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Etienne Sandier

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Description of the ground state for a model of two-component rotating Bose–Einstein condensates.

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Abstract

In the joint work with Amandine Aftalion [3], we describe the ground states of a rotating two-component Bose–Einstein condensate in two dimensions. In the regime we consider, both a one-dimensional interface between the two components, and zero-dimensional interfaces (vortices) are present and contribute to the energy. The difficulty is that the two contributions are not of the same order, and to show that they somehow decouple requires a precise localisation of the line energy.

1. Introduction

Two component Bose–Einstein condensates are one of the simplest examples of increasingly complex atomic systems for which experimental realization has become possible in recent years. From a mathematical point of view, they generalize two types of models describing phase transitions. The first one is the two-phase models of Modica–Mortola type [12] in which two phases are separated by an perimeter-minimizing interface of codimension 1. The second-one is the models with quantized vortices as the Ginzburg–Landau model of superconductivity or models of superfluids or single-component Bose–Einstein condensates.

Each of these two types of models is by now well understood from a mathematical point of view, see [1, 2, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15]. One can, from this understanding, build an intuition for what the ground states in the model we study should behave like, at least for rotation speeds which are not too high. However it is mathematically not so simple to rigourously validate this intuition and it requires, in particular, to prove in a precise quantitative way that almost minimizers of Modica–Mortola type energy functionals have their energy concentrated near the interface. We report here on joint work with Amandine Aftalion [3] where this analysis is carried out.

A two component Bose–Einstein condensate (BEC) is described by two complex valued wave functions u_1 and u_2 minimizing the following energy functional:

$$E_{\varepsilon,\delta}^{\Omega}(u_1, u_2) = \sum_{j=1}^2 \int_D \frac{1}{2} |\nabla u_j - i\Omega x^\perp u_j|^2 + \int_D W_{\varepsilon,\delta}(|u_1|^2, |u_2|^2) \, dx \quad (1.1)$$

where

$$\begin{aligned} W_{\varepsilon,\delta}(|u_1|^2, |u_2|^2) &= \frac{1}{4\varepsilon^2} (1 - |u_1|^2)^2 + \frac{1}{4\varepsilon^2} (1 - |u_2|^2)^2 + \frac{\delta}{2\varepsilon^2} |u_1|^2 |u_2|^2 - \frac{1}{4\varepsilon^2} \\ &= \frac{1}{4\varepsilon^2} (1 - |u_1|^2 - |u_2|^2)^2 + \frac{(\delta - 1)}{2\varepsilon^2} |u_1|^2 |u_2|^2. \end{aligned} \quad (1.2)$$

Minimization is with respect to (u_1, u_2) belonging to the space

$$\mathcal{H} = \left\{ (u_1, u_2) : u_j \in H^1(D, \mathbb{C}), \int_D |u_j|^2 = \alpha_j, j = 1, 2 \right\}, \quad (1.3)$$

where $\int_D u$ denotes the average of u over D and $\alpha_1 + \alpha_2 = 1$.

The parameters δ, ε and Ω are positive: Ω is the angular velocity corresponding to the rotation of the condensate, $x^\perp = (-x_2, x_1)$. We are interested in studying the behavior of the minimizers in the limit when ε is small, describing strong interactions, also called the Thomas–Fermi limit. We focus on the regime where $(\delta - 1)$ is small. More precisely, we let $\tilde{\varepsilon} = \varepsilon/\sqrt{\delta(\varepsilon) - 1}$ and we assume that, as $\varepsilon \rightarrow 0$,

$$\varepsilon \ll \tilde{\varepsilon} \ll 1.$$

We need also to assume that the rotation is not too large. We have fairly precise results for $\Omega = \Omega(\varepsilon)$ such that $\Omega(\varepsilon) \ll 1/\tilde{\varepsilon}$, and more partial results for higher rotations, but in this report we will assume that for some $\beta > 0$ there holds

$$\Omega = \beta |\log \tilde{\varepsilon}|,$$

which is the case where the phenomenology is richest.

