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Abstract

In this text, we present two recent results on the characterization of the lack of compactness of some critical Sobolev embedding. The first one derived in [5] deals with an abstract framework including Sobolev, Besov, Triebel-Lizorkin, Lorentz, Hölder and BMO spaces. The second one established in [3] concerns the lack of compactness of $H^1(\mathbb{R}^2)$ into the Orlicz space. Although the two results are expressed in the same manner (by means of defect measures) and rely on the defect of compactness due to concentration as in [17] and [18], they are actually of different nature. In fact, both in [5] and [3] it is proved that the lack of compactness can be described in terms of an asymptotic decomposition, but the elements involved in the decomposition are of completely different kinds in the two frameworks. We also highlight that contrary to semilinear cases like the wave equation studied in [2] and [9], the linearizability of the non linear wave equation with exponential growth is not directly related to the lack of compactness of $H^1(\mathbb{R}^2)$ into the Orlicz space.

1. Introduction

After the pioneering works of P. -L. Lions [17] and [18], the lack of compactness in critical Sobolev embedding was investigated for different types of examples through several angles. For instance, in [9] the lack of of compactness in the critical Sobolev embedding

$$\dot{H}^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$$

in the case where $d \geq 3$ with $0 \leq s < d/2$ and $p = 2d/(d-2s)$ is described in terms of microlocal defect measures and in [10], it is characterized by means of profiles. More generally for Sobolev spaces in the L^q frame, this question is treated in [12] by the use of nonlinear wavelet approximation theory.

Keywords: Critical Sobolev embedding, lack of compactness, BMO space, Orlicz space.

Other studies have been conducted in various work ([6], [7], [22], [25], [23],...) supplying us with a large amount of informations about solutions of nonlinear partial differential equations, both in the elliptic frame or the evolution frame. (Among other applications, one can mention [2], [15], [13], [16], [26],...).

Recently in [5], the wavelet-based profile decomposition introduced by S. Jaffard in [12] is revisited in order to treat a larger range of examples of critical embedding of functions spaces

$$X \hookrightarrow Y$$

including Sobolev, Besov, Triebel-Lizorkin, Lorentz, Hölder and BMO spaces. (One can consult [4] and the references therein for an introduction to these spaces).

For that purpose, two generic properties on the spaces X and Y was identified to build the profile decomposition in a unified way. These properties concern wavelet decompositions in the spaces X and Y supposed having the same scaling and endowed by an unconditional wavelet basis $(\psi_\lambda)_{\lambda\in\Lambda}$.

The first assumption is related to the existence of a *nonlinear projector* Q_M satisfying

$$\lim_{M\to+\infty} \max_{\|f\|_X\leq 1} \|f - Q_M f\|_Y = 0. \tag{1.1}$$

More precisely, if $(\psi_\lambda)_{\lambda\in\nabla}$ is a normalized wavelet basis in the space X (so in Y in view of the invariance by the same scaling) and

$$f = \sum_{\lambda\in\nabla} d_\lambda \psi_\lambda, \tag{1.2}$$

is the wavelet decomposition of the function f, then $Q_M f$ sometimes called *best M-term approximation* takes the general form

$$Q_M f := \sum_{\lambda\in E_M} d_\lambda \psi_\lambda, \tag{1.3}$$

where the sets $E_M = E_M(f)$ of cardinality M depend on f and satisfy

$$E_M(f) \subset E_{M+1}(f).$$

The existence of such nonlinear projector was extensively studied in nonlinear approximation theory and for many cases as Sobolev embedding of Besov spaces in Besov or Lebesgue spaces, it turns out that the set $E_M = E_M(f)$ can be chosen as the subset of ∇ that corresponds to the M largest values of $|d_\lambda|$.

