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The Calderón problem with partial data [1]

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Résumé

Nous décrivons un travail avec C.E. Kenig and G. Uhlmann [9] dans lequel nous améliorons un résultat de Bukhgeim and Uhlmann [1], en montrant qu'en dimension $n \geq 3$, la connaissance des données de Cauchy pour l'équation de Schrödinger sur des sous-ensembles possiblement très petits du bord détermine le potential de manière unique. Nous suivons la stratégie générale de [1] mais nous utilisons un ensemble plus riche de solutions du problème de Dirichlet.

Abstract

We describe a joint work with C.E. Kenig and G. Uhlmann [9] where we improve an earlier result by Bukhgeim and Uhlmann [1], by showing that in dimension $n \geq 3$, the knowledge of the Cauchy data for the Schrödinger equation measured on possibly very small subsets of the boundary determines uniquely the potential. We follow the general strategy of [1] but use a richer set of solutions to the Dirichlet problem.

1. Introduction

In this talk we describe the results and the methods from a recent joint work with C.E. Kenig and G. Uhlmann [9]. The background is the problem of electrical impedance tomography (EIT) and we here follow [15]: Let $\Omega \subset \mathbf{R}^n$ be a bounded open set. Here and throughout the whole text we will assume that $n \geq 3$.

Assume that the boundary $\partial\Omega$ is smooth. Let $\gamma(x) > 0$, $x \in \Omega$ be the electrical conductivity and $u(x)$, $x \in \Omega$ th electrical potential. Then under suitable regularity assumptions on γ and u, we have

$$\begin{cases} L_\gamma u := \operatorname{div} \gamma \operatorname{grad} u = 0, \text{ in } \Omega \\ u_{|\partial\Omega} = f, \end{cases} \tag{1.1}$$

[1] based on joint work with C.E. Kenig and G. Uhlmann

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where the function f represents the imposed voltage on the boundary. We have the corresponding voltage to current map:

$$\Lambda_\gamma : f \mapsto \gamma \partial_\nu u_{|\partial\Omega}, \tag{1.2}$$

where ν denotes the exterior unit normal. The general EIT problem is then to study what information we can get about the function γ on Ω, from informations about Λ_γ.

In this talk, we only discuss the injectivity question. Calderón [3] formulated the problem and showed among other things that the differential of $\gamma \mapsto \Lambda_\gamma$ at $\gamma =$ Const. > 0 is injective. He also introduced in this context the idea of using complex WKB-solutions. A global uniqueness result was obtained by Sylvester–Uhlmann:

Theorem 1.1 *([14]) Let $0 < \gamma_j \in C^2(\overline{\Omega})$, $j = 1, 2$. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$ in Ω.*

Using the identity:

$$\gamma^{-1/2} \circ L_\gamma \circ \gamma^{-1/2} = \Delta - q, \; q = \frac{\Delta\sqrt{\gamma}}{\sqrt{\gamma}}, \tag{1.3}$$

where Δ is the standard Laplace operator, and a result of Kohn–Vogelius, they reduced the proof to a corresponding statement for Schrödinger operators:

Let $q \in L^\infty(\Omega)$ and assume for simplicity:

$$0 \text{ is not an eigenvalue of } -\Delta + q : (H^2 \cap H_0^1)(\Omega) \to L^2(\Omega). \tag{1.4}$$

Define the Dirichlet to Neumann map

$$\mathcal{N}_q : H^{\frac{1}{2}}(\partial\Omega) \ni v \mapsto \partial_\nu u_{|\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega), \tag{1.5}$$

where u is the solution of the Dirichlet problem:

$$(-\Delta + q)u = 0 \text{ in } \Omega, \; u_{|\partial\Omega} = v. \tag{1.6}$$

Then we have the following result due to Sylvester-Uhlmann:

Theorem 1.2 *([14]) Let $q_1, q_2 \in L^\infty(\Omega)$ satisfy (1.4) for simplicity. If $\mathcal{N}_{q_1} = \mathcal{N}_{q_2}$, then $q_1 = q_2$ in Ω.*

Global uniqueness in dimension $n \geq 3$ for the closely related problem of inverse scattering at a fixed energy was proven by Novikov [11].

