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Quelques propriétés de décroissance pour l'équation des ondes amorties sur le tore

Résumé

Cet article est la version courte d'un travail en cours [1], et a fait l'objet d'un exposé du second auteur au cours des Journées "Équations aux Dérivées Partielles" (Biarritz, 2012).

On s'intéresse aux taux de décroissance de l'énergie pour l'équation des ondes amorties dans des situations où le coefficient d'amortissement b ne satisfait pas la condition de contrôle géométrique. On donne tout d'abord un lien avec la contrôlabilité de l'équation de Schrödinger associée. On montre que l'observabilité du groupe de Schrödinger implique la décroissance à taux $1/\sqrt{t}$ du semigroupe des ondes amorties (taux meilleur que le taux logarithmique *a priori* fourni par le théorème de Lebeau).

Dans un second temps, on se focalise sur le tore 2-D. Toujours en supposant que le contrôle géométrique n'est pas réalisé, on montre que le semigroupe décroît au mieux à taux $1/t$. Réciproquement, pour des coefficients d'amortissements b réguliers, on prouve la décroissance à taux $1/t^{1-\varepsilon}$, pour tout $\varepsilon > 0$.

Dans le cas où le le coefficient d'amortissement est la fonction caractéristique d'une bande (donc discontinu), on effectue des simulations numériques qui semblent exhiber un taux de décroissance strictement pire que $1/t$.

En particulier, notre étude tend à montrer que le taux de décroissance dépend fortement du taux d'annulation de b.

Abstract

This article is a proceedings version of the ongoing work [1], and has been the object of a talk of the second author during the Journées "Équations aux Dérivées Partielles" (Biarritz, 2012).

We address the decay rates of the energy of the damped wave equation when the damping coefficient b does not satisfy the Geometric Control Condition (GCC). First, we give a link with the controllability of the associated Schrödinger equation. We prove that the observability of the Schrödinger group implies that the semigroup associated to the damped wave equation decays at rate $1/\sqrt{t}$ (which is a stronger rate than the general logarithmic one predicted by the Lebeau Theorem).

Second, we focus on the 2-dimensional torus. We prove that the best decay one can expect is $1/t$, as soon as the damping region does not satisfy GCC. Conversely, for smooth damping coefficients b, we show that the semigroup decays at rate $1/t^{1-\varepsilon}$, for all $\varepsilon > 0$.

In the case where the damping coefficient is a characteristic function of a strip (hence discontinuous), we give numerical evidence of decay rates strictly worse than $1/t$. In particular, our study tends to prove that the decay rate highly depends on the way b vanishes.

1. Introduction and main results

1.1. The damped wave equation

Let (M, g) be a smooth compact connected Riemannian d-dimensional manifold without boundary (for the sake of simplicity). We denote by Δ the (non-positive) Laplace-Beltrami operator on M for the metric g. Given $b \in L^\infty(M)$, $b(x) \geq 0$ on M, we want to understand the asymptotic behaviour as $t \to +\infty$ of the solution u of the problem

$$\begin{cases} \partial_t^2 u - \Delta u + b(x)\partial_t u = 0 & \text{in } \mathbb{R}^+ \times M, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } M. \end{cases} \tag{1.1}$$

The energy of a solution is defined by

$$E(u, t) = \frac{1}{2}(\|\nabla u(t)\|_{L^2(M)}^2 + \|\partial_t u(t)\|_{L^2(M)}^2). \tag{1.2}$$

Multiplying (1.1) by $\partial_t u$ and integrating on M yields the following dissipation identity

$$\frac{d}{dt}E(u, t) = -\int_M b|\partial_t u|^2 dx,$$

which, as b is nonnegative, implies a decay of the energy. As soon as $b \geq C > 0$ on a nonempty open subset of M, the decay is strict and $E(u, t) \to 0$ as $t \to +\infty$. The question is then to know at which rate the energy goes to zero.

The first interesting issue concerns uniform stabilization: under which condition does there exist a function $F(t)$, $F(t) \to 0$, such that

$$E(u, t) \leq F(t)E(u, 0) ? \tag{1.3}$$

The answer was given by Rauch and Taylor [29] (and by Bardos, Lebeau and Rauch [5] in the case of a manifold with boundary): assuming that $b \in \mathscr{C}^0(\overline{M})$, uniform stabilisation occurs if and only if the set $\{b > 0\}$ satisfies the Geometric Control Condition (GCC). Recall that a set $\omega \subset M$ is said to satisfy GCC if there exists $L_0 > 0$ such that every geodesic γ of M with length larger than L_0 satisfies $\gamma \cap \omega \neq \emptyset$. Under this condition, one can take $F(t) = Ce^{-\kappa t}$ (for some constants $C, \kappa > 0$) in (1.3), and the energy decays exponentially. Finally, Lebeau gives in [21] the explicit (and optimal) value of the best decay rate κ in terms of the spectral abscissa of the generator of the semigroup and the mean value of the function b along the rays of geometrical optics.

In the case where $\{b > 0\}$ does not satisfy GCC, i.e. in the presence of "trapped rays" that do not meet $\{b > 0\}$, what can be said about the decay rate of the energy? As soon as $b \geq C > 0$ on a nonempty open subset of M, Lebeau shows in [21] that the energy (of smoother initial data) goes at least logarithmically to zero (see also [8]):

$$E(u, t) \leq C\big(f(t)\big)^2 \left(\|u_0\|_{H^2(M)}^2 + \|u_1\|_{H^1(M)}^2 \right), \quad \text{for all } t > 0, \qquad (1.4)$$

with $f(t) = \frac{1}{\log(2+t)}$. Note that here, $\big(f(t)\big)^2$ characterizes the decay of the energy, whereas $f(t)$ is that of the associated semigroup. Moreover, the author constructed a series of explicit examples of geometries for which this rate is optimal, including for instance the case where $M = \mathbb{S}^2$ is the two-dimensional sphere and $\{b > 0\} \cap N_\varepsilon = \emptyset$, where N_ε is a neighbourhood of an equator of $\mathbb{S}^2$. This result is generalised in [22] for a wave equation damped on a (small) part of the boundary. In this paper, the authors also make the following comment about the result they obtain:

"Notons toutefois qu'une étude plus approfondie de la localisation spectrale et des taux de décroissance de l'énergie pour des données régulières doit faire intervenir la dynamique globale du flot géodésique généralisé sur M. Les théorèmes [22, Théorème 1] et [22, Théorème 2] ne fournissent donc que les bornes *a priori* qu'on peut obtenir sans aucune hypothèse sur la dynamique, en n'utilisant que les inégalités de Carleman qui traduisent "l'effet tunnel"."

In all examples where the optimal decay rate is logarithmic, the trapped ray is a stable trajectory from the point of view of the dynamics of the geodesic flow. This means basically that an important amount of the energy can stay concentrated, for a long time, in a neighbourhood of the trapped ray, i.e. away from the damping region.

If the trapped trajectories are less stable, or unstable, one can expect to obtain an intermediate decay rate, between exponential and logarithmic. We shall say that the energy decays at rate $f(t)$ if (1.4) is satisfied. This problem has already been adressed and, in some particular geometries, several different behaviours have been exhibited. Two main directions have been investigated.

On the one hand, Liu and Rao considered in [23] the case where M is a square and the set $\{b > 0\}$ contains a vertical strip. In this situation, the trapped trajectories consist in a family of parallel vertical geodesics; these are unstable, in the sense that nearby geodesics diverge at a linear rate. They proved that the energy decays at rate

$\left(\frac{\log(t)}{t}\right)^{\frac{1}{2}}$ (i.e., that (1.4) is satisfied with $f(t) = \left(\frac{\log(t)}{t}\right)^{\frac{1}{2}}$). This was extended by Burq and Hitrik [9] to the case of partially rectangular two-dimensional domains, if the set $\{b > 0\}$ contains a neighbourhood of the non-rectangular part. In [27], Phung proved a decay at rate $t^{-\delta}$ for some (unprecised) $\delta > 0$ in a three-dimensional domain having two parallel faces. In all these situations, the only obstruction to GCC is due to a "cylinder of periodic orbits". The geometry is flat and the unstabilities of the geodesic flow around the trapped rays are relatively weak (geodesics diverge at a linear rate).