In fact it is known (see [19] for instance) that in Besov spaces $\dot{B}^\sigma_{r,r}$, we have the following norm equivalence :

$$\|f\|_{\dot{B}^\sigma_{r,r}} \sim \|(d_\lambda)_{\lambda\in\nabla}\|_{\ell^r}, \tag{1.4}$$

for $f = \sum_{\lambda\in\nabla} d_\lambda\psi_\lambda$ with wavelets normalized in $\dot{B}^\sigma_{r,r}$. Therefore, in the particular case where $X = \dot{B}^s_{p,p}$ and $Y = \dot{B}^t_{q,q}$, with $\frac{1}{p} - \frac{1}{q} = \frac{s-t}{d}$, the *nonlinear projector*

Q_M defined by (1.3) where $E_M = E_M(f)$ is the subset of ∇ of cardinality M that corresponds to the M largest values of $|d_\lambda|$ is appropriate and satisfies:

$$\sup_{\|f\|_{\dot{B}^s_{p,p}}\leq 1} \|f - Q_M f\|_{\dot{B}^t_{q,q}} \leq CM^{-\frac{s-t}{d}}. \tag{1.5}$$

Indeed, taking advantage of (1.4) and using the decreasing rearrangement $(d_m)_{m>0}$ of the $|d_\lambda|$, we get

$$\begin{aligned}
\|f - Q_M f\|_{\dot{B}^t_{q,q}} &\leq C(\textstyle\sum_{\lambda\notin E_M} |d_\lambda|^q)^{\frac{1}{q}} \\
&= (\textstyle\sum_{m>M} |d_m|^q)^{\frac{1}{q}} \\
&\leq |d_M|^{1-\frac{p}{q}} (\textstyle\sum_{m>M} |d_m|^p)^{\frac{1}{q}} \\
&\leq (M^{-1} \textstyle\sum_{m=1}^M |d_m|^p)^{\frac{1}{p}-\frac{1}{q}} (\textstyle\sum_{m>M} |d_m|^p)^{\frac{1}{q}} \\
&\leq M^{-(\frac{1}{p}-\frac{1}{q})} (\textstyle\sum_{m>0} |d_m|^p)^{\frac{1}{p}} \\
&\leq M^{-\frac{s-t}{d}} \|(d_\lambda)_{\lambda\in\nabla}\|_{\ell^p} \leq CM^{-\frac{s-t}{d}} \|f\|_{\dot{B}^s_{p,p}},
\end{aligned}$$

which achieves the proof of Assertion (1.5).

The second assumption concerns the stability of wavelet expansions in the functions space X with respect to certain operations such as "shifting" the indices of wavelet coefficients, as well as disturbing the value of these coefficients. In practice and for most cases of interest, this property derives from the fact that the X norm of a function is equivalent to the norm of its wavelet coefficients in a certain sequence space by invoking Fatou's lemma. We refer for instance to [8] and [19] for more details on the construction of wavelet bases and on the characterization of classical function spaces by expansions in such bases.

Under these assumptions, we proved in [5] that as in the previous works [9] and [12] translational and scaling invariance are the sole responsible for the defect of compactness of the embedding of $X \hookrightarrow Y$. More precisely, we established that the lack of compactness in this embedding can be described in terms of an asymptotic decomposition in the following terms: a sequence $(u_n)_{n\geq 0}$ bounded in X can be decomposed up to a subsequence extraction according to

$$u_n = \sum_{l=1}^{L} h_{l,n}^{s-d/p} \phi^l\left(\frac{\cdot - x_{l,n}}{h_{l,n}}\right) + r_{n,L} \tag{1.6}$$

where $(\phi^l)_{l>0}$ is a family of functions in X,

$$\lim_{L\to+\infty} (\limsup_{n\to+\infty} \|r_{n,L}\|_Y) = 0, \tag{1.7}$$

and where the decomposition is *asymptotically orthogonal* in the sense that for $k \neq l$

$$|\log(h_{l,n}/h_{k,n})| \to +\infty \ \text{ or } \ |x_{l,n} - x_{k,n}|/h_{l,n} \to +\infty, \ \text{ as } \ n \to +\infty.$$

The construction of the decomposition (1.6) relies on a diagonal subsequence extraction procedure and proceed in several steps. In the first step, we split the wavelet decomposition of the sequence (u_n) using the *nonlinear projector* Q_M. In the second step, by an iterative scheme based on the orthogonality property, we built approximate profiles $\phi^{l,j}$. In the third step, the exact profiles ϕ^l are constructed as the limits in X of the approximate profiles $\phi^{l,j}$ as $j \to +\infty$, making use of the second assumption and finally in the last step, (1.7) is established.

