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Abstract

We describe some recent results concerning the nonlinear L^2-stability of multi-solitons of the Korteweg-de Vries equation [4], and H^1-stability of multi-kinks of the modified Korteweg-de Vries [49]. The proof of both results is closely linked to stability properties for solitons of the integrable Gardner equation, which have been considered by Martel, Merle and Tsai [41, 40].

1. Introduction

In this notes we review some recent results concerning the stability of well-known solutions of some integrable equations [4, 48, 49]. Some of these works have been in collaboration with M.A. Alejo and L. Vega [4], from Bilbao.

More precisely, we will consider the following two integrable models: the (focusing) Korteweg-de Vries equation (KdV)

$$u_t + (u_{xx} + u^2)_x = 0, \quad u(t,x) \in \mathbb{R}, \quad (t,x) \in \mathbb{R}^2, \tag{1.1}$$

and the (defocusing) modified KdV equation (mKdV)

$$u_t + (u_{xx} - u^3)_x = 0, \quad u(t,x) \in \mathbb{R}, \quad (t,x) \in \mathbb{R}^2. \tag{1.2}$$

The KdV equation arises in Physics as a model of propagation of dispersive long waves, as was pointed out by Russel in 1834 [44]. The exact formulation of the KdV equation comes from Korteweg and de Vries (1895) [30]. This equation was studied in a numerical work by Fermi, Pasta and Ulam, and by Kruskal and Zabusky [15, 31].

From a mathematical point of view, equations (1.1) and (1.2) are *integrable models* [3, 2, 1], with a Lax pair structure and infinitely many conservation laws. Moreover, since the Cauchy problem associated to (1.1) is locally well posed in $L^2(\mathbb{R})$ (cf. Bourgain [9]), each solution is indeed global in time thanks to the *Mass* conservation

$$M[u](t) := \frac{1}{2}\int_{\mathbb{R}} u^2(t,x)dx = M[u](0). \tag{1.3}$$

Keywords: KdV equation, modified KdV equation, Gardner equation, integrability, multi-soliton, multi-kink, stability, asymptotic stability, Gardner transform.

On the other hand, equation (1.1) has solitary wave solutions called *solitons*, namely solutions of the form

$$u(t,x) = Q_c(x-ct), \quad Q_c(s) := cQ(\sqrt{c}s), \quad c>0, \tag{1.4}$$

and

$$Q(s) := \frac{3}{1+\cosh(s)}. \tag{1.5}$$

Similarly, equation (1.2) has non-localized *solitons* solutions, called *kinks*, namely solutions of the form

$$u(t,x) = \varphi_c(x+ct+x_0), \quad \varphi_c(s) := \sqrt{c}\,\varphi(\sqrt{c}s), \quad c>0,\ x_0\in\mathbb{R}, \tag{1.6}$$

and

$$\varphi(s) := \tanh(\frac{s}{\sqrt{2}}), \tag{1.7}$$

which solves

$$\varphi'' + \varphi - \varphi^3 = 0, \ \text{ in } \mathbb{R}, \quad \varphi(\pm\infty) = \pm 1, \quad \varphi' > 0. \tag{1.8}$$

The Cauchy problem associated to (1.2) is locally well posed in $\varphi_c(\cdot + ct) + H^1(\mathbb{R})$ (cf. Merle-Vega [42, Prop. 3.1]), then each solution is indeed global in time thanks to the conservation of Energy:

$$E[u](t) := \frac{1}{2}\int_{\mathbb{R}} u_x^2(t,x)dx + \frac{1}{4}\int_{\mathbb{R}} (u^2-c)^2(t,x)dx = E[u](0). \tag{1.9}$$

A simple inspection reveals that this is a non-negative quantity.

It is also important to stress that (1.2) has in addition another less regular conserved quantity, called mass:

$$M[u](t) := \frac{1}{2}\int_{\mathbb{R}} (c - u^2(t,x))dx = M[u](0). \tag{1.10}$$

Of course this quantity is well-defined for solutions $u(t)$ such that $(u^2(t)-c)$ has enough decay at infinity. In particular, one has $M[\varphi_c] < +\infty$.

The study of perturbations of solitons and kinks lead to the introduction of the concepts of *orbital and asymptotic stability*. In particular, it is natural to expect that solitons and kinks are stable in the energy space $H^1(\mathbb{R})$. Indeed, H^1-stability of KdV solitons has been considered by Benjamin and Bona-Souganidis-Strauss in [5, 8]. The asymptotic stability has been studied e.g. in Pego-Weinstein and Martel-Merle [50, 36]. On the other hand, H^1-stability of mKdV kinks has been considered initially by Zhidkov [56], see also Merle-Vega [42] for a complete proof, including an adapted well-posedness theory. We recall that their proof uses the **non-negative** character of the energy (1.9) around a kink solution φ_c, which balances the bad behavior of the mass (1.3) under general H^1-perturbations of a kink solution.