2. The case without rotation

The case where $\Omega = 0$ was studied for instance in [6], and can be cast in the framework of P.Sternberg’s generalization [15] of the analysis of Modica–Mortola [12] to two-well potentials of vector functions. We outline it below, writing F_ε for the energy with $\Omega = 0$.

The potential W_ε defined on \mathbb{R}_+^2 has exactly two minimum points $a = (1, 0)$ and $b = (0, 1)$. We may define the distance from $x \in \mathbb{R}_+^2$ to a as the energy needed to connect a to x :

$$d_\varepsilon(x, a) = \inf \left\{ \int_{-\infty}^0 \frac{1}{2} |\gamma'(t)|^2 + W_\varepsilon(\gamma(t)) dt \mid \gamma : \mathbb{R}_- \rightarrow \mathbb{R}^2, \lim_{-\infty} \gamma = a, \gamma(0) = x \right\}.$$

Similarly the distance from x to b is defined as

$$d_\varepsilon(x, b) = \inf \left\{ \int_0^{+\infty} \frac{1}{2} |\gamma'(t)|^2 + W(\gamma(t)) dt \mid \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}^n, \lim_{+\infty} \gamma = b, \gamma(0) = x \right\},$$

while we let

$$d_\varepsilon(a, b) = \inf \left\{ \int_{-\infty}^{+\infty} \frac{1}{2} |\gamma'(t)|^2 + W(\gamma(t)) dt \mid \gamma : \mathbb{R}_+ \rightarrow \mathbb{R}^n, \lim_{-\infty} \gamma = a, \lim_{+\infty} \gamma = b \right\}.$$

Then we let

$$d_\varepsilon(x) = \begin{cases} d_\varepsilon(x, a) & \text{if } d_\varepsilon(x, a) < d_\varepsilon(a, b)/2, \\ d_\varepsilon(a, b) - d_\varepsilon(x, b) & \text{if } d_\varepsilon(x, b) < d_\varepsilon(a, b)/2, \\ d_\varepsilon(a, b)/2 & \text{otherwise.} \end{cases}$$

The co-area formula then yields the following:

Proposition 2.1 ([15]). *We have, for any $u : D \rightarrow \mathbb{R}_+^2$,*

$$F_\varepsilon(u) \geq \int_0^{d_\varepsilon(a, b)} \text{per}_D(\{x \in D \mid d_\varepsilon(u(x)) < t\}) dt, \quad (2.1)$$

where $\text{per}_D(A)$ denotes the perimeter of A in D , i.e. the length of $\partial A \cap D$ if ∂A is a sufficiently nice curve.

This lower-bound is the central tool of the analysis: it allows to prove that, as $\varepsilon \rightarrow 0$, almost-minimizers $\{u_\varepsilon\}_\varepsilon$ of F_ε over \mathcal{H} are bounded in BV , and then converge modulo subsequences weakly in BV to a limit of the form $a\chi_{D_1} + b\chi_{D_2}$. The mass constraint built into the definition of \mathcal{H} translates to the fact that $|D_1| = \alpha|D|$ and $D_2 = (1 - \alpha)|D|$. Then, passing to the limit in the above lower-bound yields

$$\liminf_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon)}{m_\varepsilon} \geq \text{per}_D(D_1), \quad \text{where } m_\varepsilon = d_\varepsilon(a, b).$$

An upper bound can be constructed by modifying near the interface the function $a\chi_{D_1} + b\chi_{D_2}$, by using as the transition profile a minimizer in the infimum defining $d_\varepsilon(a, b)$. Both bounds together yield the following result,