In [3], we looked into the lack of compactness of the critical Sobolev embedding

$$H^1_{rad}(\mathbb{R}^2) \hookrightarrow \mathcal{L}, \tag{1.8}$$

where $\mathcal{L}$ denotes the Orlicz space associated to the function $\phi(s) = \mathrm{e}^{s^2} - 1$ and gave a characterization by means of an asymptotic decomposition. It was found that the profiles involved in the decomposition (1.6) are not the right concept to describe the lack of compactness in this embedding. In fact in [3], we characterized the lack of compactness of the critical Sobolev embedding (1.8) by means of an *asymptotically orthogonal* decomposition in terms of elementary concentrations under the form:

$$g_n(x) := \sqrt{\frac{\alpha_n}{2\pi}}\ \psi\Big(\frac{-\log|x|}{\alpha_n}\Big),$$

where $\underline{\alpha} := (\alpha_n)$ called the scale is a sequence of positive real numbers going to infinity and ψ called the profile belongs to the set

$$\mathcal{P} := \Big\{\ \psi \in L^2(\mathbb{R}, \mathrm{e}^{-2s}ds); \quad \psi' \in L^2(\mathbb{R}),\ \psi_{|]-\infty,0]} = 0\ \Big\}.$$

These elementary concentrations satisfying (see [3] for more details)

$$\lim_{n\to\infty} \|g_n\|_{\mathcal{L}} = \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\psi(s)|}{\sqrt{s}}$$

are the same kind as the Lions' example

$$f_{\alpha_n}(x) \quad = \quad \begin{cases} 0 & \text{if} \quad |x| \geq 1, \\ -\frac{\log|x|}{\sqrt{2\alpha_n\pi}} & \text{if} \quad \mathrm{e}^{-\alpha_n} \leq |x| \leq 1, \\ \sqrt{\frac{\alpha_n}{2\pi}} & \text{if} \quad |x| \leq \mathrm{e}^{-\alpha_n} \end{cases}$$

that we can write as

$$f_{\alpha_n}(x) = \sqrt{\frac{\alpha_n}{2\pi}}\ \mathbf{L}\Big(\frac{-\log|x|}{\alpha_n}\Big)$$

where

$$\mathbf{L}(s) \quad = \quad \begin{cases} 0 & \text{if} \quad s \leq 0, \\ s & \text{if} \quad 0 \leq s \leq 1, \\ 1 & \text{if} \quad s \geq 1. \end{cases}$$

and which satisfies $\|f_{\alpha_n}\|_{\mathcal{L}} \to \frac{1}{\sqrt{4\pi}} \max_{s>0} \frac{|\mathbf{L}(s)|}{\sqrt{s}} = \frac{1}{\sqrt{4\pi}} \quad \text{as} \quad \alpha_n \to \infty\,.$

2. Critical $2D$ Sobolev embedding

2.1. Sobolev embedding in $\mathrm{BMO} \cap L^2$ and in Orlicz space

It is well known that $H^1(\mathbb{R}^2)$ is continuously embedded in $\mathrm{BMO}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, where $BMO(\mathbb{R}^d)$ denotes the space of bounded mean oscillations which is the space of locally integrable functions f such that

$$\|f\|_{BMO} \stackrel{\text{def}}{=} \sup_{B} \frac{1}{|B|} \int_B |f - f_B|\, dx < \infty \quad \text{with} \quad f_B \stackrel{\text{def}}{=} \frac{1}{|B|} \int_B f\, dx.$$

The above supremum being taken over the set of Euclidean balls B, $|\cdot|$ denoting the Lebesgue measure.

It is also known that

$$\|u\|_{\mathcal{L}} \leq \frac{1}{\sqrt{4\pi}} \|u\|_{H^1}, \tag{2.1}$$

where $\mathcal{L}$ denotes the Orlicz space associated to the function $\phi(s) = \mathrm{e}^{s^2} - 1$. This embedding derives immediately from the following Trudinger-Moser type inequalities (see [1, 20, 27]):

Proposition 2.1.

$$\sup_{\|u\|_{H^1} \leq 1} \int_{\mathbb{R}^2} \left(\mathrm{e}^{4\pi|u(x)|^2} - 1 \right) dx := \kappa < \infty. \tag{2.2}$$