Kinks are also present in other nonlinear models, such as the sine-Gordon (SG) equation, the ϕ^4-model, and the Gross-Pitaevskii (GP) equation [1, 14]. In each case, it has been proved that their are stable for small perturbations in a suitable energy space, cf. [24, 20, 56, 17, 7]. Let us also recall that the SG and GP equations are integrable models in one dimension [1, 14].

In [4, 49], we have studied the nonlinear L^2-stability, and H^1-stability, of the *multi-soliton* and *multi-kink* solutions of KdV and mKdV respectively. The purpose of the following sections is to briefly describe these two results.

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2. The KdV case: L^2 stability

Using a completely different method, Merle and Vega [42] showed that KdV solitons are L^2-stable. To prove this result, they considered the *Miura transform*

$$M[v](t,x) := \frac{3}{2}c + \Big[\frac{3}{\sqrt{2}}v_x - \frac{3}{2}v^2\Big](t, x-3ct). \tag{2.1}$$

This nonlinear $H^1 - L^2$ transformation links solutions of (1.2) with solutions of the KdV equation (1.1). In particular, the image of the family of kink solutions (1.4) under the transformation (2.1) is the well-known soliton of KdV, with scaling $2c$ (cf. [42]):

$$M[\varphi_c(x+ct+x_0)] = Q_{2c}(x-2ct+x_0).$$

Therefore, by proving the H^1-stability of single kinks –a question previously considered by Zhidkov [56]–, and (2.1), they obtained a form of L^2-stability for the KdV soliton. Additionally, a simple statement of asymptotic stability for the kink solution was proved. Related asymptotic results for soliton-like solutions can also be found e.g. in [13, 51, 50, 29, 35, 36].

Let us describe in more detail the Merle-Vega's approach. First of all, from the fact that the image of mKdV kinks are KdV solitons, and using the fact that the soliton is a minimizer of a well-known functional, one can describe the inverse of the Miura transform (2.1) in a small L^2-vicinity of the soliton Q_c, to obtain a small H^1-vicinity of the kink solution. Since the kink solution of (1.2) is H^1-stable (see e.g. Zhidkov, Merle-Vega [56, 42]), by applying once again the Miura transform to the mKdV solution, and using a well-known unicity argument, the authors concluded the L^2-stability of the KdV soliton. The following figure describes the aforementioned approach:

$$\begin{array}{ccc}
\begin{matrix}\text{KdV}\\ u_0 \sim_{L^2} Q_c\end{matrix} & \xrightarrow[v_0=M^{-1}[u_0]]{\text{Miura}} & \begin{matrix}\text{mKdV}\\ v_0 \sim_{H^1} \varphi_{c/2}\end{matrix} \\
\begin{matrix}L^2\text{-KdV flow}\\ \text{(Bourgain)}\end{matrix} \Big\downarrow\, t>0 & & \begin{matrix}H^1\text{-mKdV flow}\\ \text{(Merle-Vega)}\end{matrix} \Big\downarrow \begin{matrix}H^1\text{-stability}\\ \text{(Zhidkov)}\end{matrix} \\
\begin{matrix}u(t) = \bar u(t)\\ u(0)=u_0\end{matrix} & \xleftarrow[\bar u(t)=M[v](t)]{\text{Miura}} & \begin{matrix}v(t) \text{ stable}\\ v(0)=v_0\end{matrix}
\end{array} \tag{2.2}$$

Fig. 1: *The Merle-Vega's approach.*

The Merle-Vega's idea has been applied to different models describing several phenomena. A similar Miura transform is available for the KP II equation, a two-dimensional generalization of the KdV equation. Mizumachi and Tzvetkov have showed the stability of solitary waves of KdV, seen as solutions of KP II, under periodic transversal perturbations [46]. Finally, we recall the L^2-stability result for solitary waves of the cubic NLS proved by Mizumachi and Pelinovsky in [45]. Other applications of the Miura transform are local well-posedness and ill-posedness results (cf. [28, 12]). For a more detailed description of this map, see [25].

Concerning the more involved case of the sum of $N(\geq 2)$ decoupled solitons, stability and asymptotic stability results are very recent. First of all, let us recall that, as a consequence of the integrability property, KdV has solutions behaving, as time goes to infinity, as the sum of N decoupled solitons. These solutions are well-known in the literature and are called *N-solitons*, or generically *multi-solitons* [23]. Indeed, any N-soliton solution has the form $u(t,x) := U^{(N)}(t,x) = U^{(N)}(x; c_j, x_j - c_j t)$, where

$$\Big\{ U^{(N)}(x; c_j, y_j) \ : \ c_j > 0, \, y_j \in \mathbb{R}, \, j = 1, \dots, N \Big\} \tag{2.3}$$

is the family of explicit N-soliton profiles (see e.g. Maddocks-Sachs [33], §3.1). In particular, this solution describes multiple soliton collisions; but since solitons for KdV equation interact in a linear fashion, there is no residual appearing after the collisions, even if the equation is nonlinear in nature:

$$\lim_{t \to \pm\infty} \Big\| U^{(N)}(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - c_j t - x_j^{\pm}) \Big\|_{H^1(\mathbb{R})} = 0,$$

with $x_j^{\pm} \in \mathbb{R}$ depending on the set (c_k). This is also a consequence of the integrability property.

