Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Roscoff, 5–9 juin 2017

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J. É. D. P. (2017), Exposé n° III, 13 p.

<http://jedp.cedram.org/item?id=JEDP_2017____A3_0>

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GDR 2434 (CNRS)

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Abstract

We explain the main steps in the proof of the existence of mixing solutions of the incompressible porous media equation for all Muskat type H^5 initial data in the fully unstable regime which appears in [4]. Also we present some numerical simulations about these solutions.

1. Introduction: The IPM system and the Muskat problem

The incompressible porous media system (IPM) models the dynamics of an incompressible fluid with density ρ and viscosity ν which relies in a porous media with permeability κ and is given by the following equations on $\mathbb{R}^d$:

$$\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \qquad \rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}), \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

$$\frac{\nu}{\kappa}\mathbf{u} = -\nabla p - g\rho \mathbf{e}_d, \tag{1.3}$$

where $\mathbf{u}$ is the velocity of the flow, p its pressure and g is the acceleration due to gravity (we will consider that the fluid is under the gravitational force). Thus this system is composed of the mass conservation law (1.1), the incompressibility condition (1.2) and the Darcy's law (1.3). This last law was obtained by Darcy, in an experimental way, in 1920 (see [19]) and reflects the fact that the porous media is trying to freeze the motion of the fluid. For a derivation of Darcy's law from the Navier-Stokes equation by using homogenization one can read [39].

In this paper we will work in $\mathbb{R}^2$ and we will take all the constant ν, κ and g in (1.3) equal to one. In two dimensions the IPM system also models the dynamics of a fluid in Hele-Shaw cell. Such a cell consist of two plates and a viscous and incompressible fluid between both of them whose velocity satisfies the Navier-Stokes equations with zero boundary conditions. In this situation, under certain assumptions, one can obtain the IPM system by sending the distance between the two plates to zero.

The Muskat problem deals with the following initial data ρ_0 for IPM: the density takes only two (constant) values

$$\rho_0(\mathbf{x}) = \begin{cases} \rho^- & \mathbf{x} \in \Omega_0^- \\ \rho^+ & \mathbf{x} \in \Omega_0^+ \end{cases} \tag{1.4}$$

where $\Omega_0^\pm$ are open and simply connected domains such that $\mathbb{R}^2 = \overline{\Omega_0^+} \cup \Omega_0^-$. The interface between these two difference domains will be denoted by $\Gamma(0) = \partial\Omega_0^+ \cap \partial\Omega_0^-$. Although more general situations can be considered we will restrict our analysis to either asymptotically flat interfaces, i.e., $\lim_{x_1 \to \pm\infty} x_2 = 0$ for $\mathbf{x} \in \Gamma(0)$ or periodic in the horizontal variable and Ω_0^+ will be the domain that relies below.

The author was partially supported by the grant MTM2014-59488-P (Spain), ICMAT Severo Ochoa project SEV-2015-556 and ERC grant 307179-GFTIPFD..

If one consider that, starting with the initial data (1.4), the (weak) solution to the IPM system has the form

$$\rho(\mathbf{x},t) = \begin{cases} \rho^- & \mathbf{x} \in \Omega^-(t) \\ \rho^+ & \mathbf{x} \in \Omega^+(t) \end{cases} \tag{1.5}$$

where $\Omega^{\pm}(t)$ depend on time and move with the flow, an equation for the interface $\Gamma(t) = \partial\Omega^+(t) \cap \partial\Omega^-(t)$ can be found. Indeed, if we parameterize the $\Gamma(t)$ by

$$\Gamma(t) = \{\mathbf{x} \in \mathbb{R}^2 \, : \, \mathbf{x} = \mathbf{z}(\alpha,t) = (z_1(\alpha,t), \, z_2(\alpha,t)), \, \alpha \in \mathbb{R}\},$$

we obtain the equation

$$\partial_t \mathbf{z}(\alpha,t) = \frac{\rho^+ - \rho^-}{2\pi} \int_{\mathbb{R}} \frac{z_1(\alpha,t) - z_1(\beta,t)}{|\mathbf{z}(\alpha,t) - \mathbf{z}(\beta,t)|^2} (\partial_\alpha \mathbf{z}(\alpha,t) - \partial_\beta \mathbf{z}(\beta,t)) d\beta. \tag{1.6}$$

$$\mathbf{z}(\alpha,0) = \mathbf{z}_0(\alpha), \tag{1.7}$$

Conversely, solutions to equation (1.6) give rise to weak solutions of the IPM system through expression (1.5) and Biot-Savart law.