Let us recall that generally, if $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a convex increasing function such that

$$\phi(0) = 0 = \lim_{s \to 0^+} \phi(s) \,\text{and}\, \lim_{s \to \infty} \phi(s) = \infty,$$

then L^ϕ the Orlicz space on $\mathbb{R}^d$ associated to the function ϕ is defined as follows:

Definition 2.2.
We say that a measurable function $u : \mathbb{R}^d \to \mathbb{C}$ belongs to L^ϕ if there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda} \right) dx < \infty$$

and we denote

$$\|u\|_{L^\phi} = \inf \left\{ \lambda > 0, \quad \int_{\mathbb{R}^d} \phi\left(\frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}. \tag{2.3}$$

Remark 2.3. *Let us observe that we may replace in* (2.3) *the number* 1 *by any positive constant. This changes the norm $\|\cdot\|_{L^\phi}$ by an equivalent norm. In what follows we shall endow the Orlicz space $\mathcal{L}$ with the norm $\|\cdot\|_{\mathcal{L}}$ where the number* 1 *is replaced by the constant κ involving in Identity* (2.2).

2.2. Comparaison between $\mathrm{BMO} \cap L^2$ and $\mathcal{L}$

It turns out that there is no comparison between $\mathcal{L}$ and BMO. More precisely, we have the following result

Proposition 2.4.

$$\mathcal{L} \not\hookrightarrow \mathrm{BMO} \cap L^2 \quad \textit{and} \quad \mathrm{BMO} \cap L^2 \not\hookrightarrow \mathcal{L}.$$

Proof. Let us consider the sequence $g_{\alpha_n}(r, \theta) = f_{\alpha_n}(r)\, \mathrm{e}^{i\theta}$ where f_{α_n} is the fundamental example introduced in Section 1, and let us set $B_{\alpha_n} = B(0, \mathrm{e}^{-\frac{\alpha_n}{2}})$. It is obvious that

$$\int_{B_{\alpha_n}} g_{\alpha_n} = 0\,.$$

In other respects, by elementary computations we get

$$\frac{1}{|B_{\alpha_n}|}\int_{B_{\alpha_n}} |g_{\alpha_n}| = \frac{\sqrt{\alpha_n}}{2\sqrt{2\pi}} + \frac{1-\mathrm{e}^{-\alpha_n}}{2\sqrt{2\pi\alpha_n}}\,.$$

Hence $\|g_{\alpha_n}\|_{\mathrm{BMO}} \to \infty \quad \text{as} \quad \alpha_n \to \infty$. Since $\|g_{\alpha_n}\|_{\mathcal{L}} = \|f_{\alpha_n}\|_{\mathcal{L}} \to \frac{1}{\sqrt{4\pi}}$, we deduce that

$$\mathcal{L} \not\hookrightarrow \mathrm{BMO} \cap L^2\,.$$

To show that $\mathrm{BMO} \cap L^2$ is not embedded in $\mathcal{L}$, we shall use the following sharp inequality established in [14]

$$\|u\|_{L^q} \le C\,q\|u\|_{\mathrm{BMO}\cap \mathrm{L}^2}, \quad q \ge 2, \tag{2.4}$$

together with the fact that (for $u \neq 0$),

$$\int_{\mathbb{R}^2} \left(\mathrm{e}^{\frac{|u(x)|^2}{\|u\|^2_{\mathcal{L}}}} - 1\right) dx \le \kappa\,. \tag{2.5}$$

To go to this end, let us suppose that $\mathrm{BMO} \cap L^2$ is embedded in $\mathcal{L}$. Then, for any integer $q \ge 1$, we get

$$\|u\|_{L^{2q}} \le \kappa^{1/2q}\,(q!)^{1/2q}\,\|u\|_{\mathcal{L}} \le C\,\kappa^{1/2q}\,(q!)^{1/2q}\,\|u\|_{\mathrm{BMO}\cap\mathrm{L}^2}$$

which contradicts (2.4) since

$$(q!)^{1/2q} \sim \mathrm{e}^{-1/2}\sqrt{q},$$

where $\sim$ is used to indicate that the ratio of the two sides goes to 1 as q goes to ∞. □

3. Lack of compactness in Sobolev embedding in Orlicz space

The embedding $H^1 \hookrightarrow \mathcal{L}$ is non compact at least for two reasons. The first reason is the lack of compactness at infinity. A typical example is $u_k(x) = \varphi(x + x_k)$ where $0 \neq \varphi \in \mathcal{D}$ and $|x_k| \to \infty$. The second reason is of concentration-type derived by P.-L. Lions [17, 18] and illustrated by the fundamental example f_{α_n} defined above.