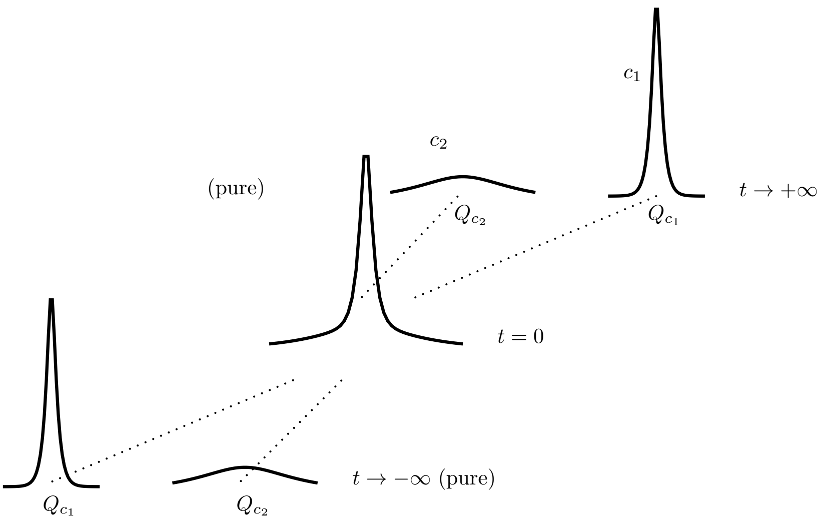

Figure 2.1: The evolution of a KdV 2-soliton solution (courtesy of Y. Martel).

In [33], Maddocks and Sachs considered the $H^N(\mathbb{R})$-stability of the N-soliton solution of KdV, by using $N+1$ conservation laws. Next, in [41, 40], Martel, Merle and Tsai improved the preceding result by proving stability and asymptotic stability of the *sum of N solitons*, *well decoupled* at the initial time, in the energy space. Their proof also applies for general nonlinearities and not only for the integrable cases, provided they have stable solitons, in the sense of Weinstein [53]. Their approach is based on the construction of N *almost conserved quantities*, related to the mass of each solitary wave, plus the total energy of the solution. As a consequence of the existence of N-soliton solutions for KdV, the above results can be extended to give a global stability property, improving the Maddock-Sachs results. See also [37, 38, 39] for global H^1-stability results in some non-integrable cases.

A natural question to consider is the generalization of the Merle-Vega's result to the case of multi-soliton solutions, namely to understand the L^2-stability of the multi-soliton solution. In [19] (see also [52]), Gesztesy-Schweinger-Simon prove that the Miura transform sends *multi-kink* solutions of (1.2) to a well defined family of *multi-soliton* solutions of (1.1), provided a criticality condition is satisfied. However, in order to prove our result, we have followed a different approach.

Indeed, in [4] we considered a *Gardner transform* [43, 16], well-known in the mathematical and physical literature since the late sixties, and which links H^1-solutions of the Gardner equation

$$v_t + (v_{xx} + v^2 - \beta v^3)_x = 0, \quad \text{in } \mathbb{R}_t \times \mathbb{R}_x, \quad \beta > 0, \tag{2.4}$$

with L^2-solutions of the KdV equation (1.1). More specifically, given any $\beta > 0$ and $v(t) \in H^1(\mathbb{R})$, solution of the Gardner equation (2.4), the Gardner transform [43, 16]

$$u(t) = M_\beta[v](t) := [v - \frac{3}{2}\sqrt{2\beta} v_x - \frac{3}{2}\beta v^2](t), \tag{2.5}$$

is an L^2-solution of KdV. Compared with the original Miura transform (2.1), it has an additional *linear* term which simplifies the proofs.

In addition, the Gardner equation is also an integrable model [16], with soliton solutions of the form

$$v(t,x) := Q_{c,\beta}(x - ct),$$

and[1]

$$Q_{c,\beta}(s) := \frac{3c}{1 + \rho\cosh(\sqrt{c}s)}, \qquad \text{with} \quad \rho := (1 - \frac{9}{2}\beta c)^{1/2}, \quad 0 < c < \frac{2}{9\beta}. \tag{2.6}$$

In particular, in the formal limit $\beta \to 0$, we recover the standard KdV soliton (1.4)-(1.5). On the other hand, the Cauchy problem associated to (2.4) is globally well-posed under initial data in the energy class $H^1(\mathbb{R})$ (cf. Kenig-Ponce-Vega [27]), thanks to the mass (1.3) and *energy* conservation laws.

