Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Biarritz, 6 juin–10 juin 2011

Massimiliano Berti

Quasi-periodic solutions of Hamiltonian PDEs

J. É. D. P. (2011), Exposé n° II, 13 p.

<http://jedp.cedram.org/item?id=JEDP_2011____A2_0>

Journées *Équations aux dérivées partielles*
Biarritz, 6 juin–10 juin 2011
GDR 2434 (CNRS)

Quasi-periodic solutions of Hamiltonian PDEs

Massimiliano Berti

Abstract

We overview recent existence results and techniques about KAM theory for PDEs.

1. Introduction

Many partial differential equations arising in physics can be seen as infinite dimensional Hamiltonian systems. Main examples are the nonlinear Schrödinger (NLS) and wave equations (NLW), the beam, the membrane and the Kirkhoff equations in elasticity theory, the Euler equations of hydrodynamics, as well as their approximate models, like KdV, Benjamin-Ono, KP equations, ...

In the last years important mathematical progresses have been achieved in the study of these evolutionary Partial Differential Equations (PDEs) adopting the "dynamical systems philosophy", focusing, in particular, on the search of invariant tori of the phase space filled by periodic and quasi-periodic solutions.

A natural setting concerns the bifurcation of quasi-periodic solutions close to linearly stable (elliptic) equilibria of a PDE. The main difficulty for the existence proof is the presence of arbitrarily "*small divisors*" in the perturbative expansion series of the expected solutions. Such small divisors arise by complex resonance phenomena between the normal mode frequencies of the system.

The main strategies which have been developed to overcome the small divisors difficulty are:

1. KAM theory,

2. Newton-Nash-Moser implicit function theorems.

The KAM approach consists in generating iteratively a sequence of canonical changes of variables of the phase space (close to the identity) which bring the

This research was supported by the European Research Council under FP7 "New Connections between dynamical systems and Hamiltonian PDEs with small divisors phenomena".

MSC 2000: 35Q55, 37K55, 37K50.

Keywords: KAM for PDE, Nash-Moser Theory, Quasi-Periodic Solutions, Small Divisors, Nonlinear Schrödinger and wave equation, Infinite Dimensional Hamiltonian Systems.

Hamiltonian into a normal form with an invariant torus at the origin. This iterative procedure requires, at each step, to invert the so called linear "homological equations". In the usual KAM scheme the normal form has constant coefficients (reducibility), hence the homological equations have *constant coefficients* and can be solved by Fourier series imposing the "second order Melnikov" non-resonance conditions. The final KAM torus is linearly stable.

Actually, Kuksin [22] and Wayne [27] developed this scheme to prove the existence of quasi-periodic solutions for one dimensional (1-d) NLW and NLS equations. These pioneering results were limited to Dirichlet boundary conditions because the eigenvalues of ∂_{xx} had to be simple. Actually, the required second order Melnikov non resonance conditions are violated in presence of multiple normal frequencies (because differences of normal frequencies appear), for example, already for periodic boundary conditions (two eigenvalues of ∂_{xx} are equals).

Then a more direct bifurcation approach was proposed by Craig and Wayne [17] for 1-d NLS and NLW with periodic boundary conditions. After a Lyapunov-Schmidt decomposition, the search of the embedded torus is reduced to solve a functional equation in scales of Banach spaces, by some Newton-Nash-Moser implicit function theorem.

The main advantage of this approach is to require only the so called "first order Melnikov" non-resonance conditions for solving the linearized equations (homological equations) at each step of the iteration. These conditions are essentially the minimal assumptions, and, in particular, do not involve differences of normal frequencies. Translated in the KAM language this corresponds to allow a non-constant coefficients normal form around the torus. The main difficulty of this strategy is that the homological equations are PDEs with *non-constant* coefficients and are small perturbations of a diagonal operator having arbitrarily small eigenvalues. Hence it is hard to estimate their inverses in high norms. Craig-Wayne [17] solved this problem for periodic solutions of 1-d analytic NLS and NLW and Bourgain [10] also for quasi-periodic solutions.

Clearly, a drawback of this approach is to prove only the existence of the invariant torus, unlike the usual KAM theory also provides a reducible normal form around it, which implies, in particular, the stability of the torus, and can be used to study nearby the dynamics of the PDE, see [4], [20].

At present, the theory for 1-d NLS and NLW has been sufficiently understood (see e.g. [23]) but much work remains in higher space dimensions, due to the more complex spectral analysis of $-\Delta + V(x)$. The main difficulties are:

1. the eigenvalues of $-\Delta + V(x)$ appear in clusters of unbounded sizes,

2. the eigenfunctions are, in general, "not localized with respect to the exponentials".

Roughly speaking, property 2 means that, if we expand the eigenfunctions of $-\Delta + V(x)$ with respect to the exponentials, the Fourier coefficients rapidly converge to zero. This property always holds in 1 space dimension (see [17]) but may fail for $d \geq 2$, see [13]. This problem has been often bypassed considering pseudo-differential PDEs where the multiplicative potential $V(x)$ is substituted by a "convolution potential" $V * (e^{\mathrm{i}j\cdot x}) := m_j e^{\mathrm{i}j\cdot x}$, $m_j \in \mathbb{R}$, $j \in \mathbb{Z}^d$ (which acts diagonally on

the exponentials). The scalars m_j are called the "Fourier multipliers" and play the role of "external parameters".