In [3], we described the lack of compactness of this embedding in terms of an asymptotic decomposition as follows:

Theorem 3.1. *Let (u_n) be a bounded sequence in $H^1_{rad}(\mathbb{R}^2)$ such that*

$$u_n \rightharpoonup 0, \tag{3.1}$$

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} = A_0 > 0, \qquad \textit{and} \tag{3.2}$$

$$\lim_{R\to\infty}\ \limsup_{n\to\infty} \int_{|x|>R} |u_n|^2\,dx = 0. \tag{3.3}$$

Then, there exists a sequence $(\underline{\alpha}^{(j)})$ of pairwise orthogonal scales and a sequence of profiles $(\psi^{(j)})$ in $\mathcal{P}$ such that, up to a subsequence extraction, we have for all $\ell \ge 1$,

$$u_n(x) = \sum_{j=1}^{\ell} \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\,\psi^{(j)}\left(\frac{-\log|x|}{\alpha_n^{(j)}}\right) + \mathrm{r}_n^{(\ell)}(x), \quad \limsup_{n\to\infty}\ \|\mathrm{r}_n^{(\ell)}\|_{\mathcal{L}} \overset{\ell\to\infty}{\longrightarrow} 0. \tag{3.4}$$

Moreover, we have the following stability estimates

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^{\ell} \|\psi^{(j)\prime}\|_{L^2}^2 + \|\nabla \mathrm{r}_n^{(\ell)}\|_{L^2}^2 + \circ(1), \quad n \to \infty. \tag{3.5}$$

The approach that we adopted to prove Theorem 3.1 uses in a crucial way the radial setting and particularly the fact we deal with bounded functions far away from the origin thanks to the well known radial estimate

$$|u(r)| \leq \frac{C}{\sqrt{r}}\, \|u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}. \tag{3.6}$$

Through a diagonal subsequence extraction, the main step consists to extract a scale (α_n) and a profile ψ such that

$$\|\psi'\|_{L^2} \geq C\, A_0,$$

where C is a universal constant. The extraction of the scale follows from the fact that for any $\epsilon > 0$

$$\sup_{s \geq 0} \left(\left| \frac{v_n(s)}{A_0 - \epsilon} \right|^2 - s \right) \to \infty, \quad n \to \infty, \tag{3.7}$$

with $v_n(s) = u_n(\mathrm{e}^{-s})$. Property (3.7) is proved by contradiction assuming that

$$\sup_{s \geq 0, n \in \mathbb{N}} \left(\left| \frac{v_n(s)}{A_0 - \epsilon} \right|^2 - s \right) \leq C < \infty,$$

which ensures by virtue of Lebesgue theorem that

$$\int_{|x|<1} \left(\mathrm{e}^{|\frac{u_n(x)}{A_0-\epsilon}|^2} - 1 \right) dx = 2\pi \int_0^\infty \left(\mathrm{e}^{|\frac{v_n(s)}{A_0-\epsilon}|^2} - 1 \right) \mathrm{e}^{-2s}\, ds \to 0, \quad n \to \infty.$$

In other respects, taking advantage of the radial estimate (3.6), we deduce that the sequence (u_n) is bounded on the set $\{|x| \geq 1\}$ which implies that

$$\int_{|x|\geq 1} \left(\mathrm{e}^{|\frac{u_n(x)}{A_0-\epsilon}|^2} - 1 \right) dx \leq C \|u_n\|_{L^2}^2 \to 0\,.$$

In conclusion, this leads to

$$\limsup_{n\to\infty} \|u_n\|_{\mathcal{L}} \leq A_0 - \epsilon,$$

which is in contradiction with Hypothesis (3.2). Fixing $\epsilon = A_0/2$, a scale (α_n) can be extracted such that

$$\frac{A_0}{2} \sqrt{\alpha_n} \leq |v_n(\alpha_n)| \leq C\sqrt{\alpha_n} + \circ(1).$$

Finally, setting

$$\psi_n(y) = \sqrt{\frac{2\pi}{\alpha_n^{(1)}}}\, v_n(\alpha_n^{(1)} y),$$

one can prove that ψ_n converges simply to a profile ψ. Since $|\psi_n(1)| \geq CA_0$, we obtain

$$CA_0 \leq |\psi(1)| = \left| \int_0^1 \psi'(\tau)\, d\tau \right| \leq \|\psi'\|_{L^2(\mathbb{R})},$$

which ends the proof of the main point.