[1]See e.g. [10, 47] and references therein for a more detailed description of solitons and integrability for the Gardner equation.

We have been interested in the image of the family of solutions (2.6) under the aforementioned, Gardner transform. Surprisingly enough, it turns out that the resulting family is **nothing but** the KdV soliton family (1.4). Indeed, a direct computation shows that for the Gardner soliton solution (2.6), one has

$$\begin{aligned} M_\beta[Q_{c,\beta}](t) &= \Big[Q_{c,\beta} - \frac{3}{2}\sqrt{2\beta}Q'_{c,\beta} - \frac{3}{2}\beta Q^2_{c,\beta}\Big](x-ct) \\ &= Q_c(x-ct-\delta), \end{aligned} \tag{2.7}$$

with $\delta = \delta(c,\beta) > 0$ provided $\beta > 0$, and Q_c the KdV soliton solution (1.4). In other words, the Gardner transform (2.5) sends the Gardner soliton towards a slightly translated KdV soliton.

In [4], we profit of this property to improve the H^1-stability and asymptotic stability properties proved by Martel-Merle-Tsai in [41], Martel-Merle [40], and Merle-Vega [42], now in the case of L^2-perturbations of the KdV multi-solitons. See [4] for the full details.

Theorem 2.1 (L^2-stability of the KdV N-soliton, [4])**.**
Let $\delta > 0$, $N \geq 2$, $0 < c_1^0 < \ldots < c_N^0$ and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$. There exists $\alpha_0 > 0$ such that if $0 < \alpha < \alpha_0$, then the following holds. Let $u(t)$ be a solution of (1.1) such that

$$\|u(0) - U^{(N)}(\cdot; c_j^0, -x_j^0)\|_{L^2(\mathbb{R})} \leq \alpha,$$

with U^N the N-soliton profile described in (2.3). Then there exist $x_j(t)$, $j = 1, \ldots, N$, such that

$$\sup_{t\in\mathbb{R}} \Big\| u(t) - U^{(N)}(\cdot; c_j^0, -x_j(t))\Big\|_{L^2(\mathbb{R})} \leq \delta. \tag{2.8}$$

The above result can be seen as a consequence of the stability of an initial datum close enough to the **sum of N decoupled solitons** of the KdV equation, and the uniform continuity of the KdV flow for L^2-data, see e.g. [41], Corollary 1.

In order to prove this result, we construct an inverse of the Gardner transform in a vicinity of the sum of KdV soliton solutions, using a fixed point argument. First of all, recall that for $\beta > 0$ small,

$$M_\beta[S_0] = R_0 + \text{l.o.t.},$$

where $S_0(x) := \sum_{j=1}^N Q_{c_j^0,\beta}(x - x_j^0 - \delta_j)$, $R_0(x) := \sum_{j=1}^N Q_{c_j^0}(x - x_j^0)$, $\delta_j = O(\beta)$, and $Q_{c,\beta}$ being the soliton solution of the Gardner equation (2.4).

Next, we look for a solution $v_0 \in H^1(\mathbb{R})$ of $M_\beta[v_0] = R_0 + z_0$, where z_0 is small in $L^2(\mathbb{R})$, $v_0 = S_0 + w_0$, and w_0 is small in $H^1(\mathbb{R})$. In other words, w_0 has to solve the nonlinear equation

$$\mathcal{L}[w_0] = (R_0 - M_\beta[S_0]) + z_0 + \frac{3}{2}\beta w_0^2, \tag{2.9}$$

with

$$\mathcal{L}[w_0] := -\frac{3}{2}\sqrt{2\beta}w_{0,x} + (1 - 3\beta S_0)w_0. \tag{2.10}$$

We may think $\mathcal{L}$ as a unbounded operator in $L^2(\mathbb{R})$, with dense domain $H^1(\mathbb{R})$. From standard energy estimates (see [4]), one has that for $\beta > 0$ small enough, any solution $w_0 \in H^1(\mathbb{R})$ of the linear problem

$$\mathcal{L}[w_0] = f, \quad f \in L^2(\mathbb{R}), \tag{2.11}$$

must satisfy, for some fixed constant $K_0 > 0$,

$$\|w_0\|_{H^1(\mathbb{R})} \leq \frac{K_0}{\sqrt{\beta}}\|f\|_{L^2(\mathbb{R})}. \tag{2.12}$$

In order to prove the existence and uniqueness of a solution of (2.11), we use (2.12) and a fixed point approach, in the spirit of [55, 26]. See [4] for the details.

In what follows, let us denote by $T := \mathcal{L}^{-1} : L^2(\mathbb{R}) \to H^1(\mathbb{R})$ the resolvent operator above mentioned. Now, from (2.9), we want to solve the nonlinear problem

$$w_0 = T[w_0] = \mathcal{L}^{-1}\Big[(R_0 - M_\beta[S_0]) + z_0 + \frac{3}{2}\beta w_0^2\Big]. \tag{2.13}$$

In order to invoke, once again, a fixed point argument in a suitable closed ball of $H^1(\mathbb{R})$. A direct argument shows that T is a contraction mapping from $\mathcal{B}$ into itself, provided β is small.

