Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Port d'Albret, 7 juin–11 juin 2010

Oana Ivanovici

Dispersive and Strichartz estimates for the wave equation in domains with boundary

J. É. D. P. (2010), Exposé n° XI, 19 p.

<http://jedp.cedram.org/item?id=JEDP_2010____A11_0>

GROUPEMENT DE RECHERCHE 2434 DU CNRS

Journées *Équations aux dérivées partielles*
Port d'Albret, 7 juin–11 juin 2010
GDR 2434 (CNRS)

Dispersive and Strichartz estimates for the wave equation in domains with boundary

Oana Ivanovici

Abstract

In this note we consider a strictly convex domain $\Omega \subset \mathbb{R}^d$ of dimension $d \geq 2$ with smooth boundary $\partial\Omega \neq \emptyset$ and we describe the dispersive and Strichartz estimates for the wave equation with the Dirichlet boundary condition. We obtain counterexamples to the optimal Strichartz estimates of the flat case; we also discuss the some results concerning the dispersive estimates.

1. Introduction

The dispersive type estimates called "Strichartz inequalities" measure the size and dispersion of solutions of the linear wave equation on a manifold (Ω, g), with (possibly empty) boundary $\partial\Omega$:

$$\begin{cases} (\partial_t^2 - \Delta_g)u(t,x) = 0, \quad x \in \Omega \\ u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x), \\ u(t,x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.1}$$

Here Δ_g denotes the Laplace-Beltrami operator on Ω.

In order to be able to perturb such equations to nonlinear equations, it is crucial that we have some efficient way to control the "size" of solutions to the linear problems in terms of the size of the initial data. Of course, one has to quantify this notion of size by specifying a suitable function space norm; it turns out that for equations like nonlinear wave, the mixed norms $L_t^p L_x^q$ are particularly useful.

Regarding solutions of the homogeneous linear equation, a basic homogeneous local estimate says that, on any smooth Riemannian manifold without boundary, solutions of the wave equation (1.1) satisfy (for $T < \infty$)

$$\|u\|_{L^q(0,T)L^r(\Omega)} \leq C_T\Big(||u_0||_{\dot{H}^\beta(\Omega)} + ||u_1||_{\dot{H}^{\beta-1}(\Omega)}\Big), \tag{1.2}$$

where, if d denotes the dimension of the manifold, we have $\beta = d(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$ (which is a scale invariant condition) and where the pair (q, r) is wave-admissible, i.e. it satisfies

$$\frac{2}{q} + \frac{d-1}{r} \leq \frac{d-1}{2}, \quad (d, q, r) \neq (3, 2, \infty). \tag{1.3}$$

When equality holds in (1.3) we say that the pair (q, r) is sharp wave-admissible. Here H^β denotes the L^2 Sobolev space over Ω. If this holds for $T = \infty$ we say there are global Strichartz estimates. Such inequalities were long ago established for Minkowski space (flat metrics) and can be generalized to any (Ω, g) because of the local character of the estimate (finite propagation speed).

Outside the admissibility range (1.2) does not hold, as can be seen from A.W.Knapp's counterexample: consider the solution to (1.1) in $\mathbb{R}^{1+d}$ with initial data equal to the indicator function of the set $\{\frac{1}{h} < |\xi| < \frac{2}{h}, h^{1/2}|\xi'| < 1, \xi = (\xi_1, \xi')\}$:

$$\int_{\mathbb{R}^d} e^{ix\xi - it|\xi|}\beta(h|\xi|)\rho(h^{1/2}|\xi'|)d\xi,$$

where $\beta \in C_0^\infty((1,2))$, $\rho \in C_0^\infty((0,1))$. For $t \simeq 1$, it "lives" on $\{|x'| \lesssim h^{1/2}, |t - x_1| \lesssim h\}$. This example provides the highest possible concentration of Euclidian waves and computing the ratio forces (1.3) to hold.

Strichartz estimates admit many variations. One which is particularly close to eigenfunction estimates is the squarefunction estimate, which says that, in the boundaryles case, the solution to (1.1) satisfies

$$\|u\|_{L^p(\Omega)L^2(0,T)} \leq C_T\Big(||u_0||_{\dot{H}^{\delta(p)}(\Omega)} + ||u_1||_{\dot{H}^{\delta(p)-1}(\Omega)} \Big), \tag{1.4}$$

where $\delta(p) = \max(d(1/2 - 1/p) - 1/2, \frac{d-1}{2}(1/2 - 1/p))$. The exponent $p = p_d := \frac{2(d+1)}{d-1}$ (at which the estimates change) is critical in dimension d, since all the other bounds follow easily from it (by Sobolev if $p > p_d$ and by interpolation with the trivial L^2 energy estimate if $2 \leq p < p_d$). The estimate (1.4) implies eigenfunction estimates since, if one takes $(u_0, u_1) = (e_\lambda, 0)$ where $e_\lambda(x)$ is an eigenfunction with eigenvalue λ, then, since $u(t, x) = \cos(t\lambda)\, e_\lambda(x)$ in this case, it follows that the estimate (1.4) for $T = 1$ implies the eigenfunction estimate $\|e_\lambda\|_{L^{p_d}(\Omega)} \leq C\lambda^{\delta(p_d)}\|e_\lambda\|_{L^2(\Omega)}$. A slightly more involved argument shows that it also implies corresponding bounds for the spectral projection operators, and hence (1.4) is sharp on every Riemannian manifold without boundary. This argument can also be reversed, in that estimates for the spectral projection operators imply estimates like (1.4).

The main motivation for the above types of Strichartz estimates comes from applications to harmonic analysis and the study of nonlinear dispersive equations. Estimates like (1.4) can be used to prove L^p multiplier theorems in harmonic analysis, while estimates like (1.2) can be used to prove existence theorems for nonlinear wave equations.

In order to prove such estimates we usually reduce the analysis to the case of a spectrally localized wave by using a Littlewood-Paley decomposition to frequencies $\frac{1}{h} \simeq 2^k$, and write $u \simeq \sum_{h=2^{-k}} u_h$, where $\mathrm{supp}(\hat{u}_h(t, .)) \subset \{|\xi| \simeq \frac{1}{h}\}$. For, we let $\psi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ be compactly supported in the interval $(\frac{1}{4}, 2)$. We introduce the operator $\psi(-h^2\Delta_g)$ using the Dynkin-Helffer-Sjöstrand formula [4] and refer to [15], [4] or [10] for a complete overview of its properties (see also [2] for compact manifolds without boundaries). Given $\psi \in C_0^\infty(\mathbb{R})$ we have

$$\psi(-h^2\Delta_g) = -\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}\tilde{\psi}(z)(z + h^2\Delta_g)^{-1}dL(z),$$

where $dL(z)$ is the Lebesque measure on $\mathbb{C}$ and $\tilde{\psi}$ is an almost analytic extension of ψ.

A classical way to prove Strichartz inequalities is to use dispersive estimates. If we denote by $e^{it\sqrt{-\Delta_{\mathbb{R}^d}}}$ the linear wave flow in the flat space, it satisfies the following dispersive inequality:

$$\|\psi(-h^2\Delta_{\mathbb{R}^d})e^{it\sqrt{-\Delta_{\mathbb{R}^d}}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq C(d)h^{-d}\min\{1,(h/|t|)^{\frac{d-1}{2}}\}. \tag{1.5}$$

The Strichartz estimates then follow by interpolation between (1.5) and the energy bounds, together with the standard squarefunction estimate which allows to recover bounds on u.

The aforementioned results for $\mathbb{R}^d$ and manifolds without boundary have been understood for sometime. Euclidean results go back to R.Strichartz's pioneering work [20], where he proved the particular case $q = r$ for the wave and Schrödinger equations. This was later generalized to mixed $L^q_t L^r_x$ norms independently by J.Ginibre and G.Velo [6] and H.Lindblad and C.Sogge [14], following earlier work by L.Kapitanski [11]. The remaining endpoints were finally settled by M.Keel and T.Tao [12]. In the case of manifolds without boundary, by finite speed of propagation it suffices to work in coordinate charts and to establish estimates for variable coefficients operators in $\mathbb{R}^d$. For operators with $C^{1,1}$ coefficients, Strichartz estimates were shown by H.Smith [17] (see also the work of D.Tataru [21] for C^α coefficients of the metric): a necessary and sufficient condition for sharp Strichartz estimates to hold is to impose the metric to have at least two bounded derivatives.