The uniqueness problem with partial data appears if we assume that we only know that $\mathcal{N}_{q_1} v = \mathcal{N}_{q_2} v$ on some fixed part of the boundary for all functions v on the boundary, or more generally for all functions v with support in some other fixed part of the boundary.

There are results by Greenleaf–Uhlmann [6] and by Izosaki–Uhlmann [8], saying that if $\mathcal{N}_{q_j}$ coincide in the above sense on some part of the boundary, then $q_1 = q_2$ on some part of Ω. Full identification of q_1 and q_2 is given by the following result of Bukhgeim–Uhlmann:

Theorem 1.3 *([1]) Let $q_1, q_2 \in L^\infty(\Omega)$ satisfy (1.4). Let $\xi_0 \in S^{n-1}$ and assume that $\mathcal{N}_{q_1}u = \mathcal{N}_{q_2}u$ in some fixed neighborhood in $\partial\Omega$ of $\{x \in \partial\Omega;\ \nu(x)\cdot\xi_0 \leq 0\}$ for all $u \in H^{\frac{1}{2}}(\partial\Omega)$. Then $q_1 = q_2$.*

We would like to get uniqueness from the equality of the $\mathcal{N}_{q_j}u$ for all u on an even smaller part of the boundary. Let $x_0 \in \mathbf{R}^n \setminus \overline{\mathrm{ch}\,(\Omega)}$, where "ch" = "convex hull of". Define the front and the back faces of Ω by

$$\begin{aligned} F(x_0) : &= \{x \in \partial\Omega;\ (x - x_0)\cdot\nu(x) \leq 0\}, \\ B(x_0) : &= \{x \in \partial\Omega;\ (x - x_0)\cdot\nu(x) \geq 0\}. \end{aligned} \tag{1.7}$$

The following is a special case of our main result:

Theorem 1.4 *([9]) Let $q_1, q_2 \in L^\infty(\Omega)$ satisfy (1.4) and assume there exists a neighborhood $\widetilde{F} \subset \partial\Omega$ of $F(x_0)$, such that*

$$\mathcal{N}_{q_1}u = \mathcal{N}_{q_2}u \ \text{ in } \widetilde{F},\ \forall u \in H^{\frac{1}{2}}(\partial\Omega). \tag{1.8}$$

Then $q_1 = q_2$.

Corollary 1.5 *([9]) Let $q_1, q_2 \in L^\infty(\Omega)$ satisfy (1.4). Let $x_0 \in \partial\Omega$ and assume that $T_{x_0}\partial\Omega \cap \partial\Omega = \{x_0\}$. Also assume that Ω is strongly starshaped with respect to x_0. Let $\widetilde{F} \subset \partial\Omega$ be a neighborhood of x_0 and assume that (1.8) holds. Then $q_1 = q_2$.*

Here we say that Ω is strongly star-shaped with respect to the boundary point x_0, if every line through x_0 which is not contained in $T_{x_0}\partial\Omega$, intersects $\partial\Omega$ at precisely two distinct points x_0 and x_1 and the intersection at x_1 is transversal.

Theorem 1.4 has the following generalization that we discovered at a later stage:

Theorem 1.6 *([9]) Let $q_1, q_2 \in L^\infty(\Omega)$ satisfy (1.4) and assume there exist neighborhoods $\widetilde{F} \subset \partial\Omega$ and $\widetilde{B} \subset \partial\Omega$ of $F(x_0)$ and $B(x_0)$ respectively, such that*

$$\mathcal{N}_{q_1}u = \mathcal{N}_{q_2}u \ \text{ in } \widetilde{F},\ \forall u \in H^{\frac{1}{2}}(\partial\Omega) \cap \mathcal{E}'(\widetilde{B}). \tag{1.9}$$

Then $q_1 = q_2$.