Proposition 2.2. *Let $\{u_\varepsilon\}_\varepsilon$ be almost minimizers of F_ε , i.e. such that $F_\varepsilon(u_\varepsilon) \simeq \min_{\mathcal{H}} F_\varepsilon$ as $\varepsilon \rightarrow 0$. Then any sequence $\{\varepsilon\}$ converging to zero has a subsequence (not relabeled) which converges weakly in BV and strongly in L^1 to a limit of the form $a\chi_{D_1} + b\chi_{D_2}$, where $|D_1| = \alpha_1|D|$ and $D_2 = \alpha_2|D|$ is a partition of D with minimal interface length $\text{per}_D(D_1)$. Moreover, $\min_{\mathcal{H}} F_\varepsilon \simeq m_\varepsilon \text{per}_D(D_1)$ and $m_\varepsilon \simeq m_0/\tilde{\varepsilon}$ as $\varepsilon \rightarrow 0$.*

A crucial step in our analysis will be to make this statement quantitative, and to show that the energy of almost-minimizers concentrate in a neighbourhood of the interface.

3. The case with a single component

As long as the rotation speed Ω is not too large, the minimizers of $E_{\varepsilon,\delta}^\Omega$ are actually almost-minimizers of F_ε , or more precisely if $(u_{1,\varepsilon}, u_{2,\varepsilon})$ are minimizers of $E_{\varepsilon,\delta}^\Omega$, then $(\rho_{1,\varepsilon}, \rho_{2,\varepsilon})$ are almost minimizers of F_ε , where $\rho_{1,\varepsilon} = |u_{1,\varepsilon}|$ and $\rho_{2,\varepsilon} = |u_{2,\varepsilon}|$. Then modulo a subsequence $(\rho_{1,\varepsilon}, \rho_{2,\varepsilon})$ converges to $a\chi_{D_1} + b\chi_{D_2}$, and it is reasonable to expect that in D_1 , where $\rho_{1,\varepsilon}$ is close to one and $\rho_{2,\varepsilon}$ is close to zero, the behaviour will be that of a single component BEC, and similarly in D_2 .

Let us summarize the analysis of single-component BEC's with rotation (see the account in [1] or [14]), which is also very similar to that of type II superconductors in the Ginzburg–Landau model of superconductivity (see [13] and the references therein). Let us concentrate on $u_{1,\varepsilon}$ in D_1 , the analysis of $u_{2,\varepsilon}$ in D_2 is exactly symmetrical. According to the aforementioned works, $u_{1,\varepsilon} : D \rightarrow \mathbb{C}$ is described in terms of its *vortices*, which may be defined in several ways, but can loosely be described as follows: they are points a_i around which $u_{1,\varepsilon}$ has the behaviour of the map

$$x \rightarrow f_\varepsilon(|x - a_i|) \frac{x - a_i}{|x - a_i|}, \quad \text{where } f_\varepsilon(|x|) = \min(1, |x|/r).$$

What is meant here by “the behaviour” is that the energy of $u_{1,\varepsilon}$ in a small ball around a_i is at least $\pi \log(1/r_\varepsilon)$, to leading order, and that $\text{curl } \rho_{1,\varepsilon}^2 \nabla \varphi_{1,\varepsilon}$, where $\varphi_{1,\varepsilon}$ is the phase of $u_{1,\varepsilon}$ (which is not globally defined), is well approximated by $2\pi \sum_i \delta_{a_i}$. An important fact, which will be crucial for us, is that the energy $\pi \log(1/r_\varepsilon)$ is accounted for by the variations of the phase of $u_{1,\varepsilon}$.

The order of magnitude of r_ε needs to be known only approximately if we are interested to the energy of a vortex at leading order, since it enters a log. For now, let us admit that replacing r_ε by $\tilde{\varepsilon}$ will give us an accurate enough estimate. We will justify this below.