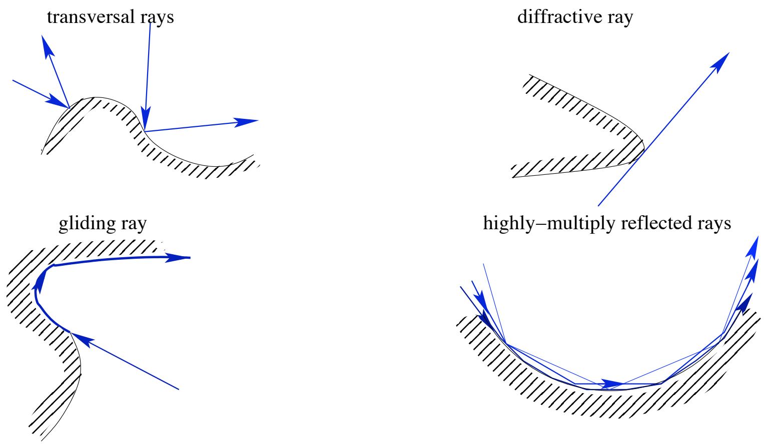
As far as the dispersive estimates are concerned, they do not hold anymore on bounded domains. Indeed, suppose that (Ω, g) is a compact manifold and that the dispersive estimate (1.5) holds true with $\Delta_{\mathbb{R}^d}$ replaced by Δ_g and for $|t| \leq T$, $T > 0$. It follows that the kernel of the operator which to u_0 associates $\psi(-h^2\Delta_{\mathbb{R}^d})e^{it\sqrt{-\Delta_{\mathbb{R}^d}}}u_0$ would belong to $L^\infty(\Omega\times\Omega) \subset L^2(\Omega\times\Omega)$, the last inclusion being a consequence of the compactness of Ω. Such an operator, on the other hand, is of Hilbert-Schmidt type, therefore it is compact. On the other hand, the application

$$L^2(\Omega) \ni u_0 \to e^{it\sqrt{-\Delta_\Omega}}u_0 \in L^2(\Omega)$$

is an isometry of $L^2(\Omega)$ and cannot be compact unless the dimension of $L^2(\Omega)$ is finite, which is obviously a contradiction.

Even though the boundaryless case has been well understood for some time, obtaining results for the case of manifolds with boundary has been surprisingly elusive. For manifolds with smooth, strictly geodesically concave boundary, because of the R.Melrose and M.Taylor parametrix for the Dirichlet wave equation, the theory for this setting was also established by H.Smith and C.Sogge in [18]. If the concavity assumption is removed, however, the presence of highly-multiply reflected geodesics and their limits, the gliding rays, prevent the construction of a similar parametrix. Besides the case involving concave boundaries mentioned before, there were no sharp estimates until quite recently.

One technique which has potential applications in these kind of problems consists in doubling the metric across its boundary to produce a boundaryless manifold with a special type of Lipschitz metric. In [19] H.Smith and C.Sogge used it together with wave packets techniques in order to prove sharp square function estimates (and hence spectral projection estimates) of the form (1.4) where now $\delta(p) = \max(2(1/2-1/p)-1/2, \frac{2}{3}(1/2-1/p))$ in dimension $d = 2$. These are always sharp for compact domains



in $\mathbb{R}^2$ because of the existence of Rayleigh whispering gallery modes. Note that these bounds are worse than the corresponding ones for the boundaryles case. However, the works by H.Smith and D.Tataru show that there is no hope to obtain sharp dispersive estimates using this approach, since the metric is not smooth enough.

The statement of the propagation of singularities of solutions to (1.1) has two main ingredients: locating singularities of a distribution, as captured by the wave front set, and describing the curves along which they propagate, namely the bicharacteristics. Both of these are closely related to an appropriate notion of "phase space", in which both the wave front set and the bicharateristics are located. On manifolds without boundary, this phase space is the standard cotangent bundle.

In the case of a non-empty boundary, for the problems we are interested in the main difficulties arise from the behavior of the singularities near the boundary. In the interior of the domain, these singularities propagate, according to a result du to L.Hörmander, along optical rays. The study of the propagation of singularities near the boundary was essentially made by R.Melrose and J.Sjöstrand who introduced the notion of "generalized bicharacteristic rays". The simplest case, corresponding to the classical geometrical optic's laws, consists of points for which the flow is transverse to the boundary (called hyperbolic points). Some difficulties may appear near the points where the rays are tangent to the boundary. The diffractive points are those through which passes an optical ray without being deviated. To describe completely the propagation we should also mention the rays which live on the boundary only, called gliding rays, which are limits of highly-multiply reflected rays.

One feels that for (1.1) the types of Strichartz estimates that are possible should reflect the geometry of (Ω, g) and especially its boundary. As mentioned above, for the problems we are interested in the main difficulties arise from the complex propagation pattern near the boundary.

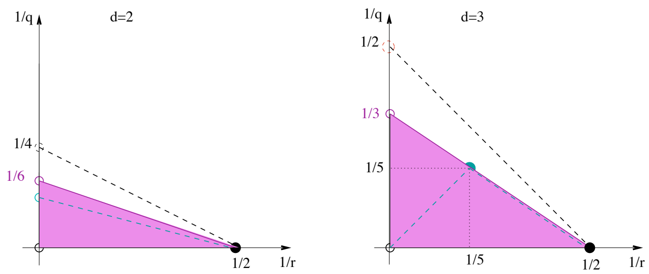


Figure 2.1: Scale invariant Strichartz estimates obtained by [1] for $d = 2, 3$

2. Results on manifolds with boundaries

2.1. Strichartz estimates

Besides the case involving concave boundaries mentioned before, there were no sharp estimates until quite recently. In [3], N.Burq, G.Lebeau and F.Planchon were able to use the square function estimates from [19] to prove Strichartz estimates without loss for solutions of (1.1) for the admissible pair ($d = 3, q = 5, r = 5$) that allowed them to show that there is global existence for the H^1-critical nonlinear wave equation for domains in $\mathbb{R}^3$.

Shortly after this, M.Blair, H.Smith and C.Sogge obtained in [1] optimal Strichartz estimates for triples which satisfy:

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \beta, \quad \begin{cases} \frac{1}{q} \leq \frac{(d-1)}{3}(\frac{1}{2} - \frac{1}{r}), & d \leq 4, \\ \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}, & d \geq 4. \end{cases} \tag{2.1}$$

The strategy in [1] consists in doubling the manifold Ω along its boundary to produce a boundaryless manifold with a special type of Lipschitz metric (with codimension-1 singulairities). The study of the general wave equation for general Lipschitz metrics has already been developed by H.Smith [17] and D.Tataru [21] by introducing new wave-packet techniques and is particularly useful in the study of eigenfunctions estimates (a recent example is give by the work [19]).

The main observation in [1] is that one can construct parametrices over large time intervals when moving to directions which are not tangential to $\partial\Omega$. Precisely, for directions of angle θ one can construct a parametrix on intervals of time of size θ, yielding to a θ-depending loss in the Strichartz estimates. On the other hand, since such directions live in a small volume cone in the frequency set, one obtains a gain in these estimates for non-sharp admissible indices (q, r) due to the frequency localization. Requiring this gain to annihilate the loss coming from summing up over θ^{-1} intervals of time yields the restriction on the range of indices (d, q, r) in (2.1).

