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Semyon Dyatlov

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Control of eigenfunctions on hyperbolic surfaces: an application of fractal uncertainty principle

Semyon Dyatlov

Abstract

This expository article, written for the proceedings of the Journées EDP (Roscoff, June 2017), presents recent work joint with Jean Bourgain [BD16] and Long Jin [DJ17]. We in particular show that eigenfunctions of the Laplacian on hyperbolic surfaces are bounded from below in L^2 norm on each nonempty open set, by a constant depending on the set but not on the eigenvalue.

1. Introduction

The purpose of this expository article is to present a recent result of Dyatlov–Jin [DJ17] on *control of eigenfunctions on hyperbolic surfaces*. The main tool is the *fractal uncertainty principle* proved by Bourgain–Dyatlov [BD16]. To keep the article short we will omit many technical details of the proofs, referring the reader to the original papers [BD16, DJ17] and to the lecture notes [Dy17]. The results discussed here belong to *quantum chaos*; see [Ma06, Ze09, Sa11] for introductions to the subject.

The setting considered is a *hyperbolic surface*, that is a compact Riemannian surface (M, g) of Gauss curvature -1 , which we fix throughout the paper. For convenience we assume that M is connected and oriented. The homogeneous *geodesic flow*

$$\varphi_t : T^*M \setminus 0 \rightarrow T^*M \setminus 0, \quad T^*M \setminus 0 := \{(x, \xi) : x \in M, \xi \in T_x^*M, \xi \neq 0\} \quad (1.1)$$

is a classical example of a strongly chaotic, or more precisely, *hyperbolic*,¹ dynamical system, see §2.2 below. We can view φ_t as a Hamiltonian flow:

$$\varphi_t = \exp(tH_p), \quad p(x, \xi) = |\xi|_g. \quad (1.2)$$

Let $-\Delta_g \geq 0$ be the Laplace–Beltrami operator on M . Our first result is

Theorem 1.1. *Fix a nonempty open set $\Omega \subset M$. Then for each eigenfunction of Δ_g*

$$u \in C^\infty(M), \quad (-\Delta_g - \lambda^2)u = 0, \quad \|u\|_{L^2(M)} = 1 \quad (1.3)$$

we have the following lower bound where the constant c_Ω depends on Ω but not on λ :

$$\|u\|_{L^2(\Omega)} \geq c_\Omega > 0. \quad (1.4)$$

Remarks. 1. For bounded λ , (1.4) follows from unique continuation principle. Thus Theorem 1.1 is really a result about the *high frequency limit* $\lambda \rightarrow \infty$.

2. Theorem 1.1 uses global information about the structure of the manifold M ; it does not hold for every Riemannian manifold. For instance, if M is the round 2-sphere and Ω lies in just one

This research was conducted during the period the author served as a Clay Research Fellow.

¹The word ‘hyperbolic’ is used in two different meanings here: a surface is hyperbolic when it has constant negative curvature and a flow is hyperbolic when it admits a stable/unstable decomposition. Luckily for us, hyperbolic surfaces do give rise to hyperbolic geodesic flows.

hemisphere, then one can construct a sequence of eigenfunctions u_n , with $\|u_n\|_{L^2(\Omega)}$ exponentially small as $\lambda_n \rightarrow \infty$.

An application of Theorem 1.1 (strictly speaking, of Theorem 1.3 below) is the following observability result for the Schrödinger equation:

Theorem 1.2. [Ji17] *Let M, Ω be as in Theorem 1.1 and fix $T > 0$. Then there exists $C = C(\Omega, T) > 0$ such that for all $f \in L^2(M)$ we have*

$$\|f\|_{L^2(M)}^2 \leq C \int_0^T \|e^{it\Delta} f\|_{L^2(\Omega)}^2 dt.$$

1.1. A semiclassical result

We now give a stronger version of Theorem 1.1 (see Theorem 1.3 below) which allows to localize u in frequency in addition to localization in position. To state it we introduce a *semiclassical quantization*

$$\text{Op}_h : a \in C_0^\infty(T^*M) \mapsto \text{Op}_h(a) : L^2(M) \rightarrow L^2(M).$$

We refer the reader to [Zw12, §14.2.2] for the definition and properties of Op_h . The operator $\text{Op}_h(a)$, acting on appropriately chosen semiclassical Sobolev spaces, can also be defined when a is not compactly supported but instead satisfies a polynomial growth bound as $\xi \rightarrow \infty$. Functions satisfying such bounds are called *symbols*.

We henceforth put $h := \lambda^{-1} > 0$, so that the equation (1.3) becomes

$$(-h^2\Delta_g - 1)u = 0, \quad \|u\|_{L^2(M)} = 1. \quad (1.5)$$

The elliptic estimate implies (see (2.2) below) that solutions to (1.5) are microlocalized on the cosphere bundle $S^*M = \{(x, \xi) : p(x, \xi) = 1\}$:

$$a|_{S^*M} \equiv 0 \implies \|\text{Op}_h(a)u\|_{L^2} = \mathcal{O}(h). \quad (1.6)$$

In particular, $\|\text{Op}_h(a)u\|$ cannot be bounded away from 0 as $h \rightarrow 0$ when $a|_{S^*M} \equiv 0$. Our next result in particular implies that such a lower bound holds for all other a :

Theorem 1.3. [DJ17, Theorem 2] *Assume that $a \in C_0^\infty(T^*M)$ and $a|_{S^*M} \not\equiv 0$. Then there exist constants $C, h_0 > 0$ depending on a but not on h , such that for all $u \in H^2(M)$ and $h < h_0$*

$$\|u\|_{L^2(M)} \leq C \|\text{Op}_h(a)u\|_{L^2(M)} + \frac{C \log(1/h)}{h} \|(-h^2\Delta_g - 1)u\|_{L^2(M)}. \quad (1.7)$$

Remarks. 1. Theorem 1.3 applies to u which are not exact eigenfunctions of Δ_g . It gives a nontrivial statement when u is an $o(h/\log(1/h))$ -quasimode for the Laplacian.

2. Theorem 1.1 follows immediately from Theorem 1.3. Indeed, take $a = a(x) \in C_0^\infty(M)$ such that $a \not\equiv 0$ and $\text{supp } a \subset \Omega$. Then (1.5), (1.6), and (1.7) together imply that $1 = \|u\|_{L^2(M)} \leq C \|\text{Op}_h(a)u\|_{L^2(M)}$. It remains to use that $\text{Op}_h(a)$ is a multiplication operator, namely $\text{Op}_h(a)u = au$.

1.2. Application to semiclassical measures

We next present an application of Theorem 1.3 to semiclassical defect measures, which are weak limits of high frequency sequences of eigenfunctions of Δ_g defined as follows:

Definition 1.1. *Consider a sequence of eigenfunctions*

$$u_j \in C^\infty(M), \quad (-h_j^2\Delta_g - 1)u_j = 0, \quad \|u_j\|_{L^2(M)} = 1, \quad h_j \rightarrow 0. \quad (1.8)$$

*Let μ be a Borel measure on T^*M . We say that u_j converges to μ weakly if*

$$\langle \text{Op}_{h_j}(a)u_j, u_j \rangle_{L^2(M)} \rightarrow \int_{T^*M} a d\mu \quad \text{as } j \rightarrow \infty \quad \text{for all } a \in C_0^\infty(T^*M). \quad (1.9)$$

*We say that μ is a **semiclassical measure** on M , if μ is the weak limit of some sequence of eigenfunctions.*

Semiclassical measures enjoy many nice properties [Zw12, Chapter 5], in particular

- every sequence of eigenfunctions has a subsequence converging weakly to some measure;

- every semiclassical measure is a probability measure supported on S^*M ;
- every semiclassical measure is invariant under the geodesic flow φ_t .