To summarize this one-component sketch of an analysis, we let $\mu_{1,\varepsilon} = 2\pi \sum_i \delta_{a_i}$ be the vortex measure for $u_{1,\varepsilon}$ and

$$j_{1,\varepsilon} = \rho_{1,\varepsilon}^2 (\nabla \varphi_{1,\varepsilon} - \Omega x^\perp),$$

so that $\text{curl } j_{1,\varepsilon} \simeq \mu_{1,\varepsilon} - 2\Omega$. We denote V_η a η -neighbourhood of the interface $\partial D_1 \cap D$, and $D_{1,\varepsilon}$ be the complement in $D_1 \setminus V_\eta$ of the vortex balls. Then we have

$$E_{\varepsilon,\delta}^\Omega(u_{1,\varepsilon}, u_{2,\varepsilon}, D_1 \setminus V_\eta) \geq \frac{1}{2} \|\mu_{1,\varepsilon}\| |\log \tilde{\varepsilon}| + \frac{1}{2} \int_{D_{1,\varepsilon}} |j_{1,\varepsilon}|^2 + \text{l.o.t.}, \quad \text{curl } j_{1,\varepsilon} \simeq \mu_{1,\varepsilon} - 2\Omega.$$

In this lower-bound, the first term accounts for the energy inside the vortex balls while the second one accounts for the energy outside of the balls.

It is noteworthy that both terms take only into account the energy of the phase $\varphi_{1,\varepsilon}$. To be more precise, $\nabla u_{1,\varepsilon} = (\nabla \rho_{1,\varepsilon} + i\rho_{1,\varepsilon} \nabla \varphi_{1,\varepsilon}) e^{i\varphi_{1,\varepsilon}}$, so that

$$|\nabla u_{1,\varepsilon} - i\Omega x^\perp u_{1,\varepsilon}|^2 = |\nabla \rho_{1,\varepsilon}|^2 + \rho_{1,\varepsilon}^2 |\nabla \varphi_{1,\varepsilon} - \Omega x^\perp|^2.$$

Only the second term is involved in the above lower-bound so that we may write, dividing the previous lower-bound by Ω^2 , remembering that $\Omega = \beta |\log \tilde{\varepsilon}|$, and passing to the limit as $\varepsilon \rightarrow 0$,

$$\liminf_{\varepsilon \rightarrow 0} \frac{1}{\Omega^2} \int_{D_1 \setminus V_\eta} \rho_{1,\varepsilon}^2 |\nabla \varphi_{1,\varepsilon} - \Omega x^\perp|^2 \geq \frac{1}{2\beta} \|\text{curl } j_1 + 2\| + \frac{1}{2} \int_{D_1 \setminus V_\eta} |j_1|^2, \quad (3.1)$$

where j_1 is the limit as $\varepsilon \rightarrow 0$ of $j_{1,\varepsilon}/\Omega$.

The proof of (3.1) for minimizers $(u_{1,\varepsilon}, u_{2,\varepsilon})$ of $E_{\varepsilon,\delta}^\Omega$ follows and uses similar results for the Ginzburg–Landau functional or single component BEC's. There is however one point which we left unjustified that is specific to this model and which requires specific arguments.

Claim. *The vortex radius r_ε can be taken to be $\tilde{\varepsilon}$ when computing the lower bound.*

4. The main result

Putting together Proposition 2.2 and (3.1) we obtain the lower-bound for $E_{\varepsilon,\delta}^\Omega(u_{1,\varepsilon}, u_{2,\varepsilon})$. The matching upper-bound follows from a construction which combines a one-dimensional interface and the insertion of vortices, while preserving the mass constraints. We arrive at the following result. Define

$$\ell_\alpha = \min_{\substack{\omega \subset D \\ |\omega| = \alpha|D|}} \text{per}_D(\omega). \quad (4.1)$$

Theorem 4.1. *Assume D is a smooth bounded domain in \mathbb{R}^2 and that $\alpha \in (0, 1)$. Assume that $\tilde{\varepsilon} = \varepsilon/\sqrt{\delta-1}$ is such that $\tilde{\varepsilon} \rightarrow 0$, $\tilde{\varepsilon} \ll \varepsilon$ as $\varepsilon \rightarrow 0$. Let $u_\varepsilon = (u_{1,\varepsilon}, u_{2,\varepsilon})$ denote a minimizer of $E_{\varepsilon,\delta}^\Omega$ (where $\Omega = \beta|\log \tilde{\varepsilon}|$ for some $\beta > 0$) under the constraint*

$$\int_D |u_{1,\varepsilon}|^2 = \alpha, \quad \int_D |u_{2,\varepsilon}|^2 = 1 - \alpha. \quad (4.2)$$