The Newton-Nash-Moser approach is, in principle, very useful to overcome problem 1, because it requires only the first order Melnikov non-resonance conditions and therefore does not exclude multiplicity of normal frequencies. Actually, developing this perspective, Bourgain [11], [13], [14] was able to prove the existence of quasi-solutions for NLW and NLS with Fourier multipliers on $\mathbb{T}^d$, $d \geq 2$.

More recently, also the KAM approach has been extended by Eliasson-Kuksin [21] for NLS on $\mathbb{T}^d$ with Fourier multipliers. The key issue is to control more accurately the perturbed frequencies after the KAM iteration and, in this way, the difference of the normal frequencies, verifying the second order Melnikov non-resonance conditions. We refer also to Procesi-Procesi [25] and Wang [26] for the cubic NLS.

The goal of this note is to present new recent extensions of these theories in the following directions:

1. **Finitely differentiable PDEs.** All the previous results are valid for analytic nonlinearities (actually polynomials in [13], [14]). This simplifies the analysis because the resonance effects are weaker for analytic nonlinearities than for finitely differentiable one's. A natural question concerns the persistence of quasi-periodic solutions for PDEs in a setting of finitely many derivatives. The theory developed in [5], [8], [6]-[7], see also [3], answers positively this question. We refer also to Delort [18] for periodic solutions of C^∞-NLS.

   Of course we cannot expect quasi-periodic solutions under too weak regularity assumptions on the nonlinearities. Actually, for finite dimensional Hamiltonian systems, it has been rigorously proved that, if the vector field is not sufficiently smooth, then all the invariant tori could be destroyed and only discontinuous Aubry-Mather invariant sets survive.

2. **PDEs defined on more general manifolds**. The dynamics of a PDE on a compact Riemannian manifold strongly depends on its geometry, in particular, via the properties of the eigenvalues and the eigenfunctions of the Laplace-Beltrami operator. All the previous results are valid for PDEs on flat tori $\mathbb{T}^d$, $d \geq 1$. In [8]-[9] we prove the existence of periodic solutions of NLS and NLW defined not only on tori, but also on compact Zoll manifolds (i.e. spheres), Lie groups and homogeneous spaces. In these cases, the eigenvalues are highly degenerate and only weak properties of localization of the eigenfunctions hold.

   Interestingly, many tools in [8]-[9] resemble the Birkhoff normal form techniques developed by Bambusi, Delort, Grebért, Szeftel [2] for PDEs on spheres and Zoll manifolds, and seem deeply related with the methods of Burq, Gérard, Tzvetkov [15] for the initial value problem.

3. **Multiplicative potential.** In [6] we prove the existence of quasi-periodic solutions of NLS with a *multiplicative* potential $V(x)$ on $\mathbb{T}^d$, $d \geq 2$ (see Theorem 2.1). Actually, the Nash-Moser approach described in section 3 requires essentially no informations about the eigenfunctions of the Laplacian with a periodic potential, which, on the contrary, seem to be unavoidable to prove also reducibility.

4. **Parameter dependence.** In KAM theory several "parameters" are usually available for verifying the required Melnikov non resonance conditions. In [6] we use only *one* external parameter -the length of the quasi-periodic frequency vector (i.e. time-scaling) for NLS in $d \geq 2$. For finite dimensional Hamiltonian systems this kind of result was proved by Eliasson [19] and Bourgain [12]. For 1-d PDEs it was proved in [4], using an explicit caracterization of the Cantor set of parameters which satisfy the non-resonance conditions at all the KAM steps, in terms of the *final* frequencies only. We also refer to [1] for an application to degenerate KAM theory.

Let us present rigorously the existence result of quasi-periodic solutions for NLS on $\mathbb{T}^d$.

2. Quasi-periodic solutions of NLS on $\mathbb{T}^d$

As a model equation we consider a Schrödinger equation

$$\mathrm{i}u_t - \Delta u + V(x)u = \varepsilon f(\omega t, x, |u|^2)u + \varepsilon g(\omega t, x) \tag{2.1}$$

with periodic boundary conditions

$$x \in \mathbb{T}^d := (\mathbb{R}/(2\pi\mathbb{Z}))^d \,,$$

where the *multiplicative* potential V is in $C^q(\mathbb{T}^d;\mathbb{R})$ for some q large enough, $\varepsilon > 0$ is small, the nonlinearity is *finitely differentiable* and quasi-periodic in time, more precisely

$$f \in C^q(\mathbb{T}^\nu \times \mathbb{T}^d \times \mathbb{R};\mathbb{R}) \,, \quad g \in C^q(\mathbb{T}^\nu \times \mathbb{T}^d;\mathbb{C}) \tag{2.2}$$

for some $q \in \mathbb{N}$ large enough, and the frequency vector $\omega \in \mathbb{R}^\nu$ is *colinear* with a fixed diophantine vector $\bar\omega \in \mathbb{R}^\nu$, namely

$$\omega = \lambda\bar\omega \,, \quad \lambda \in \Lambda := [1/2, 3/2] \subset \mathbb{R} \,, \quad |\bar\omega \cdot l| \geq \frac{\gamma_0}{|l|^{\tau_0}} \,, \quad \forall l \in \mathbb{Z}^\nu \setminus \{0\} \,, \tag{2.3}$$

for some $\gamma_0 \in (0,1)$, $\tau_0 > \nu - 1$ (for definiteness $\tau_0 := \nu$), $|l| := \max\{|l_1|, \dots, |l_\nu|\}$.