The question of which φ_t -invariant probability measures on S^*M can be semiclassical measures has received considerable attention in the quantum chaos literature. Here we only mention a few works, referring the reader to the introduction to [DJ17] for a more comprehensive overview.

The *Quantum Ergodicity (QE)* theorem of Shnirelman, Zelditch, and Colin de Verdière [Sh74, Ze87, CdV85] implies that a density 1 sequence of eigenfunctions converges to the *Liouville measure* μ_L , which is the natural volume measure on S^*M . Rundick–Sarnak [RS94] made the *Quantum Unique Ergodicity (QUE)* conjecture that μ_L is the only semiclassical measure, i.e. it is the weak limit of the entire sequence of eigenfunctions. So far QUE has only been proved for Hecke eigenfunctions on arithmetic surfaces, by Lindenstrauss [Li06].

While QUE for general hyperbolic surfaces is still an open problem, one can show that semiclassical measures satisfy lower bounds on entropy. In particular, Anantharaman–Nonnenmacher [AN07] proved that each semiclassical measure μ has Kolmogorov–Sinai entropy $H_{\text{KS}}(\mu) \geq \frac{1}{2}$. This rules out many possible weak limits, in particular the delta measure on a closed geodesic (which has entropy 0). We remark that $H_{\text{KS}}(\mu_L) = 1$, so in some sense [AN07] rules out half of the invariant measures as weak limits.

For $a \in C_0^\infty(T^*M)$, we have $\|\text{Op}_h(a)u\|_{L^2}^2 = \langle \text{Op}_h(|a|^2)u, u \rangle_{L^2} + \mathcal{O}(h)\|u\|_{L^2}^2$. Thus Theorem 1.3 implies

Theorem 1.4. *Any semiclassical measure μ on M has support equal to the entire S^*M , that is $\mu(U) > 0$ for any nonempty open $U \subset S^*M$.*

We remark that Theorem 1.4 is in some sense orthogonal to the entropy bound mentioned above. In particular, there exist φ_t -invariant probability measures μ on S^*M such that $\text{supp } \mu \neq S^*M$ but $H_{\text{KS}}(\mu) > \frac{1}{2}$, in fact the entropy of μ may be arbitrarily close to 1. The simplest way to construct these is to choose M which has a short closed geodesic, cut M along this geodesic and attach two funnels to obtain a convex co-compact surface \widetilde{M} , and define μ as the Patterson–Sullivan measure on the set of trapped geodesics on \widetilde{M} (see for instance [Bo16, §14.2]).

2. Propagation of control

In this section we explain how to reduce Theorem 1.3 to a norm bound (2.15) which will follow from the fractal uncertainty principle.

In this section $\|\bullet\|$ denotes the $L^2(M)$ -norm. We will only show a weaker version of the estimate (1.7), in the case of exact eigenfunctions:

$$(-h^2\Delta_g - 1)u = 0 \quad \implies \quad \|u\| \leq C \log(1/h) \|\text{Op}_h(a)u\|. \quad (2.1)$$

See the end of §2.3 for a discussion of how to modify the proof to remove the $\log(1/h)$ prefactor.

We start with a few basic observations and reductions. We henceforth assume that $(-h^2\Delta_g - 1)u = 0$, $\|u\| = 1$, and h is small.

By the semiclassical elliptic estimate, we have for any two symbols a, b (with the order of growth of b in ξ bounded by that of a)

$$\text{supp } b \subset \{a \neq 0\} \quad \implies \quad \|\text{Op}_h(b)u\| \leq C \|\text{Op}_h(a)u\| + \mathcal{O}(h)\|u\|. \quad (2.2)$$

This follows immediately from the algebraic properties of semiclassical quantization, writing $\text{Op}_h(b) = \text{Op}_h(b/a) \text{Op}_h(a) + \mathcal{O}(h)$. This argument also gives the bound (1.6), using that $-h^2\Delta_g - 1 = \text{Op}_h(p^2 - 1) + \mathcal{O}(h)$ where p is defined in (1.2).

Now, fix $a \in C_0^\infty(T^*M)$ such that $a|_{S^*M} \not\equiv 0$. By (1.6) the estimate (2.1) only depends on the values of a on S^*M . By (2.2) it suffices to prove (2.1) with a replaced by some symbol whose support lies in $\{a \neq 0\}$. Thus we may assume without loss of generality that a is homogeneous of degree 0 near S^*M and

$$a \equiv 1 \quad \text{near } \mathcal{U} \cap S^*M \quad (2.3)$$

where $\mathcal{U} \subset T^*M \setminus 0$ is some nonempty open conic set.

2.1. Partitions and control

Define $a_1, a_2 \in C_0^\infty(T^*M \setminus 0; [0, 1])$ such that

$$a_1 = a \quad \text{and} \quad a_1 + a_2 = 1 \quad \text{on} \quad S^*M; \quad \text{supp } a_2 \cap \mathcal{U} = \emptyset. \quad (2.4)$$

We may also assume that $a_1 + a_2 \leq 1$ everywhere. Define the pseudodifferential operators

$$A_j := \text{Op}_h(a_j) : L^2(M) \rightarrow L^2(M), \quad j = 1, 2.$$

Note that the $L^2 \rightarrow L^2$ operator norms of $A_1, A_2, A_1 + A_2$ are bounded by $1 + \mathcal{O}(h)$, see [DJ17, (2.10)]. We can choose the quantizations A_1, A_2 so that $A_1 + A_2$ commutes with $-\Delta_g$, see [DJ17, §3.1]. By (1.6) we have

$$\|u - (A_1 + A_2)u\| = \mathcal{O}(h), \quad (2.5)$$

$$\|A_1 u\| \leq \|\text{Op}_h(a)u\| + \mathcal{O}(h). \quad (2.6)$$

The bound (2.6) says that $\|A_1 u\|$ is *controlled* in the sense that it is bounded by an expression of the form $C\|\text{Op}_h(a)u\| + o(1)_{h \rightarrow 0}$. Our goal is to obtain control on u in a large region of the phase space T^*M .

Since u is an eigenfunction of the Laplacian, it is also an eigenfunction of the half-wave propagator:

$$U(t)u = e^{-it/h}u, \quad U(t) := \exp(-it\sqrt{-\Delta_g}) : L^2(M) \rightarrow L^2(M).$$

For a bounded operator $A : L^2(M) \rightarrow L^2(M)$, denote

$$A(t) := U(-t)AU(t). \quad (2.7)$$

Then (2.6) implies that $A_1(t)u$ is controlled for all t :

$$\|A_1(t)u\| = \|A_1 u\| \leq \|\text{Op}_h(a)u\| + \mathcal{O}(h). \quad (2.8)$$

On the other hand we have *Egorov's Theorem* [Zw12, Theorem 11.1]: for bounded t ,

$$A_1(t) = \text{Op}_h(a_1 \circ \varphi_t) + \mathcal{O}(h)_{L^2 \rightarrow L^2}. \quad (2.9)$$

Therefore, $\|\text{Op}_h(a_1 \circ \varphi_t)u\|$ is controlled for any bounded t .

