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Entropy of eigenfunctions of the Laplacian in dimension 2

Gabriel Rivière

Abstract

We study asymptotic properties of eigenfunctions of the Laplacian on compact Riemannian surfaces of Anosov type (for instance negatively curved surfaces). More precisely, we give an answer to a question of Anantharaman and Nonnenmacher [4] by proving that the Kolmogorov-Sinai entropy of a semiclassical measure μ for the geodesic flow g^t is bounded from below by half of the Ruelle upper bound. (This text has been written for the proceedings of the 37$^{\text{èmes}}$ Journées EDP (Port d'Albret-June, 7-11 2010))

1. Motivations and results

Consider a smooth, compact, connected and Riemannian manifold M which has no boundary and which is of finite dimension d. In this talk, the main result will give an information on the asymptotic behavior of eigenfunctions of the Laplace Beltrami operator Δ on M in the case of a chaotic geodesic flow.

The geodesic flow g^t on the cotangent bundle T^*M is defined as the Hamiltonian flow corresponding to $H(x,\xi) := \frac{\|\xi\|_x^2}{2}$, where $\|.\|_x$ is the norm on T_x^*M induced by the metric on M. Using pseudodifferential calculus with a small parameter $\hbar > 0$ [12], the quantum operator corresponding to H is $-\hbar^2\Delta$. A way to look at eigenfunctions of Δ in the large eigenvalue limit is to understand the eigenfunctions $\psi_\hbar$ of $-\frac{\hbar^2\Delta}{2}$ associated to the eigenvalue[1] 1 in the semiclassical limit $\hbar \to 0$, i.e. look at the solutions of

$$-\hbar^2\Delta\psi_\hbar = \psi_\hbar.$$

Using again $\hbar$-pseudodifferential calculus, one can associate to every observable a in a good class of symbols an operator $\mathrm{Op}_\hbar(a)$ acting on $L^2(M)$. Using these operators, one can define a distribution $\mu_\hbar$ on T^*M:

$$\forall a \in \mathcal{C}_o^\infty(T^*M),\ \mu_\hbar(a) = \int_{T^*M} a(x,\xi)d\mu_\hbar(x,\xi) := \langle \psi_\hbar, \mathrm{Op}_\hbar(a)\psi_\hbar\rangle_{L^2(M)}.$$

[1] As M is compact, a sequence of such semiclassical parameters $\hbar$ is a discrete subsequence that tends to 0.

This quantity allows to describe the state $\psi_\hbar$ in function of the variables of position and impulsion (x,ξ). In order to understand the asymptotic behavior of the eigenstates of Δ, we will describe the properties of the distribution $\mu_\hbar$ as $\hbar$ tends to 0. One can show that any accumulation point of the sequence $(\mu_\hbar)_{\hbar\to 0^+}$ is a probability measure which is invariant under the geodesic flow g^t and which is supported in the unit cotangent bundle $S^*M := \{(x,\xi) : \|\xi\|_x^2 = 1\}$ [9]. A semiclassical measure is defined as any accumulation of a sequence $(\mu_\hbar)_{\hbar\to 0}$ as defined previously. We will denote $\mathcal{M}_{sc}(S^*M, g^t)$ the set of semiclassical measures. From the point of view of ergodic theory [25], we have constructed from eigenfunctions of the Laplacian a subset of the set $\mathcal{M}(S^*M, g^t)$ of g^t-invariant probability measures of the dynamical system (S^*M, g^t). One can then ask about the form of the subset $\mathcal{M}_{sc}(S^*M, g^t)$ in function of the geometric and dynamical properties of the manifold M. Our main concern in this talk is the case where the geodesic flow g^t on S^*M is of chaotic nature. A typical assumption verified by a chaotic system is that the desintegration L of the Liouville measure on S^*M is ergodic for the geodesic flow, i.e.

$$\forall a \in \mathcal{C}^0(S^*M),\ L \text{ a.e.},\ \lim_{T\to+\infty} \frac{1}{T}\int_0^T a\circ g^s(\rho)ds = \int_{S^*M} a dL.$$

It means that almost surely, the time average of an observable along an orbit of the geodesic flow is equal to the space average of this observable. This assumption is satisfied by manifolds of negative curvature (or more generally if the geodesic flow is of Anosov type on S^*M [15]). Under this ergodicity assumption, one can prove the well-known Shnirelman-Zelditch-Colin de Verdière theorem [24], [27], [10]. It tells us that for a given orthonormal basis of eigenvectors, *almost all* the associated distributions converge to the Liouville measure on S^*M. This theorem raises the question to know whether the Liouville measure is the only element of $\mathcal{M}_{sc}(S^*M, g^t)$ for ergodic systems. More precisely, Rudnick and Sarnak conjectured that for manifolds of negative curvature, L is the only semiclassical measure[2] [21].
This conjecture remains widely open in this general setting and our goal will be more to describe some quantitative properties of the elements of $\mathcal{M}_{sc}(S^*M, g^t)$. Before describing these results, we would like to underline that this conjecture is specific to manifolds of negative curvature. For instance, for a linear symplectomorphism A of the torus $\mathbb{T}^2$, one can construct a subset $\mathcal{M}_{sc}(\mathbb{T}^2, A)$ of semiclassical measures in the subset of A-invariant probability measures on the torus [7]. In this context, one can prove an analogue of Shnirelman's theorem if the Lebesgue measure Leb is ergodic for A on $\mathbb{T}^2$. Yet, de Bièvre, Faure and Nonnenmacher proved that Quantum Unique Ergodicity fails in this setting [13] and in particular, $\frac{1}{2}\delta_0 + \frac{1}{2}$Leb (where δ_0 is the Dirac measure carried by $(0,0)$) is a semiclassical measure for $A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. This provides an highly chaotic dynamical systems (Anosov type for instance) for which Quantum Unique Ergodicity is not true. Other counterexamples have been constructed in [16] and [14] for symplectic dynamical systems of Anosov type.

[2]This conjecture is known as the Quantum Unique Ergodicity Conjecture.

1.1. Main result

Regarding the last examples for quantum chaos, it is clear that we can not hope to answer the Quantum Unique Ergodicity Conjecture by making only dynamical assumptions. However, we can try to understand which quantitative properties on the set of semiclassical measures we can get by using only dynamical properties on (S^*M, g^t). In the following (except in paragraph 1.3), we will make the assumption that the geodesic flow is strongly chaotic and we will suppose it satisfies the Anosov property (it is the case for manifolds of negative curvature). It means that, for every ρ in S^*M, one has

$$T_\rho S^*M = E^u(\rho) \oplus E^s(\rho) \oplus \mathbb{R}X_H(\rho),$$

where X_H is the Hamiltonian vector field associated to H, E^u is the unstable space and E^s the stable one [15]. We underline that these three subspaces are stable under the tangent map $d_\rho g^t$ and that in the case of surfaces, they are 1-dimensional subspace.