The dynamics of the linear Schrödinger equation

$$\mathrm{i}u_t - \Delta u + V(x)u = 0$$

is well understood. All its solutions are the linear superpositions of normal mode oscillations, namely

$$u = \sum_j a_j e^{\mathrm{i}\mu_j t}\psi_j(x) \,, \; a_j \in \mathbb{C} \,, \quad \text{where} \quad (-\Delta + V(x))\psi_j(x) = \mu_j\psi_j(x) \,,$$

hence periodic, quasi-periodic or almost periodic in time. The eigenfunctions $\psi_j(x)$ of $-\Delta + V(x)$ form a Hilbert basis in $L^2(\mathbb{T}^d)$ and the eigenvalues $\mu_j \to +\infty$ as $j \to +\infty$.

- *Question:* for $\varepsilon \neq 0$ small enough, do there exist quasi-periodic solutions of (2.1)?

Note that, if $g(\omega t, x) \not\equiv 0$, then $u = 0$ is not a solution of (2.1) for $\varepsilon \neq 0$.

Then we look for $(2\pi)^{\nu+d}$-periodic solutions $u(\varphi, x)$ of

$$\mathrm{i}\omega \cdot \partial_\varphi u - \Delta u + V(x)u = \varepsilon f(\varphi, x, |u|^2)u + \varepsilon g(\varphi, x) \tag{2.4}$$

in some Sobolev space

$$H^s := H^s(\mathbb{T}^\nu \times \mathbb{T}^d; \mathbb{C}) \quad \text{with} \quad s \in [s_0, q]\,, \quad s_0 > \frac{d+\nu}{2}\,.$$

The functions in H^s are characterized in Fourier series

$$u(\varphi, x) := \sum_{(l,j)\in\mathbb{Z}^\nu\times\mathbb{Z}^d} u_{l,j} e^{\mathrm{i}(l\cdot\varphi + j\cdot x)}$$

by the condition

$$\|u\|_s^2 := K_0 \sum_{(l,j)\in\mathbb{Z}^{\nu+d}} |u_{l,j}|^2 \langle l,j\rangle^{2s} < +\infty \quad \text{where} \quad \langle l,j\rangle := \max(1,|l|,|j|)\,. \tag{2.5}$$

The constant K_0 is fixed (large enough) so that $|u|_{L^\infty} \le \|u\|_{s_0}$ and the interpolation inequality

$$\|u_1 u_2\|_s \le \frac{1}{2}\|u_1\|_{s_0}\|u_2\|_s + \frac{C(s)}{2}\|u_1\|_s\|u_2\|_{s_0}\,, \quad \forall s \ge s_0\,,\ u_1, u_2 \in H^s\,,$$

holds with $C(s) \ge 1$ and $C(s) = 1, \forall s \in [s_0, s_1]$ for some $s_1 := s_1(d,\nu)$ (defined along the proof).

The above question turns into a bifurcation problem for equation (2.4) from the trivial solution $(u,\varepsilon) = (0,0)$. The main difficulty is that the unperturbed linear operator $\mathrm{i}\omega\cdot\partial_\varphi - \Delta + V(x)$ possesses arbitrarily small eigenvalues $-\omega\cdot l + \mu_j$, called "small divisors". As a consequence, its inverse operator, if any, is unbounded and the standard implicit function theorem can not be applied.

The following theorem is proved in [6]-[7] by a Nash-Moser implicit function approach.

Theorem 2.1. (NLS) *Assume (2.3). Then*

Existence: *There are $s := s(d,\nu)$, $q := q(d,\nu) \in \mathbb{N}$, such that: $\forall V \in C^q$ satisfying*

$$-\Delta + V(x) \ge \beta_0 I\,,\ \beta_0 > 0\,, \tag{2.6}$$

$\forall f, g \in C^q$, there exist $\varepsilon_0 > 0$, a map

$$u \in C^1([0,\varepsilon_0]\times\Lambda; H^s) \quad \textit{with} \quad u(0,\lambda) = 0\,,$$

and a Cantor like set $\mathcal{C}_\infty \subset [0,\varepsilon_0]\times\Lambda$ of asymptotically full Lebesgue measure, i.e.

$$|\mathcal{C}_\infty|/\varepsilon_0 \to 1 \quad \text{as} \quad \varepsilon_0 \to 0\,,$$

such that, $\forall(\varepsilon,\lambda) \in \mathcal{C}_\infty$, $u(\varepsilon,\lambda)$ is a solution of (2.4) with $\omega = \lambda\bar\omega$.

Regularity: *Moreover, if V, f, g are of class C^∞ then $u(\varepsilon,\lambda) \in C^\infty(\mathbb{T}^d\times\mathbb{T}^\nu, \mathbb{C})$.*

Remark 2.1. *Theorem 2.1 remains true for nonlinearities $f(\omega t, x, u, \bar u)$ which are Hamiltonian, but not gauge invariant as in* (2.1).

An analogous result holds for the NLW equation

$$u_{tt} - \Delta u + V(x)u = \varepsilon f(\omega t, x, u)\,, \quad x \in \mathbb{T}^d\,,$$

assuming that $\bar\omega \in \mathbb{R}^\nu$ also satisfies

$$\Big|\sum_{1\le i\le j\le\nu} \bar\omega_i\bar\omega_j m_{ij}\Big| \ge \frac{\gamma_0}{1+|m|^{\tau_0}}$$

for all the integers m_{ij} which are not all naught.