The above observations imply the required bound (2.1), even without the $\log(1/h)$ prefactor, in cases when \mathcal{U} satisfies the *geometric control condition*, namely there exists $T > 0$ such that for each $(x, \xi) \in S^*M$, the geodesic segment $\{\varphi_t(x, \xi) \mid 0 \leq t \leq T\}$ intersects \mathcal{U} . Indeed, in this case there exists a finite set of times $t_1, \dots, t_N \in [0, T]$ such that

$$\tilde{a} := \sum_{\ell=1}^N a_1 \circ \varphi_{t_\ell} > 0 \quad \text{on} \quad S^*M.$$

We control each $\|\text{Op}_h(a_1 \circ \varphi_{t_\ell})u\|$ and thus the sum $\|\text{Op}_h(\tilde{a})u\|$. By (1.6) and (2.2) we have $1 = \|u\| \leq C\|\text{Op}_h(\tilde{a})u\| + \mathcal{O}(h)$, giving (2.1).

2.2. Long time propagation

Many sets $\mathcal{U} \subset S^*M$ do not satisfy the geometric control condition (for instance it is enough for \mathcal{U} to miss just one closed geodesic on M). To obtain (2.1) we then need to use Egorov's Theorem (2.9) (more precisely, its version for A_2) for times t which grow as $h \rightarrow 0$.

One has to take extra caution: for t which are too large, the derivatives of the symbol $a_2 \circ \varphi_t$ grow too fast with h and the quantization $\text{Op}_h(a_2 \circ \varphi_t)$ no longer makes sense (i.e. no longer has the standard properties). One can also think of that problem as $a_2 \circ \varphi_t$ localizing to a set which is so thin in some directions that such localization contradicts the uncertainty principle.

It turns out that for hyperbolic surfaces, Egorov's Theorem still holds when $|t| \leq \rho \log(1/h)$ for any fixed $0 \leq \rho < 1$. To explain this, we first study the growth of derivatives of $a_2 \circ \varphi_t$ as $|t| \rightarrow \infty$. The geodesic flow φ_t is *hyperbolic* on the level sets $\{p = \text{const}\}$, namely it has a stable/unstable decomposition. More precisely, there exists a smooth frame H_p, U_+, U_-, D (see [DJ17, §2.1]) on $T^*M \setminus 0$ such that:

- H_p is the generator of the flow, that is $\varphi_t = \exp(tH_p)$;

- $D = \xi \cdot \partial_\xi$ is the generator of dilations in the fibers of $T^*M \setminus 0$, and it is invariant under φ_t : $(\varphi_t)_*D = D$;
- U_+ is the *stable horocyclic field*, which decays exponentially along the flow: $(\varphi_t)_*U_+ = e^{-t}U_+$;
- U_- is the *unstable horocyclic field*, which grows exponentially along the flow: $(\varphi_t)_*U_- = e^tU_-$.

We see that $H_p(a_2 \circ \varphi_t)$ and $D(a_2 \circ \varphi_t)$ are bounded uniformly in t ; $U_+(a_2 \circ \varphi_t)$ and $U_-(a_2 \circ \varphi_{-t})$ are bounded when $t \geq 0$; and $U_-(a_2 \circ \varphi_t)$ and $U_+(a_2 \circ \varphi_{-t})$ grow like e^t when $t \geq 0$. In particular we have (see [DJ17, §2.3])

$$\begin{aligned} a_2 \circ \varphi_t &\in S_{L_s, \rho}^{\text{comp}}(T^*M \setminus 0), \quad 0 \leq t \leq \rho \log(1/h); \\ a_2 \circ \varphi_t &\in S_{L_u, \rho}^{\text{comp}}(T^*M \setminus 0), \quad -\rho \log(1/h) \leq t \leq 0 \end{aligned}$$

where L_s, L_u are the *weak stable/unstable* Lagrangian foliations with tangent spaces

$$L_s = H_p \oplus U_+, \quad L_u = H_p \oplus U_-$$

and the class $S_{L, \rho}^{\text{comp}}(T^*M \setminus 0)$, $L \in \{L_u, L_s\}$, consists of symbols $b(x, \xi; h)$ such that

- b is compactly supported in (x, ξ) inside an h -independent subset of $T^*M \setminus 0$;
- $\sup |b| \leq C$, where C is independent of h ;
- for each vector fields $Y_1, \dots, Y_m, Z_1, \dots, Z_k$ on $T^*M \setminus 0$ such that Y_1, \dots, Y_m are tangent to L , and each $\varepsilon > 0$ we have

$$\sup |Y_1 \cdots Y_m Z_1 \cdots Z_k b| \leq Ch^{-\rho k - \varepsilon}.$$

In other words, b only grows by an arbitrarily small power of h when differentiated along L , but may grow by a factor up to $h^{-\rho-}$ when differentiated in other directions.

Each symbol $b \in S_{L, \rho}^{\text{comp}}(T^*M \setminus 0)$ can be quantized to an operator

$$\text{Op}_h^L(b) : L^2(M) \rightarrow L^2(M),$$

see [DJ17, Appendix]. Each of the resulting classes $S_{L_s, \rho}^{\text{comp}}, S_{L_u, \rho}^{\text{comp}}$ satisfies standard properties of semiclassical quantization. However, when $\rho > 1/2$ and $b_1 \in S_{L_s, \rho}^{\text{comp}}, b_2 \in S_{L_u, \rho}^{\text{comp}}$ the product $\text{Op}_h^{L_s}(b_1) \text{Op}_h^{L_u}(b_2)$ does not lie in any pseudodifferential class – this observation will be important for the fractal uncertainty principle later.

As promised in the beginning of this subsection, a version of Egorov's Theorem (2.9) continues to hold, see [DJ17, Proposition A.8]:

$$\begin{aligned} A_2(t) &= \text{Op}_h^{L_s}(a_2 \circ \varphi_t) + \mathcal{O}(h^{1-\rho-})_{L^2 \rightarrow L^2}, \quad 0 \leq t \leq \rho \log(1/h), \\ A_2(t) &= \text{Op}_h^{L_u}(a_2 \circ \varphi_t) + \mathcal{O}(h^{1-\rho-})_{L^2 \rightarrow L^2}, \quad -\rho \log(1/h) \leq t \leq 0. \end{aligned} \tag{2.10}$$

2.3. Control by propagation and proof of Theorem 1.3

We now explain how to control all but a small portion of u by using (2.8). Fix $0 < \rho < 1$ very close to 1, to be chosen later, and choose

$$N_1 := \lceil \rho \log(1/h) \rceil.$$

Using the notation (2.7), define the operators

$$\begin{aligned} A_{\mathcal{X}}^\pm &:= A_2(\mp N_1) \cdots A_2(\mp 1) A_2(0), \\ A_{\mathcal{Y}}^\pm &:= \sum_{j=0}^{N_1} A_{\mathcal{Y}, j}^\pm, \quad A_{\mathcal{Y}, j}^\pm = A_2(\mp N_1) \cdots A_2(\mp(j+1)) A_1(\mp j) (A_1 + A_2)^j. \end{aligned}$$

Since $A_1 + A_2$ commutes with Δ_g , we have $A_1(t) + A_2(t) = A_1 + A_2$, therefore

$$A_{\mathcal{X}}^\pm + A_{\mathcal{Y}}^\pm = (A_1 + A_2)^{N_1+1}.$$

Thus by (2.5)

$$u = A_{\mathcal{X}}^{\pm}u + A_{\mathcal{Y}}^{\pm}u + \mathcal{O}(h^{1-}). \quad (2.11)$$