Invariant probability measures are typical objects from ergodic theory so we can try to characterize them using tools from ergodic theory. More precisely, we will use Kolmogorov-Sinai entropy [25]. This quantity associates a nonnegative number $h_{KS}(\mu, g)$ to every probability measure μ invariant under g^t. The entropy tells us about what the measure μ sees of the separation of points under g^t (see paragraph 2.1 for a precise definition): the more the entropy is positive, the more the measure understands the complexity of g^t. Our main result gives a lower bound on the entropy of semiclassical measures in the case of Anosov surfaces [19]:

Theorem 1.1. *Let M be a compact, smooth, Riemannian* surface *without boundary. Suppose the geodesic flow $(g^t)_t$ satisfies the Anosov property. Then,*

$$\forall \mu \in \mathcal{M}_{sc}(S^*M, g^t),\ h_{KS}(\mu, g) \geq \frac{1}{2} \int_{S^*M} \log J^u(\rho) d\mu(\rho), \tag{1.1}$$

where $J^u(\rho)$ is the unstable Jacobian (for the induced volume) at point ρ, i.e. $J^u(\rho) := \det\left(d_\rho g^1_{|E^u(\rho)}\right)$.

1.2. Comments

This result follows earlier results and questions on the entropy of semiclassical measures by Anantharaman and Nonnenmacher [4]. In [2], Anantharaman has shown that the Kolmogorov-Sinai entropy of a semiclassical measure is positive under the Anosov assumption. Her result forbids that eigenfunctions of the Laplacian concentrate only on closed orbits of the geodesic flow in the large eigenvalue limit. In order to comment our result (which provides an explicit lower bound in the case of surfaces), let us recall that the Margulis-Ruelle's inequality tells us that for an invariant probability measure μ of $\mathcal{M}(S^*M, g^t)$, one has [22]

$$h_{KS}(\mu, g) \leq \int_{S^*M} \log J^u(\rho) d\mu(\rho),$$

with equality if and only if $\mu = L$ [17]. So, answering Rudnick and Sarnak's question would be equivalent to get rid of the factor $1/2$ in the inequality of our theorem. We can also underline that the inequality (1.1) is sharp in the case of the counterexamples in [13], [16], [14]. So, under dynamical assumptions only, this result seems optimal. In order to illustrate theorem 1.1, we can draw two corollaries. The first

one can be derived from the Margulis-Ruelle's inequality and the affine properties of the entropy [25]:

Corollary 1.2. *Suppose the assumptions of theorem 1.1 are satisfied. If μ is in $\mathcal{M}_{sc}(S^*M,(g^t))$ and if μ is of the form $tL+(1-t)\mu_\gamma$ (where μ_γ is a probability measure carried by a closed orbit γ), then one has*

$$t \geq \frac{\mu_\gamma(\log J^u)}{\mu_\gamma(\log J^u)+L(\log J^u)}.$$

A second nice corollary can be deduced from Young's equality ([26] and appendix A):

Corollary 1.3. *Suppose the assumptions of theorem 1.1 are satisfied. If μ is in $\mathcal{M}_{sc}(S^*M,(g^t))$, then one has*

$$\dim_H \mu := \inf\{\dim_H Y : \mu(Y)=1\} \geq 2,$$

where $\dim_H Y$ is the Hausdorff dimension of Y.

In terms of eigenfunctions of the Laplacian, this result tells us that in the large eigenvalue limit, eigenfunctions concentrate on a set which has at least Hausdorff dimension equal to 2. Finally, we underline that theorem 1.1 answers a question raised by Anantharaman and Nonnenmacher in [4]. In fact, in the case of a manifold of dimension d, they proved (with Koch [3]) that

$$\forall \mu \in \mathcal{M}_{sc}(S^*M,g^t),\ h_{KS}(\mu,g) \geq \int_{S^*M} \log J^u(\rho) d\mu(\rho) - \frac{(d-1)\lambda_{\max}}{2}, \qquad (1.2)$$

where $\lambda_{\max} := \lim_{t\to\pm\infty} \frac{1}{t} \log \sup_{\rho\in S^*M} |d_\rho g^t|$ is the maximal expansion rate of the geodesic flow. The term $(d-1)\lambda_{\max}$ bounds the quantity $\log J^u$ on S^*M and comes from the fact that the range of validity for the semiclassical approximation is given by time scales of order[3] $|\log\hbar|/(2\lambda_{\max})$ [5], [8]. We can underline that if $\lambda_{\max}$ is very large, then the bound obtained by Anantharaman, Koch and Nonnenmacher can be negative (which is not completely nice as the entropy is a nonnegative quantity). Regarding the different counterexamples and the case of constant negative curvature, they asked the question to know if the inequality of theorem 1.1 is true or not [4]. Our result gives a positive answer to this question in the case of dimension $d=2$ but the higher dimensional case remains an open question.

1.3. Extension of the entropic bound in the nonpositively curved case

In [20], we also showed that theorem 1.1 remains true for surfaces of nonpositive curvature. To do this, we can use an analogue of the unstable Jacobian J^u for these surfaces and use the fact that these surfaces has a strucure which is very similar to the Anosov case (existence of a stable/unstable foliation for instance) [23]. These properties allow to prove inequality (1.1) in this weakly chaotic situation. An interesting feature of this result is that it is still an open question to know if the Liouville measure is ergodic for the geodesic flow (even if the genus is larger than 1). So in this situation, we are able to prove an entropic bound on the eigenfunctions of the Laplacian even if we do not have (a priori) any Shnirelman's property.

[3]This time is also known as the Ehrenfest time.