Remark 2.1. The restrictions on the indices in [1] are naturally imposed by the local nature of the paramerix construction in [19]. In dimension $d \geq 3$ the result is certainly not optimal (Lebeau's applications give larger sets of admissible indices), and this is a consequence of the fact that in higher dimensions the approach in [1] does not allow to describe the dispersion effect in the $d - 2$ tangential variables. Moreover, by doubling the metric one cannot see more than one reflection while

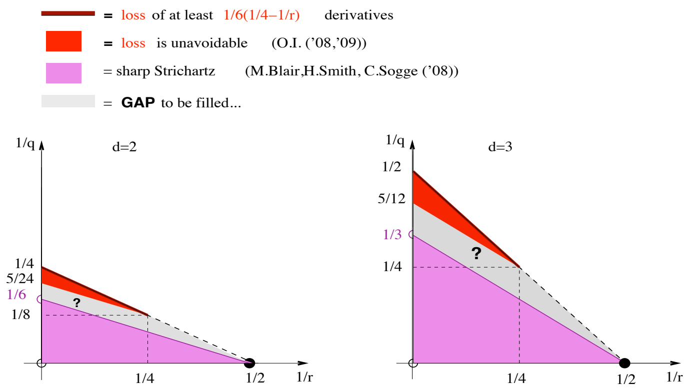


Figure 2.2: Counterexample to the Strichartz estimates inside a bounded domain [8, 9]

Lebeau's description of the wave front set of u_a in [13] (see also a forthcoming work in collaboration with F.Planchon) show precisely that the worst situation in the dispersion appears after the second reflection of the wave on the boundary.

One of the results discussed in this paper is work in the opposite direction: we show by explicit computations that the (local in time) Strichartz estimates (1.2) for the wave equation suffer losses when compared to the usual case $\Omega = \mathbb{R}^2$, at least for a subset of the usual range of admissible indices (1.3) and this is due to micro-local phenomena such as caustics generated in arbitrarily small time near the boundary. Precisely, the main results in [8, 9] state that if (1.2) holds for sharp admissible pairs then one must have $r \leq 4$ and also place a lower bound on the Sobolev index that it would be required. We proved the following:

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a regular and bounded domain, (d, q, r) be a sharp wave admissible exponent with $r > 4$ and let $\beta = d(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$. Then the quotient*

$$\frac{\|u\|_{L^q([0,1],L^r(\Omega))}}{\|u_0\|_{\dot{H}^{\beta+\frac{1}{6}(\frac{1}{4}-\frac{1}{r})}(\Omega)} + \|u_1\|_{\dot{H}^{\beta+\frac{1}{6}(\frac{1}{4}-\frac{1}{r})-1}(\Omega)}} \tag{2.2}$$

takes arbitrarily large values for suitable initial data (u_0, u_1) of (1.1), i.e. a loss of at least $\frac{1}{6}(\frac{1}{4} - \frac{1}{r})$ derivatives is unavoidable in the Strichartz estimates for the linear wave flow.

Remark 2.3. Unfortunately, yet there is a gap between the negative results from this counterexample and the known positive results from [1]. From (2.2) we can deduce for instance that the scale-invariant Strichartz estimates fail for $\frac{3}{q} + \frac{1}{r} > \frac{15}{24}$, whereas the results in [1] state that the Strichartz estimates (1.2) hold if $\frac{3}{q} + \frac{1}{r} \leq \frac{1}{2}$. This concise statement shows one explicit gap in our knowledge that remains to be filled.

Remark 2.4. A very interesting and natural question would be to determine the sharp range of exponents (q, r) for the Strichartz estimates in any dimension $d \geq 2$; to prove such sharp results we need to understand first the types of concentration phenomena that can occur for eigenfunctions. For instance, the above counterexample seems to us to be optimal, since after many explicit computations involving eigenmodes localized at different (small, frequency depending) distances away from the boundary we always obtained losses at most like in (2.2); in particular, at least in the $2D$ Friedlander' model domain it seems not possible to improve the construction in [8].

Remark 2.5. The result is striking because of the following reason: for quite a long while people tended to feel that (1.2) should hold when the boundary is assumed to be concave for all exponents satisfying the admissibility condition (1.3) (excluding the forbidden endpoints when $d = 2, 3$). In this case the singularities stay close to the boundary and can reflect on the boundary large number of times. Thus one might guess that stronger focusing may occur near the boundary. Indeed, such a focusing does occur and as a consequence of it the pointwise decay estimates for solutions to the wave equation fail, as do the estimates (1.4).

While the Rayleigh whispering gallery modes easily rule out (1.4) with $\gamma(p)$ as in the boundaryless case, they do not rule out the aforementioned Strichartz estimates. Since the Rayleigh waves seemed to have the maximum amount of concentration and they did not saturate (1.2), people felt that the latter should hold for all admissible exponents. Here we give a short review on WGM: [1]

2.1.1. Whispering gallery modes

In mathematical terms, the WGM can be described by some eigenfunctions of the Laplace operator inside the unit disk, whose gradients are almost tangent to the boundary. Indeed, let $(e_{n,m})$ be a complete system of (nonnormalized) eigenfunctions on the unit disk given by

$$e_{n,m}(r,\theta) = e^{in\theta} J_n(\lambda_{n,m} r), \quad J_n(\lambda_{n,m}) = 0, \quad (\lambda_{n,m})_m \nearrow,$$

where r, θ are polar coordinates, J_n is the nth Bessel function and $\lambda_{n,m}$ the mth positive zero of J_n ($\lambda_{n,m}$ is the frequency of $e_{n,m}$). The Bessel function can be defined as

$$J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(z\sin\theta + n\theta)} d\theta$$

and solves the differential equation $z^2 J_n'' + z J_n' + (z^2 - n^2) J_n = 0$, which follows from separation of variables when the eigenvalue equation for Δ is written in polar coordinates.

[1] In Peking, near a famous historical memorial, the Temple of Heaven, there is a miraculous stone wall, which forms an almost closed cylinder. The "miracle" consists in the fact that sounds uttered in low voice in one of the directions along the wall return back after some time to the person who uttered them. It seems that somebody invisible behind the back of the person pronounce the same sounds by the person's voice.

The modern physical explanation of this effect was proposed by Rayleigh as early as over a century ago. He explained the effect on the basis of his own observations made in an ancient gallery located under the dome of St. Paul's Cathedral in London. This gave the name "whispering gallery waves" for these waves. He found that the sound "clutches" to the wall surface and "creeps" along it. The concave surface of the dome does not allow the beam cross section to expand as fast as during propagation in free space. While in the latter case the beam cross section increases and the radiation intensity decreases proportionally to the square of the distance from a source, the radiation in the whispering gallery propagates within a narrow layer adjacent to the wall surface. As a result, the sound intensity inside this layer decreases only direct proportionally to the distance, i.e. much slower than in the free space (see also [16]).

Because the treatment of the transversal part is independent of the geometry of the boundary, one would expect that only eigenfunctions make trouble with gradients almost tangential to the boundary. For the $e_{n,m}$ this means that the ratio of angular frequency n to radial frequency (which depends monotonically on $\lambda_{n,m}$ and therefore on m) should be big and thus m small. In fact, after collecting some facts on the asymptotic behavior of Bessel functions one can see the eigenfunctions which correspond to tangent waves are those with $m = 1$ that we denote $f_n = e_{n,1}$. The corresponding eigenfunction's asymptotic is given by $\lambda_{n,1} \simeq_{n\to\infty} n + \alpha n^{1/3} + O(n^{-2/3})$, $\alpha > 0$. In contrast, the oscillation of the $e_{0,m}$ is purely radial and the estimates (1.4) and (1.2) are easily verified directly. D.Grieser [7] showed that a Bessel eigenfunction f_λ of eigenvalue λ satisfies

$$\|f_\lambda\|_{L^p(\Omega)} \simeq \lambda^{\frac{2}{3}(\frac{1}{2}-\frac{1}{p})}\|f_\lambda\|_{L^2(\Omega)},$$

which contradicts the above estimate (1.4) for $d = 2$ for $6 \le p < 8$. Grieser's counterexample involves the classical Rayleigh whispering gallery modes, which have L^2 mass that is concentrated in a $\simeq \lambda^{-2/3}$ neighborhood of the boundary. This is a greater concentration than the $\simeq \lambda^{-1/2}$ concentration around stable elliptic orbits in the boundaryless case, and hence the L^p estimates on manifolds with boundary must be worse than those in the boundaryless case.