Now we invoke the Martel, Merle and Tsai [41, 40] stability results –in the special case of the Gardner equation– to the solution of the Cauchy problem associated to the initial data $v_0 \in H^1(\mathbb{R})$, which implies the stability of $v(t)$. After this point, the proof follows closely the ideas of [42], giving the desired result. We finish with the following diagram, which describes the approach we have followed.

$$\begin{array}{ccc}
\begin{array}{c}\text{KdV}\\ u_0 \sim_{L^2} R_0\end{array} & \xrightarrow[v_0=M_\beta^{-1}[u_0]]{\text{Gardner}} & \begin{array}{c}\text{Gardner}\\ v_0 \sim_{H^1} S_0\end{array} \\
\begin{array}{c}L^2\text{-KdV flow}\\ \text{(Bourgain)}\end{array} \Big\downarrow\; t>0 & \begin{array}{c}H^1\text{-Gardner flow}\\ \text{(K-P-V)}\end{array} & \Big\downarrow \begin{array}{c}H^1\text{-stability}\\ \text{(Martel-Merle)}\end{array} \\
u(t) = \bar{u}(t) & \xleftarrow[\bar{u}(t)=M_\beta[v](t)]{\text{Gardner}} & v(t) \text{ stable}
\end{array}$$

Fig. 2: *The Gardner's approach.*

See [4] for a detailed proof.

3. The mKdV case: multi-kinks stability

Let us come back to the equation (1.2). In addition to the previously mentioned kink solution (1.4), mKdV has *multi-kink* solutions, as a consequence of the integrability property and the Inverse Scattering theory (Grosse [21, 22]). In addition, these solution can be recovered by a completely different approach, using the Miura transform (2.1), see Gesztesy-Schweinger-Simon [18, 19], or the monograph by Thaller [52]. Indeed, according to Gesztesy-Schweinger-Simon [19], there exist at least two different forms of multi-kink solutions for (1.2), such that its image by the Miura transform is the same KdV multi-soliton.

Surprisingly, the existence, uniqueness and stability of multi-kinks is closely related to the solitons of the Gardner equation (cf. Definitions 3.1 and 3.4), and more generally, *dynamical properties of defocusing gKdV equations are closely related to those of suitable focusing counterparts.*

Indeed, let $v = v(t, y) \in C(\mathbb{R}, H^1(\mathbb{R}))$ be a solution of (2.4). Then

$$u(t, x) := b - \sqrt{\beta}\, v\Big(t, x + \frac{t}{3\beta}\Big), \quad b := \frac{1}{3\sqrt{\beta}}, \tag{3.1}$$

solves the mKdV equation (1.2). In terms of the Miura and Gardner transform, it reads as follows

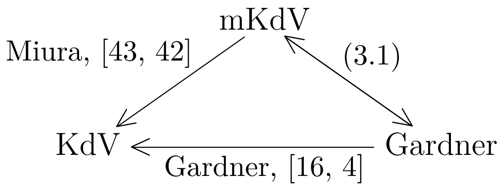

Figure 3.1: Transformation (3.1) in terms of Miura (2.1) and Gardner transforms.

Let us recall that, for t fixed, (3.1) is a diffeomorphism which **preserves regularity**. Note in addition that u in (3.1) is an L^∞-function with nonzero limits at infinity. This analysis motivates the following alternative approach for the multi-kink solution:

Definition 3.1 (*Even* multi-kink solutions, see also [21, 22, 19, 52])**.**
Let $\beta > 0$, scaling parameters $0 < c_1^0 < c_2^0 < \ldots < c_N^0 < \frac{2}{9\beta}$ and $x_1^-, \ldots, x_N^- \in \mathbb{R}$ be fixed numbers. We say that a solution $U_e(t)$ of (1.2) is an even multi-kink *if it satisfies*

$$\lim_{t\to-\infty} \Big\| U_e(t) - b + \sqrt{\beta} \sum_{j=1}^N Q_{c_j^0,\beta}(\cdot + \tilde{c}_j t + x_j^-) \Big\|_{H^1(\mathbb{R})} = 0, \tag{3.2}$$

$$\lim_{t\to+\infty} \Big\| U_e(t) - b + \sqrt{\beta} \sum_{j=1}^N Q_{c_j^0,\beta}(\cdot + \tilde{c}_j t + x_j^+) \Big\|_{H^1(\mathbb{R})} = 0, \tag{3.3}$$

with $b = \frac{1}{3\sqrt{\beta}}$, $\tilde{c}_j := \frac{1}{3\beta} - c_j^0 > 0$, $x_j^+ \in \mathbb{R}$ depending only on (x_k^-) and (c_k^0), and $Q_{c,\beta}$ being solitons of the Gardner equation (2.4).