1.4. A first intuition about the proof

The procedure developed in [3] uses a result known as the entropic uncertainty principle [18]. To use this principle in the semiclassical limit, we need to understand the precise link between the classical evolution and the quantum one for large times. Typically, we have to understand Egorov theorem for large range of times of order $t \sim |\log \hbar|$ (i.e. have a uniform remainder term for a large range of times). For a general symbol a in $\mathcal{C}_o^\infty(T^*M)$, we can only expect to have a uniform Egorov property for times t in the range of times $[-\frac{1}{2}|\log\hbar|/\lambda_{\max}, \frac{1}{2}|\log\hbar|/\lambda_{\max}]$ [8], [5]. However, if we only consider this range of times, we do not take into account that the unstable Jacobian can be very different between two points of S^*M. Our strategy to prove the main theorem relies on the fact that the range of times for which the Egorov property holds depends also on the support of the symbol $a(x,\xi)$ we consider. For particular families of symbol of small support (that depends on $\hbar$), we can show that we have a "local" Egorov theorem with an allowed range of times that depends on the support of our symbol. To make this heuristic idea work, we first try to reparametrize the flow in order to have a uniform expansion rate on the manifold. We define $\overline{g}^\tau(\rho) := g^t(\rho)$ where

$$\tau := -\int_0^t \log J^u(g^s\rho)ds. \tag{1.3}$$

This new flow $\overline{g}$ has the same trajectories as g. However, the *velocity of motion* along the trajectory at ρ is $|\log J^u(\rho)|$-greater for $\overline{g}$ than for g. We underline here that the unstable direction is of dimension 1 (as M is a surface) and it is crucial because it implies that $\log J^u$ exactly measures the expansion rate in the unstable direction at each point. As a consequence, this new flow $\overline{g}$ has a uniform expansion rate. Once this reparametrization is done, we use the following formula to recover t knowing τ:

$$t_\tau(\rho) = \inf\left\{s>0 : -\int_0^s \log J^u(g^{s'}\rho)ds' \geq \tau\right\}. \tag{1.4}$$

The number $t_\tau(\rho)$ can be thought of as a stopping time corresponding to ρ. We consider now $\tau = \frac{1}{2}|\log\hbar|$. For a given symbol $a(x,\xi)$ localized near a point ρ, $t_{\frac{1}{2}|\log\hbar|}(\rho)$ is exactly the range of times for which we can expect Egorov to hold. This new flow seems in a way more adapted to our problem. Moreover, we can define a $\overline{g}$-invariant measure $\overline{\mu}$ corresponding to μ. The measure $\overline{\mu}$ is absolutely continuous with respect to μ and verifies $\frac{d\overline{\mu}}{d\mu}(\rho) = \log J^u(\rho)/\int_{S^*M}\log J^u(\rho)d\mu(\rho)$. We can apply the classical result of Abramov

$$h_{KS}(\mu,g) = \left|\int_{S^*M}\log J^u(\rho)d\mu(\rho)\right| h_{KS}(\overline{\mu},\overline{g}).$$

To prove our main result, we would have to show that $h_{KS}(\overline{\mu},\overline{g}) \geq 1/2$. However, the flow $\overline{g}$ has no reason to be a Hamiltonian flow to which corresponds a quantum propagator $\overline{U}$. As a consequence, there is no particular reason that this inequality should be a consequence of [4]. In the quantum case, there is also no obvious reparametrization we can make as in the classical case.

In this talk, we will explain how it is possible to reparametrize the quantum propagator starting from a symbolic interpretation of the quantum dynamics. In order to have an artificial discrete reparametrization of the geodesic flow, we will have to introduce a suspension set [1]. It turns out that in this setting, it is possible to

define discrete analogues of quantities (1.3) and (1.4) (see theorem 3.2 for instance). Finally, we will briefly explain how one can prove a lower bound on the entropy of this reparametrized dynamic and then using Abramov theorem [1], we will deduce the expected lower bound on the entropy of a semiclassical measure.

2. Kolmogorov-Sinai entropy and quantum entropy

2.1. Kolmogorov-Sinai entropy

In this first paragraph, we would like to recall some definitions and properties of entropy that can be found for instance in [25]. Consider a probability space $(X, \mathcal{B}, \mu)$, a finite set I and $P := (P_\alpha)_{\alpha\in I}$ a finite measurable partition of X, i.e. a finite subset of measurable subsets of X that form a partition of X. Each of the P_α is called an atom of the partition. Using the convention $0\log 0 = 0$, we define the entropy of a partition as

$$H(\mu, P) := -\sum_{\alpha\in I} \mu(P_\alpha)\log\mu(P_\alpha) \geq 0. \tag{2.1}$$

For two given measurable partitions $P := (P_\alpha)_{\alpha\in I}$ and $Q := (Q_\beta)_{\beta\in K}$, we say that P is a refinement of Q if every element of Q can be written as the union of elements of P and we can verify that $H(\mu, Q) \leq H(\mu, P)$. In the other case, we introduce the join $P\vee Q := (P_\alpha \cap Q_\beta)_{\alpha\in I,\beta\in K}$ and we have $H(\mu, P\vee Q) \leq H(\mu, P) + H(\mu, Q)$ (subadditivity property). We fix now an application T of X which preserves μ. The n-th refined partition $\vee_{i=0}^{n-1} T^{-i}P$ of P with respect to T is the partition made of the atoms $(P_{\alpha_0} \cap \cdots \cap T^{-(n-1)}P_{\alpha_{n-1}})_{\alpha\in I^n}$. We introduce the entropy of this refined partition

$$H_n(\mu, T, P) = -\sum_{|\alpha|=n} \mu(P_{\alpha_0} \cap \cdots \cap T^{-(n-1)}P_{\alpha_{n-1}})\log\mu(P_{\alpha_0} \cap \cdots \cap T^{-(n-1)}P_{\alpha_{n-1}}). \tag{2.2}$$

Using the subadditivity property, one has, for every integers (n, m),

$$H_{n+m}(\mu, T, P) \leq H_n(\mu, T, P) + H_m(T^n\sharp\mu, T, P) = H_n(\mu, T, P) + H_m(\mu, T, P). \tag{2.3}$$

We underline that this last property has been obtained using crucially the fact that μ is T-invariant. A classical argument for subadditive sequences allows to define

$$h_{KS}(\mu, T, P) := \lim_{n\to\infty} \frac{H_n(\mu, T, P)}{n}. \tag{2.4}$$

This quantity is called the entropy of (T, μ) with respect to the partition P. The Kolmogorov-Sinai entropy $h_{KS}(\mu, T)$ of (μ, T) is defined as the supremum of $h_{KS}(\mu, T, P)$ over all the finite partitions P of X. In our case, the entropy is always finite (thanks to the Margulis-Ruelle's inequality for instance). We can also mention that, if for every sequence of indices $(\alpha_0, \cdots, \alpha_{n-1})$, $\mu(P_{\alpha_0} \cap \cdots \cap T^{-(n-1)}P_{\alpha_{n-1}}) \leq Ce^{-\beta n}$ where C is some positive constant, then $h_{KS}(\mu, T) \geq \beta$: the entropy measures the exponential decrease of the μ-volume of the atoms of the refined parition.