Then $(|u_{1,\varepsilon}|, |u_{2,\varepsilon}|)$ subsequentially converges weakly in BV to $(\chi_{\omega_\alpha}, \chi_{\omega_\alpha^c})$, where ω_α is a minimizer of $\text{per}_D(\omega)$ under the constraint $|\omega| = \alpha|D|$.

Moreover, let

$$j_{1,\varepsilon} = (iu_{1,\varepsilon}, \nabla u_{1,\varepsilon}) - \Omega x^\perp |u_{1,\varepsilon}|^2, \quad j_{2,\varepsilon} = (iu_{2,\varepsilon}, \nabla u_{2,\varepsilon}) - \Omega x^\perp |u_{2,\varepsilon}|^2,$$

then $(j_{1,\varepsilon}/\Omega, j_{2,\varepsilon}/\Omega)$ converges weakly in L^2 to $(j_{1,\beta}, j_{2,\beta})$, where

$$j_{1,\beta} = \underset{\text{div } j=0}{\text{argmin}} J_\beta(j, \omega_\alpha), \quad J_\beta(j, \omega_\alpha) = \frac{1}{2} \int_{\omega_\alpha} |j|^2 + \frac{1}{2\beta} \int_{\omega_\alpha} |\text{curl } j + 2|,$$

and $j_{2,\beta}$ is defined similarly, replacing ω_α by ω_α^c . Moreover

$$\min E_{\varepsilon,\delta}^\Omega = m_\varepsilon \ell_\alpha + \Omega^2 \left(\min_{\text{div } j=0} J_\beta(j, \omega_\alpha) + \min_{\text{div } j=0} J_\beta(j, \omega_\alpha^c) \right) + o(|\log \tilde{\varepsilon}|^2). \quad (4.3)$$

A further description of minimizers is obtained by studying the minimizers of J_β , as in [13, 14]. It is found that for these minimizers the measure $\text{curl } j + 2$ (which we recall corresponds to a limiting density of vortices) is a constant density on a subset of either ω_α or $D \setminus \omega_\alpha$. The density is constant equal to 2 but the subset expands as β increases.

5. Localisation of the line energy

In this section we focus on the proof of the claim we have made, namely that the radius of the vortex cores can be taken to be $\tilde{\varepsilon}$ in our analysis. What this means really is that $|\log r_\varepsilon| \simeq |\log \tilde{\varepsilon}|$ as $\varepsilon \rightarrow 0$, i.e. that the vortex energy is accurate to leading order if we replace r_ε by $\tilde{\varepsilon}$. Of course the important thing to prove is the lower-bound part of this statement. Once it is proved a construction allows to obtain a matching upper-bound in a straightforward way.

This means we need to bound from above the radius of the vortex core. For this we use Proposition 2.1. The idea is the following: a vortex core in D_1 is a small inclusion where $(\rho_{1,\varepsilon}, \rho_{2,\varepsilon}) \simeq (0, 1)$ while outside the vortex core one has $(\rho_{1,\varepsilon}, \rho_{2,\varepsilon}) \simeq (1, 0)$. Therefore the cost of such a core in terms of F_ε is of the order of $r_\varepsilon \times m_\varepsilon$ if r_ε is its radius. In fact Proposition 2.1 implies that the vortex cores in D_1 have radius r_ε bounded as follows

$$r_\varepsilon \leq C\tilde{\varepsilon}F_\varepsilon(\rho_{1,\varepsilon}, \rho_{2,\varepsilon}, D_1),$$

where we have used the estimate $m_\varepsilon \simeq m_0/\tilde{\varepsilon}$.