Here we may notice that Green's formula implies the identity $\mathcal{N}_q^* = \mathcal{N}_{\overline{q}}$ and it follows that we can permute $\widetilde{F}$ and $\widetilde{B}$ in (1.9) and still get the same conclusion.

In the remainder of this text we give an outline of the proof of Theorem 1.4 and in the last section we indicate the additional argument used to get the more general Theorem 1.6. (A complete exposition is available in [9].)

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2. Carleman estimate

We here only recall the main ideas (cf. [2], [10]). Let $P_0 = -h^2\Delta$, where $h > 0$ is a small semi-classical parameter. The weighted L^2-estimate

$$\|e^{\phi/h}u\| \leq C\|e^{\phi/h}P_0u\|$$

is of course equivalent to the unweighted estimate for a conjugated operator:

$$\|v\| \leq C\|e^{\phi/h}P_0e^{-\phi/h}v\|.$$

The semi-classical principal symbol of P_0 is $p(x,\xi) = \xi^2$, and that of the conjugated operator $e^{\phi/h}P_0e^{-\phi/h}$ is

$$p(x, \xi + i\phi'(x)) = a(x,\xi) + ib(x,\xi),$$

where

$$a(x,\xi) = \xi^2 - \phi'(x)^2,\ b(x,\xi) = 2\xi \cdot \phi'(x).$$

Write the conjugated operator as $A + iB$, with A and B formally selfadjoint and with a and b as their associated principal symbols. Then

$$\|(A + iB)u\|^2 = \|Au\|^2 + \|Bu\|^2 + (i[A,B]u|u).$$

The principal symbol of $i[A,B]$ is $h\{a,b\}$ and in order to get enough positivity we require that

$$a(x,\xi) = b(x,\xi) = 0 \Rightarrow \{a,b\} \geq 0.$$

It is then indeed possible to get an apriori estimate for the conjugated operator. Since we need these estimates to fit nicely with the construction of WKB-solutions, we are led to consider especially the limiting case, as it appears in the following

Definition. ϕ is a limiting Carleman weight (LCW) on some open set Ω, if $\phi'(x)$ is non-vanishing there and we have

$$a(x,\xi) = b(x,\xi) = 0 \Rightarrow \{a,b\} = 0.$$

Being an LCW is quite difficult but such weights do exist.

Example 1. $\phi(x) = x \cdot \xi_0$ with $\xi_0 \in S^{n-1}$.

Example 2. $\phi(x) = C \ln |x - x_0|$, $x \neq x_0$.

When trying to investigate more systematically the LCWs we find that the notion is indeed quite rigid and for instance the level surfaces have to be spheres (in dimension ≥ 3)[2]. These slightly more general LCWs have not yet (to my knowledge) been applied to inverse problems. Generalizing an estimate of [1] in the case of linear LCWs, we have

[2]preliminary manuscript

Proposition 2.1 *Let $\phi \in C^\infty(\mathrm{neigh}\,(\overline{\Omega}))$ be an LCW, $P = -h^2\Delta + h^2 q$, $q \in L^\infty(\Omega)$. Then, for $u \in C^\infty(\overline{\Omega})$, with $u_{|\partial\Omega} = 0$, we have*

$$\begin{aligned} -\tfrac{h^3}{C}((\phi'_x \cdot \nu)e^{\phi/h}\partial_\nu u | e^{\phi/h}\partial_\nu u)_{\partial\Omega_-} + \tfrac{h^2}{C}(\|e^{\phi/h}u\|^2 + \|e^{\phi/h}\nabla u\|^2) \\ \leq Ch^3((\phi'_x \cdot \nu)e^{\phi/h}\partial_\nu u | e^{\phi/h}\partial_\nu u)_{\partial\Omega_+} + \|e^{\phi/h}Pu\|^2, \end{aligned} \tag{2.1}$$

where norms and scalar products are in $L^2(\Omega)$ unless a subscript A (like for instance $A = \partial\Omega_-$) indicates that they should be taken in $L^2(A)$. Here

$$\partial\Omega_\pm = \{x \in \partial\Omega;\ \pm\nu(x)\cdot\phi'(x) \geq 0\}.$$

Remark. If ϕ is an LCW, then so is $-\phi$.