In [9], the authors argue that the optimal decay in their geometry should be of the form $\frac{1}{t^{1-\varepsilon}}$, for all $\varepsilon > 0$. They provide conditions on the damping coefficient $b(x)$ under which one can obtain such decay rates, and wonder whether this is true in general. Our main theorem (see Theorem 5 below) extends these results to more general damping functions b on the two-dimensional torus.

On the other hand, Christianson [12] proved that the energy decays at rate $e^{-C\sqrt{t}}$ for some $C > 0$, in the case where the trapped set is a hyperbolic closed geodesic. Schenck [30] proved an energy decay at rate e^{-Ct} on manifolds with negative sectional curvature, if the trapped set is "small enough" in terms of topological pressure (for instance, a small neighbourhood of a closed geodesic), and if the damping is "large enough" (that is, starting from a damping function b, βb will work for any $\beta > 0$ sufficiently large). In these two papers, the geodesic flow enjoys very strong unstability properties around the trapped set: the trapped set has non-zero Lyapunov exponents (exponential unstability), and is uniformly hyperbolic.

These cases confirm the idea that the decay rates of the energy strongly depends on the stability of trapped trajectories.

One may now want to compare these geometric situations to situations where the Schrödinger group is observable (or, equivalently, controllable), i.e. for which there exist $C > 0$ and $T > 0$ such that, for all $u_0 \in L^2(M)$, we have

$$\|u_0\|_{L^2(M)}^2 \leq C \int_0^T \|\sqrt{b}\ e^{-it\Delta} u_0\|_{L^2(M)}^2 dt. \tag{1.5}$$

The conditions under which this property holds are also known to be related to stability of the geodesic flow. In particular, the works [5], [23], [9] and [12, 30] can be seen as counterparts for damped wave equations of the articles [20], [17, 18], [10] and [3], respectively, in the context of observation of the Schrödinger group.

Our main results are twofold. First, we clarify the link between the observability of the Schrödinger equation and polynomial decay for the damped wave equation. More precisely, we prove that the observability of the Schrödinger equation implies a polynomial decay at rate $\frac{1}{\sqrt{t}}$ for the damped wave equation.

Second, we study precisely the damped wave equation on the flat torus $\mathbb{T}^2$ if GCC fails. We give the following *a priori* lower bound on the decay rate, revisiting the argument of [9]: (1.1) is not stable at a better rate than $\frac{1}{t}$, provided that GCC is not satisfied. In this situation, the Schrödinger group is known to be observable (see [18], [19] and the more recent works [2] and [11]). Thus, one cannot hope to have a decay better than polynomial in our previous result, i.e. under the mere assumption that the Schrödinger flow is observable.

The remainder of the paper is devoted to studying the gap between the *a priori* lower and upper bounds given respectively by $\frac{1}{t}$ and $\frac{1}{\sqrt{t}}$ on flat tori. For *smooth* nonvanishing damping coefficients $b(x)$, we prove that the energy decays at rate $\frac{1}{t^{1-\varepsilon}}$ for all $\varepsilon > 0$. This result holds without making any dynamical assumption on the damping coefficient, but only on the order of vanishing of b. It generalises a result of [9], which holds in the case where b is invariant in one direction. Our analysis is, again, inspired by the recent microlocal approach proposed in [2] and [11] for the observability of the Schrödinger group. More precisely, we follow here several ideas and tools introduced in [24] and [2].

In the situation where b is a characteristic function of a strip of the torus (hence discontinuous), we provide several numerical simulations (in the spirit of [4]) showing that the decay rate should be of type $\frac{1}{t^\alpha}$ with $\alpha < 1$. This raises the question of the optimality of the *a priori* upper bound $\alpha = \frac{1}{2}$, proved in the first part of the paper in a very general setting.

All these results support the idea that the stabilization problem for the wave equation is not only sensitive to the global properties of the geodesic flow, but also to the rate at which the damping function vanishes (which is not the case for the observability problem for the Schödinger equation).

1.2. Main results of the paper

1.2.1. Resolvent estimates, observability and an *a priori* decay rate

We recall that the damped wave equation (1.1) can be equivalently recast on $H^1(M) \times L^2(M)$ as a first order system

$$\begin{cases} \partial_t U = \mathcal{A}U, \\ U|_{t=0} = {}^t(u_0, u_1), \end{cases} \quad U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ \Delta & -b \end{pmatrix}, \quad D(\mathcal{A}) = H^2(M) \times H^1(M).$$
(1.6)

Definition 1. Let f be a function such that $f(t) \to 0$ when $t \to +\infty$. We say that System (1.1) is stable at rate $f(t)$ if there exists a constant $C > 0$ such that for all $(u_0, u_1) \in D(\mathcal{A})$, (1.4) is satisfied. If it is the case, for all $k > 0$, there exists a constant $C_k > 0$ such that for all $(u_0, u_1) \in D(\mathcal{A}^k)$, we have (see for instance [6, page 767])

$$E(u,t)^{\frac{1}{2}} \le C_k \big(f(t)\big)^k \|\mathcal{A}^k(u_0, u_1)\|_{H^1 \times L^2}, \quad \text{for all } t > 0.$$

Theorem 2. *Suppose that there exist $C > 0$ and $T > 0$ such that, for all $u_0 \in L^2(M)$, (1.5) is satisfied. Then System (1.1) is stable at rate $\frac{1}{\sqrt{t}}$.*

Note that the gain of the $\log(t)^{\frac{1}{2}}$ with respect to [23, 9] is not essential in our work. It is due to the optimal characterization of polynomially decaying semigroups obtained by Borichev and Tomilov [7].

As a first application of Theorem 2 we obtain a different proof of the polynomial decay results for wave equations of [23] and [9] as consequences of the associated control results for the Schrödinger equation of [17] and [10] respectively.

Moreover, Theorem 2 provides also several new stability results for System (1.1) in particular geometric situations; namely, in all following situations, the Schrödinger

group is proved to be observable, and Theorem 2 gives the polynomial stability at rate $\frac{1}{\sqrt{t}}$ for (1.1):

- For any nonvanishing $b(x) \geq 0$ in the 2-dimensional square (resp. torus), as a consequence of [18] (resp. [24, 11]); for any nonvanishing $b(x) \geq 0$ in the d-dimensional rectangle (resp. d-dimensional torus) as a consequence of [19] (resp. [2]);

- If M is the Bunimovich stadium and $b(x) > 0$ on the neighbourhood of one half disc and on one point of the opposite side, as a consequence of [10];

- If M is a d-dimensional manifold of constant negative curvature and the set of trapped trajectories (as a subset of S^*M, see [3, Theorem 2.5] for a precise definition) has Hausdorff dimension lower than d, as a consequence of [3];

Moreover, Lebeau gives in [21, Théorème 1 (ii)] several 2-dimensional examples for which the decay rate $\frac{1}{\log(2+t)}$ is optimal. For all these geometrical situations, Theorem 2 implies that the Schrödinger group is not observable.