The only a-priori bound we have for $F_\varepsilon(\rho_{1,\varepsilon}, \rho_{2,\varepsilon}, D_1)$ is stated in (4.3) and is proved by the construction of a test-function. It is of the order of $1/\tilde{\varepsilon}$ hence is useless since it leads to bounding r_ε by a constant. However, inspecting (4.3), we see that the leading order in the energy is the interface energy $m_\varepsilon \ell_\alpha$, which is localized to leading order near the interface. If we could prove it is localized up to a constant near the interface, then the energy F_ε in $D_1 \setminus V_\eta$ would be bounded above by $C\Omega^2$, i.e. $C|\log \tilde{\varepsilon}|^2$, for any η -neighbourhood V_η of the interface. Then we would deduce that $r_\varepsilon \leq C\tilde{\varepsilon}|\log \tilde{\varepsilon}|^2$, and then that

$$|\log r_\varepsilon| \geq |\log \tilde{\varepsilon}|(1 - o(1)).$$

In fact a localization up to $|\log \tilde{\varepsilon}|^2$ or even $|\log \tilde{\varepsilon}|^p$ for any p would be sufficient. What we prove is the following.

Theorem 5.1. *Let D be a bounded smooth domain in \mathbb{R}^2 and $\alpha \in (0, 1)$. Denote by $\{\varepsilon\}$ a sequence of real numbers tending to 0. Let $\{(\rho_{1,\varepsilon}, \rho_{2,\varepsilon})\}_\varepsilon$ be such that*

$$F_\varepsilon(\rho_{1,\varepsilon}, \rho_{2,\varepsilon}) \leq m_\varepsilon \ell_\alpha + \Delta_\varepsilon, \quad \rho_{1,\varepsilon}^2 + \rho_{2,\varepsilon}^2 \leq 1 + C\tilde{\varepsilon}, \quad (5.1)$$

where ℓ_α is defined in (4.1). Then, assuming $\Delta_\varepsilon \ll m_\varepsilon \ell_\alpha$ as $\varepsilon \rightarrow 0$, there exists a subsequence (not relabeled) $\{\varepsilon\}$ such that $\{(\rho_{1,\varepsilon}, \rho_{2,\varepsilon})\}_\varepsilon$ converges to $(\chi_{\omega_\alpha}, \chi_{\omega_\alpha^c})$, where ω_α is a minimizer of (4.1).

Moreover writing $\gamma_\alpha = \partial\omega_\alpha \cap D$, for any $\eta > 0$ there exists $C > 0$ such that if ε is small enough (depending on η) and V_η is an η -neighbourhood of γ_α we have

$$F_\varepsilon(\rho_{1,\varepsilon}, \rho_{2,\varepsilon}, D \setminus V_\eta) \leq C(\Delta_\varepsilon + |\log \tilde{\varepsilon}|). \quad (5.2)$$

The proof of this result occupies the rest of this section. It is based on the lower-bound (2.1) for $F_\varepsilon(\rho_{1,\varepsilon}, \rho_{2,\varepsilon})$ which we can localise in V_η . Then we prove the equivalent of the above Theorem for the minimisation of perimeter, i.e. that for almost-minimizers of (4.1), the interface is located in a neighbourhood of a minimal interface, except maybe for an amount of length bounded by a constant times the excess length. This would be sufficient if, in (2.1), we knew that

$$\{x \in D \mid d_\varepsilon(u(x)) < t\}$$

is a competitor in (4.1), i.e. has measure $\alpha|D|$ for every $t \in (0, m_\varepsilon)$.