2.2. Quantum entropy

We can also define a quantum analogue of entropy. To do this, we fix an Hilbert space $\mathcal{H}$ and call a partition of identity $(\tau_\alpha)_{\alpha\in I}$ a finite family of operators that

satisfies

$$\sum_{\alpha\in I}\tau_\alpha^*\tau_\alpha = \mathrm{Id}_{\mathcal{H}}. \tag{2.5}$$

We define then the entropy of a unit vector ψ with respect to this partition

$$h_\tau(\psi) := -\sum_{\alpha\in I}\|\tau_\alpha\psi\|^2\log\|\tau_\alpha\psi\|^2. \tag{2.6}$$

Finally, this entropy satisfies an entropic uncertainty principle [18] due to Maassen and Uffink (see [4] for the generalized version presented here):

Theorem 2.1. *Let O_β be a family of bounded operators and U an unitary operator on the Hilbert space $(\mathcal{H},\|.\|)$. Let δ' be some positive number. Consider two partitions of identity $(\tau_\alpha)_{\alpha\in I}$ and $(\pi_\beta)_{\beta\in K}$ and ψ an unit vector in $\mathcal{H}$ satisfying*

$$\|(Id - O_\beta)\pi_\beta\psi\| \leq \delta'.$$

Suppose also that the cardinal of these two partitions is bounded by $\mathcal{N}$. One has then

$$h_\tau(U\psi) + h_\pi(\psi) \geq -2\log\left(c_O(U) + \mathcal{N}\delta'\right),$$

where $c_O(U) = \max\limits_{\alpha\in I,\beta\in K}\left(\|\tau_\alpha U\pi_\beta^* O_\beta\|\right)$ ($\|\tau_\alpha U\pi_\beta^* O_\beta\|$ is the norm operator on $\mathcal{H}$).

3. Outline of the proof

In this talk, we will give a sketch of proof of our main result. We will emphasize on the strategy and on the main ideas of the proof and we refer the reader [19] for the details. Consider $(\psi_{\hbar_k})$ a sequence of normalized eigenfunctions of the Laplacian associated to the eigenvalues $-1/\hbar_k^{-2}\to+\infty$ and such that the corresponding sequence of distributions $\mu_{\hbar_k}$ converges to μ as k tends to infinity. In order to simplify the notations, we will denote $\hbar\to 0$ the fact that k tends to infinity, $\psi_\hbar$ the eigenfunction and $-\hbar^{-2}$ the eigenvalue.
We fix $\epsilon>0$ and define the following *Ehrenfest time*:

$$n_E(\hbar) := [(1-\epsilon)|\log\hbar|]. \tag{3.1}$$

We underline that this time does not depend on the maximal expansion rate $\lambda_{\max}$ of the geodesic flow (which was not the case in [5] or [8]).

3.1. Symbolic interpretation of semiclassical measures

In order to compute the entropy of μ, we introduce a smooth partition of unity[4] $\mathcal{P} = (P_i)_{i=1,..K}$ of the manifold M

$$\forall x\in M,\ \sum_{i=1}^K P_i^2(x) = 1.$$

We make the assumption that the diameter of the partition is small enough (in a sense that is precised in [19]). We can associate a partition of identity on $L^2(M)$ to

[4]The smoothness is required to use tools from semiclassical analysis.

this classical partition. If we introduce $\hat{P}_i$ the multiplication operator by $P_i(x)$ on $L^2(M)$, then one has

$$\sum_{i=1}^{K} \hat{P}_i^* \hat{P}_i = \mathrm{Id}_{L^2(M)}. \tag{3.2}$$

This relation will allow us to introduce a probability space associated to $\psi_\hbar$ and to give a symbolic interpretation of the quantum dynamic. This symbolic setting will provide a discretized version of the phase space in the semiclassical limit. To do this, we introduce the space $\Sigma_+ := \{1, \cdots, K\}^{\mathbb{N}}$ of sequences with values in $\{1, \cdots, K\}$. For $(\alpha_0, \cdots, \alpha_{n-1})$ fixed in $\{1, \cdots, K\}^n$, we define the cylinder

$$[\alpha_0, \cdots, \alpha_{n-1}] := \{(x_k)_{k\in\mathbb{N}} \in \Sigma_+ : \ \forall 0 \leq k \leq n-1, \ x_k = \alpha_k\}.$$

We fix $\eta > 0$ and we introduce the measure of a cylinder

$$\mu_\hbar^{\Sigma_+}([\alpha_0, \cdots, \alpha_{n-1}]) := \|\hat{P}_{\alpha_{n-1}}((n-1)\eta) \cdots \hat{P}_{\alpha_0}\psi_\hbar\|^2,$$

where $\hat{A}(t) := e^{-\frac{\imath t\hbar\Delta}{2}} \hat{A} e^{\frac{\imath t\hbar\Delta}{2}}$. Using relation (3.2), all the compatibility conditions needed to define a probability measure are satisfied [25].

Remark. This symbolic coding is a standard procedure in ergodic theory. In particular, it is important to underline that, for a fixed n, one has

$$\lim_{\hbar\to 0} \mu_\hbar^{\Sigma_+}([\alpha_0, \cdots, \alpha_{n-1}]) = \mu\left(P^2_{\alpha_0} \times P^2_{\alpha_1} \circ g^\eta \times \cdots P^2_{\alpha_{n-1}} \circ g^{(n-1)\eta}\right).$$

If we forget the fact that we considered a smooth partition, the quantity that appears in the semiclassical limit is exactly the one used to compute the Kolmogorov-Sinai entropy (see relation (2.2)). The previous limit also tells us that the sequence of measures $\mu_\hbar^{\Sigma_+}$ converges weakly to μ^{Σ_+} which is defined on cylinders by

$$\mu^{\Sigma_+}([\alpha_0, \cdots, \alpha_{n-1}]) := \mu\left(P^2_{\alpha_0} \times P^2_{\alpha_1} \circ g^\eta \times \cdots P^2_{\alpha_{n-1}} \circ g^{(n-1)\eta}\right).$$

In order to define a dynamical system on Σ_+, we introduce the shift map σ_+

$$\sigma_+((x_n)_{n\in\mathbb{N}}) := (x_{n+1})_{n\in\mathbb{N}}.$$

As the measure μ is g^t-invariant, one can verify that the measure μ^{Σ_+} is σ_+-invariant. On the other hand, the measure $\mu_\hbar^{\Sigma_+}$ is not a priori Σ_+-invariant. Using results on composition of $\hbar$-pseudodifferentials operators and Egorov theorem [12], one has yet, for every n and n_0 fixed,

$$\mu_\hbar^{\Sigma_+}\left(\sigma_+^{-n}[\alpha_0, \cdots, \alpha_{n_0-1}]\right) = \mu_\hbar^{\Sigma_+}([\alpha_0, \cdots, \alpha_{n_0-1}]) + \mathcal{O}_{n,n_0}(\hbar),$$