The proof of Theorem 2 relies on the following characterization of polynomial decay for System (1.1). For $z \in \mathbb{C}$, we define on $L^2(M)$ the operator

$$P(z) = -\Delta + z^2 + zb,$$

with domain $D(P(z)) = H^2(M)$. We recall (see for instance [21, 1]) that $z \in \mathrm{Sp}(\mathcal{A})$ if and only if $P(z)$ is not invertible. Moreover, as long as $b \geq C > 0$ on a non-empty open set, one has

$$\mathrm{Sp}(\mathcal{A}) \subset \left(\left(-\frac{1}{2}\|b\|_{L^\infty(M)}, 0 \right) + i\mathbb{R} \right) \cup \left([-\|b\|_{L^\infty(M)}, 0] + 0i \right),$$

so that, in particular, $P(is)$ is invertible for all $s \in \mathbb{R}$, $s \neq 0$. We shall make use of the following criteria to study the different polynomial decay rates.

Proposition 3. *Suppose that $b \geq C > 0$ on a non-empty open set. Then, for all $\alpha > 0$, the five following assertions are equivalent:*

$$System~(1.1)~is~stable~at~rate~\frac{1}{t^\alpha}, \tag{1.7}$$

$$There~exist~C > 0~and~s_0 \geq 0~such~that~for~all~s \in \mathbb{R}, |s| \geq s_0, \\ \|(is - \mathcal{A})^{-1}\|_{\mathcal{L}(H^1 \times L^2)} \leq C|s|^{\frac{1}{\alpha}}, \tag{1.8}$$

$$There~exist~C > 0~and~s_0 \geq 0~such~that~for~all~z \in \mathbb{C},~satisfying~|z| \geq s_0, \\ and~|\mathrm{Re}(z)| \leq \frac{1}{C|\mathrm{Im}(z)|^{\frac{1}{\alpha}}},~we~have~\|(z - \mathcal{A})^{-1}\|_{\mathcal{L}(H^1 \times L^2)} \leq C|\mathrm{Im}(z)|^{\frac{1}{\alpha}}, \tag{1.9}$$

$$There~exist~C > 0~and~s_0 \geq 0~such~that~for~all~s \in \mathbb{R}, |s| \geq s_0,~\|P(is)^{-1}\|_{\mathcal{L}(L^2)} \leq C|s|^{\frac{1}{\alpha}-1}, \tag{1.10}$$

$$There~exists~C > 0~and~s_0 \geq 0~such~that~for~all~s \in \mathbb{R}, |s| \geq s_0~and~u \in H^2(M), \\ \|u\|_{L^2}^2 \leq C\left(|s|^{\frac{2}{\alpha}-2}\|P(is)u\|_{L^2}^2 + |s|^{\frac{1}{\alpha}}\|\sqrt{b}u\|_{L^2}^2 \right). \tag{1.11}$$

This proposition is proved in [1] as a consequence of the characterization of polynomial decay for general semigroups in terms of resolvent estimates given in [7], providing (essentially) the equivalence between (1.7) and (1.8). See also [6] for general decay rates in Banach spaces. Note in particular that the proof of a decay rate is reduced to the proof of a resolvent estimate on the imaginary axes. By the way, this estimate implies the existence of a "spectral gap" between the spectrum of $\mathcal{A}$ and the imaginary axes, given by (1.9).

Note also that the estimates (1.8), (1.10) and (1.11) can be equivalently restricted to $s > 0$, since $P(-is)\overline{u} = \overline{P(is)u}$.

Remark finally that if GCC is satisfied, one has (in a general compact manifold M) for some $C > 1$ and all $|s| \geq s_0$ the estimate

$$\|P(is)^{-1}\|_{\mathcal{L}(L^2(M))} \leq C|s|^{-1}. \tag{1.12}$$

instead of (1.15). Estimate (1.12) is in turn equivalent to uniform stabilization.

1.2.2. Decay rates for the damped wave equation on the torus

The main results of this article deal with the decay rate for Problem (1.1) on the torus $M = \mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$ endowed with the standard flat metric.

First, we give an *a priori* lower bound for the decay rate of the damped wave equation, on $\mathbb{T}^2$, when GCC is "strongly violated", i.e. assuming that supp(b) does not satisfy GCC (instead of $\{b > 0\}$).

Theorem 4. *Suppose that there exists $(x_0, \xi_0) \in T^*\mathbb{T}^2$, $\xi_0 \neq 0$, such that*

$$\overline{\{b > 0\}} \cap \{x_0 + \tau\xi_0, \tau \in \mathbb{R}\} = \emptyset.$$

Then there exist two constants $C > 0$ and $\kappa_0 > 0$ such that for all $n \in \mathbb{N}$,

$$\|P(in\kappa_0)^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \geq C. \tag{1.13}$$

As a consequence of Proposition 3, polynomial stabilization at rate $\frac{1}{t^{1+\varepsilon}}$ for $\varepsilon > 0$ is not possible if there is a strongly trapped ray (i.e. that does not intersect supp(b)). More precisely, Theorem 4 with [1, Lemma 4.6] and [6, Proposition 1.3] yield $m_1(t) \geq \frac{C}{1+t}$ where $m_1(t)$ denotes the best decay rate.

Then, the main goal of this paper is to explore the gap between the *a priori* upper bound $\frac{1}{\sqrt{t}}$ for the decay rate, given by Theorem 2, and the *a priori* lower bound $\frac{1}{t}$ of Theorem 4. Our results are twofold (somehow in two opposite directions) and concern either the case of smooth damping functions b, or the case $b = \mathbb{1}_U$, with $U \subset \mathbb{T}^2$.

The case of smooth damping coefficients. Our main result deals with the case of smooth damping coefficients. Without any geometric assumption, but with an additional hypothesis on the order of vanishing of the damping function b, we prove a weak converse of Theorem 4.

Theorem 5. *There exists $\varepsilon_0 > 0$ satisfying the following property. Suppose that b is a nonnegative nonvanishing function on $\mathbb{T}^2$ satisfying $\sqrt{b} \in \mathscr{C}^\infty(\mathbb{T}^2)$ and that there exist $\varepsilon \in (0, \varepsilon_0)$ and $C_\varepsilon > 0$ such that*

$$|\nabla b(x)| \leq C_\varepsilon b^{1-\varepsilon}(x), \quad for \ x \in \mathbb{T}^2. \tag{1.14}$$

Then, there exist $C > 0$ and $s_0 \geq 0$ such that for all $s \in \mathbb{R}$, $|s| \geq s_0$,

$$\|P(is)^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \leq C|s|^{\delta}, \quad \text{with } \delta = 8\varepsilon. \tag{1.15}$$

As a consequence of Proposition 3, in this situation, the damped wave equation (1.1) *is stable at rate $\frac{1}{t^{\frac{1}{1+\delta}}}$.*

Following carefully the steps of the proof, one sees that $\varepsilon_0 = \frac{1}{76}$ works, but the proof is not optimised with respect to this parameter, and it is likely that it could be much improved.

One of the main difficulties in understanding the decay rates is that there is no general monotonicity property of the type "$b_1(x) \leq b_2(x)$ for all $x \implies$ the decay rate associated to the damping b_2 is larger (or smaller) than the decay rate associated to the damping b_1". This makes a significant difference with observability or controllability problems of the type (1.5).

Assumption (1.14) is only a local assumption in a neighbourhood of $\partial\{b > 0\}$ (even if it is stated here globally on $\mathbb{T}^2$). Far from this set, i.e. on each compact set $\{b \geq b_0\}$ for $b_0 > 0$, the constant C_ε can be choosen uniformly, depending only on b_0, and not on ε. Hence, ε somehow quantifies the vanishing rate of the damping function b.

An interesting situation is when the smooth function b vanishes like $e^{-\frac{1}{x^\alpha}}$ in smooth local coordinates, for some $\alpha > 0$. In this case, Assumption (1.14) is satisfied for any $\varepsilon > 0$, and the associated damped wave equation (1.1) is stable at rate $\frac{1}{t^{1-\delta}}$ for any $\delta > 0$. This shows that the lower bound given by Theorem 4, as well as the decay rate $\frac{1}{t}$, are sharp in general. This phenomenon had already been remarked by Burq and Hitrik in [9] in the case where b is invariant in one direction. Our results show that on the torus, the geometric conditions are less important than the vanishing rate of b, as long as the decay rate is concerned. Whether or not the decay rate $\frac{1}{t}$ is achieved for some damping coefficient b is to our knowledge an open problem.