Of course this is not the case, hence a quantitative statement in this direction needs to be proved, which is precisely the following

$$\delta A_\varepsilon := |\{x \in D \mid C|\log \tilde{\varepsilon}| < d_\varepsilon(u(x)) < m_\varepsilon - C|\log \tilde{\varepsilon}|\}| \leq C\tilde{\varepsilon}(|\log \tilde{\varepsilon}| + \Delta_\varepsilon). \quad (5.3)$$

First we introduce some notation. We let $u = (\rho_{1,\varepsilon}, \rho_{2,\varepsilon})$ and we define

$$\gamma_t = \{x \in D \mid d_\varepsilon(u(x)) = t\}, \quad v(t) = \int_{\gamma_t} \frac{2W_\varepsilon(u(x))}{|\nabla(d_\varepsilon \circ u)(y)|} d\ell(y), \quad a(t) = \int_{\gamma_t} \frac{d\ell(y)}{|\nabla(d_\varepsilon \circ u)(y)|},$$

where $d\ell$ denotes the line element on the curve γ_t .

We have, using the coarea formula, and letting $I_\varepsilon = [C|\log \tilde{\varepsilon}|, m_\varepsilon - C|\log \tilde{\varepsilon}|]$,

$$\delta A_\varepsilon = \int_{t \in I_\varepsilon} a(t) dt.$$

It can be shown that $W_\varepsilon(u) \geq (C\tilde{\varepsilon})^{-1} \min(d_\varepsilon(u), m_\varepsilon - d_\varepsilon(u))$. Therefore

$$a(t) \leq \frac{1}{2} \frac{\tilde{\varepsilon}}{t} |\gamma_t| v(t),$$

where $|\gamma_t|$ denotes the length of the curve γ_t . It follows that

$$\delta A_\varepsilon \leq \frac{1}{2} \int_{t \in I_\varepsilon} \frac{\tilde{\varepsilon}}{t} |\gamma_t| v(t) dt \quad (5.4)$$

On the other hand, using again the coarea formula,

$$\begin{aligned} F_\varepsilon(u) &= \frac{1}{2} \int_D |\nabla u|^2 + \int_D W_\varepsilon(u) \\ &\geq \int_0^{m_\varepsilon} \int_{\gamma_t} \frac{1}{2} \frac{|\nabla u|^2}{|\nabla(d_\varepsilon \circ u)|} + \frac{W_\varepsilon(u)}{|\nabla(d_\varepsilon \circ u)|} d\ell dt. \end{aligned}$$

Using Jensen's inequality and the fact that $|\nabla d_\varepsilon \circ u| \leq |\nabla d_\varepsilon(u)| |\nabla u| = \sqrt{2W_\varepsilon(u)} |\nabla u|$, we have

$$\int \frac{|\nabla u|^2}{|\nabla(d_\varepsilon \circ u)|} \geq \left(\int \frac{|\nabla(d_\varepsilon \circ u)|}{|\nabla u|^2} \right)^{-1} \geq \frac{1}{v(t)}.$$

It follows that

$$F_\varepsilon(u) \geq \int_0^{m_\varepsilon} \frac{|\gamma_t|}{2} \left(v(t) + \frac{1}{v(t)} \right) dt.$$

We may then subtract $m_\varepsilon \ell_\alpha$ and obtain, in view of the hypothesis of the theorem

$$\begin{aligned} \Delta_\varepsilon &\geq F_\varepsilon(u) - m_\varepsilon \ell_\alpha \\ &\geq \int_0^{m_\varepsilon} \frac{|\gamma_t|}{2} \left(v(t) + \frac{1}{v(t)} \right) dt - m_\varepsilon \ell_\alpha - C\tilde{\varepsilon} \\ &\geq \int_{t \in I_\varepsilon} \frac{|\gamma_t|}{2} \left(v(t) + \frac{1}{v(t)} \right) - \ell_\alpha dt - C|\log \tilde{\varepsilon}|. \end{aligned} \quad (5.5)$$

Let $\delta(t) = \frac{|\gamma_t|}{2} \left(v(t) + \frac{1}{v(t)} \right) - \ell_\alpha$. We wish to bound from above the integrand in (5.4), possibly in terms of $\delta(t)$. We distinguish several cases, C denotes a generic constant independent of ε .