Remark. Using the Hahn-Banach theorem, we get an existence result for the adjoint equation $e^{-\phi/h}Pe^{\phi/h}u = v$.

3. Special null solutions

Let ϕ be an LCW and write $p(x, \phi'(x) + \xi) = a(x,\xi) + ib(x,\xi)$. Then we know that a and b are in involution on their common zero set, and in this case it is well-known and exploited in [5] that we can find plenty of local solutions to the Hamilton-Jacobi system

$$\begin{cases} a(x,\psi'(x)) = 0 \\ b(x,\psi'(x)) = 0 \end{cases} \Leftrightarrow \begin{cases} \psi'^2 = \phi'^2 \\ \psi'\cdot\phi' = 0 \end{cases}. \tag{3.1}$$

We need the following more global statement:

Proposition 3.1 *Let $\phi \in C^\infty(\mathrm{neigh}\,(\overline{\Omega}))$ be an LCW, where Ω is a domain in $\mathbf{R}^n$ and define the hypersurface $G = p^{-1}(C_0)$ for some fixed value of C_0. Assume that each integral curve of $\phi'\cdot\partial_x$ through a point in Ω also intersects G and that the corresponding projection map $\Omega \to G$ is proper. Then we get a solution of (3.1) in $C^\infty(\Omega)$ by solving first $g'(x)^2 = \phi'(x)^2$ on G and then defining ψ by $\psi_{|G} = g$, $\phi'(x)\cdot\partial_x\psi = 0$. The vector fields $\phi'\cdot\partial_x$ and $\psi'\cdot\partial_x$ commute.*

This result will be applied with a new domain Ω that contains the original one. Next consider the WKB-problem

$$P_0(e^{\frac{1}{h}(-\phi+i\psi)}a(x)) = e^{\frac{1}{h}(-\phi+i\psi)}\mathcal{O}(h^2). \tag{3.2}$$

The transport equation for a is of Cauchy-Riemann type along the two-dimensional integral leaves of $\{\phi'\cdot\partial_x, \psi'\cdot\partial_x\}$. We have solutions that are smooth and everywhere $\neq 0$. (See [5]).

The existence result for $e^{\phi/h}Pe^{-\phi/h}$ mentioned in one of the remarks after Proposition 2.1 permits us to to replace the right hand side of (3.2) by zero, more precisely, we can find $r = \mathcal{O}(h)$ in the semi-classical Sobolev space H^1 equipped with the norm $\|r\| = \|\langle hD\rangle r\|$, such that

$$P(e^{\frac{1}{h}(-\phi+i\psi)}(a + r)) = 0. \tag{3.3}$$

4. The Bukhgeim-Uhlmann argument.

Here we repeat the argument of [1] with richer spaces of null-solutions. Let ϕ be an LCW for which the constructions of Section 3 are available. Let $q_1, q_2 \in L^\infty(\Omega)$ be as in Theorem 1.4 with

$$\mathcal{N}_{q_1}(f) = \mathcal{N}_{q_2}(f) \text{ in } \partial\Omega_{-,\epsilon_0},\ \forall f \in H^{\frac{1}{2}}(\partial\Omega), \tag{4.1}$$

where

$$\begin{aligned} \partial\Omega_{-,\epsilon_0} &= \{x \in \partial\Omega;\ \nu(x)\cdot\phi'(x) < \epsilon_0\} \\ \partial\Omega_{+,\epsilon_0} &= \{x \in \partial\Omega;\ \nu(x)\cdot\phi'(x) \geq \epsilon_0\}. \end{aligned}$$