Remark 2.2. *It is also true that $\forall \varepsilon \in (0, \varepsilon_0)$ fixed, there is a Cantor set $\mathcal{C}_\infty(\varepsilon) \subset [1/2, 3/2]$ of λ's, with asymptotically full Lebesgue measure as $\varepsilon \to 0$, such that the conclusion of Theorem 2.1 holds.*

It is clear that the existence of solutions for just a Cantor like set of parameters is *not* a technical restriction! In a complementary region, chaotic motions and Arnold diffusion phenomena shall occur. In some sense Theorem 2.1 is complementary to the results in [16].

Remark 2.3. *The positivity condition* (2.6) *is used for the measure estimates, see section 3. Note that, for autonomous and gauge invariant NLS, it is verified after a gauge-transformation $u \mapsto e^{-\mathrm{i}\sigma t}u$ for σ large enough.*

The novelties of Theorem 2.1 are that we prove the existence of quasi-periodic solutions with:

1. finitely *differentiable* nonlinearities, see (2.2),
2. a *multiplicative* (finitely differentiable) potential $V(x)$, see (2.6),
3. a *pre-assigned* (diophantine) direction of the tangential frequencies, see (2.3).

3. Ideas of the proof

Theorem 2.1 is proved by Nash-Moser iteration and a multiscale analysis of the linearized operators.

Vector NLS. We prove Theorem 2.1 finding solutions of the "vector" NLS equation

$$\begin{cases} \mathrm{i}\omega \cdot \partial_\varphi u^+ - \Delta u^+ + V(x)u^+ = \varepsilon f(\varphi, x, u^- u^+)u^+ + \varepsilon g(\varphi, x) \\ -\mathrm{i}\omega \cdot \partial_\varphi u^- - \Delta u^- + V(x)u^- = \varepsilon f(\varphi, x, u^- u^+)u^- + \varepsilon \bar{g}(\varphi, x) \end{cases} \tag{3.1}$$

where

$$\mathbf{u} := (u^+, u^-) \in \mathbf{H}^s := H^s \times H^s$$

(the second equation is obtained by formal complex conjugation of the first one). In the system (3.1) the variables u^+, u^- are independent. However (3.1) reduces to the scalar NLS equation (2.1) in the set

$$\mathcal{U} := \left\{ \mathbf{u} := (u^+, u^-) \ : \ \overline{u^+} = u^- \right\}$$

in which u^- is the complex conjugate of u^+ (and viceversa). In (3.1) we choose, for example, the following smooth extension of $f(\varphi, x, \cdot)$ to $\mathbb{C}$, $f(\varphi, x, z) := (1 - \mathrm{i})f(\varphi, x, \mathrm{Re}(z)) + \mathrm{i}f(\varphi, x, \mathrm{Re}(z) + \mathrm{Im}(z))$, $z \in \mathbb{C}$.

Linearized equations. The main step of the Nash-Moser scheme concerns the invertibility of (any finite dimensional restriction of) the family of linearized operators at any $\mathbf{u} \in \mathbf{H}^s \cap \mathcal{U}$, namely

$$\mathcal{L}(\mathbf{u}) := L_\omega - \varepsilon T_1 \tag{3.2}$$

where

$$L_\omega := \begin{pmatrix} \mathrm{i}\omega \cdot \partial_\varphi - \Delta + V(x) & 0 \\ 0 & -\mathrm{i}\omega \cdot \partial_\varphi - \Delta + V(x) \end{pmatrix}, \ T_1 := \begin{pmatrix} p(\varphi, x) & q(\varphi, x) \\ \bar{q}(\varphi, x) & p(\varphi, x) \end{pmatrix},$$

and

$$p(\varphi,x):=f(\varphi,x,|u^+|^2)+f'(\varphi,x,|u^+|^2)|u^+|^2\,,\ q(\varphi,x):=f'(\varphi,x,|u^+|^2)(u^+)^2\,,\ (3.3)$$

with f' denoting the derivative with respect to $s\mapsto f(\varphi,x,s)$. The functions p,q depend also on ε,λ through $\mathbf{u}$. Note that $p(\varphi,x)$ is real valued and so the operator $\mathcal{L}(\mathbf{u})$ is *symmetric* in $\mathbf{H}^0$, i.e. $(\mathcal{L}(\mathbf{u})h,k)_0=(h,\mathcal{L}(\mathbf{u})k)_0$ for all h,k in the domain of $\mathcal{L}(\mathbf{u})$. As a consequence, the eigenvalues of all its finite dimensional restrictions vary smoothly with respect to one dimensional parameter.

The operator $\mathcal{L}(\mathbf{u})$ in (3.2) can also be written as

$$\mathcal{L}(\mathbf{u})=D_\omega+T\,,\qquad T:=T_2-\varepsilon T_1\,,$$

where D_ω is the constant coefficient operator

$$D_\omega:=\begin{pmatrix}\mathrm{i}\omega\cdot\partial_\varphi-\Delta+m & 0\\ 0 & -\mathrm{i}\omega\cdot\partial_\varphi-\Delta+m\end{pmatrix},\qquad T_2:=\begin{pmatrix}V_0(x) & 0\\ 0 & V_0(x)\end{pmatrix},$$

m is the average of $V(x)$ and $V_0(x):=V(x)-m$ has zero mean value.