- If $\ell_\alpha \leq |\gamma_t|v(t)/4$ then $\delta(t) \geq |\gamma_t|v(t)/2$ and therefore, using the fact that $t \geq C|\log \tilde{\varepsilon}|$, if ε is small enough then

$$\frac{\tilde{\varepsilon}}{t} |\gamma_t|v(t) \leq v(t) \leq C\tilde{\varepsilon}\delta(t).$$

- If $|\gamma_t|v(t) \leq 4\ell_\alpha$ then, since ℓ_α is independent of ε ,

$$\frac{\tilde{\varepsilon}}{t} |\gamma_t|v(t) \leq C\frac{\tilde{\varepsilon}}{t}.$$

It follows, in view of (5.5) and since $I_\varepsilon = [C|\log \tilde{\varepsilon}|, m_\varepsilon - C|\log \tilde{\varepsilon}|]$, that

$$\delta A_\varepsilon \leq C\tilde{\varepsilon} \int_{t \in I_\varepsilon} \frac{1}{t} + \delta_+(t) dt \leq C\tilde{\varepsilon} \left(|\log \tilde{\varepsilon}| + \Delta_\varepsilon + \int_{t \in I_\varepsilon} \delta_-(t) dt \right). \quad (5.6)$$

It remains to bound the last integral on the right-hand side. For this we note that, since $\delta(t) \geq |\gamma(t)| - \ell_\alpha$, we have

$$\delta_-(t) \leq (\ell_\alpha - |\gamma(t)|)_+.$$

But $\ell_\alpha - |\gamma_t| \leq \ell_\alpha - \ell_\beta$, where $\beta = |\{d_\varepsilon \circ u < t\}|/|D|$, in view of the definition (4.1). Since the isoperimetric profile function $\alpha \rightarrow \ell_\alpha$ is lipschitz we deduce that

$$\ell_\alpha - |\gamma_t| \leq C|\alpha|D| - |\{d_\varepsilon \circ u < t\}|.$$

It is not hard to show that there exists t_0 such that $|\alpha|D| - |\{d_\varepsilon \circ u < t_0\}| \leq C\tilde{\varepsilon}|\log \tilde{\varepsilon}|$, therefore for any $t \in I_\varepsilon$ we have

$$\begin{aligned} \delta_-(t) &\leq C|\alpha|D| - |\{d_\varepsilon \circ u < t\}| \\ &\leq |\alpha|D| - |\{d_\varepsilon \circ u < t_0\}| + |\{d_\varepsilon \circ u < t_0\}| - |\{d_\varepsilon \circ u < t\}| \\ &\leq C\tilde{\varepsilon}|\log \tilde{\varepsilon}| + \delta A_\varepsilon. \end{aligned} \quad (5.7)$$

Together with (5.6) we deduce that

$$\delta A_\varepsilon \leq C\tilde{\varepsilon} (|\log \tilde{\varepsilon}| + \Delta_\varepsilon + \delta A_\varepsilon),$$

which implies (5.3) if $\tilde{\varepsilon}$ is small enough.

6. Conclusion

In the above analysis, a precise study of the energy without rotation allowed us, using the existing knowledge on the Ginzburg–Landau model of superconductivity or on single component BEC's, to describe ground states of a model for two-component BEC with rotation. In a suitable asymptotic regime, the problem reduces to first a partition problem for the domain which determines the subdomains occupied by each component, and then a separate analysis on each subdomain where only one component is present.

This picture is valid as long as the interface energy dominates the vortex energy. For rotations of the order of or higher than $1/\tilde{\varepsilon}$ this is no longer true and then we are only able to determine the leading order vortex energy for minimizers, but not to determine whether components remain in subdomains determined by a minimal partition problem.

Another interesting question is whether the localisation of the interface energy, which involves an error of order $|\log \tilde{\varepsilon}|$ is optimal. Even for the Modica–Mortola functional this is an open question, even though very precise results of Murray–Leoni ([11]) exist on the value of the minimal energy.

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ETIENNE SANDIER
UNIVERSITÉ PARIS-EST
CRÉTEIL
FRANCE
sandier@u-pec.fr