where the remainder depends on n and n_0. So, for fixed scales of times, the measure is "almost invariant" under σ_+, i.e. it is "invariant" modulo terms that tend to 0 in the semiclassical limit. In order to make the more precise analysis as possible, we need to understand for which range of times n and n_0 (depending on $\hbar$), the measure is 'almost invariant' modulo small terms in $\hbar$. To do this, one can apply Egorov property for large times [5], [8] and verify that, for $|n+n_0|\eta \leq n_E(\hbar)/\lambda_{\max}$ [4],

$$\mu_\hbar^{\Sigma_+}\left(\sigma_+^{-n}[\alpha_0, \cdots, \alpha_{n_0-1}]\right) = \mu_\hbar^{\Sigma_+}([\alpha_0, \cdots, \alpha_{n_0-1}]) + \mathcal{O}(\hbar^{2\epsilon}),$$

where the remainder is uniform for n and n_0 in the allowed interval. The measure is "invariant" (modulo small terms in $\hbar$) as long as we remain in a scale of times of order $n_E(\hbar)/\lambda_{\max}$. In other words, the semiclassical approximation is valid for the discretized system until the Ehrenfest time. The main default of this last property

is that it does not take into account the variations of the unstable Jacobian. All the points in the phase space are treated in the same way, i.e. like if they all have a Lyapunov exponent equal to $\lambda_{\max}$. This limit of the semiclassical approximation appeared in [4] and [3] and was the reason of the apparition of the correction term $-\frac{d-1}{2}\lambda_{\max}$ in inequality (1.2).

3.2. Suspension of the quantum dynamic

In order to solve the problem we have just mentioned, we can, in the case of **dimension 2**, reparametrize the map σ_+ (and so implicitly the geodesic flow and the quantum dynamic) in order to take into account the variations of the unstable Jacobian. In fact, in this case, **the unstable direction is of dimension 1** and the Jacobian J^u measures then exactly how vectors are expanded in the unstable direction for positive times[5]. We will use this map to reparametrize the map σ_+. To do this, we introduce a roof function on Σ_+ by setting, for a sequence $\alpha := (\alpha_0, \alpha_1, \cdots)$ (finite or not),

$$f(\alpha) := \eta \sup \left\{ \log J^u(\rho) : \rho \in \mathrm{supp}(P_{\alpha_0}) \cap g^{-\eta}\mathrm{supp}(P_{\alpha_1}) \cap S^*M \right\}.$$

Remark. We have introduced a small parameter $\eta > 0$. The use of this parameter is quite important even if the reason for this will not really be made precised in this talk. Again, we refer the reader to [19] for more details (sections 3 and 5). We also make the assumption that $f(\alpha) > 0$ (we can restrict to this case: see also [19]). If the set over which the supremum is taken is empty, we choose the supremum of $\log J^u$ over S^*M. Last, we underline that choosing a step of time η and the Jacobian of the map at time 1 is not really symmetric. However, it is not really important and it presents the advantage of having a clear dependence of f in η, i.e. a linear one.

Now that we have defined a roof function on Σ_+, we can use a classical process of dynamical systems to reparametrize the map: we make a suspension of $(\Sigma_+, \mu_\hbar^{\Sigma_+}, \sigma_+)$. We first define the so-called suspension set

$$\overline{\Sigma}_+ := \{(x,t) \in \Sigma_+ \times \mathbb{R}_+ : 0 \leq t < f(x)\}.$$

This set is naturally endowed with the probability measure

$$\overline{\mu}_\hbar^{\Sigma_+} := \frac{\mu_\hbar^{\Sigma_+} \times \mathrm{Leb}}{\int_{\Sigma_+} f d\mu_\hbar^{\Sigma_+}}.$$

There exists also a natural semiflow $\overline{\sigma}_+$ associated to σ_+:

$$\forall s \geq 0,\ \overline{\sigma}_+^s(x,t) := \left(\sigma_+^{n-1}(x), s + t - \sum_{j=0}^{n-2} f\left(\sigma_+^j x\right) \right),$$

where n is the unique integer such that $\sum_{j=0}^{n-2} f\left(\sigma_+^j x\right) \leq s + t < \sum_{j=0}^{n-1} f\left(\sigma_+^j x\right)$. This new system $(\overline{\Sigma}_+, \overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+)$ reparametrizes $(\Sigma_+, \mu_\hbar^{\Sigma_+}, \sigma_+)$ and takes into account the

[5] This is not the case in higher dimension.

variations of $\log J^u$. We underline that the measure $\overline{\mu}_\hbar^{\Sigma_+}$ converges weakly to the measure

$$\overline{\mu}^{\Sigma_+} := \frac{\mu^{\Sigma_+} \times \mathrm{Leb}}{\int_{\Sigma_+} f d\mu^{\Sigma_+}}.$$

Moreover, this last measure is $\overline{\sigma}_+^t$-invariant. Again, the measure $\overline{\mu}_\hbar^{\Sigma_+}$ is not a priori $\overline{\sigma}_+^t$-invariant. However, compared with the case of $\mu_\hbar^{\Sigma_+}$, we will be able to prove the "almost-invariance" of the measure $\overline{\mu}_\hbar^{\Sigma_+}$ for larger scales of times than $n_E(\hbar)/\lambda_{\max}$. In order to observe this phenomenon, we have first to construct a partition for which we can compute the measure of the atoms of the partition and of its refinements.

3.2.1. Construction of an adapted partition

The construction of the partition is a little bit technical and we need to make good choices in order to do the necessary computations. We will just give here a good partition and we refer the refer the reader to [19] (section 5) for more explanations on the different choices we made. First, we define, for $t \geq 0$ (large enough) the family of indices

$$I(t) := \left\{ \alpha = (\alpha_0, \cdots, \alpha_k) : k \geq 3, \sum_{i=1}^{k-2} f\left(\sigma_+^i \alpha\right) \leq t < \sum_{i=1}^{k-1} f\left(\sigma_+^i \alpha\right) \right\}.$$

We underline[6] that for $\alpha \in I(1)$ of length $k(\alpha)+1$, there exists an unique integer $k'(\alpha) \leq k(\alpha)$ such that

$$\sum_{j=0}^{k'-2} f\left(\sigma_+^j \alpha\right) \leq 1 < \sum_{j=0}^{k'-1} f\left(\sigma_+^j \alpha\right).$$