In the exponential basis $\mathcal{L}(\mathbf{u})$ is represented by the infinite dimensional self-adjoint matrix

$$A(\varepsilon,\lambda):=D_\omega+T \tag{3.4}$$

of 2×2 complex matrices, where

$$D_\omega:=\mathrm{diag}_{i\in\mathbb{Z}^b}\begin{pmatrix}-\omega\cdot l+\|j\|^2+m & 0\\ 0 & \omega\cdot l+\|j\|^2+m\end{pmatrix},\ i:=(l,j)\in\mathbb{Z}^b:=\mathbb{Z}^\nu\times\mathbb{Z}^d\,, \tag{3.5}$$

with $\|j\|^2:=j_1^2+\ldots+j_d^2$, and $T:=(T_i^{i'})_{i,i'\in\mathbb{Z}^b}$, $T_i^{i'}:=-\varepsilon(T_1)_i^{i'}+(T_2)_i^{i'}$,

$$(T_1)_i^{i'}=\begin{pmatrix}p_{i-i'} & q_{i-i'}\\ (\overline{q})_{i-i'} & p_{i-i'}\end{pmatrix},\quad (T_2)_i^{i'}=\begin{pmatrix}(V_0)_{j-j'} & 0\\ 0 & (V_0)_{j-j'}\end{pmatrix}, \tag{3.6}$$

where $p_i,q_i,(V_0)_j$ denote the Fourier coefficients of $p(\varphi,x),q(\varphi,x),V_0(x)$.

Note that the matrix T is *Töplitz*, namely $T_i^{i'}$ depends only on the difference of the indices $i-i'$. Moreover, since the functions p,q in (3.3), as well as the potential V, are in H^s, then $T_i^{i'}\to0$ as $|i-i'|\to\infty$ at a polynomial rate.

We introduce the one-parameter family of infinite dimensional matrices

$$A(\varepsilon,\lambda,\theta):=A(\varepsilon,\lambda)+\theta Y:=D_\omega+T+\theta\,Y\qquad\text{where}\qquad Y:=\mathrm{diag}_{i\in\mathbb{Z}^b}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

The reason for adding θY is the crucial covariance property (3.7) below.

The core of the proof of Theorem 2.1 is a polynomial off-diagonal decay for the inverse of the $(2N+1)^b$-dimensional sub-matrices of $A(\varepsilon,\lambda,\theta)$ centered at (l_0,j_0) denoted by

$$A_{N,l_0,j_0}(\varepsilon,\lambda,\theta):=A_{|l-l_0|\le N,|j-j_0|\le N}(\varepsilon,\lambda,\theta)\,.$$

If $l_0=0$ we use the simpler notation

$$A_{N,j_0}(\varepsilon,\lambda,\theta):=A_{N,0,j_0}(\varepsilon,\lambda,\theta)\,.$$

If also $j_0=0$, we write $A_N(\varepsilon,\lambda,\theta):=A_{N,0}(\varepsilon,\lambda,\theta)$, and, for $\theta=0$, we denote $A_{N,j_0}(\varepsilon,\lambda):=A_{N,j_0}(\varepsilon,\lambda,0)$.

Since the matrix T is Töplitz, the following *covariance* property holds:

$$A_{N,l_0,j_0}(\varepsilon,\lambda,\theta)=A_{N,j_0}(\varepsilon,\lambda,\theta+\lambda\bar\omega\cdot l_0)\,. \tag{3.7}$$

Matrices with off-diagonal decay. In the space of matrices

$$\mathcal{M}_C^B := \Big\{ M = (M_k^{k'})_{k' \in B, k \in C} \,,\ M_k^{k'} \in \mathbb{C} \Big\},$$

where B, C are finite subsets of $\mathbb{Z}^b \times \{0,1\}$ (the indices $0,1$ are introduced to distinguish the $\pm$ sign in matrices like (3.5)), we consider the s-norm

$$|M|_s^2 := K_0 \sum_{n \in \mathbb{Z}^b} [M(n)]^2 \langle n \rangle^{2s} \quad \text{where} \quad \langle n \rangle := \max(1, |n|)\,,$$

$$[M(n)] := \begin{cases} \max\limits_{i - i' = n, i \in \overline{C}, i' \in \overline{B}} |M_i^{i'}| & \text{if} \ \ n \in \overline{C} - \overline{B} \\ 0 & \text{if} \ \ n \notin \overline{C} - \overline{B} \end{cases}$$

with $\overline{B} := \mathrm{proj}_{\mathbb{Z}^b} B$, $\overline{C} := \mathrm{proj}_{\mathbb{Z}^b} C$, and $K_0 > 0$ is introduced in (2.5).

The s-norm is designed to estimate the off-diagonal decay of matrices like T in (3.6): if $p, q, V \in H^s$ then

$$|T_1|_s \leq K \|(q,p)\|_s\,, \quad |T_2|_s \leq K \|V\|_s\,.$$

The set of (square) matrices with finite s-norm form an algebra. Hence products and powers of matrices with finite s-norm will exhibit the same off-diagonal decay. We refer to section 3 of [6] for more details.

Improved Nash-Moser iteration. We construct inductively better and better approximate solutions

$$\mathbf{u}_n \in H_n := \Big\{ \mathbf{u} = (u^+, u^-) \in \mathbf{H}^s \,:\, \mathbf{u} = \sum_{|(l,j)| \leq N_n} \mathbf{u}_{l,j}\, e^{\mathrm{i}(l \cdot \varphi + j \cdot x)} \,,\ \mathbf{u}_{l,j} \in \mathbb{C}^2 \Big\}$$

of the NLS equation (3.1), solving, by a Nash-Moser iterative scheme, the "truncated" equations

$$(P_n) \qquad P_n \Big(L_\omega \mathbf{u} - \varepsilon (f(\mathbf{u}) + g) \Big) = 0\,, \quad \mathbf{u} \in H_n\,,$$

where $P_n : \mathbf{H}^s \to H_n$ denote the orthogonal projectors onto H_n and $N_n := N_0^{2^n}$, see Theorem 7.1 in [6].