Using (2.5) again together with (2.8), we have for all j

$$\|A_{\mathcal{Y},j}^{\pm}u\| \leq C\|A_1(j)u\| + \mathcal{O}(h^{1-}) \leq C\|\text{Op}_h(a)u\| + \mathcal{O}(h^{1-}).$$

Summing these up we see that $A_{\mathcal{Y}}^{\pm}u$ is controlled as follows:

$$\|A_{\mathcal{Y}}^{\pm}u\| \leq C\log(1/h)\|\text{Op}_h(a)u\| + \mathcal{O}(h^{1-}). \quad (2.12)$$

Together with (2.11) this gives

$$\|u - A_{\mathcal{X}}^{-}A_{\mathcal{X}}^{+}u\| \leq C\log(1/h)\|\text{Op}_h(a)u\| + \mathcal{O}(h^{1-}).$$

Using the properties of $S_{L,\rho}^{\text{comp}}$ calculus and Egorov's theorem for long time (2.10), we get (see [DJ17, Lemma 5.9])

$$A_{\mathcal{X}}^{+} = \text{Op}_h^{L^u}(a_{+}) + \mathcal{O}(h^{1-\rho-})_{L^2 \rightarrow L^2}, \quad A_{\mathcal{X}}^{-} = \text{Op}_h^{L^s}(a_{-}) + \mathcal{O}(h^{1-\rho-})_{L^2 \rightarrow L^2},$$

where the symbols a_{\pm} are given by

$$a_{+} = \prod_{j=0}^{N_1} (a_2 \circ \varphi_{-j}) \in S_{L^u, \rho}^{\text{comp}}(T^*M \setminus 0), \quad a_{-} = \prod_{j=0}^{N_1} (a_2 \circ \varphi_j) \in S_{L^s, \rho}^{\text{comp}}(T^*M \setminus 0). \quad (2.13)$$

Thus we obtain

$$\|u - \text{Op}_h^{L^s}(a_{-})\text{Op}_h^{L^u}(a_{+})u\| \leq C\log(1/h)\|\text{Op}_h(a)u\| + \mathcal{O}(h^{1-\rho-}). \quad (2.14)$$

The key component of the proof is the following estimate, following from the fractal uncertainty principle: if we take ρ sufficiently close to 1 then

$$\|\text{Op}_h^{L^s}(a_{-})\text{Op}_h^{L^u}(a_{+})\|_{L^2 \rightarrow L^2} \leq Ch^{\beta} \quad (2.15)$$

where $\beta > 0$ depends only on the set \mathcal{U} from (2.3).² Together (2.14) and (2.15) imply (2.1), where we take h small enough to remove the $\mathcal{O}(h^{\beta} + h^{1-\rho-})$ error.

To remove the $\log(1/h)$ prefactor in (2.1), we have to revise the definitions of $A_{\mathcal{X}}^{\pm}, A_{\mathcal{Y}}^{\pm}$. Roughly speaking, we expand $A_{\mathcal{X}}^{\pm}$ to include the products $A_{w_{N_1}}(\mp N_1) \cdots A_{w_1}(\mp 1)A_{w_0}(0)$ where $w_0, \dots, w_{N_1} \in \{1, 2\}$ are such that at most αN_1 of the digits w_j are equal to 1, where $\alpha > 0$ is a constant (the choice $\alpha = 0$ would correspond to the previous definition of $A_{\mathcal{X}}$). If α is small enough depending on β from (2.15), then the norm of $A_{\mathcal{X}}^{-}A_{\mathcal{X}}^{+}$ still goes to 0 with h .

Correspondingly we let $A_{\mathcal{Y}}^{\pm}$ include all the products $A_{w_{N_1}}(\mp N_1) \cdots A_{w_1}(\mp 1)A_{w_0}(0)$ where at least αN_1 of the digits w_j are equal to 1. Using an argument similar to [An08] (and further revising the definitions of $A_{\mathcal{X}}^{\pm}, A_{\mathcal{Y}}^{\pm}$), we can show that for an α -independent constant C we have the following stronger version of (2.12):

$$\|A_{\mathcal{Y}}^{\pm}u\| \leq C\alpha^{-1}\|\text{Op}_h(a)u\| + \mathcal{O}(h^{1/8-}),$$

which removes the $\log(1/h)$ prefactor in (2.1). See [DJ17, §4] for details.

3. Reduction to fractal uncertainty principle

In this section we explain how to prove the operator norm bound (2.15) based on a fractal uncertainty principle, Proposition 3.3.

²Note that the product $\text{Op}_h^{L^s}(a_{-})\text{Op}_h^{L^u}(a_{+})$ is not in any pseudodifferential calculus. This makes sense in light of (2.15): if we could write $\text{Op}_h^{L^s}(a_{-})\text{Op}_h^{L^u}(a_{+}) = \text{Op}_h(a_{-}a_{+})$ then the norm (2.15) would converge to $\sup|a_{-}a_{+}| = 1$ as $h \rightarrow 0$.

3.1. Porosity

Let a_{\pm} be the symbols defined in (2.13). In this subsection we establish *porosity* of their supports in the stable/unstable directions. We first define porosity of subsets of \mathbb{R} , which holds for instance for fractal sets of dimension < 1 :

Definition 3.1. *Let $\Omega \subset \mathbb{R}$ be a closed set, $0 < \alpha_0 \leq \alpha_1$, and $\nu \in (0, 1)$. We say that Ω is ν -porous on scales α_0 to α_1 if for each interval I of size $|I| \in [\alpha_0, \alpha_1]$, there exists a subinterval $J \subset I$ such that $|J| = \nu|I|$ and $J \cap \Omega = \emptyset$.*

In other words, Ω has holes at all points on all scales between α_0 and α_1 .

The following statement shows that $\text{supp } a_+$ is porous in the stable direction and $\text{supp } a_-$ is porous in the unstable direction:

Lemma 3.2. *There exist $\nu > 0$, $C_0 > 0$ depending only on M, \mathcal{U} such that for each $(x, \xi) \in T^*M \setminus 0$, the sets*

$$\Omega_{\pm}(x, \xi) = \{s \in \mathbb{R} \mid \exp(sU_{\pm})(x, \xi) \in \text{supp } a_{\pm}\} \subset \mathbb{R}$$

are ν -porous on scales $C_0 h^{\rho}$ to 1.

Proof. We consider the case of Ω_- . For an interval I and $(x, \xi) \in T^*M \setminus 0$, define the unstable horocyclic segment

$$e^{IU_-}(x, \xi) := \{e^{sU_-}(x, \xi) \mid s \in I\} \subset T^*M \setminus 0.$$

Note that the geodesic flow maps horocyclic segments to other horocyclic segments:

$$\varphi_t(e^{IU_-}(x, \xi)) = e^{(e^t I)U_-}(\varphi_t(x, \xi)).$$

Since \mathcal{U} is open, nonempty, and conic and the horocycle flow $\exp(sU_-)$ is uniquely ergodic on S^*M , there exists $T = T(M, \mathcal{U}) \geq 1$ and $\nu > 0$ such that each horocyclic segment of length T has a piece of length $10\nu T$ which lies in \mathcal{U} . Thus for each $(x, \xi) \in S^*M$ and each interval $\tilde{I} \subset \mathbb{R}$ with $T \leq |\tilde{I}| \leq 10T$, there exists a subinterval $\tilde{J} \subset \tilde{I}$ with $|\tilde{J}| = \nu|\tilde{I}|$ and $e^{\tilde{J}U_-}(x, \xi) \subset \mathcal{U}$.