Before concluding this section, let us comment the results of Theorem 3.1.

Remarks 3.2.
• *It should be emphasized that, contrary to the case of Sobolev embedding in Lebesgue spaces, where the asymptotic decomposition derived by P. Gérard in* [10] *leads to*

$$\|u_n\|_{L^p}^p \to \sum_{j\geq 1} \|\psi^{(j)}\|_{L^p}^p,$$

Theorem 3.1 induces to

$$\|u_n\|_{\mathcal{L}} \to \sup_{j\geq 1} \left(\lim_{n\to\infty} \|g_n^{(j)}\|_{\mathcal{L}} \right), \tag{3.8}$$

where $g_n^{(j)} = \sqrt{\frac{\alpha_n^{(j)}}{2\pi}}\, \psi^{(j)} \left(\frac{-\log|x|}{\alpha_n^{(j)}} \right)$*. A detailed proof of this fact is given in* [3].
• *Let us also observe that each elementary concentration* $g_n^{(j)}$ *is supported in the unit disc. This is due to the fact that in the radial case, any bounded sequence in* $H^1(\mathbb{R}^2)$ *is compact away from the origin in Orlicz space.*
• *As it is mentioned above, the elementary concentration* $g_n^{(j)}$ *are completely different from the profiles involving in the characterization of the lack of compactness in* [5]. *In fact, one can prove that for any* $0 < a < b$ *and any sequence* (h_n) *of nonnegative real numbers*

$$\int_{a<h_n|\xi|<b} |\widehat{\nabla g_n^{(j)}}(\xi)|^2\, d\xi \to 0, \quad n\to\infty.$$

Actually, the scales $\alpha_n^{(j)}$ *do not correspond to scales in point of view frequencies but to values taken by the functions* $g_n^{(j)}$ *in consistent sets of size.*

4. Qualitative study of nonlinear wave equation

The two-dimensional nonlinear Klein-Gordon equation

$$\Box u + u + f(u) = 0, \quad u : \mathbb{R}_t \times \mathbb{R}_x^2 \to \mathbb{R}, \tag{4.1}$$

where

$$f(u) = u \left(\mathrm{e}^{4\pi u^2} - 1 \right)$$

have been studied for the sake of several physical models and global well posedness is established in subcritical and critical cases (see [11] and the references therein for a survey on the subject). Here and contrary to higher dimensions where the criticality depends on the nonlinearity, the notion of criticality depends on the size of the initial energy E_0 with respect to 1. More precisely, denoting by

$$E_0 := \|u_1\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2 + \frac{1}{4\pi} \|\mathrm{e}^{4\pi u_0^2} - 1\|_{L^1},$$

we define as follows the various regimes:

Definition 4.1. *The Cauchy problem associated to Equation (4.1) with initial data* $(u_0, u_1) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ *is said to be subcritical if*

$$E_0 < 1.$$

It is said critical if $E_0 = 1$ *and supercritical if* $E_0 > 1$.

Let us emphasize that the solutions of the two-dimensional nonlinear Klein-Gordon equation formally satisfy the conservation of energy

$$\begin{aligned} E(u,t) &= \|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \frac{1}{4\pi}\,\|\mathrm{e}^{4\pi u(t)^2} - 1\|_{L^1} \\ &= E(u,0) := E_0. \end{aligned} \tag{4.2}$$

As in earlier works of P. Gérard [9] and H. Bahouri-P. Gérard [2], we undertook in [3] a qualitative study of the solutions of two-dimensional nonlinear Klein-Gordon equation. This study was conducted following the approach introduced by P. Gérard in [9] which consists to compare the evolution of oscillations and concentration effects displayed by sequences of solutions of the nonlinear Klein-Gordon equation (4.1) and solutions of the linear Klein-Gordon equation.