Typical smooth functions not satisfying Assumption (1.14) are for instance functions vanishing like $\sin(\frac{1}{x})^2 e^{-\frac{1}{x}}$. We do not have any idea of the decay rate achieved in this case (except for the *a priori* bounds $\frac{1}{\sqrt{t}}$ and $\frac{1}{t}$).

Theorem 5 generalises the result of [9], which only holds if b is assumed to be invariant in one direction. Our proof is based on ideas and tools developed in [24, 2] and especially on two-microlocal semiclassical measures. One of the key technical points appears in Section 3.3: we have to construct, for each trapped direction, a cutoff function invariant in that direction and adapted to the damping coefficient b. We do not know how to adapt this technical construction to tori of higher dimension, $d > 2$; hence we do not know whether Theorem 5 holds in higher dimension (although we have no reason to suspect it should not hold). Only in the particular case where b is invariant in $d-1$ directions can our methods (or those of [9]) be applied to prove the analogue of Theorem 5.

The case of discontinuous damping functions. In the case $b = \mathbb{1}_U$, for some $U \subset \mathbb{T}^2$, we provide in Section 4 numerical simulations in which U is a strip. More precisely, on the square $(0,1)^2$, we take $b = \kappa \mathbb{1}_{[0,\sigma]}$, with $\kappa > 0$ and $\sigma \in (0,1)$.

Even in this very simple case the decay rate seems difficult to compute explicitly. Numerical simulations tend to show that the decay rate is $\frac{1}{t^\alpha}$, with α between $1/2$ and 1. An important point is that, in all simulations, we find α around $\frac{1}{1.4}$. This leads to think that the decay rates on the torus can be worse than $\alpha = 1$.

These conclusions are drawn by computing numerically the spectrum of the operator $\mathcal{A}$ and using Criterium (1.9). The particular shape of the spectrum (see [4] and Figures 4.1 and 4.4), separated in several branches, is very helpful to obtain precise estimates.

2. Proof of Theorems 2 and 4: *a priori* bounds

2.1. Proof of Theorem 2

This proof is elementary and we hence reproduce it. We express the observability condition as a resolvent estimate (also known as the Hautus test), as introduced by Burq and Zworski [10], and further developed by Miller [26] and Ramdani, Takahashi, Tenenbaum and Tucsnak [28]. In particular [26, Theorem 5.1] yields that the observability inequality (1.5) holds in some time $T > 0$ if and only if there exists a constant $C > 0$ such that we have

$$\|u\|^2_{L^2(M)} \leq C\Big(\|(-\Delta - \lambda)u\|^2_{L^2(M)} + \|\sqrt{b}u\|^2_{L^2(M)}\Big), \quad \text{for all } \lambda \in \mathbb{R} \text{ and } u \in H^2(M).$$

Hence, we obtain for $s \geq 1$ and for some $C > 0$ and all $u \in H^2(M)$,

$$\begin{aligned}
\|u\|^2_{L^2(M)} &\leq C\Big(\|(-\Delta - s^2 + isb - isb)u\|^2_{L^2(M)} + \|\sqrt{b}u\|^2_{L^2(M)}\Big)\\
&\leq C\Big(\|P(is)u\|^2_{L^2(M)} + s^2\|bu\|^2_{L^2(M)} + \|\sqrt{b}u\|^2_{L^2(M)}\Big)\\
&\leq C\Big(\|P(is)u\|^2_{L^2(M)} + s^2\|\sqrt{b}u\|^2_{L^2(M)}\Big).
\end{aligned}$$

Proposition 3 (1.11) is satisfied for $\alpha = \frac{1}{2}$, and yields the polynomial stability at rate $\frac{1}{\sqrt{t}}$ for (1.1). $\qquad\square$

2.2. Proof of Theorem 4

Under the assumption

$$\overline{\{b > 0\}} \cap \{x_0 + \tau\xi_0, \tau \in \mathbb{R}\} = \emptyset, \tag{2.1}$$

for some $(x_0, \xi_0) \in T^*\mathbb{T}^2$, $\xi_0 \neq 0$, we construct in this section $s_n \to +\infty$ and a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{O}(1)$-quasimodes for the family of operators $P(is_n)$ in the limit $n \to +\infty$.

We denote by (x_1, x_2) the coordinates in $\mathbb{T}^2$, and $(\xi_1, \xi_2) \in \mathbb{R}^2$ the associated cotangent variables. We give the (very simple) proof of Theorem 4 in the particular simpler geometric situation where $\xi_0 = (0, 1)$ (see Figure 2.1). Every geometric situation is actually similar to that case (see [1] for the general geometric setting).

Under Assumption (2.1), there exists a cutoff function $\chi = \chi(x_1) \in \mathscr{C}^\infty_c(\mathbb{T}^1)$ such that $\chi b = 0$ on $\mathbb{T}^2$ and χ does not vanish identically. We set $\varphi_n(x_1, x_2) := \chi(x_1)e^{inx_2}$ and $s_n = n$, and we have

$$P(is_n)\varphi_n = \underbrace{-\Delta(\chi(x_1)e^{inx_2}) - n^2\chi(x_1)e^{inx_2}}_{=\chi''e^{inx_2}} + \underbrace{inb\chi(x_1)e^{inx_2}}_{=0} = \chi''(x_1)e^{inx_2}.$$
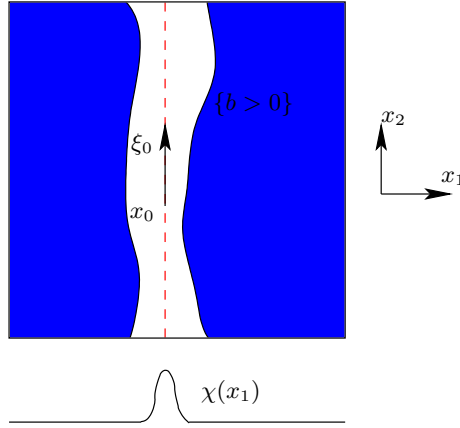
Figure 2.1. Trapped ray and cutoff function $\chi \in \mathscr{C}_c^{\infty}(\mathbb{T}^1)$ such that $\chi b = 0$

As a consequence, $\|P(is_n)\varphi_n\|_{L^2(\mathbb{T}^2)} = C_0\|\chi''\|_{L^2(\mathbb{T}^1)}$, with $\|\varphi_n\|_{L^2(\mathbb{T}^2)} = C_0\|\chi\|_{L^2(\mathbb{T}^1)}$. This yields finally $\|P(is_n)^{-1}\|_{\mathcal{L}(L^2)} \geq C$. $\qquad\square$

3. On the proof of Theorem 5

Since it is very technical, we only sketch the main steps of the proof of Theorem 5. Our strategy is to prove Estimate (1.10) with $\alpha = \frac{1}{1+\delta}$ (which, according to Proposition 3, is equivalent to the statement of Theorem 5). Let us first recast (1.10) with $\alpha = \frac{1}{1+\delta}$ in the semiclassical setting: taking $h = s^{-1}$, we are left to prove that there exist $C > 1$ and $h_0 > 0$ such that for all $h \leq h_0$, for all $u \in H^2(\mathbb{T}^2)$, we have

$$\|u\|_{L^2(\mathbb{T}^2)} \leq Ch^{-\delta}\|P(i/h)u\|_{L^2(\mathbb{T}^2)}. \tag{3.1}$$

We prove this inequality by contradiction, using the notion of semiclassical measures. The idea of developing such a strategy for proving energy estimates, together with the associate technology, originates from Lebeau [21].