By (2.4) and (2.13) we have for all $j = 0, 1, \dots, N_1$

$$\text{supp } a_- \cap \varphi_{-j}(\mathcal{U}) = \emptyset.$$

Put $C_0 := 10T$. Assume that $(x, \xi) \in T^*M \setminus 0$ and $I \subset \mathbb{R}$ satisfies $C_0 h^{\rho} \leq |I| \leq 1$. We need to find a subinterval $J \subset I$ with $|J| = \nu|I|$, $J \cap \Omega_-(x, \xi) = \emptyset$.

Choose $j \in \{0, 1, \dots, N_1\}$ such that $T \leq e^j |I| \leq 10T$ and put $\tilde{I} := e^j I$. Then there exists an interval $\tilde{J} \subset \tilde{I}$ such that $|\tilde{J}| = \nu|\tilde{I}|$ and $e^{\tilde{J}U_-}(\varphi_j(x, \xi)) \subset \mathcal{U}$. Put $J := e^{-j}\tilde{J}$, then $J \subset I$, $|J| = \nu|I|$, and

$$\text{supp } a_- \cap e^{JU_-}(x, \xi) \subset \text{supp } a_- \cap \varphi_{-j}(e^{\tilde{J}U_-}(\varphi_j(x, \xi))) = \emptyset.$$

Thus $J \cap \Omega_-(x, \xi) = \emptyset$ and the proof is finished. \square

Lemma 3.2 shows that the intersection of Ω_{\pm} with each stable/unstable horocycle is porous. This argument can be upgraded to show that in fact the projection of Ω_{\pm} onto the stable/unstable directions is porous, see [DJ17, Lemma 5.10]. Splitting a_{\pm} into finitely many pieces via a partition of unity, we may assume that for some $\nu > 0$ depending only on M, \mathcal{U}

- $\text{supp } a_{\pm} \subset V$ where $V \subset T^*M \setminus 0$ is a small h -independent open set;
- $\psi_{\pm} : V \rightarrow \mathbb{R}$ are smooth functions, homogeneous of order 0 and such that

$$\ker d\psi_{\pm} = \text{span}(H_p, D, U_{\mp}),$$

that is for $(x, \xi) \in V \cap S^*M$ the value $\psi_+(x, \xi)$ determines which weak unstable leaf passes through (x, ξ) and $\psi_-(x, \xi)$ determines which weak stable leaf passes through (x, ξ) ;

- $\text{supp } a_{\pm} \subset \psi_{\pm}^{-1}(X_{\pm})$ where $X_{\pm} \subset [0, 1]$ are ν -porous on scales h^{ρ} to 1 (we changed the value of ρ slightly to get rid of C_0).

3.2. Reduction to FUP for Fourier transform

We now briefly explain how to reduce the norm bound (2.15) to a one-dimensional fractal uncertainty principle for the semiclassical Fourier transform, Proposition 3.3, referring to [BD16, DJ17] for details.

The main tool is a microlocal normal form given by Fourier integral operators (see [DZ16, §2.2]). If $\varkappa : V \rightarrow T^*\mathbb{R}^2$ is a symplectomorphism onto its image, then we can associate to \varkappa a pair of bounded h -dependent operators

$$B : L^2(M) \rightarrow L^2(\mathbb{R}^2), \quad B' : L^2(\mathbb{R}^2) \rightarrow L^2(M)$$

such that for each h -independent $a \in C_0^\infty(V)$ we have

$$\text{Op}_h(a) = B' \text{Op}_h(a \circ \varkappa^{-1}) B + \mathcal{O}(h)_{L^2 \rightarrow L^2}.$$

Moreover, BB' and $B'B$ are semiclassical pseudodifferential operators. Same is true for the $S_{L,\rho}^{\text{comp}}$ calculus, thus (2.15) follows from the bound (suppressing the foliation L in the quantization Op_h^L)

$$\|\text{Op}_h(b_-) \text{Op}_h(b_+)\|_{L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)} \leq Ch^\beta, \quad b_\pm := a_\pm \circ \varkappa^{-1}. \quad (3.1)$$

We would like to choose \varkappa which ‘straightens out’ the weak stable/unstable foliations, which can be expressed in terms of the functions ψ_\pm . Imagine first that we could find \varkappa such that

$$\psi_- \circ \varkappa^{-1} = x_1, \quad \psi_+ \circ \varkappa^{-1} = \xi_1. \quad (3.2)$$

Then we get

$$\text{supp } b_- \subset \{x_1 \in X_-\}, \quad \text{supp } b_+ \subset \{\xi_1 \in X_+\}. \quad (3.3)$$

We use the right quantization

$$\text{Op}_h(b)v(x) = (2\pi h)^{-2} \int_{\mathbb{R}^4} e^{\frac{i}{h}\langle x-y, \xi \rangle} b(y, \xi) v(y) dy d\xi$$

Then the first part of (3.3) implies that for all $v \in L^2(\mathbb{R}^2)$

$$\|\text{Op}_h(b_-)v\|_{L^2} \leq C\|v\|_{L^2(X_- \times \mathbb{R})}.$$

The second one shows that $\text{Op}_h(b_+)v$ lives at semiclassical frequencies in $X_+ \times \mathbb{R}$: if $\mathcal{F}_h f(\xi) = \widehat{f}(\xi/h)$ is the semiclassical Fourier transform then

$$\text{supp } \mathcal{F}_h(\text{Op}_h(b_+)v) \subset X_+ \times \mathbb{R}.$$

Therefore (3.1) reduces to the following estimate for all $v \in L^2(\mathbb{R}^2)$:

$$\text{supp } \widehat{v} \subset h^{-1}X_+ \times \mathbb{R} \implies \|v\|_{L^2(X_- \times \mathbb{R})} \leq Ch^\beta \|v\|_{L^2(\mathbb{R}^2)}. \quad (3.4)$$

Since (3.4) does not depend on the x_2 variable, we can remove it and reduce to the following one-dimensional statement (applied with $f(x_1) = v(x_1, x_2)$ and all x_2 , and quantifying how close ρ should be to 1):

Proposition 3.3. *For each $\nu > 0$ there exist $C, \beta > 0$ such that for all $f \in L^2(\mathbb{R})$, $0 < \rho \leq 1$, and $X, Y \subset [0, 1]$ which are ν -porous on scales h^ρ to 1,*

$$\text{supp } \widehat{f} \subset h^{-1} \cdot Y \implies \|f\|_{L^2(X)} \leq Ch^{\beta-2(1-\rho)} \|f\|_{L^2(\mathbb{R})}. \quad (3.5)$$

Unfortunately there is no symplectomorphism \varkappa that simultaneously straightens out the stable and the unstable foliations, that is (3.2) can never hold. Instead one can reduce (2.15) to an estimate of the form (3.5) when the Fourier transform is replaced by a different oscillatory integral operator, see [DJ17, §5.2]. The resulting statement can be reduced to the original version of (3.5) (with the Fourier transform) – see [BD16, §4.2].

4. Proof of fractal uncertainty principle

We finally explain some ideas behind the proof of the fractal uncertainty principle, Proposition 3.3. First of all we may assume that $\rho = 1$. Indeed, otherwise X, Y can be written as unions of $h^{\rho-1}$ sets which are ν -porous on scales h to 1 (specifically take intersections of X, Y with the unions of intervals $\bigsqcup_{j \in \mathbb{Z}} [h^\rho j + \frac{h}{2}\ell, h^\rho j + \frac{h}{2}(\ell + 1)]$ where $\ell = 0, 1, \dots, [2h^{\rho-1}]$), and it remains to use the triangle inequality.