As far as the Stricartz estimates are concerned, in [8] we proved the following

Proposition 2.6. *Restricted to the gallery modes* $\{e^{in\theta}J_n(\lambda_{n,0}r)\}_{n\in\mathbb{Z}}$*, the wave flow* $\exp(it\sqrt{-\Delta_g})$ *satisfies sharp dispersive/Strichartz estimates.*

Here we used the asymptotic behavior $J_n(\lambda_{n,1}r) \simeq \phi(\lambda^{2/3}(1-r))$, $n \simeq \lambda$ and that if the initial data writes as a superposition of whispering gallery modes

$$f(r,\theta) = \sum_{n\in[\lambda,2\lambda]} e^{in\theta}J_n(\lambda_{n,1}r),$$

then applying the wave flow yields

$$\exp(it\sqrt{-\Delta_g})f \simeq \phi(\lambda^{2/3}(1-r)) \sum_{n\in[\lambda,2\lambda]} e^{in(\theta-t)}e^{-i\alpha t n^{1/3}}.$$

Remark 2.7. It is worth noticing that applying the semi-classical Schrödinger evolution shows that a loss of derivatives is unavoidable for the Strichartz estimates. However, while dealing with the wave operator, this strategy fails as the gallery modes satisfy the usual Strichartz estimates.

Remark 2.8. The result in [8, 9] is related to earlier work of Gilles Lebeau [13] and involves conormal cusp waves. In the next section we will give a brief idea of the proof of the counterexample and show the similitudes with G.Lebeau's announced approach in [13]. The last section of this note will be entirely devoted to G.Lebeau's announced result, whose detailed proof is available in a forthcoming work in collaboration with Fabrice Planchon.

2.2. Dispersive estimates

G.Lebeau was the first who described in [13] the dispersive estimates on small time intervals for the solutions of (1.1) inside a strictly convex domain (Ω, g) of dimension $d \ge 2$. He considered the equation (1.1) with initial data $(u_0, u_1) = (\delta_a, 0)$, where

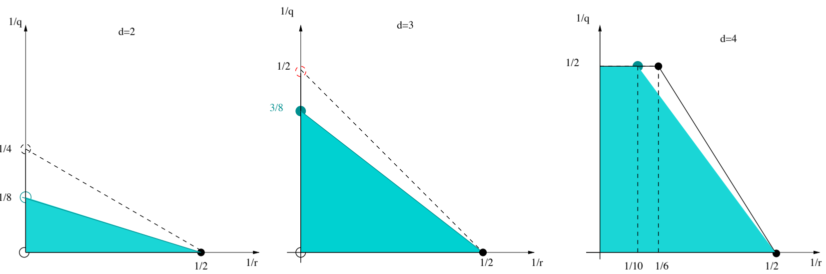


Figure 2.3: Scale-invariant Stricharz estimates obtained from Theorem 2.9

$a \in \Omega$ is a point sufficiently close to the boundary and announced the following result [13][Theorem 1.1]:

Theorem 2.9. *If u_a denotes the solution to (1.1), then there exists $T > 0$, $C > 0$ s.t. for every $h \in (0,1]$ and $t \in (0,T]$ the solution u_a satisfies*

$$|\psi(hD_t)u(t,x)| \lesssim h^{-d}\min(1,(h/t)^{\frac{d-2}{2}+\frac{1}{4}}). \tag{2.3}$$

Recall that if $u_{a,\mathbb{R}^d} = (2\pi)^{-d}\int \cos(t|\xi|)e^{i(x-a)\xi}d\xi$ is the Green function in the Euclidian space, i.e. it solves (1.1) in $\mathbb{R}^d$ with the same data, the classical dispersive estimates reads as follows

$$\|u_{a,\mathbb{R}^d}(t,.)\|_{L^\infty(\mathbb{R}^d)} \lesssim h^{-d}\min\{1,(h/t)^{\frac{d-1}{2}}\}.$$

The estimate (2.3) means that, compared to the dispersive estimate in the free space, there is a loss of a power of $1/4$ of h/t inside a strictly convex domain, and this is due to micro-local phenomena such as caustics generated in arbitrarily small time near the boundary. A main point of the proof consists in a detailed description of the "sphere" of center a and radius t, i.e. the set of points of Ω which can be reached following all the optical rays starting from a of length t.

Lebeau's announced result is optimal for the dispersion and this is due to the presence of swallowtail type singularities in the wave front set of u_a, as we will show in the last part of this note. Notice that it still an open problem to determine the (scale-invariant) admissible indices (d,q,r) in the Strichartz estimates. A consequence of (2.3) would be the validity of the Strichartz estimates for (1.1) for the range of indices (d,q,r) satisfying

$$\frac{1}{q} \leq (\frac{d-2}{2}+\frac{1}{4})(\frac{1}{2}-\frac{1}{r}).$$

As a remark, even if in dimension $d=2$ the range of admissible indices for which sharp Strichartz hold has been recently generalized in [1], in dimension $d \geq 3$ the result in Theorem 2.9 greatly improves the range of indices for which sharp Strichartz do hold.

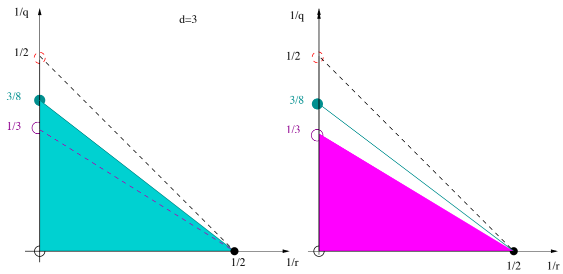


Figure 2.4: Improved range of indices for which the Strichartz estimates hold for $d = 3$

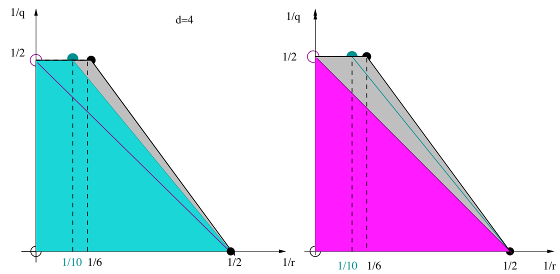


Figure 2.5: Improved range of indices for which the Strichartz estimates hold for $d = 4$

3. Sketch of the proofs

3.1. Choice of a suitable model domain

We consider here the Friedlander's model domain of the two-dimensional half-space $\Omega = \{(x, y)|x > 0, y \in \mathbb{R}\}$ with Laplace operator given by $\Delta_g = \partial_x^2 + (1 + x)\partial_y^2$ and we denote by $p(t, x, y, \tau, \xi, \eta) = \xi^2 + (1 + x)\eta^2 - \tau^2$ the symbol of the wave operator $\Box = \partial_t^2 - \Delta_g$. The characteristic set of $\Box$ is the closed conic set $\{(t, x, y, \tau, \xi, \eta)|p(t, x, y, \tau, \xi, \eta) = 0\}$, denoted $\mathrm{Char}(p)$. We define the semi-classical wave front set $WF_h(u)$ of a distribution u on $\mathbb{R}^3$ to be the complement of the set of points $(\rho = (t, x, y), \zeta = (\tau, \xi, \eta)) \in \mathbb{R}^3 \times (\mathbb{R}^3 \setminus 0)$ for which there exists a symbol $a(\rho, \zeta) \in \mathcal{S}(\mathbb{R}^6)$ such that $a(\rho, \zeta) \neq 0$ and for all integers $m \geq 0$ the following holds