We assume that (3.1) is not satisfied, and will obtain a contradiction at the end of Section 3.4. Setting

$$P_b^{h_n} = -h_n^2\Delta - 1 + ih_n b(x) = h_n^2 P(i/h_n),$$

there exists hence a sequence (h_n, v_n) satisfying, as $n \to \infty$,

$$\begin{cases} h_n \to 0^+, \\ \|v_n\|_{L^2(\mathbb{T}^2)} = 1, \\ h_n^{-2-\delta}\|P_b^{h_n}v_n\|_{L^2(\mathbb{T}^2)} \to 0. \end{cases}$$

From now on, we drop the subscript n of the sequences above, and write h in place of h_n and v_h in place of v_n. We study sequences (h, v_h) such that $h \to 0^+$ and

$$\begin{cases} \|v_h\|_{L^2(\mathbb{T}^2)} = 1 \\ \|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta}), \quad \text{as } h \to 0^+. \end{cases} \tag{3.2}$$

Such a sequence will sometimes be called a family of "$o(h^{2+\delta})$-quasimodes" of P_b^h. In particular, this last equation also yields the key information

$$(bv_h, v_h)_{L^2(\mathbb{T}^2)} = h^{-1}\operatorname{Im}(P_b^h v_h, v_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta}), \quad \text{as } h \to 0^+. \tag{3.3}$$

We can associate (see for instance [15, 16]) to a subsequence of (h, v_h) a semiclassical measure, i.e. a nonnegative Radon Measure on $T^*\mathbb{T}^2 = \mathbb{T}^2 \times \mathbb{R}^2$ such that

$$(\operatorname{Op}_h(a)v_h, v_h)_{L^2(\mathbb{T}^2)} \to \langle \mu, a \rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}_c^0(T^*\mathbb{T}^2)} \quad \text{for all } a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2).$$

To obtain a contradiction, hence proving (3.1), we shall prove both that $\mu(T^*\mathbb{T}^2) = 1$, and that $\mu = 0$ on $T^*\mathbb{T}^2$.

3.1. Zero-th and first order informations on μ

The geodesic flow on the torus $\phi_\tau : T^*\mathbb{T}^2 \to T^*\mathbb{T}^2$ for $\tau \in \mathbb{R}$ is the flow generated by the Hamiltonian vector field associated to the symbol $\frac{1}{2}(|\xi|^2 - 1)$, i.e. by the vector field $\xi \cdot \partial_x$ on $T^*\mathbb{T}^2$. Explicitely, we have $\phi_\tau(x, \xi) = (x + \tau\xi, \xi)$, for $\tau \in \mathbb{R}$ and $(x, \xi) \in T^*\mathbb{T}^2$. Note that ϕ_τ preserves the ξ-component, and, in particular every energy layer $\{|\xi|^2 = C > 0\} \subset T^*\mathbb{T}^2$.

Now, we describe the first properties of the measure μ implied by (3.2).

Proposition 6. *We have*

1. $\operatorname{supp}(\mu) \subset \{|\xi|^2 = 1\}$ *(hence is compact in $T^*\mathbb{T}^2$)*,

2. $\mu(T^*\mathbb{T}^2) = 1$,

3. μ *is invariant by the geodesic flow, i.e. $(\phi_\tau)_*\mu = \mu$,*

4. $\langle \mu, b \rangle_{\mathcal{M}_c(T^*\mathbb{T}^2), \mathscr{C}^0(T^*\mathbb{T}^2)} = 0$, *where $\mathcal{M}_c(T^*\mathbb{T}^2)$ denotes the space of compactly supported measures on $T^*\mathbb{T}^2$.*

In other words, μ is an invariant probability measure on $T^*\mathbb{T}^2$ supported by the cospheres and vanishing on $\{b > 0\}$.

These are standard arguments. In particular, we recover all informations required to prove the Bardos-Lebeau-Rauch-Taylor uniform stabilization theorem under GCC. But we do not use here the second order informations of (3.2) (we only use $\|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h)$); this will be the key point to prove Theorem 5.

We denote by $\mathcal{M}^+(T^*\mathbb{T}^2)$ the set of finite, nonnegative measures on $T^*\mathbb{T}^2$. We say that Γ is a "rational direction" if $\Gamma = \mathbb{R}\xi_0$ for $\xi_0 \in \mathbb{R}^2 \setminus \{0\}$ such that $k \cdot \xi_0 = 0$ for some $k \in \mathbb{Z}^2 \setminus \{0\}$. In particular, for any $x_0 \in \mathbb{T}^2$, the line $x_0 + \Gamma$ (when taken to the quotient in $\mathbb{T}^2$) is periodic if and only if Γ is a rational direction. If not, $x_0 + \Gamma$ is dense in $\mathbb{T}^2$.

Given a direction $\Gamma = \mathbb{R}\xi_0$ for $\xi_0 \in \mathbb{R}^2 \setminus \{0\}$, and a function $a(x, \xi)$ on $T^*\mathbb{T}^2$, we define $\langle a \rangle_\Gamma(x, \xi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T a(x + t\xi_0, \xi)dt$.

The restrictions of the measure μ enjoy the following properties, proved in [24] or [2, Section 2].

Lemma 7. *For any direction Γ, we have*

- $\mu|_{\mathbb{T}^2 \times \Gamma} \in \mathcal{M}^+(T^*\mathbb{T}^2)$;

- $\mu|_{\mathbb{T}^2 \times \Gamma}$ *is invariant by the geodesic flow;*

- $\langle \mu|_{\mathbb{T}^2 \times \Gamma}, a \rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}_c^0(T^*\mathbb{T}^2)} = \langle \mu|_{\mathbb{T}^2 \times \Gamma}, \langle a \rangle_\Gamma \rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}_c^0(T^*\mathbb{T}^2)}$ *for all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$.*

We hence have the following key decomposition formula of the measure μ:

$$\mu = \sum_{\Gamma \text{ rational direction}} \mu|_{\mathbb{T}^2 \times \Gamma}. \tag{3.4}$$

Indeed, if the direction Γ is not rational, $\mu|_{\mathbb{T}^2 \times \Gamma}$ vanishes identically as a consequence of Lemma 7.

Our task now is to prove that the restriction $\mu_\Gamma := \mu|_{\mathbb{T}^2 \times \Gamma}$ vanishes for any rational direction Γ. For the sake of simplicity, we shall only consider here the vertical direction $\Gamma = \mathbb{R}\xi_0$, with $\xi_0 = (0, 1)$ (as we did in Section 2.2): see Figure 3.1. Any other periodic direction can be dealt with similarly (see [1]).

This proof is achieved in three main steps, that we sketch now.

3.2. First step: concentration rate towards Γ

Our first task when studying the measure μ_Γ is to understand at which rate a family of $o(h^{2+\delta})$-quasimodes of P_b^h (i.e., satisfying (3.2)) can concentrate towards the direction Γ. We denote by $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ a function satisfying $\chi = 1$ in a neighbourhood of the origin and recall that Γ is given by $\{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2, \xi_1 = 0\}$. Using only the additional information that $\sqrt{b} \in \mathscr{C}^\infty(\mathbb{T}^2)$, we obtain the following concentration rate of the quasimodes v_h.

Lemma 8. *For all $0 < \alpha \leq \frac{3+\delta}{4}$ and $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$, we have*

$$\langle \mu_\Gamma, a \rangle = \lim_{h \to 0} \left(\mathrm{Op}_h \left(a(x, \xi) \chi(\xi_1/h^\alpha) \right) v_h, v_h \right)_{L^2}.$$

This means basically that for any sequence of $o(h^{2+\delta})$-quasimodes of P_b^h oscillating at frequency $|\xi| = \frac{1}{h}$ and concentrating on the direction Γ, the concentration occurs at a rate $|\xi_1| \leq \frac{1}{h^{\frac{1-\delta}{4}}}$.