The main step is to prove that the finite dimensional matrices $\mathcal{L}_n := \mathcal{L}_n(\mathbf{u}_{n-1}) := P_n \mathcal{L}(\mathbf{u}_{n-1})_{|H_n}$ are invertible for "most" parameters $(\varepsilon, \lambda) \in [0, \varepsilon_0] \times \Lambda$ and satisfy

$$|\mathcal{L}_n^{-1}|_s = O(N_n^{\tau' + \delta s})\,, \quad \delta \in (0,1)\,,\ \tau' > 0\,,\ \forall s > 0\,. \tag{3.8}$$

The bound (3.8) implies the interpolation estimates

$$\|\mathcal{L}_n^{-1} h\|_s \leq C(s)(N_n^{\tau' + \delta s} \|h\|_{s_0} + N_n^{\tau' + \delta s_0} \|h\|_s)\,, \quad \forall s \geq s_0\,,$$

which are sufficient for the Nash-Moser convergence, see section 7 in [6]. Note that the exponent $\tau' + \delta s$ in (3.8) grows with s, unlike the usual Nash-Moser theory where the "tame" exponents are s-independent. Actually the conditions (3.8) are *optimal* for the convergence, as a famous counter-example of Lojasiewicz-Zehnder [24] shows: if $\delta = 1$ the Nash-Moser iterative scheme does not converge.

L^2-bounds. The first step is to show that, for "most" parameters $\lambda \in \Lambda$, the eigenvalues of $\mathcal{L}_n := P_n \mathcal{L}(\mathbf{u}_{n-1})_{|H_n}$ are in modulus bounded from below by $O(N_n^{-\tau})$ and so

$$\|\mathcal{L}_n^{-1}\|_0 = O(N_n^\tau)\,. \tag{3.9}$$

The proof is based on an eigenvalue variation argument using that $-\Delta + V(x) \geq \beta_0 I > 0$ is positive definite, see (2.6). Dividing $\mathcal{L}_n$ by λ, and setting $\xi := 1/\lambda$, we observe that the derivative with respect to ξ satisfies

$$\partial_\xi(\xi\mathcal{L}_n) = P_n \begin{pmatrix} -\Delta + V(x) & 0 \\ 0 & -\Delta + V(x) \end{pmatrix}_{|H_n} + O(\varepsilon\|T_1\|_0 + \varepsilon\|\partial_\lambda T_1\|_0) \overset{(2.6)}{\geq} \frac{\beta_0}{2},$$

for ε small, i.e. it is positive definite. So, the eigenvalues $\mu_{l,j}(\xi,\varepsilon)$ (which depend C^1-smoothly on ξ for fixed ε) of the self-adjoint matrix $\xi\mathcal{L}_n$ satisfy

$$\partial_\xi \mu_{l,j}(\xi,\varepsilon) \geq \frac{\beta_0}{2}, \quad \forall |(l,j)| \leq N_n,$$

which easily implies (3.9) except in a set of λ's of measure $O(N_n^{-\tau+d+\nu})$, see Lemma 6.7 in [6]

Remark 3.1. *The L^2-estimate* (3.9) *alone implies only that*

$$\|\mathcal{L}_n^{-1}\|_s \leq N_n^{s+d+\nu}\|\mathcal{L}_n^{-1}\|_0 = O(N_n^{s+d+\nu+\tau}), \ \forall s > 0,$$

which has the form (3.8) *with* $\delta = 1$.

In order to prove the sublinear decay (3.8) for the Green functions we have to exploit (mild) "separation properties" of the small divisors: not all the eigenvalues of $\mathcal{L}_n$ are $O(N_n^{-\tau})$ small. We have to worry only about the SINGULAR sites (l,j) such that

$$|\pm\omega\cdot l + |j|^2 + m| \leq \Theta, \tag{3.10}$$

where $\Theta \geq 1$ is a fixed constant, depending, in particular, on V.

Remark 3.2. *For periodic solutions* ($\nu = 1$) *the singular sites are "separated at infinity" (see* [17]*), namely the distance between distinct singular sites increases when the Fourier indices tend to infinity. This property never holds for quasi-periodic solutions* ($\nu \geq 1$)*, neither for finite dimensional systems.*

Multiscale Step. The bounds (3.8) follow by an inductive application of a "multiscale argument".

A matrix $A \in \mathcal{M}_E^E$, $E \subset \mathbb{Z}^b \times \{0,1\}$, with $\mathrm{diam}(E) \leq N$ is called N**-good** if

$$\|A^{-1}\|_s \leq N^{\tau'+\delta s}, \quad \forall s \in [s_0, s_1],$$

for some $s_1 := s_1(d,\nu)$ large. Otherwise we say that A is N-bad.

The aim of the multiscale step is to deduce that a matrix $A \in \mathcal{M}_E^E$ with

$$\mathrm{diam}(E) \leq N' = N^\chi \quad \text{with} \quad \chi \gg 1,$$

is N'-good, knowing

- **(H1) (Off-diagonal decay)** $|A - \mathrm{Diag}(A)\|_{s_1} \leq \Upsilon$ where $\mathrm{Diag}(A) := (\delta_{kk'}A_k^{k'})_{k,k'\in E}$.

Condition (H1) means that A is "polynomially localized" close to the diagonal. For the matrix A in (3.4) the constant $\Upsilon = O(\|V\|_{s_1} + \varepsilon\|(p,q)\|_{s_1})$ and Θ, defined in (3.10), must be $\Theta \gg \Upsilon$.