The proof of Proposition 3.3 is given in [BD16, Theorem 4]; the only difference is that [BD16] required X, Y to be Ahlfors–David regular of dimension $\delta < 1$. As explained in [DJ17, §5.1] each ν -porous set can be embedded into a regular set of dimension $\delta = \delta(\nu) < 1$. Moreover, the proof in [BD16] can be adapted directly to ν -porous sets, and this is the approach we take here.

4.1. An iteration argument

We start by exploiting the porosity of X to reduce Proposition 3.3 to a lower bound on f on a union of intervals, Proposition 4.1. Fix large $K \in \mathbb{N}$ such that $2^{-K-1} \leq h \leq 2^{-K}$. For $k \in 0, 1, \dots, K$, consider the partition

$$\mathbb{R} = \bigcup_{I \in \mathcal{I}(k)} I, \quad \mathcal{I}(k) := \{[2^{-k}j, 2^{-k}(j+1)]: j \in \mathbb{Z}\}.$$

Since X is ν -porous on scales h to 1, for each $I \in \mathcal{I}(k)$ there exists a subinterval

$$I' \subset I, \quad |I'| = \nu|I| = 2^{-k}\nu, \quad I' \cap X = \emptyset.$$

Denote

$$U'_k := \bigcup_{I \in \mathcal{I}(k)} I' \subset \mathbb{R}, \quad U'_k \cap X = \emptyset.$$

The following is the key lower bound of f on U'_k , proved later in this section:

Proposition 4.1. *There exists $c = c(\nu) > 0$ such that for all $f \in L^2(\mathbb{R})$ and all k ,*

$$\text{supp } \widehat{f} \subset h^{-1} \cdot Y + [-2^k, 2^k] \implies \|f\|_{L^2(U'_k)} \geq c\|f\|_{L^2(\mathbb{R})}. \quad (4.1)$$

We now explain how Proposition 4.1 implies Proposition 3.3; see [BD16, §3.4] for more details. Denote $X_k := \mathbb{R} \setminus U'_k$, then $X \subset X_k$ and Proposition 4.1 implies

$$\text{supp } \widehat{f} \subset h^{-1} \cdot Y + [-2^k, 2^k] \implies \|\mathbf{1}_{X_k} f\|_{L^2} \leq \sqrt{1 - c^2} \|f\|_{L^2}. \quad (4.2)$$

We have

$$\|f\|_{L^2(X)} \leq \|\mathbf{1}_{X_K} \cdots \mathbf{1}_{X_1} \mathbf{1}_{X_0} f\|_{L^2}.$$

We would like to use (4.2) to show that for all k ,

$$\|\mathbf{1}_{X_k} \cdots \mathbf{1}_{X_0} f\|_{L^2} \leq \sqrt{1 - c^2} \|\mathbf{1}_{X_{k-1}} \cdots \mathbf{1}_{X_0} f\|_{L^2}.$$

This works well for $k = 0$, since $\text{supp } \widehat{f} \subset h^{-1} \cdot Y$. However, the next values of k do not work since we have no restriction on the Fourier support of $\mathbf{1}_{X_{k-1}} \cdots \mathbf{1}_{X_0} f$. To fix this we choose large k_0 depending on c, ν and replace $\mathbf{1}_{X_k}$ by its mollification

$$\chi_k := \mathbf{1}_{X_k} * \psi_k, \quad \psi_k(x) = 2^{k+k_0} \psi(2^{k+k_0} x)$$

where ψ is a nonnegatives Schwartz function with $\text{supp } \widehat{\psi} \subset [-\frac{1}{2}, \frac{1}{2}]$, and $\int \psi = 1$. Note that $0 \leq \chi_k \leq 1$ and $\text{supp } \widehat{\chi}_k \subset \text{supp } \widehat{\psi}_k \subset [-2^{k+k_0-1}, 2^{k+k_0-1}]$.

For k divisible by k_0 , put $f_k := \chi_k \chi_{k-k_0} \cdots \chi_{k_0} \chi_0 f$. Then

$$\text{supp } \widehat{f}_{k-k_0} \subset \text{supp } \widehat{\chi}_{k-k_0} + \cdots + \text{supp } \widehat{\chi}_0 + \text{supp } \widehat{f} \subset h^{-1} \cdot Y + [-2^k, 2^k].$$

We can arrange to have $\chi_k \leq \frac{1}{2}$ on U'_k . Then (4.2) applies to f_{k-k_0} , implying

$$\|f_k\|_{L^2} = \|\chi_k f_{k-k_0}\|_{L^2} \leq \sqrt{1 - c^2/10} \|f_{k-k_0}\|_{L^2}. \quad (4.3)$$

Shrinking the intervals I' by a factor of 2, we can ensure that X_k contains the $2^{-k-2}\nu$ -neighborhood of X . Then for large k_0 , we have $\chi_k \geq 1 - 2^{-k_0}$ on X . Iterating (4.3) we get (assuming for simplicity that K is divisible by k_0)

$$\|f\|_{L^2(X)} \leq (1 - 2^{-k_0})^{-K/k_0-1} \|f_K\|_{L^2} \leq 2 \left(\frac{\sqrt{1 - c^2/10}}{1 - 2^{-k_0}} \right)^{K/k_0} \|f\|_{L^2}.$$

Taking k_0 large enough, we can make this bounded by $(1 - c^2/100)^{K/k_0} \|f\|_{L^2}$. Since c, k_0 are independent of h and $K \sim \log(1/h)$ this gives (3.5) and finishes the proof of Proposition 3.3.

4.2. Unique continuation estimates

We now discuss how to prove Proposition 4.1, which is a *unique continuation estimate* for functions of restricted Fourier support.

It suffices to consider the case $k = 0$. Indeed, otherwise we define the rescaled function $\tilde{f}(x) := 2^{-k/2} f(2^{-k}x)$. The Fourier transform of \tilde{f} is supported on $2^{-k} \text{supp } \hat{f} \subset \tilde{h}^{-1} \cdot Y + [-1, 1]$ where $\tilde{h} := 2^k h$. Moreover $\|\tilde{f}\|_{L^2(U'_k)} = \|f\|_{L^2(U'_k)}$ and $2^k U'_k$ satisfies same assumptions as U'_0 . It remains to apply (4.1) with $k := 0$, $h := \tilde{h}$, and $f := \tilde{f}$. We henceforth assume that $k = 0$ and write $\mathcal{I} := \mathcal{I}(0)$ and $U' := U'_0$.

In this section we show a bound of the type (4.1) which replaces the requirement on $\text{supp } \hat{f}$ with a Fourier decay condition. See [BD16, §§3.2,3.3] for details.

Lemma 4.2. *Assume that θ is a positive even function on \mathbb{R} , θ is decreasing on $[0, \infty)$, and $\xi\theta(\xi)$ is increasing on $[0, \infty)$. Then there exists a constant C depending only on ν, θ such that for all $K \geq 10$ and all $g \in L^2(\mathbb{R})$ we have*

$$\|g\|_{L^2(\mathbb{R})} \leq \frac{C}{\theta(K)} \|g\|_{L^2(U')}^\kappa \cdot \|e^{|\xi|\theta(\xi)} \hat{g}(\xi)\|_{L^2(\mathbb{R})}^{1-\kappa} + \frac{C e^{-\kappa K \theta(K)}}{\theta(K)} \|e^{|\xi|\theta(\xi)} \hat{g}(\xi)\|_{L^2(\mathbb{R})} \quad (4.4)$$

where $\kappa = e^{-C/\theta(K)}$.