Inspired by the definition of the suspension flow, we divide the interval $[0, f(\alpha)[$ in the following way:

$$I_{k'(\alpha)-2}(\alpha) = \left[0, \sum_{j=0}^{k'(\alpha)-1} f\left(\sigma_+^j \alpha\right) - 1\right[, \cdots I_{p-2}(\alpha) = \left[\sum_{j=0}^{p-2} f\left(\sigma_+^j \alpha\right) - 1, \sum_{j=0}^{p-1} f\left(\sigma_+^j \alpha\right) - 1\right[,$$

$$\cdots, I_{k(\alpha)-2}(\alpha) = \left[\sum_{j=0}^{k(\alpha)-2} f\left(\sigma_+^j \alpha\right) - 1, f(\alpha)\right[,$$

for $k'(\alpha) \leq p \leq k(\alpha)$. If $k(\alpha) = k'(\alpha)$, we define$I_{k'(\alpha)-2}(\alpha) = I_{k(\alpha)-2}(\alpha) = [0, f(\alpha)[$. We define a partition of $\overline{\Sigma}_+$ using these subintervals:

$$\overline{\mathcal{C}}_+ := \left\{ \overline{\mathcal{C}}_{\alpha,p} = [\alpha_0, \cdots, \alpha_k] \times I_{p-2}(\alpha) : \alpha \in I(1), \text{ et } k'(\alpha) \leq p \leq k(\alpha) \right\}.$$

The advantage of the subdivision of the interval $[0, f(\alpha)[$ is to know the precise action of $\overline{\sigma}_+^1$ on every atom of the partition.

[6]We remark that as f is proportional to η, the set $I(1)$ is nonempty for η small enough [19].

3.2.2. Invariance of the measure until times of order $n_E(\hbar)$

For this choice of partition (there could be others), we can prove the following proposition (sections 6 and 7 in [20]):

Proposition 3.1. *Let n, n_0 be two positive integers satisfying $n + n_0 \leq n_E(\hbar)$. For every atom $A := \overline{\mathcal{C}}_{\gamma_0,p_0} \cap \cdots \cap \overline{\sigma}_+^{-(n_0-1)} \overline{\mathcal{C}}_{\gamma_{n_0-1},p_{n_0-1}}$ of the refined partition $\vee_{j=0}^{n_0-1} \overline{\sigma}_+^{-j} \overline{\mathcal{C}}_+$, one has*

$$\overline{\mu}_\hbar^{\Sigma_+}\left(\overline{\sigma}_+^{-n} A\right) = \overline{\mu}_\hbar^{\Sigma_+}(A) + \mathcal{O}(\hbar^{5\epsilon}),$$

where the constant of the remainder is uniform for n_0 and n in the allowed interval.

By making this choice of partition, we can compute precise expressions for $\overline{\mu}_\hbar^{\Sigma_+}\left(\overline{\sigma}_+^{-n} A\right)$ and $\overline{\mu}_\hbar^{\Sigma_+}(A)$ and show that they are equal modulo terms of lower order. We underline that we have now "almost invariance" of the measure for *scales of times that do not depend on the maximal expansion rate* $\lambda_{\max}$ of the geodesic flow. In order to prove this proposition, we need to consider operators of the form $\hat{P}_{\alpha_n}(n\eta) \cdots \hat{P}_{\alpha_0}$ of length n that can be more or less long depending on the value of $\log J^u$ along the trajectory associated to the word $(\alpha_0, \cdots, \alpha_n)$. The proof of the proposition relies in particular on the following theorem of pseudodifferential calculus [19] (section 7):

Theorem 3.2. *Let $(Q_i)_{i=1}^K$ be a family of smooth functions on T^*M such that, for every $1 \leq i \leq K$, Q_i belongs to $\mathcal{C}^\infty(suppP_{\alpha_i} \cap \mathcal{E})$ (where $\mathcal{E}$ is a small neighborhood of S^*M [19]) and $0 \leq Q_i \leq 1$. Consider a family of indices $(\alpha_1, \cdots, \alpha_l)$ satisfying*

$$\sum_{j=1}^{l-1} f(\alpha_{j+1}, \alpha_j) \leq \frac{n_E(\hbar)}{2}.$$

Then, for every $1 \leq j \leq l$, $Op_\hbar(Q_{\alpha_1})(j\eta) Op_\hbar(Q_{\alpha_2})((j-1)\eta) \cdots Op_\hbar(Q_{\alpha_j})(\eta)$ is an $\hbar$-pseudo-differential operator of the class $\Psi_\nu^{-\infty,0}(M)$ where $\nu < 1/2$ (section 7 and appendix of [19]) and of principal symbol

$$Q_{\alpha_j} \circ g^\eta \cdots Q_{\alpha_2} \circ g^{(j-1)\eta} Q_{\alpha_1} \circ g^{j\eta}.$$

Remark. In order to prove this theorem, we crucially use the fact that the **dimension is 2**. In fact, the remainders that appear in the different formulas of stationary phase (for the composition of pseudodifferential operators, for the Egorov theorem) contain derivatives of g^t. For instance, we have to estimate the norm of $d_\rho g^t$ for ρ in the support of $Q_{\alpha_j} \circ g^\eta \cdots Q_{\alpha_2} \circ g^{(j-1)\eta} Q_{\alpha_1} \circ g^{j\eta}$. As the unstable direction is of dimension 1, for $t \geq 0$, this norm is controlled in term of the unstable Jacobian (norms are preserved in the direction of the flow and contracted in the stable one). For the allowed family of indices, we can verify that the loss of derivatives is at most $\hbar^{-\frac{1}{2}}$ which is the maximal allowed loss of derivatives in the stationary phase formulas [12].

3.3. Lower bound on the entropy of the suspension system

Our construction has provided a dynamical system adapted to the variation of the unstable Jacobian over the manifold. Our goal now is now to bound its Kolmogorov-Sinai entropy using the entropic uncertainty principle (theorem 2.1). Before we give our main estimate on the entropy, we underline that we have constructed a system that looks at positive times only. We could also have introduced a system that would

look at negative times (see [19]-section 4 for a more precise definition) and that we would denote $(\overline{\Sigma}_-, \overline{\mu}_\hbar^{\Sigma_-}, \overline{\sigma}_-)$. Using the entropic uncertainty principle, we are then able to prove the following proposition:

Proposition 3.3. *Using the notations of paragraph 2.1, one has, for $\hbar$ small enough,*

$$\frac{1}{n_E(\hbar)}\left(H_{n_E(\hbar)}\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+, \overline{\mathcal{C}}_+\right) + H_{n_E(\hbar)}\left(\overline{\mu}_\hbar^{\Sigma_-}, \overline{\sigma}_-, \overline{\mathcal{C}}_-\right)\right) \geq (1-5\epsilon). \tag{3.3}$$

This lower bound on the entropy of the reparametrized dynamical system at time $n_E(\hbar)$ is a key point of our proof. We provide some details on the strategy we should follow to prove such an inequality and we refer the reader to section 5 of [19] for more details.