The idea of the proof of Lemma 8 is to consider 2-microlocal semiclassical measures at scale α (in the spirit of [25] and [14]). More precisely, we introduce the following symbol class.

For $\Gamma = \mathbb{R}\xi_0$, with $\xi_0 = (0, 1)$, we say that $a \in S_\Gamma^1$ if $a = a(x_1, \xi, \eta) \in \mathscr{C}^\infty(\mathbb{T}^1 \times \mathbb{R}^2 \times \mathbb{R})$ is independent from the x_2-variable and

1. there exists a compact set $K_a \subset \mathbb{T}^1 \times \mathbb{R}^2$ such that, for all $\eta \in \mathbb{R}$, the function $(x_1, \xi) \mapsto a(x_1, \xi, \eta)$ is compactly supported in K_a;

2. a is homogeneous of order zero at infinity in the variable $\eta \in \mathbb{R}$; i.e., there exists $R_0 > 0$ (depending on a) and two functions $a_{\mathrm{hom}}(x_1, \xi, \pm 1) \in \mathscr{C}_c^\infty(\mathbb{T}^1 \times \mathbb{R}^2)$ such that

$$a(x_1, \xi, \eta) = a_{\mathrm{hom}}\left(x_1, \xi, \frac{\eta}{|\eta|}\right), \quad \text{for} \quad |\eta| \geq R_0 \text{ and } (x_1, \xi) \in \mathbb{T}^1 \times \mathbb{R}^2.$$
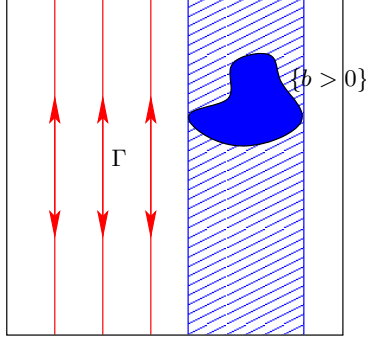
Figure 3.1. Direction Γ, and support of the functions b and $\langle b \rangle_\Gamma$

Up to a subsequence, there exists for any $\alpha \in (0,1)$ a 2-microlocal semiclassical measures ν_α at scale α, defined for symbols $a \in S_\Gamma^1$ by

$$\left\langle \nu_\alpha, a_{\mathrm{hom}}\left(x_1, \xi, \frac{\eta}{|\eta|}\right) \right\rangle = \lim_{h \to 0} \left(\mathrm{Op}_h\left(a\left(x_1, \xi, \frac{\xi_1}{h^\alpha}\right) \left(1 - \chi\left(\frac{\xi_1}{h^\alpha}\right)\right)\right) v_h, v_h \right)_{L^2}.$$

We then prove that

- $\langle \nu_\alpha, \langle b \rangle_\Gamma \rangle = 0$,

- $\partial_{x_1} \nu_\alpha = 0$ as soon as $0 < \alpha \leq \frac{3+\delta}{4}$ (transverse propagation law).

As a consequence of these two properties, we obtain $\nu_\alpha = 0$, which proves Lemma 8.

The limitation in the range of α in the transverse propagation law (and hence in Lemma 8) is due to the antiadjoint part of P_b^h, namely the damping term.

As a consequence of Lemma 8 and regarding the measure μ_Γ, we can replace the study of the sequence v_h by that of $w_h := \mathrm{Op}_h\left(\chi\left(\frac{\xi_1}{h^\alpha}\right)\right) v_h$, that moreover satisfies $\|P_b^h w_h\|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta})$, for a suitable range of (small) parameters α and δ.

3.3. Second step: construction of an invariant cutoff function

The main problem we have to face in the proof of Theorem 5 is the following: in Lemma 8, we would like to prove a transverse propagation law for the measures ν_α with α up to 1 (and not only $0 < \alpha \leq \frac{3+\delta}{4}$). This would essentially be enough to prove that μ_Γ vanishes (see Section 3.4 and [2]). The failure of the transverse propagation law is due to the damping coefficient b, i.e. the antiadjoint part of the operator P_b^h, only satisfying $\|\sqrt{b} v_h\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{1+\delta}{2}})$ (see (3.3)). To overcome this difficulty, we have to introduce (and hence, construct) an h-dependent cutoff function χ_h being invariant in the x_2-direction, and such that b is of order h on $\mathrm{supp}(\chi_h)$.

Proposition 9. *For $\delta = 8\varepsilon$, and $\varepsilon < \varepsilon_0$, there exists $\alpha \in (0, \frac{3+\delta}{4})$, such that for any constant $c_0 > 0$, there exists a cutoff function $\chi_h \in \mathscr{C}^\infty(\mathbb{T}^2)$ valued in $[0,1]$, such that*
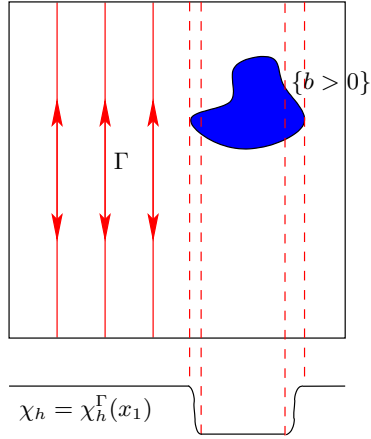
1. *$\chi_h = \chi_h(x_1)$ does not depend x_2,*

Figure 3.2. Shape of the invariant cutoff function χ_h

2. $\|(1 - \chi_h)w_h\|_{L^2(\mathbb{T}^2)} = o(1)$,

3. $b \leq c_0 h$ on $\mathrm{supp}(\chi_h)$,

4. $\|\chi_h' w_h\|_{L^2(\mathbb{T}^2)} = o(1)$ and $\|\chi_h'' w_h\|_{L^2(\mathbb{T}^2)} = o(1)$.

The shape of χ_h is illustrated on Figure 3.2.

The construction of this cutoff function is a crucial step for the proof of transverse propagation for 2-microlocal semiclassical measures associated with w_h at scale 1 (see Section 3.4). All assumptions on b are used here, and especially Assumption 1.14, together with the $o(h^{2+\delta})$ precision for the quasimodes w_h.

Note that in the case where $b(x) = b(x_1)$ is invariant in one direction, one can take $\chi_h = \chi(\frac{b}{c_0 h})$ as done in [9].

3.4. Third step: propagation for 2-microlocal semiclassical measures and end of the proof

Once the new quasimodes w_h and the cutoff function χ_h are introduced, we can follow the stategy of [2], and study 2-microlocal semiclassical measures at scale $\alpha = 1$ (see [13, 24, 2]). These measures again aim at understanding the possible concentration rate for the sequence of quasimodes w_h towards the direction Γ. The key step is again a transverse propagation result for these measures.

Still taking $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ such that $\chi = 1$ in a neighbourhood of the origin, we define for symbols $a \in S_\Gamma^1$,

$$\left\langle \tilde{\nu}_1, a_{\mathrm{hom}}\left(x_1, \xi, \frac{\eta}{|\eta|}\right) \right\rangle = \lim_{R \to +\infty} \lim_{h \to 0} \left(\mathrm{Op}_h\left(\left(1 - \chi\left(\frac{\xi_1}{Rh}\right)\right) a\left(x_1, \xi, \frac{\xi_1}{h}\right)\right) w_h, w_h \right)_{L^2(\mathbb{T}^2)},$$

$$\langle \tilde{\rho}_1, a \rangle = \lim_{R \to +\infty} \lim_{h \to 0} \left(\mathrm{Op}_h\left(\chi\left(\frac{\xi_1}{Rh}\right) a\left(x_1, \xi, \frac{\xi_1}{h}\right)\right) w_h, w_h \right)_{L^2(\mathbb{T}^2)}.$$

Note that the measure $\tilde{\nu}_1$ is very similar to the measures ν_α defined in Section 3.2 (the class of symbols a used in these two definitions is the same). The structure of the distribution $\tilde{\rho}_1$ is more complicated and we refer the interested reader to [13, 2, 1].