- **(H2) (L^2-bound)** $\|A^{-1}\|_0 \leq (N')^\tau$.

Condition (H2) is usually verified with an exponent $\tau \geq d+\nu$ large, imposing lower bounds on the modulus of the eigenvalues of A.

In order to prove an off-diagonal decay for A^{-1}, we need assumptions concerning the N-dimensional submatrices centered along the diagonal of A. We define an index $k \in E$ to be

1. REGULAR for A if $|A_k^k| \geq \Theta$. Otherwise, k is SINGULAR.
2. (A,N)-REGULAR if there is $F \subset E$ such that $\mathrm{diam}(F) \leq 4N$, $\mathrm{d}(k, E\backslash F) \geq N$ and A_F^F is N-good.
3. (A,N)-GOOD if it is regular for A or (A,N)-regular. Otherwise we say that k is (A,N)-BAD.

We suppose that

- **(H3) (Separation properties)** There is a partition of the (A,N)-bad sites $B = \cup_\alpha \Omega_\alpha$ with
$$\mathrm{diam}(\Omega_\alpha) \leq N^{C_1}\,, \quad \mathrm{d}(\Omega_\alpha, \Omega_\beta) \geq N^2\ ,\ \forall \alpha \neq \beta\,, \tag{3.11}$$
for some $C_1 := C_1(d,\nu) \geq 2$.

The goal of the multiscale proposition is to deduce that A is N'-good, from (H1)-(H2)-(H3), under suitable relations between the constants χ, C_1, δ, s_1, see Proposition 4.1 in [6] for a precise statement. The proof is based on "resolvent identity" arguments, showing that A can be "quasi block-diagonalized" on subsbaces which, in Fourier space, are supported on the bad-clusters Ω_α. The main conditions on the exponents are $C_1 < \delta\chi$ and $2s_1 \gg \chi\tau$. The first means that the size N^{C_1} of any bad clusters Ω_α is small with respect to the size $N' := N^\chi$ of the matrix A. The second means that s_1 is large enough to have a sufficiently fast off diagonal decay outside the resonant clusters Ω_α.

Separation properties of small divisors. We apply the previous multiscale step to the matrix $A_{N_{n+1}}(\varepsilon,\lambda)$. The key property to verify is (H3). It is sufficient to prove the separation properties (3.11) for the N_n-BAD sites of $A(\varepsilon,\lambda)$, namely the indices (l_0,j_0) which are singular and for which there exists a site (l,j), with $|(l,j)-(l_0,j_0)| \leq N$, such that $A_{N_n,l,j}(\varepsilon,\lambda)$ is N_n-bad.

Such separation properties are obtained for all the parameters (ε,λ) which are N_n-good, namely such that

$$\forall\, j_0 \in \mathbb{Z}^d\,, \quad B_{N_n}(j_0;\varepsilon,\lambda) \ := \ \Big\{\theta \in \mathbb{R}\,:\, A_{N_n,j_0}(\varepsilon,\lambda,\theta) \text{ is } N_n - \text{bad}\Big\} \subset \bigcup_{q=1,\dots,N_n^{2d+\nu+4}} I_q$$
$$\text{where } I_q \text{ are disjoint intervals with } |I_q| \leq N_n^{-\tau}\,. \tag{3.12}$$

We first use the covariance property (3.7) and the "complexity" information (3.12) to bound the number of "bad" time-Fourier components. Indeed

$$A_{N_n,l_0,j_0}(\varepsilon,\lambda) \text{ is } N_n\text{-bad} \ \Leftrightarrow\ A_{N_n,j_0}(\varepsilon,\lambda,\omega\cdot l_0) \text{ is } N_n\text{-bad} \ \Leftrightarrow\ \omega\cdot l_0 \in B_{N_n}(j_0;\varepsilon,\lambda)\,.$$

Then, using that ω is Diophantine, the complexity bound (3.12) implies that, for each fixed j_0, there are at most $CN_n^{3d+2\nu+4}$ sites (l_0,j_0), $|l_0| \leq N_{n+1}$, which are N_n-bad, see Corollary 5.1 in [6].

Next, we prove that a N_n^2-"chain" of singular sites, i.e. a sequence of integers

$$k_1, k_2, \dots, k_L \;\text{ satisfying }\; (3.10) \;\text{ with }\; |k_{i+1} - k_i| \le N_n^2 \,,$$

which are also N_n-bad, has a "length" L bounded by $L \le N_n^{C(d,\nu)}$, see Lemma 5.2 in [6]. The proof uses ideas similar to [14]. This implies a partition of the $(A_{N_{n+1}}(\varepsilon,\lambda), N_n)$-bad sites as in (3.11) at order N_n, see Proposition 5.1 in [6]

Measure and "complexity" estimates. In order to conclude the inductive proof we have to verify that "most" parameters (ε,λ) are N_n-good. We prove first that, except a set of measure $O(\varepsilon_0 N_n^{-1})$, all parameters $(\varepsilon,\lambda) \in [0,\varepsilon_0]\times\Lambda$ are N_n-good in a weak sense, namely

$$\forall\, j_0 \in \mathbb{Z}^d\,, \quad B^0_{N_n}(j_0;\varepsilon,\lambda) \;:=\; \left\{\theta\in\mathbb{R} \,:\, \|A^{-1}_{N_n,j_0}(\varepsilon,\lambda,\theta)\|_0 > N_n^\tau\right\} \subset \bigcup_{q=1,\dots,N_n^{2d+\nu+4}} I_q$$

$$\text{where } I_q \text{ are disjoint intervals with } |I_q| \le N_n^{-\tau}\,.$$

The proof is again based on simple eigenvalue variation arguments, using that $-\Delta + V(x)$ is positive definite, see section 6 in [6].