Remark. Letting $K \rightarrow \infty$ we see that if

$$e^{-C/\theta(K)} K \theta(K) - \log \theta(K) \rightarrow \infty \quad \text{as } K \rightarrow \infty \quad (4.5)$$

then any $g \not\equiv 0$ with $e^{|\xi|\theta(\xi)} \hat{g}(\xi) \in L^2$ cannot identically vanish on U' ; this can be used to show that such g cannot be compactly supported. Note that (4.5) holds when $\theta(K) = 1$, this is reasonable since functions with exponentially decaying Fourier transform are real analytic. On the other hand (4.5) does not hold when $\theta(K) = K^{-\varepsilon}$ and $\varepsilon > 0$; this corresponds to existence of compactly supported functions in Gevrey classes. We will use later that (4.5) holds with $\theta(K) = (\log K)^{-\delta}$ when $\delta < 1$.

Sketch of the proof. Put $R := \|e^{|\xi|\theta(\xi)} \hat{g}(\xi)\|_{L^2}$. We split g into low and high frequencies:

$$g = g_{\text{low}} + g_{\text{high}}, \quad \text{supp } \hat{g}_{\text{low}} \subset [-K, K], \quad \text{supp } \hat{g}_{\text{high}} \subset \mathbb{R} \setminus [-K, K].$$

Then g_{high} is estimated in terms of R :

$$\|g_{\text{high}}\|_{L^2} \leq e^{-K\theta(K)} R. \quad (4.6)$$

As for g_{low} , we claim that it satisfies the bound

$$\|g_{\text{low}}\|_{L^2(\mathbb{R})} \leq \frac{C}{\theta(K)} \|g_{\text{low}}\|_{L^2(U')}^\kappa \cdot R^{1-\kappa}. \quad (4.7)$$

We only show the following local version: for all $I \in \mathcal{I}$,

$$\|g_{\text{low}}\|_{L^2(I)} \leq \frac{C}{\theta(K)} \|g_{\text{low}}\|_{L^2(I')}^\kappa \cdot R^{1-\kappa}. \quad (4.8)$$

The bound (4.7) is obtained similarly, summing over all the intervals I in the end; see the proof of [BD16, Lemma 3.2].

To prove (4.8), we extend g_{low} holomorphically to \mathbb{C} ; this is possible since \hat{g}_{low} is compactly supported. Consider the following domain in the complex plane:

$$\Sigma := \{z \in \mathbb{C} : |\text{Im } z| < r\} \setminus I', \quad r := \theta(K).$$

For each $t \in I \setminus I'$, let μ_t be the harmonic measure of Σ centered at t ; it is a probability measure on the boundary

$$\partial\Sigma = \Sigma_+ \sqcup \Sigma_- \sqcup I', \quad \Sigma_{\pm} := \{\operatorname{Im} z = \pm r\}.$$

In other words, for each harmonic function u on Σ its value $u(t)$ is equal to the integral of the boundary values of u over μ_t . Since $\log |g_{\text{low}}|$ is subharmonic, we have

$$\log |g_{\text{low}}(t)| \leq \int_{\partial\Sigma} \log |g_{\text{low}}(z)| d\mu_t(z). \quad (4.9)$$

We have $\mu_t(I') \geq \kappa$ for all $t \in I \setminus I'$. This can be explained for instance using the stochastic interpretation of harmonic measure: μ_t is the probability distribution of the point through which the Brownian motion starting at t will exit the domain Σ . Since t is distance 1 away from I' and Σ has height $2r$, the probability that the Brownian motion starting at t hits I' before Σ_{\pm} is bounded below by $e^{-C/r}$.

Now (4.9) implies (using that $\sup_{I'} |g_{\text{low}}| \leq \sup_{\Sigma_+ \sqcup \Sigma_-} |g_{\text{low}}|$)

$$\sup_{I'} |g_{\text{low}}| \leq \left(\sup_{I'} |g_{\text{low}}| \right)^{\kappa} \cdot \left(\sup_{\Sigma_+ \sqcup \Sigma_-} |g_{\text{low}}| \right)^{1-\kappa}. \quad (4.10)$$

Using estimates on the density of μ_t one can replace the sup-norms in (4.10) by L^2 -norms, with small changes – see the proof of [DZ16, Lemma 3.2]. This yields (4.8), where we use that

$$\|g_{\text{low}}\|_{L^2(\Sigma_{\pm})} \leq \|e^{r|\xi|} \widehat{g}_{\text{low}}(\xi)\|_{L^2(\mathbb{R})} \leq R.$$

Armed with (4.6) and (4.7) we write

$$\begin{aligned} \|g\|_{L^2} &\leq \|g_{\text{low}}\|_{L^2} + \|g_{\text{high}}\|_{L^2} \leq \frac{C}{\theta(K)} \|g_{\text{low}}\|_{L^2(U')}^{\kappa} \cdot R^{1-\kappa} + \|g_{\text{high}}\|_{L^2} \\ &\leq \frac{C}{\theta(K)} (\|g\|_{L^2(U')}^{\kappa} + \|g_{\text{high}}\|_{L^2}^{\kappa}) \cdot R^{1-\kappa} + \|g_{\text{high}}\|_{L^2} \\ &\leq \frac{C}{\theta(K)} \|g\|_{L^2(U')}^{\kappa} \cdot R^{1-\kappa} + \frac{C e^{-\kappa K \theta(K)}}{\theta(K)} R \end{aligned}$$

which gives (4.4). □

4.3. An adapted multiplier and end of the proof

We finally explain how to use the Fourier support condition $\operatorname{supp} \widehat{f} \subset h^{-1} \cdot Y + [-1, 1]$. To do that we construct a compactly supported multiplier adapted to Y :

Lemma 4.3. *There exists constants³ $c_1 > 0$, $\delta < 1$ depending only on ν and a function $\psi \in L^2(\mathbb{R})$ such that, defining*

$$\theta(\xi) := (\log(10 + |\xi|))^{-\delta},$$

we have

1. $\operatorname{supp} \psi \subset [-\nu/10, \nu/10]$;
2. $\|\widehat{\psi}\|_{L^2(-1/2, 1/2)} \geq c_1$;
3. $|\widehat{\psi}(\xi)| \leq (1 + |\xi|)^{-10} \exp(-c_1 |\xi| \theta(\xi))$ for all $\xi \in h^{-1} \cdot Y + [-2, 2]$;
4. $|\widehat{\psi}(\xi)| \leq (1 + |\xi|)^{-10}$ for all ξ .

Before explaining the proof of Lemma 4.3 we use it to give

³In the notation of [BD16], one should replace δ by $\frac{1+\delta}{2}$.