First observation

We introduce a slightly smaller time than $n_E(\hbar)$, i.e.

$$T_E(\hbar) := (1-2\epsilon)n_E(\hbar).$$

We underline that the following collection of subsets forms a partition of $\overline{\Sigma}_+$:

$$\overline{\mathcal{C}}_+^\hbar := \left\{[\alpha_0, \cdots, \alpha_k] \times [0, f(\alpha)[\colon \alpha \in I(T_E(\hbar))\right\}.$$

This new partition depends on $\hbar$ and a crucial (but nontrivial) property of this partition is that **the partition $\vee_{j=0}^{n_E(\hbar)-1}\overline{\sigma}_+^{-j}\overline{\mathcal{C}}_+$ is a refinement of the partition** $\overline{\mathcal{C}}_+^\hbar$. In particular, from paragraph 2.1, we know that

$$H\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\mathcal{C}}_+^\hbar\right) \leq H\left(\overline{\mu}_\hbar^{\Sigma_+}, \vee_{j=0}^{n_E(\hbar)-1}\overline{\sigma}_+^{-j}\overline{\mathcal{C}}_+\right) = H_{n_E(\hbar)}\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+, \overline{\mathcal{C}}_+\right).$$

The quantity $H\left(\overline{\mu}_\hbar^{\Sigma_-}, \overline{\mathcal{C}}_-^\hbar\right) + H\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\mathcal{C}}_+^\hbar\right)$ is the one for which we will be able to give a lower bound on the entropy using the entropic uncertainty principle.

Second observation

In order to apply the entropic uncertainty principle (theorem 2.1), we underline that the family

$$\left(\hat{P}_{\alpha_k}(k\eta)\cdots\hat{P}_{\alpha_1}(\eta)\hat{P}_{\alpha_0}\right)_{\alpha\in I(T_E(\hbar))}$$

is a partition of identity for $L^2(M)$. We can apply the entropic uncertainty principle for this partition (and for its analogue for negative times). It turns out that this strategy do not allow us to find our result, i.e. a lower bound on the entropy of the partition $\overline{\mathcal{C}}_+^\hbar$. In fact, if we apply theorem 2.1 directly to these partitions, the terms that correspond to the Lebesgue part of the measure $\overline{\mu}_\hbar^{\Sigma_+}$ are missing in the sum. In order to solve this problem, we can apply the entropic uncertainty principle several times. To do this, we observe two things:

- the Lebesgue part of the measure of a cylinder only depends on the two first letters (α_0, α_1) of the word;
- if (γ_0, γ_1) is fixed, $\left(\hat{P}_{\alpha_k}(k\eta)\cdots\hat{P}_{\alpha_3}(3\eta)\hat{P}_{\alpha_2}(2\eta)\right)$ (where α is in $I(T_E(\hbar))$ and $(\alpha_0, \alpha_1) = (\gamma_0, \gamma_1)$) is a partition of identity.

The entropic uncertainty principle can be applied for each of these partitions. We obtain K^2 inequalities and we can make a sum of these lower bounds (multiplying also by appropriate coefficients). It gives the lower bound on the entropy we wanted to bound. We will not give more details but these are the main tools we used in [19] to give a lower bound on $H\left(\overline{\mu}_\hbar^{\Sigma_-}, \overline{\mathcal{C}}_-^\hbar\right) + H\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\mathcal{C}}_+^\hbar\right)$.

Third observation

In the entropic uncertainty principle, the lower bound is the norm of an operator and this quantity can be bounded using tools from [2] and [4]. We briefly recall a central result of [4]. To do this, we fix $\mathcal{K} > 0$ (large enough). One can construct a family of symbols $\chi^{(n)}$ localized in a small neighborhood of S^*M (of size depending on $\hbar$ and n) and these symbols can be quantized using a nonstandard procedure [4]. Anantharaman and Nonnenmacher proved then that there exist two constants c (depending only on M) and $C_\mathcal{K}$ (depending on M and $\mathcal{K}$) such that for all family of indices $(\alpha_0, \cdots, \alpha_n)$ satisfying $n\eta \leq \mathcal{K}|\log\hbar|$, one has

$$\left\|\hat{P}_{\alpha_n}(n\eta)\cdots\hat{P}_{\alpha_1}(\eta)\hat{P}_{\alpha_0}\mathrm{Op}_\hbar(\chi^{(n)})\right\|_{\mathcal{L}(L^2(M))} \leq C_\mathcal{K}\hbar^{-\frac{1}{2}-c\delta}\exp\left(-\frac{1}{2}\left(\sum_{j=0}^{n-1} f(\sigma_+^j\alpha)\right)\right),$$

where $\delta > 0$ is fixed. When we apply the entropic principle with the previous partitions, the norm that appears in the lower bound is the one of $\hat{P}_{\alpha_n}(n\eta)\cdots\hat{P}_{\alpha_1}(\eta)\hat{P}_{\alpha_0}\mathrm{Op}_\hbar(\chi^{(n)})$ where α in $I(\mathbf{2}T_E(\hbar))$. The apparition of the factor 2 in front of $T_E(\hbar)$ is important and it comes from the fact that we have considered positive and negative times. For this scale of times ($\alpha \in I(\mathbf{2}T_E(\hbar))$), one has, thanks to the Anantharaman-Nonnenmacher's result,

$$\left\|\hat{P}_{\alpha_n}(n\eta)\cdots\hat{P}_{\alpha_1}(\eta)\hat{P}_{\alpha_0}\mathrm{Op}_\hbar(\chi^{(n)})\right\|_{\mathcal{L}(L^2(M))} \leq C_\mathcal{K}\hbar^{\frac{1}{2}-2\epsilon-c\delta},$$

where $\delta > 0$ can be picked arbitrarily small. This upper bound tells us that the atom of the refined quantum partition has an exponential decrease for α in $I(\mathbf{2}T_E(\hbar))$. If we have used the Anantharaman-Nonnenmacher's bound for larger times, we would have obtain a better exponent but the property of invariance of the measure (we will use to make the next arguments work) is no longer true for this range of times. The power $\frac{1}{2}$ is the better exponent we can expect in the proof that we present here.