$$\Box v + v = 0 \tag{4.3}$$

More precisely, if (φ_n, ψ_n) is a sequence of data in $H^1 \times L^2$ supported in some fixed ball and satisfying

$$\varphi_n \rightharpoonup 0 \quad \text{in } H^1, \quad \psi_n \rightharpoonup 0 \quad \text{in } L^2, \tag{4.4}$$

such that

$$E^n \leq 1, \quad n \in \mathbb{N} \tag{4.5}$$

where E^n stands for the energy of (φ_n, ψ_n) given by

$$E^n = \|\psi_n\|_{L^2}^2 + \|\nabla\varphi_n\|_{L^2}^2 + \frac{1}{4\pi}\,\|\mathrm{e}^{4\pi\varphi_n^2} - 1\|_{L^1},$$

we consider (u_n) and (v_n) the sequences of finite energy solutions of (4.1) and (4.3) such that

$$(u_n, \partial_t u_n)(0) = (v_n, \partial_t v_n)(0) = (\varphi_n, \psi_n).$$

Arguing as in [9], we introduce the following definition

Definition 4.2. *Let T be a positive time. The sequence (u_n) is said linearizable on $[0,T]$, if*

$$\sup_{t\in[0,T]} E_c(u_n - v_n, t) \longrightarrow 0 \quad \textit{as} \quad n \to \infty$$

where $E_c(w,t)$ denotes the kinetic energy defined by:

$$E_c(w,t) = \int_{\mathbb{R}^2} \left[|\partial_t w|^2 + |\nabla_x w|^2 + |w|^2\right](t,x)\,dx. \tag{4.6}$$

Similarly to the case of dimension $d \geq 3$ (see [9]), we proved that in the subcritical case (i.e. the case where $\limsup_{n\to\infty} E^n < 1$), the sequence (u_n) is linearizable on any time interval $[0,T]$. In other respects, we proved that in the critical case (i.e. the case where $\limsup_{n\to\infty} E^n = 1$), the sequence (u_n) is linearizable on $[0,T]$ provided that the sequence (v_n) satisfies

$$\limsup_{n\to\infty} \|v_n\|_{L^\infty([0,T];\mathcal{L})} < \frac{1}{\sqrt{4\pi}}, \tag{4.7}$$

where $\mathcal{L}$ denotes the Orlicz space $\mathcal{L}$.

Denoting by $w_n = u_n - v_n$, we can easily verify that w_n is the solution of the nonlinear wave equation

$$\Box w_n + w_n = -f(u_n)$$

with null Cauchy data.

Under energy estimate, we obtain

$$\|w_n\|_T \leq \|f(u_n)\|_{L^1([0,T],L^2(\mathbb{R}^2))},$$

where $\|w_n\|_T^2 \stackrel{\text{def}}{=} \sup_{t\in[0,T]} E_c(w_n,t)$. Therefore, to prove that the sequence (u_n) is linearizable on $[0,T]$, it suffices to establish that

$$\|f(u_n)\|_{L^1([0,T],L^2(\mathbb{R}^2))} \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

Thanks to finite propagation speed, the sequence $f(u_n)$ is uniformly supported in a fixed compact subset of $[0,T]\times\mathbb{R}^2$. So, to prove that the sequence $(f(u_n))$ converges strongly to 0 in $L^1([0,T],L^2(\mathbb{R}^2))$, we just follow the strategy of P. Gérard in [9] which reduce the problem to demonstrate that this sequence is bounded in $L^{1+\epsilon}([0,T],L^{2+\epsilon}(\mathbb{R}^2))$, for some nonnegative ϵ. This is done in the subcritical case by classical arguments thanks to Strichartz estimates and Trudinger-Moser inequality.