Let us emphasize that $\tilde{c}_N < \tilde{c}_{N-1} < \ldots < \tilde{c}_1$, which means that bigger Gardner solitons are actually slower than the smaller ones. Note also that they move from the right to the left, as time evolves.

The denomination **multi-kink** above comes from the fact the these solutions can be seen asymptotically as the *sum* of several kinks $\pm\varphi_c$ of the form (1.4). For instance, with our notation, given $\beta > 0$ and $0 < c < \frac{2}{9\beta}$, an expression for the 2-kink solution is given by [19, p. 505] (see also [52, p. 273])

$$U_e(t, x) = b - [\varphi_{c/2}(x + \tilde{c}t + 2x_0) - \varphi_{c/2}(x + \tilde{c}t)], \tag{3.4}$$

with

$$b := \frac{1}{3\sqrt{\beta}}, \quad \tilde{c} := \frac{1}{3\beta} - c, \quad x_0 := \frac{1}{2\sqrt{c}} \log\Big(\frac{\sqrt{2} + 3\sqrt{\beta c}}{\sqrt{2} - 3\sqrt{\beta c}}\Big) > 0,$$

and φ_c as in (1.4). Note that both kinks $\pm\varphi_{c/2}$ have the **same velocity** $\tilde{c}$. After a quick computation, one can see that (3.4) can be written, as in (3.2)-(3.3), using the Gardner soliton (2.6):

$$U_e(t,x) = b - \sqrt{\beta}Q_{c,\beta}(x + \tilde{c}t + x_0),$$

which will be helpful for our purposes. From this fact one says that, in general, the function U_e represents a $2N$**-kink solution**. In terms of our point of view, it will represent N different Gardner solitons attached to the non-zero constant b.

The existence of a solution U_e satisfying (3.2) is a simple consequence of (3.1) and the behavior of the N-soliton solution $V^{(N)}(t)$ of the Gardner equation (see also [21, 22, 19] and [52, pp. 272-273] for the standard deduction). Moreover, this solution is unique, in the sense considered by Martel [34].

From (3.1) we can define

$$U_e(t) := b + \sqrt{\beta}V^{(N)}(t, \cdot + \frac{t}{3\beta}). \tag{3.5}$$

Therefore, as a conclusion of the preceding analysis, and using (3.1), we get the uniqueness of the corresponding solution U_e.

Theorem 3.2 (Uniqueness of even multi-kink solutions, [49])**.**
Let $\beta > 0$, $0 < c_1^0 < c_2^0 < \ldots < c_N^0 < \frac{2}{9\beta}$ and $x_1^-, \ldots, x_N^- \in \mathbb{R}$ be fixed numbers. Then the associated even multi-kink U_e defined in (3.5) is the unique solution of (1.2) satisfying (3.2).

The second problem that we considered is the **stability** of the multi-kink U_e. From the Martel-Merle-Tsai and Martel-Merle [41, 40] results and Definition 3.1 we claim the following

Theorem 3.3 (Stability of *even* multi-kink solutions, [49])**.**
The family of multi-kink solutions $U_e(t)$ from Definition 3.1 and (3.5) is global-in-time H^1-stable, and asymptotically stable as $t \to \pm\infty$.

Theorems 3.2 and 3.3 can be deduced from Martel [34] and Martel-Merle-Tsai [41, 40]. We recall that, without using transformation 3.1, these results were unable to be tackled down by using any direct method.

There is a second type of multi-kink solutions for (1.2), which is actually the best known one. Here, the standard kink φ_c in (1.4) and the Gardner equation play once again a crucial and surprising rôle. Indeed, let $\beta > 0$ be a fixed parameter and suppose that one has a solution of (1.2) of the form (the reader may compare with (3.1))

$$u(t,x) := \varphi_c(x + ct) + \sqrt{\beta}\tilde{u}(t, x + \frac{t}{3\beta}), \quad c := \frac{1}{9\beta}, \tag{3.6}$$

and $\tilde{u}(t) \in H^1(\mathbb{R})$. Then $\tilde{u}(t,y)$ satisfies the equation

$$\tilde{u}_t + (\tilde{u}_{yy} + \tilde{u}^2 - \beta\tilde{u}^3)_y = 3((\varphi_c^2 - c)\tilde{u} + \sqrt{\beta}(\varphi_c + \sqrt{c})\tilde{u}^2)_y, \tag{3.7}$$

with $\varphi_c = \varphi_c(y - 2ct)$. In particular, if the support of $\tilde{u}(t)$ is mainly localized in the region where $\varphi_c \sim -\sqrt{c}$, namely $y \ll 2ct$, then the right hand side above is a **small**

perturbation of the left hand side, a Gardner equation with parameter $\beta > 0$. As an admissible function $\tilde{u}$, we can take e.g. a sum of Gardner solitons:

$$\tilde{u}(t,y) \sim \sum_{j=1}^{N-1} Q_{c_j,\beta}(y - c_j t), \qquad 0 < c_1 < c_2 \ldots < c_{N-1} < \frac{2}{9\beta} = 2c,$$

with support localized in the region $c_1 t \lesssim y \lesssim c_{N-1}t$, for $t \gg 1$. In particular, one has $c_{N-1}t \ll 2ct$ for $t \gg 1$, which is a necessary condition for the existence of a solution of the form (3.6).