Let

$$u_2 = e^{\frac{1}{h}(\phi+i\psi_2)}(a_2 + r_2)$$

solve

$$(\Delta - q_2)u_2 = 0 \text{ in } \Omega,$$

with $\|r_2\|_{H^1} = \mathcal{O}(h)$. Let u_1 solve

$$(\Delta - q_1)u_1 = 0 \text{ in } \Omega,\ u_{1|\partial\Omega} = u_{2|\partial\Omega}.$$

Then according to the assumptions in the theorem, we have $\partial_\nu u_1 = \partial_\nu u_2$ in $\partial\Omega_{-,\epsilon_0}$ if $\epsilon_0 > 0$ has been fixed sufficiently small and we choose $\phi(x) = \ln|x - x_0|$.

Put $u = u_1 - u_2$, $q = q_2 - q_1$, so that

$$(\Delta - q_1)u = qu_2,\ u_{|\partial\Omega} = 0,\ \operatorname{supp}(\partial_\nu u_{|\partial\Omega}) \subset \partial\Omega_{+,\epsilon_0}. \tag{4.2}$$

For $v \in H^1(\Omega)$ with $\Delta v \in L^2$, we get from Green's formula

$$\begin{aligned} \int_\Omega q u_2 \overline{v} dx &= \int_\Omega (\Delta - q_1)u\overline{v}dx \\ &= \int_\Omega u\,\overline{(\Delta - \overline{q}_1)v}dx + \int_{\partial\Omega_{+,\epsilon_0}} (\partial_\nu u)\overline{v}S(dx). \end{aligned} \tag{4.3}$$

Similarly to u_2, we choose

$$v = e^{-\frac{1}{h}(\phi+i\psi_1)}(a_1 + r_1),$$

with

$$(\Delta - \overline{q}_1)v = 0.$$

Then

$$\int_\Omega q e^{\frac{i}{h}(\psi_1+\psi_2)}(a_2+r_2)(\overline{a_1+r_1})dx = \int_{\partial\Omega_{+,\epsilon_0}} \partial_\nu u\, e^{-\frac{1}{h}(\phi-i\psi_1)}(\overline{a_1+r_1})S(dx). \tag{4.4}$$

Assume that ψ_1, ψ_2 are slightly h-dependent with

$$\frac{1}{h}(\psi_1+\psi_2) \to f,\ h \to 0.$$

The left hand side of (4.4) tends to

$$\int_\Omega q e^{if} a_2 \overline{a}_1 dx,$$

when $h \to 0$. The modulus of the right hand side is

$$\leq \|a_1 + r_1\|_{\partial\Omega_{+,\epsilon_0}} \Big(\int_{\partial\Omega_{+,\epsilon_0}} e^{-2\phi/h} |\partial_\nu u|^2 S(dx) \Big)^{\frac{1}{2}}.$$

Here the first factor is bounded when $h \to 0$. In the Carleman estimate (2.1) we can replace ϕ by $-\phi$ and make the corresponding permutation of $\partial\Omega_-$ and $\partial\Omega_+$. Applying this variant to the equation (4.2), we see that the second factor tends to 0, when $h \to 0$. Thus,

$$\int_\Omega e^{if(x)} a_2(x) \overline{a_1(x)} q(x) dx = 0.$$

Here we can arrange so that f, a_2, a_1 are real-analytic and so that a_1, a_2 are non-vanishing. Moreover if f can be attained as a limit of $(\psi_1 + \psi_2)/h$ when $h \to 0$, so can λf for any $\lambda > 0$. Thus we get the conclusion

$$\int_\Omega e^{i\lambda f(x)} a_2(x) \overline{a_1(x)} q(x) dx = 0. \tag{4.5}$$