Finally, the multiscale Proposition step, and the fact that the separation properties of the N_n-bad sites of $A(\varepsilon,\lambda,\theta)$ hold uniformly in $\theta\in\mathbb{R}$, imply inductively that most of the parameters (ε,λ) are actually N_n-good (in the strong sense), concluding the inductive argument, see Lemma 7.6 in [6].

References

[1] Bambusi D., Berti M., Magistrelli E., *Degenerate KAM theory for partial differential equations*, J. Differential Equations 250, 3379-3397, 2011.

[2] Bambusi D., Delort J.M., Grebért B., Szeftel J., *Almost global existence for Hamiltonian semilinear Klein-Gordon equations with small Cauchy data on Zoll manifolds*, Comm. Pure Appl. Math. 60, 11, 1665-1690, 2007.

[3] Berti M., *Nonlinear Oscillations of Hamiltonian PDEs*, Progr. Nonlinear Differential Equations Appl. 74, H. Brézis, ed., Birkhäuser, Boston, 1-181, 2008.

[4] Berti M., Biasco L., *Branching of Cantor manifolds of elliptic tori and applications to PDEs*, Comm. Math. Phys, 305, 3, 741-796, 2011.

[5] Berti M., Bolle P., *Sobolev Periodic solutions of nonlinear wave equations in higher spatial dimension*, Archive for Rational Mechanics and Analysis, 195, 609-642, 2010.

[6] Berti M., Bolle P., *Quasi-periodic solutions with Sobolev regularity of NLS on $\mathbb{T}^d$ with a multiplicative potential*, to appear on the Journal European Math. Society.

[7] Berti M., Bolle P., *Quasi-periodic solutions of nonlinear Schrödinger equations on $\mathbb{T}^d$*, Rend. Lincei Mat. Appl. 22, 223-236, 2011.

[8] Berti M., Bolle P., Procesi M., *An abstract Nash-Moser theorem with parameters and applications to PDEs*, Ann. I. H. Poincaré, 27, 377-399, 2010.

[9] Berti M., Procesi M., *Nonlinear wave and Schrödinger equations on compact Lie groups and homogeneous spaces*, Duke Math. J., 159, 3, 479-538, 2011.

[10] Bourgain J., *Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE*, Internat. Math. Res. Notices, no. 11, 1994.

[11] Bourgain J., *Construction of periodic solutions of nonlinear wave equations in higher dimension*, Geom. Funct. Anal. 5, no. 4, 629-639, 1995.

[12] Bourgain J., *On Melnikov's persistency problem*, Internat. Math. Res. Letters, 4, 445 - 458, 1997.

[13] Bourgain J., *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations*, Annals of Math. 148, 363-439, 1998.

[14] Bourgain J., *Green's function estimates for lattice Schrödinger operators and applications*, Annals of Mathematics Studies 158, Princeton University Press, Princeton, 2005.

[15] Burq N., Gérard P., Tzvetkov N., *Bilinear eigenfunction estimates and the nonlinear Schrödinger equation on surfaces*, Invent. Math., 159, 187-223, 2005.

[16] Colliander J., Keel M., Staffilani G., Takaoka H., Tao T., *Weakly turbolent solutions for the cubic defocusing nonlinear Schrödinger equation*, 181, 1, 39-113, Inventiones Math., 2010.

[17] Craig W., Wayne C. E., *Newton's method and periodic solutions of nonlinear wave equation*, Comm. Pure Appl. Math. 46, 1409-1498, 1993.

[18] Delort J.M., *Periodic solutions of nonlinear Schrödinger equations: a paradifferential approach*, to appear in Analysis and PDEs.

[19] Eliasson L.H., *Perturbations of stable invariant tori for Hamiltonian systems*, Ann. Sc. Norm. Sup. Pisa., 15, 115-147, 1988.

[20] Eliasson L. H., Kuksin S., *On reducibility of Schrödinger equations with quasiperiodic in time potentials*, Comm. Math. Phys, 286, 125-135, 2009.

[21] Eliasson L. H., Kuksin S., *KAM for nonlinear Schrödinger equation*, Annals of Math., 172, 371-435, 2010.

[22] Kuksin S., *Hamiltonian perturbations of infinite-dimensional linear systems with imaginary spectrum*, Funktsional Anal. i Prilozhen. 2, 22-37, 95, 1987.

[23] Kuksin S., *Analysis of Hamiltonian PDEs*, Oxford Lecture series in Math. and its applications, 19, Oxford University Press, 2000.

[24] Lojasiewicz S., Zehnder E., *An inverse function theorem in Fréchet-spaces*, J. Funct. Anal. 33, 165-174, 1979.

[25] Procesi C., Procesi M., *A normal form for the Schrödinger equation with analytic non-linearities*, to appear on Comm. Math. Phys.

[26] Wang W. M., *Supercritical nonlinear Schrödinger equations I: quasi-periodic solutions*, preprint 2010.

[27] Wayne E., *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Comm. Math. Phys. 127, 479-528, 1990.

Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli Studi Napoli Federico II, Via Cintia, Monte S. Angelo, I-80126, Napoli, Italy
m.berti@unina.it