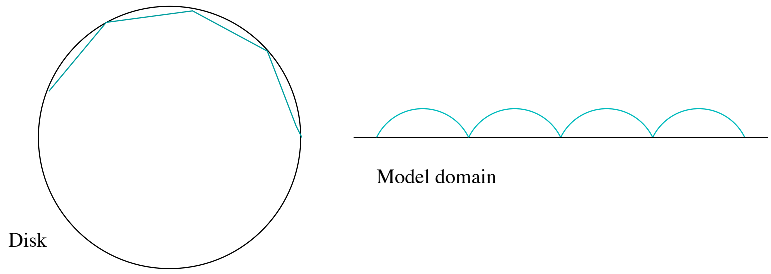
$$\|a(\rho, hD_\rho)u\|_{L^2} \leq c_m h^m.$$

Let $\rho = \rho(\sigma)$, $\zeta = \zeta(\sigma)$ be a bicharacteristic of $p(\rho, \zeta)$, i.e. such that (ρ, ζ) satisfies

$$\frac{d\rho}{d\sigma} = \frac{\partial p}{\partial \zeta}, \quad \frac{d\zeta}{d\sigma} = -\frac{\partial p}{\partial \rho}, \quad p(\rho(0), \zeta(0)) = 0. \tag{3.1}$$

Assume that the interior of Ω is given by the inequality $\gamma(\rho) > 0$, in this case $\gamma(\rho = (t, x, y)) = x$. Then $\rho = \rho(\sigma)$, $\zeta = \zeta(\sigma)$ is tangential to $\mathbb{R} \times \partial\Omega$ if

$$\gamma(\rho(0)) = 0, \quad \frac{d}{d\sigma}\gamma(\rho(0)) = 0. \tag{3.2}$$



We say that a point (ρ, ζ) on the boundary is a gliding point if it is a tangential point and if in addition

$$\frac{d^2}{d\sigma^2}\gamma(\rho(0)) < 0. \tag{3.3}$$

This is equivalent (see for example [5]) to saying that $(\rho, \zeta) \in T^*(\mathbb{R} \times \partial\Omega) \setminus 0$ is a gliding point if

$$p(\rho, \zeta) = 0, \quad \{p, \gamma\}|_{(\rho,\zeta)} = 0, \quad \{\{p, \gamma\}, p\}|_{(\rho,\zeta)} > 0, \tag{3.4}$$

where $\{.,.\}$ denotes the Poisson braket. We say that a point (ρ, ζ) is hyperbolic if $x = 0$ and $\tau^2 > \eta^2$, so that there are two distinct nonzero real solutions ξ to $\xi^2 + (1 + x)\eta^2 - \tau^2 = 0$.

Remark 3.1. It is clear that, together with metric inherited from the Δ_g, Ω becomes a strictly convex domain: in fact, after the change of variables $x = 1 - r$, $y = \theta$, one can easily see that Δ_g becomes equal, modulo first order terms, to the operator $\partial_r^2 + \frac{1}{r^2}\partial_\theta^2$ inside the unit disk $\{r \leq 1\}$.

3.2. The counterexample - Sketch of the proof of Theorem 2.2

The key feature of the manifold leading to the counterexample is the strict geodesic convexity of the boundary, i.e. the presence of highly-multiply reflected rays. The particular manifold studied in[8] is one for which the eigenmodes can be explicitly expressed in terms of Airy's function, and the phase for the oscillatory integrals to be evaluated have precise form: the counterexample is constructed, essentially, as a superposition of traveling cusp solutions to the wave equation (1.1).

The eigenfunctions of Δ_g can be written as $e^{iy\eta}Ai(|\eta|^{2/3}x-\omega_k)$, where the Dirichlet condition dictates that $-\omega_k$ be the zeroes of the Airy function $Ai(-\omega_k) = 0$. Rewriting the mode in the form $e^{iy\eta}Ai(|\eta|^{2/3}(x - a))$, the eigenvalue is $\lambda_a := |\eta|(1 + a)^{1/2}$, which means that such a wave moves with velocity $(1 + a)^{1/2}$ in the tangential direction y. The associated eigenfunction reads as

$$\int_\eta e^{it\lambda_a} e^{iy\eta} Ai(\eta^{2/3}(x - a))d\eta. \tag{3.5}$$

Imposing the Dirichlet boundary condition at this level yields a condition on the parameter a, precisely, a must depend in this case on some zeros ω_k of the Airy function. A simple computation shows that such a wave "lives" essentially in the regime of the whispering gallery modes (since, if it is localized at frequency $\frac{1}{h}$, one can easily see that it will remain essentially supported in a neighborhood of size $h^{2/3}$ of the boundary), therefore, in view of Proposition 2.6, it will satisfy sharp estimates.

To construct waves that do not disperse requires superimposing waves with the same value of a. If one ignores the boundary condition for the moment, the superposition of such waves over a range $\eta \in [\frac{1}{h}, \frac{2}{h}]$ would give, as can be seen by the asymptotic of the Airy function, a solution living in an h-neighborhood of the cusp

$$y - (1+a)^{1/2}t = \pm|a-x|^{3/2}, \quad x \in [0,a]. \tag{3.6}$$

Indeed, this follows by performing stationary phase arguments in the integral defining the wave (3.5), where we have expressed the Airy function in terms of its integral formula

$$\int_{\xi,\eta} e^{i\eta(y-t(1+a)^{1/2}+\xi(x-a)+\frac{\xi^3}{3})} d\xi d\eta. \tag{3.7}$$

We denote by Φ the phase function in (3.7). Then the Lagrangian manifold associated to the phase function Φ (which contains the wave front set WF_h of this wave) is given by

$$\Lambda_\Phi = \{(t,x,y,\tau = \partial_t\Phi, \xi = \partial_x\Phi = \eta s, \eta = \partial_y\Phi) | \partial_s\Phi = 0, \partial_\eta\Phi = 0\} \subset T^*\mathbb{R}^3 \setminus 0. \tag{3.8}$$

Let $\pi : \Lambda_\Phi \to \mathbb{R}^3$ be the natural projection and let Σ denote the set of its singular points. The points where the Jacobian of $d\pi$ vanishes lie over the caustic set, thus the fold set is given by $\Sigma = \{\xi = 0\}$ and the caustic is defined by $\pi(\Sigma) = \{x + (1 - \frac{\tau^2}{\eta^2}) = 0\} = \{x = a\}$.

Remark 3.2. The motivation of this construction comes from the fact that near the caustic set $\pi(\Sigma)$ one notices the cusp type singularity (3.6) for which one can compute explicitly the $L^r(\Omega)$ norms. The key observation is that if the parameter a is very small this cusp type parametrix provides a loss in the Strichartz estimates (which increases when a gets smaller). In particular, if a could be chosen to be $\simeq h^{2/3}$, then the associated loss would involve the optimality of the work [1] and the gap between the positive results and the counterexample given by Theorem 2.2 would be filled.

Therefore the goal is to construct a similar solution (with a cusp type singularity) that satisfies boundary conditions at $x = 0$, while taking a as small as possible depending on h. Rather then attempt to deal with the zeroes of the Airy function, the boundary condition are met by taking a superposition of localized cusp solutions, each term in the sum being chosen to cancel off the boundary value of the previous term. The relation between amplitudes will be dictated by the billiard ball maps. Roughly speaking, the billiard ball maps $\delta^\pm : T^*(\mathbb{R} \times \partial\Omega) \to T^*(\mathbb{R} \times \partial\Omega)$, defined on the hyperbolic region, continuous up to the boundary, smooth in the interior, are defined at a point of $T^*(\mathbb{R} \times \partial\Omega)$ by taking the two rays that lie over this point, in the hypersurface Char(p), and following the null bicharacteristic through these points until you pass over $\{x = 0\}$ again, projecting such a point onto $T^*(\mathbb{R} \times \partial\Omega)$ (a gliding point being "a diffractive point viewed from the other side of the boundary", there is no bicharacteristic in $T^*(\mathbb{R} \times \partial\Omega)$ through it, but in any neighborhood of a gliding point there are hyperbolic points).