5. Analytic wavefront sets.

Under the assumptions of Theorem 1.4, let $H \subset \mathbf{R}^n$ be an affine hyperplane separating x_0 from $\overline{\mathrm{ch}\,(\Omega)}$. Write $\mathbf{R}^n \setminus H = H_+ \cup H_-$, where H_- and H_+ are the open half-spaces that contain x_0 and $\overline{\Omega}$ respectively. Let $R > 0$ be sufficiently large so that $\overline{\Omega}$ is contained in the open ball $B(x_0, R)$ of center x_0 and radius R. We choose the LCW $\phi(x) = \ln|x - x_0|$. The corresponding functions ψ used in the special null-solutions above are then characterized by the fact that they should solve the standard Hamilton-Jacobi equation: $\psi'^2 = \phi'^2$ on $\partial B(x_0, R) \cap H_+$ and that they are positively homogeneous of degree 0 in $x - x_0$. Let $x_1, x_2 \in H_- \cap \partial B(x_0, R)$ be close to antipodal on $\partial B(x_0, R)$ and with their antipodal points also belonging to H_-. Then we can choose $\psi(x) = \psi_j(x) = \mathrm{dist}_{(\phi')^2 dx^2}(x_j, x)$ on $\partial B(x_0, R)$, where dx^2 is the induced Euclidean metric on $\partial B(x_0, R)$.

If x_1, x_2 are precisely antipodal, we get $\psi_1 + \psi_2 = \mathrm{Const.}$. If we move x_j slightly in directions roughly parallel to H, and modifying ψ_j by adding suitable constants, we can produce limiting fs as above with the property that $f'(x) \neq 0$ near $\overline{\Omega}$.

Varying the available parameters x_0, x_1, x_2, and multiplying by Gaussians in the parameters and integrating, we can get an *FBI-transform* that can be used to show that $(x, f'(x)) \notin WF_a(q1_\Omega)$. The analytic wavefront version of Holmgren's uniqueness theorem ([7, 12]) can then be used to show that $q = 0$ on Ω, which concludes the (outline of the) proof of Theorem 1.4.

6. The extra ingredient for getting Theorem 1.6

Let ϕ be an LCW defined in a neighborhood of $\overline{\Omega}$ and assume (1.9) where now $\widetilde{F}$ and $\widetilde{B}$ denote fixed neighborhoods of $\partial\Omega_+$ and $\partial\Omega_-$ respectively. The idea is now to add a reflected term to u_2 in Section 4. Start as there with the WKB solution $e^{\frac{1}{h}(\phi+i\psi_2)}a_2$, and add a reflected term to get

$$e^{\frac{1}{h}(\phi+i\psi_2)}a_2 - e^{i\frac{\Phi}{h}}b,$$

where b is a symbol of order 0 and we want

$$\begin{aligned} -h^2\Delta(e^{i\frac{\Phi}{h}}b(x;h)) &= e^{i\frac{\Phi}{h}}\mathcal{O}(\mathrm{dist}\,(x,\partial\Omega)^\infty + h^\infty), \\ e^{i\frac{\Phi}{h}}b_{|\partial\Omega} &= e^{\frac{1}{h}(\phi+i\psi_2})\chi a_{2|\partial\Omega}, \end{aligned}$$

where $\chi \in C_0^\infty(\partial\Omega_-)$ is equal to 1 away from a small neighborhood of $\partial(\partial\Omega_-)$. Here Φ should solve

$$(\Phi_x')^2 = \mathcal{O}(\mathrm{dist}\,(x,\partial\Omega)^\infty),\ \Phi_{|\partial\Omega_-} = (\psi_2 - i\phi)_{|\partial\Omega_-}.$$

One solution is of course $\psi_2 - i\phi$, and we choose the second one (unique up to term $\mathcal{O}(\mathrm{dist}\,(\cdot,\partial\Omega_-)^\infty)$).

The term $e^{i\Phi/h}$ is subdominant away from the boundary. Use the existence theorem as before, to get a null solution of $\Delta - q_2$ of the form

$$u_2 = e^{\frac{1}{h}(\phi+i\psi_2)}(a_2 + r_2) - e^{i\frac{\Phi}{h}}b,$$

with

$$r_{2|\partial\Omega\cap\mathrm{supp}\,\chi} = 0.$$

Then everything works as before.

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