Finally, the same argument works in the case of a solution of the form $u(t,x) := \varphi_c(x+ct) - \sqrt{\beta}\hat{u}(t, x + \frac{t}{3\beta})$, and the equation for $\hat{u}(t,y)$,

$$\hat{u}_t + (\hat{u}_{yy} + \hat{u}^2 - \beta\hat{u}^3)_y = 3((\varphi_c^2 - c)\hat{u} + \sqrt{\beta}(\sqrt{c} - \varphi_c)\hat{u}^2)_y,$$

provided $\hat{u}$ is supported mainly in the region $\{\varphi_c \sim \sqrt{c}\}$. These two new ideas allow us to consider the following definition of a multi-kink solution, from the point of view of the Gardner equation:

Definition 3.4 (*Odd* multi-kink solutions, [19, 52]**).**
Let $N \geq 2$, $\beta > 0$, scaling parameters $0 < c_1^0 < c_2^0 < \ldots < c_{N-1}^0 < \frac{2}{9\beta}$ and $x_1^0, \ldots, x_N^0 \in \mathbb{R}$ be fixed numbers. We say that a solution $U_o(t)$ of (1.2) is an odd multi-kink *solution if it satisfies*

$$\lim_{t\to-\infty} \Big\| U_o(t) - \varphi_{c_N^0}(\cdot + c_N^0 t + x_N^-) + \sqrt{\beta}\sum_{j=1}^{N-1} Q_{c_j^0,\beta}(\cdot + \tilde{c}_j t + x_j^-)\Big\|_{H^1(\mathbb{R})} = 0, \quad (3.8)$$

$$\lim_{t\to+\infty} \Big\| U_o(t) - \varphi_{c_N^0}(\cdot + c_N^0 t + x_N^+) - \sqrt{\beta}\sum_{j=1}^{N-1} Q_{c_j^0,\beta}(\cdot + \tilde{c}_j t + x_j^+)\Big\|_{H^1(\mathbb{R})} = 0, \quad (3.9)$$

with $c_N^0 := \frac{1}{9\beta}$, $\tilde{c}_j := \frac{1}{3\beta} - c_j^0 > 0$ and $x_j^{\pm} \in \mathbb{R}$ depending only on (c_k^0). Finally, φ_c is a kink solution (1.4) with scaling c, and $Q_{c,\beta}$ is a soliton solution of the Gardner equation (2.4).

One can also say that U_o is composed by $(2N-1)$ single kinks, in other words, it is a $(2N-1)$**-kink solution**. From the point of view of the Gardner equation, this solution represents a big kink, solution of mKdV, with attached $(\pm)$Gardner solitons ordered according their corresponding velocities $\tilde{c}_j$. Note finally that solitons move from the right to the left.

In [49] we present a new proof of existence of $U_o(t)$, which gives in addition a uniqueness property and uniform estimates in time. The uniqueness is, of course, modulo the $2N$-parameter family (c_j^0, x_j^-). By extending the results of Martel [34] to the case of multi-kink solutions, we get the following

Theorem 3.5 (Existence and uniqueness of odd multi-kink solutions, [49]**).**
Let $N \geq 2$, $\beta > 0$, $c_N^0 = \frac{1}{9\beta}$, $0 < c_1^0 < c_2^0 < \ldots < c_{N-1}^0 < \frac{2}{9\beta}$ and $x_1^-, \ldots, x_N^- \in \mathbb{R}$ be fixed numbers. There exists a unique solution $U_o(t)$ of (1.2) satisfying (3.8).

Our final result is a positive answer to the open question of stability of odd multi-kinks.