One also proves that the measure μ_Γ can be essentially decomposed as $\mu_\Gamma = \tilde{\nu}_1 + \tilde{\rho}_1$ (when tested against symbols of the form $a = a(x_1, \xi)$).

With the introduction of the cutoff function χ_h, we are then able to prove that

- $\langle \tilde{\nu}_1, \langle b \rangle_\Gamma \rangle = 0$,

- $\partial_{x_1} \tilde{\nu}_1 = 0$ (transverse propagation law),

with similar properties for the distribution $\tilde{\rho}_1$.

This yields $\tilde{\nu}_1 = 0$ and $\tilde{\rho}_1 = 0$. Because of the decomposition of the measure μ_Γ in terms of $\tilde{\nu}_1$ and $\tilde{\rho}_1$, we then obtain $\mu_\Gamma = 0$. This result holds for all periodic direction Γ. Coming back to the decomposition (3.4) of the measure μ in terms of its restrictions μ_Γ in all periodic directions, we finally have $\mu = 0$. This is a contradiction with $\mu(T^*\mathbb{T}^2) = 1$ (see Proposition 6), and concludes the proof of Theorem 5.

4. Numerical simulations in the case of a discontinuous damping

In this section, we explain numerical experiments done with rough damping coefficient, to understand the influence of the smoothness of b. These are inspired from the ones of Asch and Lebeau [4], whom we thank for having sent us their matlab code.

4.1. Description of the method

Our approach is the following: compute the spectrum of the damped wave operator and try to estimate how close to the imaginary axes eigenvalues can be. This is done to disprove in this case a decay at rate $1/t^\alpha$, with α close to 1.

Here, we choose the function b to be invariant in one direction $b(x_1, x_2) = b(x_1)$, $(x_1, x_2) \in [0,1]^2$, to take a advantage of the particular shape of the spectrum, separated in several branches. We compute the Dirichlet eigenfunctions on the square $[0,1]^2$, and refer to [4, Section 4.1] for comments on the very particular shape of the spectrum in this geometry.

The computations are done with the following method. The square $[0,1]^2$ is approximated by the cartesian net of equidistant $N \times N$ points. Setting $\theta = \frac{1}{N-1}$, the nodes of the grid are given by $(x_1^i, x_2^j) = (i\theta, j\theta)$ for $i, j \in \{0, \ldots, N-1\}$. We use here the usual five-points finite difference approximation of the flat Laplace operator on $[0,1]^2$, given by

$$\Delta_\theta = (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j})/\theta^2,$$

where $u_{i,j}$ approximates $u(x_1^i, x_2^j)$. The discretised damped wave operator is in this case the following approximation of the operator $\mathcal{A}$:

$$\mathcal{A}_\theta = \begin{pmatrix} 0 & \mathrm{Id}_\theta \\ \Delta_\theta & -b_\theta \end{pmatrix},$$

where b_θ is the diagonal matrix given by $(b_\theta)_{kk} = b(x_1^i, x_2^j)$ (the value of k being determined by the ordering of the points of the grid), and Id_θ is the $N^2 \times N^2$

identity matrix. Each block in the matrix $\mathcal{A}_\theta$ has size N^2, so that $\mathcal{A}_\theta$ is a square $2N^2 \times 2N^2$ matrix.

The computation of the eigenfunctions is done with the matlab function `eigs` (see [4, Section 3.2] for a brief description of the method).

The idea we follow in this section is to use criterium (1.9) of Proposition 3 which is equivalent to a decay at rate $\frac{1}{t^\alpha}$. It shows in particular that decay at rate $\frac{1}{t^\alpha}$ implies the existence of a constant $K > 0$ so that there is no eigenvalue in the set

$$\mathcal{C}(\alpha, K) := \left\{ z \in \mathbb{C}, 0 \geq \mathrm{Re}(z) \geq \frac{1}{K |\mathrm{Im}(z)|^{\frac{1}{\alpha}}} \right\}.$$

Our objective is to disprove decay at rate α exhibiting a sequence of eigenvalues inside $\mathcal{C}(\alpha, K)$ for any K, i.e. a sequence of eigenvalues converging towards the positive imaginary axes at a rate stronger than $\mathrm{Re}(z) \sim \frac{1}{|\mathrm{Im}(z)|^{\frac{1}{\alpha}}}$.

Our procedure is the following :

1. We compute and plot the spectrum of the matrix $\mathcal{A}_\theta$, see Figure 4.1;

2. We isolate the branch closest to the positive imaginary axes, see Figure 4.2;

3. We plot $\log(-\mathrm{Re}(z))$ as a function of $\log(\mathrm{Im}(z))$, see Figure 4.3.

Since we expect to have $\mathrm{Re}(z) = -\frac{1}{C_1 \mathrm{Im}(z)^{\frac{1}{\beta}}}$ on this branch, we also expect to obtain (for asymptotically large eigenvalues) a line of equation

$$\log(-\mathrm{Re}(z)) = -\frac{1}{\beta} \log(\mathrm{Im}(z)) - \log(C_1), \qquad \text{for } z \in \mathrm{Sp}(\mathcal{A}_\theta).$$

We obtain indeed a very nice line, as illustrated by Figure 4.3. The slope of the line, $-\frac{1}{\beta}$ is calculated here with an interpolation method. Obtaining $\frac{1}{\beta} > 1$ in this procedure leads to think that there are asymptotically infinitely many eigenvalues in the set $\mathcal{C}(1, K)$ for any positive K. In particular (using (1.7)$\Rightarrow$(1.9) in Proposition 3), this means that the decay rate is strictly less than $1/t$. This is what we observe in the following numerical simulations.

4.2. Numerical results

In this experiment, the damping function is $b(x_1, x_2) = 2\mathbb{1}_{[0,1/2]}(x_1)$, and we change the number of discretization points N. We obtain the following results:

Discretization size $N =$	20	30	40	50
Slope $-1/\beta$	-1.411	-1.389	-1.385	-1.386

This table suggests that the computations made here are not very sensitive to the discretization step. Figures 4.1, 4.2 and 4.3 illustrate the strategy developed above.