In our model case the analysis is simplified by the presence of a large commutative group of symmetries, the translations in the tangential and time variables (y,t), and the billiard ball maps have specific formulas

$$\delta^{\pm}(y,t,\eta,\tau)=\Big(y\pm 4(\frac{\tau^2}{\eta^2}-1)^{1/2}\pm\frac{8}{3}(\frac{\tau^2}{\eta^2}-1)^{3/2},t\mp 4(\frac{\tau^2}{\eta^2}-1)^{1/2}\frac{\tau}{\eta},\eta,\tau\Big). \quad (3.9)$$

The composite relation $(\Lambda_{\Phi|_{x=0}})^{\circ n}$ can be obtained using the graphs of the iterates $(\delta^{\pm})^n$, namely

$$(\delta^{\pm})^n(y,t,\eta,\tau)=\Big(y\pm 4n(\frac{\tau^2}{\eta^2}-1)^{1/2}\pm\frac{8}{3}n(\frac{\tau^2}{\eta^2}-1)^{3/2},t\mp 4n(\frac{\tau^2}{\eta^2}-1)^{1/2}\frac{\tau}{\eta},\eta,\tau\Big). \quad (3.10)$$

All these graphs, of the powers of $\delta^{\pm}$, are disjoint away from $\pi(\Sigma)$ and in order to find microlocal representations of the associated Fourier integral operators it is necessary to find a parametrization of each. We see that

$$y\eta+t\tau+\frac{4}{3}\eta(\frac{\tau^2}{\eta^2}-1)^{3/2},$$

are parametrizations of $\Lambda_{\Phi|_{x=0}}$, thus the iterated Lagrangians $(\Lambda_{\Phi|_{x=0}})^{\circ n}$ are parametrized by

$$y\eta+t\tau+\frac{4}{3}n\eta(\frac{\tau^2}{\eta^2}-1)^{3/2},$$

and the corresponding phase functions associated to $(\Lambda_{\Phi})^{\circ n}$ will be given by

$$\begin{aligned}\Phi_n&=\Phi+\frac{4}{3}n\eta(\frac{\tau^2}{\eta^2}-1)^{3/2}\\&=\eta(y-t(1+a)^{1/2}+\xi(x-a)+\frac{\xi^3}{3}+\frac{4}{3}na^{3/2}).\end{aligned}$$

Therefore we shall construct a parametrix for (1.1) under the form

$$U_h(t,x,y)=\sum_{n=0}^{N}u_h^n(t,x,y),\quad u_h^n(t,x,y)=\int e^{\frac{i}{h}\Phi_n}g_h^n(t,\xi,\eta)d\eta d\xi,$$

where $\Phi_n=\Phi+\frac{4}{3}n\eta a^{3/2}$ are the phase functions defined above such that $\Lambda_{\Phi_n}=(\Lambda_\Phi)^{\circ n}$ and where the symbols g_h^n are chosen such that on the boundary the Dirichlet condition to be satisfied. At $x=0$ the phases have two critical, non-degenerate points, thus each u_h^n writes as a sum of two trace operators, $\mathrm{Trace}_{\pm}(u_h^n)$, localized respectively for $y-(1+a)^{1/2}t+\frac{4}{3}na^{3/2}$ near $\pm\frac{2}{3}na^{3/2}$, and in order to obtain a contribution $O_{L^2}(h^\infty)$ on the boundary we chose the symbol g_h^{n+1} such that $\mathrm{Trace}_-(g_h^n)+\mathrm{Trace}_+(g_h^{n+1})=O_{L^2}(h^\infty)$. This is possible by Egorov theorem, iff $N\ll a^{3/2}/h$.

Moreover, if at $t=0$ one considers symbols localized in a small neighborhood of the caustic set, then one can show that the respective "pieces of cusps" propagate until they reach the boundary but short after that their contribution becomes $O_{L^2}(h^\infty)$, since as t increases, ξ takes greater values too and thus one quickly quits a neighborhood of the Lagrangian Λ_Φ (where $\xi^2=a-x$) which contains the semiclassical wave front set $WF_h(u_h)$. This argument is valid for all u_h^n, hence we would like to take advantage of the fact that at a given time t we have nontrivial contributions coming from no more than two successive cusps so that we can estimate the $L^q([0,1],L^r(\Omega))$ norm of the sum U_h to be of the size of the sum of the norms of each u_h^n. We will be able to do this if we impose the symbols g_h^n to have compact supports in a fixed-sized neighborhood of $\pi(\Sigma)$ uniformly with respect to $n\in\{0,..,N\}$.

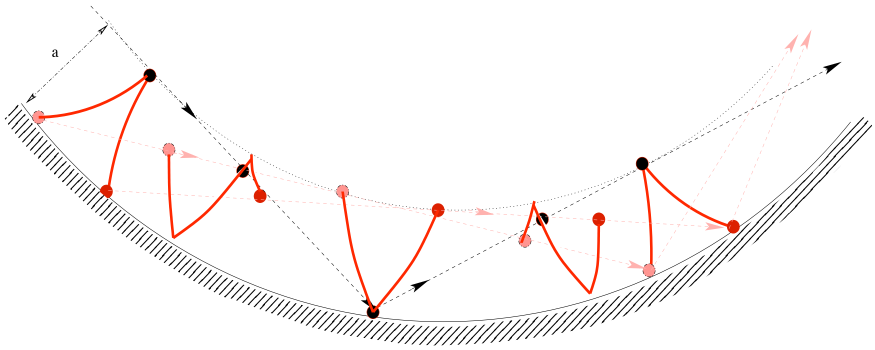


Figure 3.1: Propagation of the cusp: preserving ray optics picture requires $a > h^{1/2}$

Indeed, this last condition is crucial since, after the reflection on the boundary, the n-th piece of cusp u_h^n will continue to give a nontrivial contribution on a time interval directly proportional to the size of the support of g_h^n. If these supports become larger with n, then u_h^n starts to interact with other cusps around and we are not allowed to use the almost orthogonality of the supports in time anymore.

Under these assumptions, if we denote I_n the time interval (of size $\sqrt{a}$) on which u_h^n is essentially supported and let $J_n \subset I_n$ be smaller intervals chosen so that for $t \in J_n$ there is only the cusp u_h^n which has a nontrivial contribution, then we have the following:

$$\begin{aligned} \|U_h\|^q_{L^q([0,1],L^r(\Omega))} &\geq \sum_k \int_{J_k} \|\sum_n u_h^n(t,.)\|^q_{L^r(\Omega)} dt \\ &\simeq \sum_k \int_{J_k} \|u_h^k(t,.)\|^q_{L^r(\Omega)} dt \\ &\simeq N a^{1/2} \|u_h^0(0,.)\|^q_{L^r(\Omega)} \\ &\simeq \|u_h^0(0,.)\|^q_{L^r(\Omega)}. \end{aligned}$$

Remark 3.3. These last conditions on the symbols, together with the assumption of finite time (which implies that $N \times$ the size of the support in time I_n of u_h^n should be equal to one) allows to estimate the number of reflections N and gives a lower bound for the parameter a as follows:

$$a > h^{1/2} \quad \text{and} \quad N \simeq \frac{1}{\sqrt{a}}.$$

Since we were looking for the smallest possible values of a we take $a \simeq h^{1/2}$ and compute the $L^r(\Omega)$ norm of the initial cusp u_h^0. We obtain the following estimates, uniformly with respect to $t \in J_0 \subset I_0$

- If $2 \leq r < 4$ then

$$\frac{\|u_h^0(t,.)\|_{L^r(\Omega)}}{\|u_h^0(0,.)\|_{L^2(\Omega)}} \simeq h^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})}, \tag{3.11}$$

- If $r > 4$ then

$$\frac{\|u_h^0(t,.)\|_{L^r(\Omega)}}{\|u_h^0(0,.)\|_{L^2(\Omega)}} \simeq h^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})+\frac{1}{6}(\frac{1}{4}-\frac{1}{r})}. \tag{3.12}$$

Since $\beta(d=2,q,r)=\frac{3}{2}(\frac{1}{2}-\frac{1}{r})$, with (q,r) sharp admissible, is the order of the Sobolev space for which the Strichartz estimates hold in the free space, we notice that for $r > 4$ we have obtained in (3.12) a lower bound of $h^{-\beta(2,q,r)+\frac{1}{6}(\frac{1}{4}-\frac{1}{r})}$, and therefore (since all the functions are spectrally localized at frequency $\frac{1}{h}$) we deduce that a loss in the Stricharz estimates (compared to the free case) of at least $\frac{1}{6}(\frac{1}{4}-\frac{1}{r})$ derivatives must occur. On the other hand, the estimate (3.11) simply says that for $2 \leq r < 4$ the above contruction doesn't contradict the estimates of the free space.