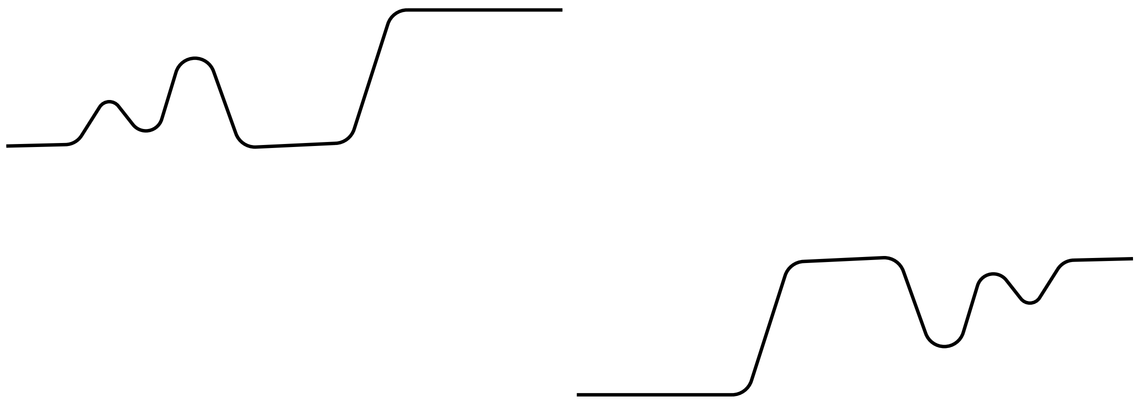

Figure 3.2: A schematic design of the evolution in time of a 5-kink solution of mKdV, composed of a big kink $\varphi_{c_3^0}$ and two Gardner solitons, $Q_{c_1^0,\beta}$ and $Q_{c_2^0,\beta}$, $0 < c_1^0 < c_2^0 < 2c_3^0$, and $\beta = \frac{1}{9c_3^0}$. Below, the behavior as $t \to -\infty$; above, the behavior as $t \to +\infty$. Each part is ordered according to their respective velocity $-\tilde{c}_j := c_j^0 - 3c_3^0 < 0$, $j = 1, 2$. Note that $-\tilde{c}_1 < -\tilde{c}_2 < -c_3^0$, which means that the smallest soliton $Q_{c_1^0,\beta}$ is actually the fastest one.

Theorem 3.6 (Stability of odd multi-kink solutions, [49]).
The family of multi-kink solutions $U_o(t)$ from Definition 3.4 is global-in-time H^1-stable.

The proof of this result is based in the approach introduced in [41] in order to describe the stability in $H^1(\mathbb{R})$ of N decoupled solitons. However, in this opportunity we face several new problems since the kink solution and the Gardner solitons are in strong interaction through the dynamics. Moreover, the mass (1.10) cannot be used to control the Gardner solitons, as has been done in [41]. In this sense, the transformation (3.6) is the first step –and the more important one– to understand the interaction among kinks as actually localized, soliton-like interactions.

Let us be more precise. Using the energy (see (1.9)) of the solution $u(t)$, one controls with no additional difficulties the kink solution. This is a consequence of the non negative character of the linearized operator around the kink solution, see [56, 42] for more details. However, this quantity is not enough to control the behavior of the Gardner solitons. We overcome this difficulty by using the transformation (3.6), which introduces a new function $\tilde{u}(t)$, almost solution of a Gardner-like equation (cf. (3.7)). It turns out that the perturbative terms on the right hand side of (3.7) can be controlled provided the solitons are far from the center of the main kink solution, which holds true if we assume that the initial configuration is well prepared. Additionally, we introduce a new, **almost conserved mass** for the portion on the left of the solution $\tilde{u}$, which allows to control each Gardner soliton by separated, since the standard mass (1.10) is bad behaved for H^1-perturbations of a kink solution. This approach is completely general and can be adapted to the gKdV case. No additional hypotheses are needed, only the single stability of each generalized soliton component of the multi-kink solution.

4. Final remarks

We point out that the preceding results, starting from transformations (3.1)-(3.6) and the theory developed by Zhidkov in [56], can be made even more general and include a wide range of non-integrable, defocusing gKdV equations.

Indeed, in [49] we have introduced the notion of **generalized, even and odd multi-kink solutions**. These new objects have to match with those considered in Definitions 3.1 and 3.4, for the special case of the integrable mKdV model. We proved the existence, uniqueness and stability of these new solutions in the case of *well-prepared* initial data. Next, we have considered some particular collision problems, in the spirit of [37, 38, 47] (note that the collision problem makes sense since we consider non-integrable equations). The method used is the same as in the previous results. We emphasize that the main idea is to exploit the properties contained in Figure 4.1.

$$u \in \text{ defocusing gKdV (even)} \longleftarrow (u \sim b + \tilde{u}) \longrightarrow \tilde{u} \in \text{focusing gKdV}$$

$$u \in \text{ defocusing gKdV (odd)} \longleftarrow (u \sim \varphi_c + \tilde{u}) \longrightarrow \tilde{u} \in \text{focusing gKdV}$$

Figure 4.1: The generalized transformations (3.1)-(3.6) linking a defocusing gKdV with a focusing gKdV equation.

We also recall that the collision problem in the case of odd multi-kink solutions remains an interesting open question. Finally, let us mention that a similar transformation to (3.1)-(3.6) can be introduced in the cases of the ϕ^4 and sine-Gordon models, with different results [49].

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Departamento de Ingeniería Matemática y CMM, Universidad de Chile, Casilla 170-3, Correo 3, Santiago
Chile
cmunoz@dim.uchile.cl