric decomposition of Σ, and identify $|\widetilde{\Sigma}|$, $|\Sigma|$ and M. Then F is weakly $\widetilde{\Sigma}$-constructible and Z and W are $\widetilde{\Sigma}$-constructible. Set $U = \bigcup_{\widetilde{\sigma} \in \widetilde{\Delta}_Z} U(\widetilde{\sigma})$. Then $U \subset W$ by Lemma B.4. Moreover, condition (B.3) is satisfied by Lemma B.6. Hence, Lemma B.7 implies that $R\varGamma(U; F) \to R\varGamma(Z; F)$ is an isomorphism. $\square$

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Andrea D'Agnolo, Dipartimento di Matematica, Università di Padova
via Trieste 63, 35121 Padova, Italy
E-mail : `dagnolo@math.unipd.it`

Masaki Kashiwara, Kyoto University Institute for Advanced study, Research Institute for Mathematical Sciences, Kyoto University,
Kyoto 606-8502, Japan
& Korea Institute for Advanced Study,
Seoul 02455, Korea
E-mail : `masaki@kurims.kyoto-u.ac.jp`