Remark 3.4. We conjecture that for $2 \leq r \leq 4$ there are no losses in the Strichart inequalities inside a strictly convex domain. This a consequence of the proof of Theorem 2.9 that will be sketched in the next section together with some refined computations involving the Pearcey type integrals. This is a work in progress in collaboration with Fabrice Planchon.

Remark 3.5. Notice that like in the Knapp's counterexample, this construction involves wave packets of size $h^{1/2}$; this localization will also play an essential role in the proof of Theorem 2.9.

3.3. The dispersive estimates - Sketch of the proof of Theorem 2.9

This last section is devoted to the proof of Theorem 2.9 which has been announced in [13]. Precisely, in [13] Gilles Lebeau sketched the main steps of the proof and gave a full description of the geometry behind. However, many details are missing and therefore, our forthcoming work in collaboration with Fabrice Planchon is intended to complete the analytical part of G.Lebeau's result.

While working in this direction, the first thing to understand is the type of concentration phenomena such as caustics that may occur near the boundary. *What are caustics?* Caustics are envelopes of light rays that appear in a given problem. At the caustic point the intensity of light is singularly large, causing different physical phenomena. An example of a caustic is given in Picture 3.2, where the caustic is the smooth curve $\pi(\Sigma)$ corresponding to $x = a$. Each ray is tangent to the caustic at a given point. If one assigns a direction on the caustic, it induces a direction on each ray. Each point outside the caustic lies on a ray which has left the caustic and also lies on a ray approaching the caustic. Each curve of constant phase has a cusp where it meets the caustic.

Mathematically, caustics could be characterized as points were usual bounds on oscillatory integrals are no longer valid. Oscillatory integrals with caustics have enjoyed much attention: it is well known that their asymptotic behavior is governed by the number and the order of their critical points which are real. Let us recall some background about them. Let us consider an oscillatory integral

$$u_h(z) = \frac{1}{(2\pi h)^{1/2}} \int_\zeta e^{\frac{i}{h}\Phi(z,\zeta)} g(z,\zeta,h) d\zeta, \quad z \in \mathbb{R}^d, \quad \zeta \in \mathbb{R}, \quad h \in (0,1]. \tag{3.13}$$

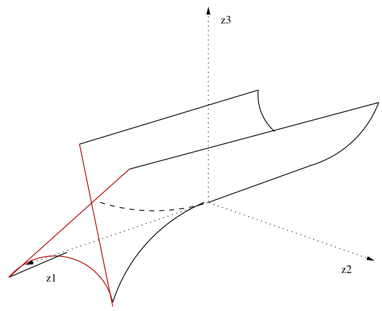


Figure 3.2: The caustic for the swallowtail catastrophe.

We assume that Φ is smooth and that $g(.,h)$ is compactly supported in z and in ζ. If there are no critical points of the map $\zeta \to \Phi(z,\zeta)$, so that $\partial_\zeta \Phi \neq 0$ everywhere in an open neighborhood of the support of $g(.,h)$, then the repeated integration by parts shows that $|u_h(z)| = O(h^N)$, for any $N > 0$. If there are non-degenerate critical points, where $\partial_\zeta \Phi = 0$ but $\det(\partial^2_{\zeta_j\zeta_k}\Phi) \neq 0$, then the method of stationary phase applies and gives $\|u_h(z)\|_{L^\infty} = O(1)$. It follows that as long as all critical points are non-degenerate, $u_h(z) \in L^r(\mathbb{R}^d)$, $2 \leq r \leq \infty$, with the norms bounded uniformly in $h \in (0,1]$.

If there are degenerate critical points, known as caustics, then $\|u_h(z)\|_{L^\infty}$ is no longer uniformly bounded. The order of a caustic κ is defined as the infimum of κ' so that $\|u_h(z)\|_{L^\infty} = O(h^{-\kappa'})$.

Remark 3.6. For example, the phase function $\Phi_F(z,\zeta) = \frac{\zeta^3}{3} + z_1\zeta + z_2$ corresponds to a fold with order $\kappa = \frac{1}{6}$. It turns out that the phase function which appears in the proof of Theorem 2.2 and corresponds to a cusp, homogeneous in η, writes under the form $\eta(\frac{\xi^3}{3} + z_1\xi + z_2)$. This is the reason why the loss we obtained there is precisely of $\frac{1}{6}$ derivatives. Notice that the canonical form of a cusp type singularity is given by a phase function which is a polynomial of degree 4, namely of the form $\Phi_C(z,\zeta) = \frac{\zeta^4}{4} + z_1\frac{\zeta^2}{2} + z_2\zeta + z_3$ whose order is $\kappa = \frac{1}{4}$; its associated integral is called the Pearcey's function. This integral plays a crucial role in the proof of Theorem 2.9 together with the swallowtail integral (which is an oscillatory integral with four coalescing saddle points) whose canonical form is given by a polynomial of degree 5, $\Phi_S(z,\zeta) = \frac{\zeta^5}{5} + z_1\frac{\zeta^3}{3} + z_2\frac{\zeta^2}{2} + z_3\zeta + z_4$.

The caustic surface of the swallowtail is defined by the condition that two or more *real* saddle points are equal. In the event that two simple saddle points undergo confluence when $z \to z_0$, then the uniform asymptotic behavior of (3.13) contains terms involving the Airy function and its derivatives multiplied by powers of $h^{-\frac{1}{2}+\frac{1}{3}}$. If three simple saddles coalesce as $z \to z_0$, then the uniform asymptotic behavior of (3.13) can be described by terms containing the Pearcey function and its first-order derivatives, each multiplied by a power of $h^{-\frac{1}{2}+\frac{1}{4}}$. The swallowtail enters in picture when four simple saddle points of (3.13) undergo confluence as $z \to z_0$.

Let us now mention the main ideas of the proof of Theorem 2.9. The first step consists into a decomposition in wave packets depending on the number of reflections on the boundary. It turns out that it will be sufficient to prove the estimates (2.3) for the following initial data:

$$u_0(x,y)=\frac{1}{(2\pi h)^2}\int e^{\frac{i}{h}((x-a)\xi+y\eta)}\psi(\eta)\rho(\frac{\xi}{h^{1/4}})d\xi d\eta,$$

where ψ,ρ are smooth functions compactly supported in a neighborhood of 1 and 0, respectively, $\psi\in C_0^\infty(\frac{1}{2},2)$, $\rho\in C_0^\infty(-\frac{1}{2},\frac{1}{2})$. If the initial distance a to the boundary is sufficiently small, namely $a\leq h^{\frac{1}{2}}$, we proceed like in [13], writing a parametrix u for (1.1) and using the fact that the essential support of the Fourier transform $\hat{u}$ remains small, together with the elementary estimate [13][(2.24)]. In order to construct the parametrix u we use the FBI transform for t close 0 so that we can describe the wave before the reflection on the boundary; we then solve the Airy equation explicitly with data on the boundary given by the trace of $\hat{u}$ on $\partial\Omega$. We repeat this construction a number of times $N\simeq\frac{1}{\sqrt{a}}$. We obtain a parametrix of the form

$$U_h(t,x,y)=\sum_{n=0}^{N}u_n(t,x,y),$$

with

$$u_n(t,x,y)=\int e^{\frac{i}{h}\eta\phi_n(t,x,y,\xi)}g_h^n(x,y,t,\xi)\tilde{\psi}(\eta)\tilde{\rho}(h^{-1/2}(\frac{a}{\xi}-\frac{\xi}{4}))d\xi d\eta,$$

where $\tilde{\psi},\tilde{\rho}$ are smooth functions essentially supported in a neighborhood of 1 and 0, respectively. The symbols g_h^n are chosen so that u_n to have almost orthogonal supports in time and so that the Dirichlet condition to be satisfied. For $a>h^{\frac{1}{2}}$ we study the asymptotic behavior of the parametrices u_n. We have the following Lemma, which is the equivalent of [13][Lemma 3.7]:

Lemma 3.7. *For every $n\in\{1,..,N\}$, the phase ϕ_n has saddle points of order at most 3; for each $n\in\{1,..,N\}$ there exists a unique time $t=t_{S,n}$ for which $\phi_n(t)$ has a critical point ξ_S of order 3. For $t\neq t_{S,n}$ the phase $\phi_n(t)$ has only critical points of order at most 2. Moreover, for each n we have $16a(n-1)n<t_{S,n}^2<16an(n+1)$.*

From the above Lemma it follows, using Arnold's classification, that ϕ_n is a Pearcey type integral with order $\frac{1}{4}$. Writing the asymptotic expansion of $u_n(t)$ near $t_{S,n}$, we deduce that a loss of $\frac{1}{4}$ powers of $\frac{|t|}{h}$ is unavoidable in the L^∞ norm of u_n.

Theorem 3.8. *The loss of $\frac{1}{4}$ powers of $\frac{|t|}{h}$ in the dispersive estimates (2.3) is optimal in any dimension $d\geq 2$.*

Indeed, the optimality follows from the fact that there is a swallowtail type singularity in the wavefront set $WF_h(u_n)$ for each $n\in\{1,..,N\}$. This loss occurs only from the dispersion in the normal variable x, therefore is enough to prove the result in dimension $d=2$.

Remark 3.9. In the first picture of Section 3.3 we can see the propagation of the wavefront set of U_h; the second picture is a zoomed version of the first one and shows in detail the formation of the swallowtail singularity for packets moving to directions tangent to the boundary.

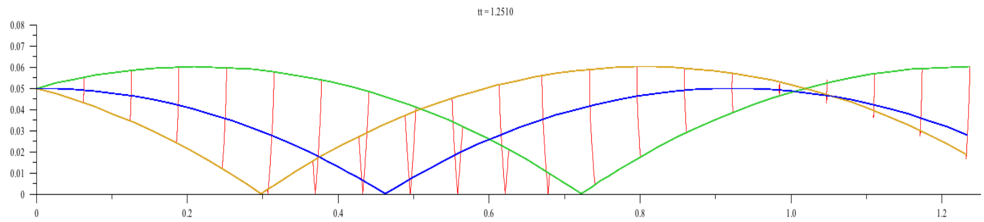


Figure 3.3: Propagation of the wavefront

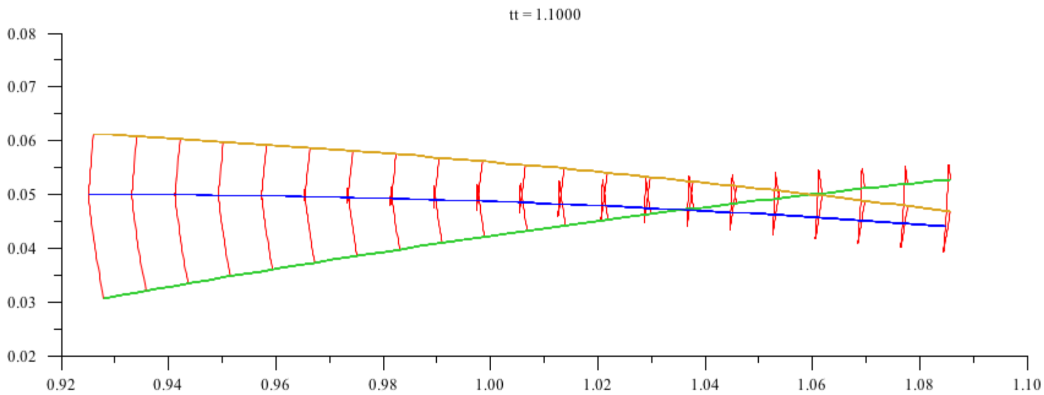


Figure 3.4: The formation of a swallowtail singularity just after the first reflection (zoomed image)

References

[1] Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge. Strichartz estimates for the wave equation on manifolds with boundary. *to appear in Ann.Inst.H.Poincaré, Anal.Non Liréaire.*

[2] Nicolas Burq, Patrick Gérard, and Nicolay Tzvetkov. Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math.*, 126(3):569–605, 2004.

[3] Nicolas Burq, Gilles Lebeau, and Fabrice Planchon. Global existence for energy critical waves in 3-D domains. *J. Amer. Math. Soc.*, 21(3):831–845, 2008.

[4] E. B. Davies. The functional calculus. *J. London Math. Soc. (2)*, 52(1):166–176, 1995.

[5] Gregory Eskin. Parametrix and propagation of singularities for the interior mixed hyperbolic problem. *J. Analyse Math.*, 32:17–62, 1977.

[6] J. Ginibre and G. Velo. Generalized Strichartz inequalities for the wave equation. In *Partial differential operators and mathematical physics (Holzhau, 1994)*, volume 78 of *Oper. Theory Adv. Appl.*, pages 153–160. Birkhäuser, Basel, 1995.

[7] Daniel Grieser. l^p bounds for eigenfunctions and spectral projections of the Laplacian near concave boundaries. Thesis, UCLA, 1992. `http://www.staff.uni-oldenburg.de/daniel.grieser/wwwpapers/diss.pdf`.

[8] Oana Ivanovici. Counter example to Strichartz estimates for the wave equation in domains, 2008. to appear in Math. Annalen, `arXiv:math/0805.2901`.

[9] Oana Ivanovici. Counterexamples to the Strichartz estimates for the wave equation in domains II, 2009. `arXiv:math/0903.0048`.

[10] Oana Ivanovici and Fabrice Planchon. Square function and heat flow estimates on domains, 2008. `arXiv:math/0812.2733`.

[11] L. V. Kapitanskiĭ. Some generalizations of the Strichartz-Brenner inequality. *Algebra i Analiz*, 1(3):127–159, 1989.

[12] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[13] Gilles Lebeau. Estimation de dispersion pour les ondes dans un convexe. In *Journées "Équations aux Dérivées Partielles" (Evian, 2006)*. 2006. see `http://www.numdam.org/numdam-bin/fitem?id=JEDP_2006____A7_0`.

[14] Hans Lindblad and Christopher D. Sogge. On existence and scattering with minimal regularity for semilinear wave equations. *J. Funct. Anal.*, 130(2):357–426, 1995.

[15] Francis Nier. A variational formulation of Schrödinger-Poisson systems in dimension $d \leq 3$. *Comm. Partial Differential Equations*, 18(7-8):1125–1147, 1993.

[16] A.N. Oraevsky. Whispering-gallery waves. *Quantum Electronics*, 32(5):377–400, 2002.

[17] Hart F. Smith. A parametrix construction for wave equations with $C^{1,1}$ coefficients. *Ann. Inst. Fourier (Grenoble)*, 48(3):797–835, 1998.

[18] Hart F. Smith and Christopher D. Sogge. On the critical semilinear wave equation outside convex obstacles. *J. Amer. Math. Soc.*, 8(4):879–916, 1995.

[19] Hart F. Smith and Christopher D. Sogge. On the L^p norm of spectral clusters for compact manifolds with boundary. *Acta Math.*, 198(1):107–153, 2007.

[20] Robert S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

[21] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. III. *J. Amer. Math. Soc.*, 15(2):419–442 (electronic), 2002.

Université de Nice Sophia-Antipolis, Laboratoire J.A.Dieudonné,
Parc Valrose 06108 Nice Cedex 02 FRANCE
`Oana.Ivanovici@math.unice.fr`