To handle with the critical case and estimate $\|f(u_n)\|^{1+\epsilon}_{L^{1+\epsilon}([0,T],L^{2+\epsilon}(\mathbb{R}^2))}$ for ϵ small enough, we split $f(u_n)$ as follows applying Taylor's formula

$$f(u_n) = f(v_n + w_n) = f(v_n) + f'(v_n)\, w_n + \frac{1}{2}\, f''(v_n + \theta_n\, w_n)\, w_n^2,$$

for some $0 \leq \theta_n \leq 1$. Strichartz inequality (see [21] for more details) yields

$$\begin{aligned} \|w_n\|_{\mathrm{ST}(I)} &\lesssim \|f(v_n)\|_{L^1([0,T];L^2(\mathbb{R}^2))} + \|f'(v_n)\, w_n\|_{L^1([0,T];L^2(\mathbb{R}^2))} \\ &+ \|f''(v_n+\theta_n\, w_n)\, w_n^2\|_{L^1([0,T];L^2(\mathbb{R}^2))} \\ &\lesssim I_n + J_n + K_n\,, \end{aligned} \tag{4.8}$$

where $I=[0,T]$ and $\|v\|_{\mathrm{ST}(I)} := \sup_{(q,r)\ \text{admissible}} \|v\|_{L^q(I;\mathrm{B}^1_{r,2}(\mathbb{R}^2))}$.

The term I_n is the easiest term to treat. Indeed, by Assumption (4.7) we have

$$\|v_n\|_{L^\infty([0,T];\mathcal{L})} \leq \frac{1}{\sqrt{4\pi(1+\eta)}}, \tag{4.9}$$

for some η and n large enough. This leads by similar arguments to the ones used in the proof of the subcritical case to the fact that

$$\|f(v_n)\|_{L^1([0,T],L^2(\mathbb{R}^2))} \to 0.$$

The second term J_n satisfies

$$J_n \leq \varepsilon_n\, \|w_n\|_{\mathrm{ST}(I)}, \tag{4.10}$$

where $\varepsilon_n \to 0$. Indeed by Hölder inequality, we are reduced to prove that $(f'(v_n))$ converges to 0 in $L^{1+\eta}([0,T];L^{2+2\eta}(\mathbb{R}^2))$, for η small enough. This is achieved arguing exactly in the same manner as for $(f(v_n))$ since the sequences $(f(v_n))$ and

$(f'(v_n))$ are similar.

For the last (more difficult) term we will establish that

$$K_n \le \varepsilon_n \|w_n\|^2_{\mathrm{ST}(I)}, \quad \varepsilon_n \to 0, \tag{4.11}$$

provided that

$$\limsup_{n\to\infty} \|w_n\|_{L^\infty([0,T];H^1)} \le \frac{1-L\sqrt{4\pi}}{2}. \tag{4.12}$$

In fact by Hölder inequality, Strichartz estimate and convexity argument, we get

$$\begin{aligned} K_n &\le \|w_n^2\|_{L^{1+\frac{1}{\eta}}([0,T];L^{2+\frac{2}{\eta}}(\mathbb{R}^2))} \|f''(v_n+\theta_n w_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta}(\mathbb{R}^2))} \\ &\le \|w_n\|^2_{\mathrm{ST}(I)} \left(\|f''(v_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta}(\mathbb{R}^2))} + \|f''(u_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta}(\mathbb{R}^2))}\right) . \end{aligned}$$

According to the previous step, we are then led to prove that for η small enough

$$\|f''(u_n)\|_{L^{1+\eta}([0,T];L^{2+2\eta}(\mathbb{R}^2))} \to 0 . \tag{4.13}$$

Arguing exactly as in the subcritical case, il suffices to establish that the sequence $(f''(u_n))$ is bounded in $L^{1+\eta_0}([0,T];L^{2+2\eta_0}(\mathbb{R}^2))$ for some $\eta_0>0$. This derives from the fact that under Assumption (4.12), we have

$$\begin{aligned} \limsup_{n\to\infty} \|u_n\|_{L^\infty([0,T];\mathcal{L})} &\le \limsup_{n\to\infty} \|v_n\|_{L^\infty([0,T];\mathcal{L})} + \limsup_{n\to\infty} \|w_n\|_{L^\infty([0,T];\mathcal{L})} \\ &\le L + \frac{1}{\sqrt{4\pi}} \|w_n\|_{L^\infty([0,T];H^1)} \\ &\le \frac{1}{2}\left(L+\frac{1}{\sqrt{4\pi}}\right) < \frac{1}{\sqrt{4\pi}} . \end{aligned}$$

This leads to the result by classical arguments.

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Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris-Est Créteil, 61 avenue du Général de Gaulle, 94010 Créteil cedex, France
hbahouri@math.cnrs.fr