3.4. Subadditivity of the entropy

We fix two integers n and n_0 in $\mathbb{N}$. Using only classical properties of the entropy (property (2.3)), we find

$$H_{n+n_0}\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+, \overline{\mathcal{C}}_+\right) \leq H_n\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+, \overline{\mathcal{C}}_+\right) + H_{n_0}\left(\overline{\sigma}^n\sharp\overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+, \overline{\mathcal{C}}_+\right).$$

As the map $x \mapsto -x\log x$ is continuous on the interval $[0,1]$ and as the measures are "almost invariant" under $\overline{\sigma}_+$ (proposition 3.1), one has that, for $n + n_0 \leq n_E(\hbar)$,

$$H_{n_0}\left(\overline{\sigma}^n\sharp\overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+, \overline{\mathcal{C}}_+\right) = H_{n_0}\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+, \overline{\mathcal{C}}_+\right) + o_{n_0}(1), \text{ as } \hbar \to 0.$$

We write $n_E(\hbar) = qn_0 + r$ with $0 \leq r < n_0$ (n_0 is fixed). We use the last two properties to obtain that

$$H_{n_E(\hbar)}\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+, \overline{\mathcal{C}}_+\right) \leq qH_{n_0}\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+, \overline{\mathcal{C}}_+\right) + H_r\left(\overline{\mu}_\hbar^{\Sigma_+}, \overline{\sigma}_+, \overline{\mathcal{C}}_+\right) + o_{n_0}(q) \text{ as } \hbar \to 0.$$

This last inequality allows to verify

$$\frac{H_{n_E(\hbar)}\left(\overline{\mu}_\hbar^{\Sigma_+},\overline{\sigma}_+,\overline{\mathcal{C}}_+\right)}{n_E(\hbar)} \leq \frac{H_{n_0}\left(\overline{\mu}_\hbar^{\Sigma_+},\overline{\sigma}_+,\overline{\mathcal{C}}_+\right)}{n_0} + o_{n_0}(1) \text{ as } \hbar \to 0.$$

This result holds also for $(\overline{\Sigma}_-, \overline{\mu}_\hbar^{\Sigma_-}, \overline{\sigma}_-)$ and if we combine them to inequality (3.3), we have that, as $\hbar$ tends to 0,

$$\frac{1}{n_0}\left(H_{n_0}\left(\overline{\mu}_\hbar^{\Sigma_+},\overline{\sigma}_+,\overline{\mathcal{C}}_+\right) + H_{n_0}\left(\overline{\mu}_\hbar^{\Sigma_-},\overline{\sigma}_-,\overline{\mathcal{C}}_-\right)\right) + o_{n_0}(1) \geq (1-5\epsilon). \tag{3.4}$$

3.5. The conclusion: applying the Abramov theorem

To conclude the proof of theorem 1.1, we first let $\hbar$ tends to 0 in inequality (3.4) and we find that

$$\frac{1}{n_0}\left(H_{n_0}\left(\overline{\mu}^{\Sigma_+},\overline{\sigma}_+,\overline{\mathcal{C}}_+\right) + H_{n_0}\left(\overline{\mu}^{\Sigma_-},\overline{\sigma}_-,\overline{\mathcal{C}}_-\right)\right) \geq (1-5\epsilon).$$

We underline that the previous inequality is a lower bound on the entropy of a smooth partition of M (and not a true partition). If we take some precautions ([2], [4], [19]-section 4), we can obtain a lower bound on the entropy of a true partition. Suppose that we have done this transformation: we can let then n_0 tends to infinity. This gives us

$$h_{KS}\left(\overline{\mu}^{\Sigma_+},\overline{\sigma}_+,\overline{\mathcal{C}}_+\right) + h_{KS}\left(\overline{\mu}^{\Sigma_-},\overline{\sigma}_-,\overline{\mathcal{C}}_-\right) \geq (1-5\epsilon).$$

Now, we have to use the last *key argument* of the proof: the **Abramov theorem** [1]. This theorem gives a link between the entropy of a dynamical system and the entropy of some of its suspensions. In our case, it tells us that

$$h_{KS}(\mu^{\Sigma_+},\sigma_+,\mathcal{P}) = h_{KS}\left(\overline{\mu}^{\Sigma_+},\overline{\sigma}_+,\overline{\mathcal{C}}_+\right) \times \int_{\Sigma_+} f d\mu^{\Sigma_+}.$$

We can then observe two things. The first one is that $h_{KS}(\mu^{\Sigma_+},\sigma_+,\mathcal{P})$ is bounded from above by $h_{KS}(\mu, g^\eta)$ which is equal to $\eta h_{KS}(\mu, g)$ [1]. The second is that, as the diameter of the partition tends to 0,

$$\int_{\Sigma_+} f d\mu^{\Sigma_+} \to \eta \int_{S^*M} \log J^u(\rho) d\mu(\rho).$$

Finally, we have

$$2\eta h_{KS}(\mu,g) = \eta(h_{KS}(\mu,g) + h_{KS}(\mu,g^{-1})) \geq (1-5\epsilon)\eta \int_{S^*M} \log J^u(\rho) d\mu(\rho).$$

As it holds for any $\epsilon > 0$, the conclusion follows.□

Appendix A. Dimension, entropy and Lyapunov exponents

In this short appendix, we would like to explain how Young's results relate the Hausdorff dimension of a measure to its entropy and to its Lyapunov exponents [26]. This will allow us to derive corollary 1.3.

Suppose M is a Riemannian surface of Anosov type and let μ be an element in $\mathcal{M}(S^*M, g^t)$. One can define the positive Lyapunov exponent of μ at point ρ as follows [6]:

$$\chi^+_\mu(\rho) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T \log J^u(g^t\rho)dt.$$

By Birkhoff's theorem, this quantity is well defined for μ-almost every ρ in S^*M. If μ is not an ergodic measure, this quantity can be a non trivial function of ρ. In the case of an ergodic measure μ, this Lyapunov exponent is constant μ-almost everywhere. Adapting an argument of Young in the case of Anosov geodesic flow on surfaces [26], one has that, for an ergodic measure μ,

$$\dim_H \mu := \inf\{\dim_H Y : \mu(Y) = 1\} = 1 + 2\frac{h_{KS}(\mu, g)}{\chi^+_\mu},$$

where $\dim_H Y$ is the Hausdorff dimension of a set Y. One can combine this result and our lower bound on the entropy of semiclassical measures to derive a lower bound on $\dim_H \mu$ when μ is in $\mathcal{M}_{sc}(S^*M, g^t)$. To do this, we consider a semiclassical measure μ and write its ergodic decomposition [11]

$$\mu = \int_{S^*M} \mu^\rho d\mu(\rho),$$

where for μ-a.e. ρ, μ^ρ is ergodic. Let Y be a measurable set such that $\mu(Y) = 1$. Then, for μ-a.e. ρ in S^*M, one has $\mu^\rho(Y) = 1$. From our lower bound on the entropy and from the properties of the ergodic decomposition, we know that

$$\int_{S^*M} h_{KS}(\mu^\rho, g)d\mu(\rho) = h_{KS}(\mu, g) \geq \frac{1}{2}\int_{S^*M} \chi^+_{\mu^\rho} d\mu(\rho).$$

Combining this observation to Young's formula, we know that on a set of μ-positive measure, $\dim_H(\mu^\rho) \geq 2$ and in particular, $\dim_H Y \geq 2$.

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