This experiment leads to conjecture that, if the associated damped wave equation decays at rate $1/t^\alpha$ then we should have $\alpha \leq \frac{1}{1.38} < 1$.
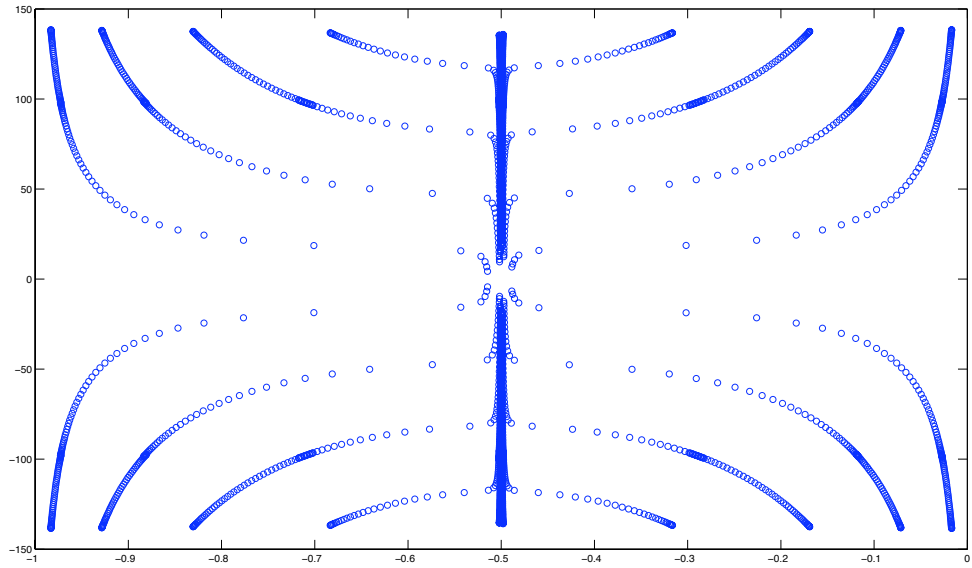
Figure 4.1. Full spectrum of the operator $\mathcal{A}_\theta$.
Discretization $N = 50$, damping $b(x_1, x_2) = 2\mathbb{1}_{[0,1/2]}(x_1)$.
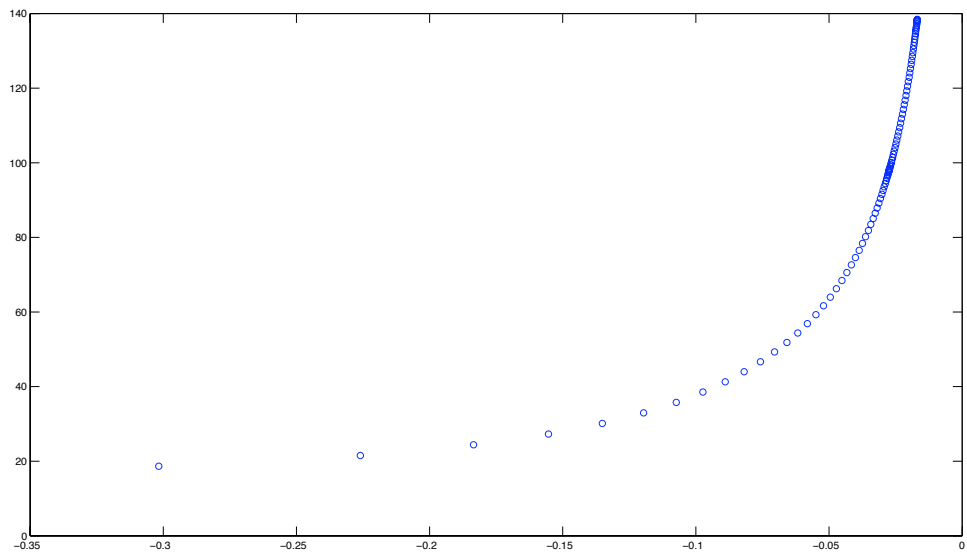

Figure 4.2. Branch of the spectrum of the operator $\mathcal{A}_\theta$ closest to the positive imaginary axes.
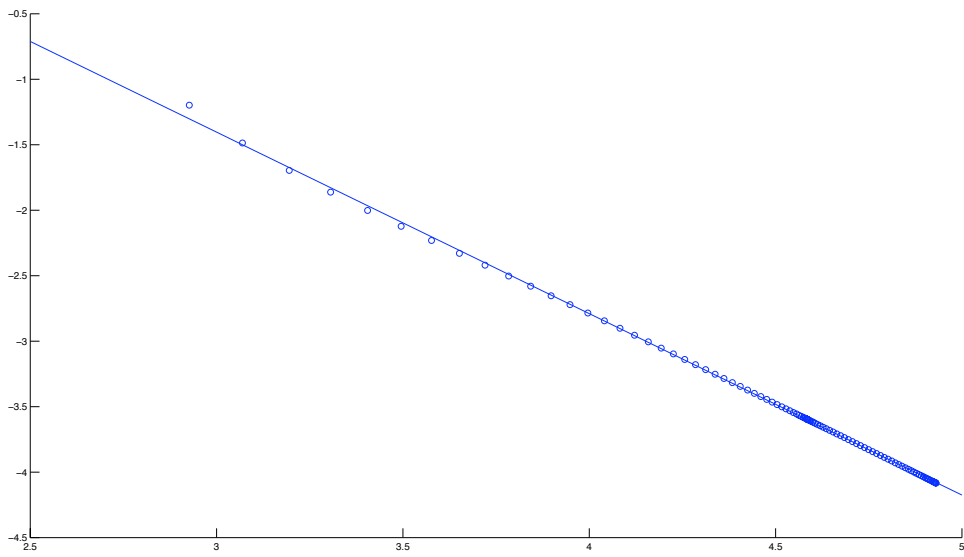Discretization $N = 50$, damping $b(x_1, x_2) = 2\mathbb{1}_{[0,1/2]}(x_1)$.

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Figure 4.3. $\log(-\operatorname{Re}(z))$ as a function of $\log(\operatorname{Im}(z))$ for $z \in \operatorname{Sp}(\mathcal{A}_\theta)$, z on the branch selected in Figure 4.2.
Discretization $N = 50$, damping $b(x_1, x_2) = 2\mathbb{1}_{[0,1/2]}(x_1)$.

As an illustration, we also plot on Figure 4.4 the shape of the spectrum for the damping coefficient $b(x_1, x_2) = 2\mathbb{1}_{[0,0.3]}(x_1)$.
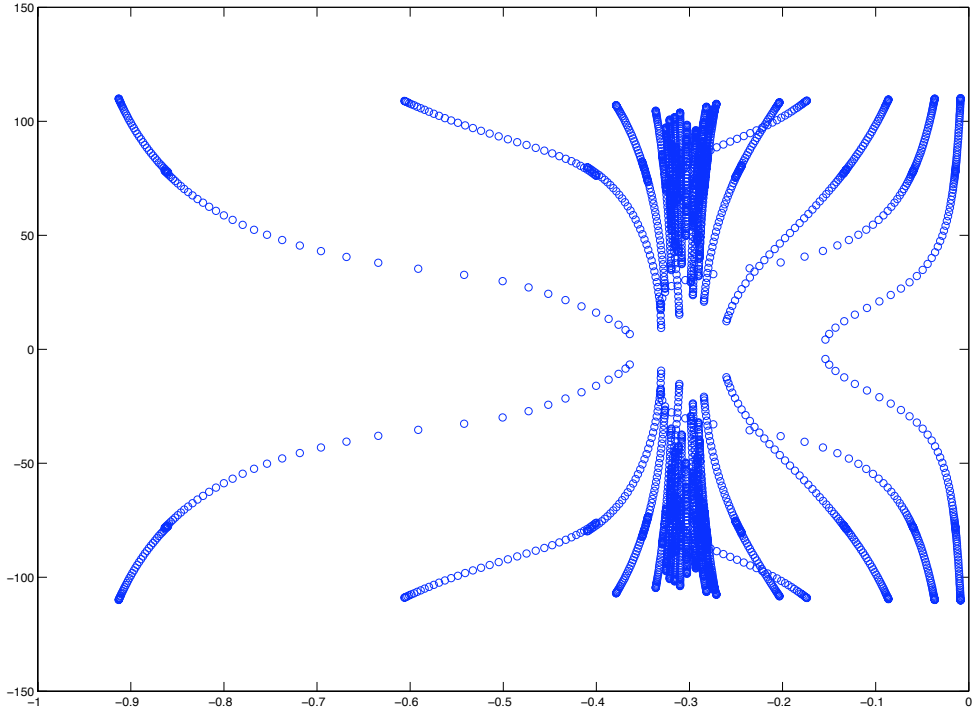

Figure 4.4. Full spectrum of the operator $\mathcal{A}_\theta$.
Discretization $N = 40$, damping $b(x_1, x_2) = 2\mathbb{1}_{[0,0.3]}(x_1)$.

To conclude, we stress that we do not provide a precise numerical analysis of the problem. The numerical results presented here should hence be handled with care, and furnish only heuristical hints. In particular, only the low-frequency eigenvalues of the operator $\mathcal{A}_\theta$ correspond to a precise approximation of those of $\mathcal{A}$. The high-frequency spectrum of $\mathcal{A}$ is not well approximated by that of $\mathcal{A}_\theta$.

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