Sketch of the proof of Proposition 4.1. For each $I \in \mathcal{I}$, let $I'' \subset I'$ be the interval with the same center and $|I''| = \frac{1}{2}|I'| = \nu/2$. Let U'' be the union of all I'' . Take the function ψ from Lemma 4.3 and define

$$g := f * \psi, \quad \widehat{g} = \widehat{f} \cdot \widehat{\psi}.$$

Since $\text{supp } \widehat{f} \subset h^{-1} \cdot Y + [-1, 1]$ we have (using the Sobolev space $H^{-10}(\mathbb{R})$)

$$\|e^{c_1|\xi|\theta(\xi)}\widehat{g}(\xi)\|_{L^2} \leq \|f\|_{H^{-10}}.$$

Applying Lemma 4.2 with U'' in place of U' , we get for all K

$$\|g\|_{L^2(\mathbb{R})} \leq \frac{C}{\theta(K)} \|g\|_{L^2(U'')}^\kappa \cdot \|f\|_{H^{-10}}^{1-\kappa} + \frac{Ce^{-c_1\kappa K\theta(K)}}{\theta(K)} \|f\|_{H^{-10}}. \quad (4.11)$$

Now, the support condition on ψ implies that $g = (\mathbf{1}_{U'}f) * \psi$ on U'' , therefore $\|g\|_{L^2(U'')} \leq \|\mathbf{1}_{U'}f\|_{H^{-10}}$. We can apply the same argument to $f_\eta(x) = e^{ix\eta}f(x)$, $|\eta| \leq 1$, since $\text{supp } \widehat{f}_\eta \subset h^{-1} \cdot Y + [-2, 2]$. Then (4.11) bounds $\|f_\eta * \psi\|_{L^2} = \|\widehat{f}(\xi - \eta)\widehat{\psi}(\xi)\|_{L^2}$. Integrating the square of this over η and using property (2) of ψ we bound $\|\widehat{f}\|_{L^2(-1/2, 1/2)}$:

$$\|\widehat{f}\|_{L^2(-1/2, 1/2)} \leq \frac{C}{\theta(K)} \|\mathbf{1}_{U'}f\|_{H^{-10}}^\kappa \cdot \|f\|_{H^{-10}}^{1-\kappa} + \frac{Ce^{-c_1\kappa K\theta(K)}}{\theta(K)} \|f\|_{H^{-10}}. \quad (4.12)$$

Finally, we see that (4.12) holds also when f is replaced by $f_\ell(x) = e^{i\ell x}f(x)$ and $\ell \in \mathbb{Z}$, $|\ell| \leq h^{-1}$. Indeed, $\text{supp } \widehat{f}_\ell$ lies in $h^{-1}(Y + \ell h) + [-1, 1]$ and $Y + \ell h$ is still ν -porous.⁴ Adding the resulting inequalities (see the proof of [BD16, Proposition 3.3]) we get

$$\|f\|_{L^2} \leq \frac{C}{\theta(K)} \|f\|_{L^2(U')}^\kappa \cdot \|f\|_{L^2}^{1-\kappa} + \frac{Ce^{-c_1\kappa K\theta(K)}}{\theta(K)} \|f\|_{L^2}. \quad (4.13)$$

As remarked after Lemma 4.2, the function $\theta(K) = c_1(\log(10 + K))^{-\delta}$ satisfies (4.5). Therefore if K is large (but independent of h) the last term on the right-hand side of (4.13) can be removed, yielding (4.1). \square

We finally explain how to prove Lemma 4.3. We ignore the $|\xi|^{-10}$ prefactor in property (3) (one can always convolve ψ with a function in $C_0^\infty(\mathbb{R})$ to regain it). We also replace $h^{-1} \cdot Y + [-2, 2]$ with just $h^{-1} \cdot Y$ for simplicity, but the argument below applies to $h^{-1} \cdot Y + [-2, 2]$ as well.

Denote $\widetilde{Y} := h^{-1} \cdot Y$. We first construct a ‘reasonable’ weight function

$$w : \mathbb{R} \rightarrow [0, \infty), \quad w(\xi) \geq |\xi|\theta(\xi) \quad \text{for } \xi \in \widetilde{Y}. \quad (4.14)$$

Assume for simplicity that $h = 2^{-K}$ and consider a dyadic partition of \widetilde{Y} :

$$\widetilde{Y} = \bigcup_{k=0}^K (\widetilde{Y} \cap J_k), \quad J_0 = [-1, 1], \quad J_k = [-2^k, -2^{k-1}] \cup [2^{k-1}, 2^k] \text{ for } k \geq 1.$$

We next take a minimal covering of each $\widetilde{Y} \cap J_k$ by intervals of size $2^k\theta(2^k)$:

$$\widetilde{Y} \cap J_k \subset \bigcup_{\ell=1}^{N_k} J_{k\ell}, \quad |J_{k\ell}| = 2^k\theta(2^k).$$

It is finally time to use the fact that Y is ν -porous: we claim that there exists $\varepsilon = \varepsilon(\nu) > 0$ such that

$$N_k \leq C(\theta(2^k))^{-(1-\varepsilon)}. \quad (4.15)$$

Indeed, put $m := \lceil 2/\nu \rceil$. Divide $J := [2^{k-1}, 2^k]$ into m intervals of size $m^{-1}|J|$ each. It follows from ν -porosity of Y that at least one of these intervals does not intersect \widetilde{Y} . Divide each of the remaining intervals into m intervals of size $m^{-2}|J|$ each and repeat the process, at j -th step keeping only $(m-1)^j$ intervals of size $m^{-j}|J|$ each. We stop once $m^{-j} \sim \theta(2^k)$, then the number of resulting intervals is bounded by (4.15) where $\varepsilon = 1 - \frac{\log(m-1)}{\log m}$.

⁴This means that the argument actually uses many multipliers ψ , one for each value of ℓ .

Now, define nonnegative weights $w_{k\ell} \in C_0^\infty(\mathbb{R})$ such that $\sup |w'_{k\ell}| \leq 100$, $w_{k\ell}$ is supported on the $2^k\theta(2^k)$ -neighborhood of $J_{k\ell}$, and $w_{k\ell} = 2^k\theta(2^k)$ on $J_{k\ell}$. Put

$$w := 10 \sum_{k=0}^K \sum_{\ell=1}^{N_k} w_{k\ell},$$

then w satisfies (4.14) for large enough $|\xi|$ and one can change it for $|\xi| \leq C$ to make sure (4.14) holds everywhere.

Given (4.14), it remains to construct ψ which satisfies properties (1) and (2) in Lemma 4.3 and $|\widehat{\psi}| \leq \exp(-c_1 w)$. One way is to use the Beurling–Malliavin Theorem [BM62], see [BD16, §3.1]. This theorem asserts that there exists a function

$$\psi \in L^2(\mathbb{R}), \quad \psi \not\equiv 0, \quad |\psi| \leq \exp(-w)$$

if the derivative w' is bounded and the Poisson integral of w converges:

$$\int_{\mathbb{R}} \frac{w(\xi)}{1 + \xi^2} d\xi \leq \infty. \quad (4.16)$$

Our weight w has derivative bounded by an h -independent constant. Same is true of its Poisson integral: using (4.15) we estimate

$$\int_{\mathbb{R}} \frac{w(\xi)}{1 + \xi^2} d\xi \leq C \sum_{k=0}^K N_k (\theta(2^k))^2 \leq C \sum_{k=0}^K (\theta(2^k))^{1+\varepsilon} \leq C \sum_{k=0}^K k^{-\delta(1+\varepsilon)}$$

which is bounded if we choose $\delta < 1$ such that $\delta(1 + \varepsilon) > 1$. A compactness argument using the Beurling–Malliavin theorem (see [BD16, Lemma 2.11]) now gives Lemma 4.3.

The Beurling–Malliavin theorem is one of the deepest statements in harmonic analysis. It is thus worth noting that we do not need the full strength of this theorem here. This is due to the fact that the Hilbert transform of our weight w has derivative bounded by some h -independent constant, and there is a direct construction of the function ψ for such weights. See [JZ17] for details.

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DEPARTMENT OF MATHEMATICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
77 MASSACHUSETTS AVENUE
CAMBRIDGE, MA 02139, USA
dyatlov@math.mit.edu