The local and global well-posedness of equation (1.6) has been studied intensively in the last decades. In order to summarize the main results we will define three different regimes: the stable regime in which the denser fluid is below, $\rho^+ > \rho^-$ and the interface can be parameterize by the graph of function; the fully unstable regime in which the denser fluid is above, $\rho^+ < \rho^-$ and again the interface can be parameterize by the graph of function; and the partial unstable regime in which there is a part of the interface where the denser fluid is above (and also a part of it where denser fluid is below). The local in time existence of solutions in the stable regime was first shown in [16] for an initial data in H^k, with $k \geq 4$ in dimension 2 and 3. In addition several global existence results have been proven for small and for medium size initial data (see [13], [17], [11], [12], [1]). For a survey about these results we recommend [26]. However, in the fully unstable regime the problem is ill-possed in Sobolev spaces ([16], [5] and [35]) and although there is not a proof of it there is a strong analytical evidence that is also ill-possed in the partial unstable regime.

Then, the situation before [4] was that any solution for the IPM system by starting with an initial data of Muskat type in the fully unstable regime was known. However, this is a quite realistic situation from a physical point of view, and one can find several experiment about the evolution of this kind configurations in a Hele-Shaw cell. In [4] we addresses this problem by construction mixing solutions. The rest of the paper is devoted to explaining what a mixing solution is and giving an sketch of the main ideas of the proof of its existence. In section 5 we present some numerical results investigating the behaviour of these solutions.

2. Mixing solution

As comment in the introduction we would like to find some solutions for the IPM system with an initial data of Muskat type in the fully unstable regime whose interface belongs to some Sobolev space (non analytic in general). In the figure 2.1 we can see a draw of such an initial data. In particular, we are interesting in showing the existence of mixing solutions whose definition is the following

Definition 1. *The density $\rho(\boldsymbol{x},t)$ and the velocity $\boldsymbol{u}(\boldsymbol{x},t)$ are a "mixing solution" of the IPM system if they are a weak solution and also there exist, for every $t \in [0,T]$, open simply connected domains $\Omega^{\pm}(t)$ and $\Omega_{mix}(t)$ with $\overline{\Omega^+} \cup \overline{\Omega^-} \cup \Omega_{mix} = \mathbb{R}^2$ such that, for almost every $(x,t) \in \mathbb{R}^2 \times [0,T]$, the following holds:*

$$\rho(x,t) = \begin{cases} \rho^{\pm} & in\ \Omega^{\pm}(t) \\ (\rho - \rho^+)(\rho - \rho^-) = 0 & in\ \Omega_{mix}(t) \end{cases} .$$

For every $r > 0, x \in \mathbb{R}^2, 0 < t < T$ $B((x,t), r) \subset \cup_{0<t<T} \Omega_{mix}(t)$ it holds that

$$\int_B (\rho - \rho^+) \int_B (\rho - \rho^-) \neq 0 \, .$$

This definition is motivated by the solutions that were constructed by L. Székelyhidi Jr., in [36]. In that paper the author showed the existence of mixing solutions with a initial data of Muskat type in the fully unstable regime in which the interface was flat (see also [32]).

The theorem that has been proven in [4] is the following

Theorem 1. *Let* $\Gamma(0) = \{\boldsymbol{x} \in \mathbb{R}^2, : \boldsymbol{x} = (x, f_0(x))\}$ *with* $f_0(x) \in H^5$. *Let us suppose that* $\rho^+ < \rho^-$. *Then there exist infinitely many "mixing solutions" starting with the initial data of Muskat type given by* $\Gamma(0)$ *(in the fully unstable regime) for the IPM system.*

We would like to make two remarks about a mixing solution:

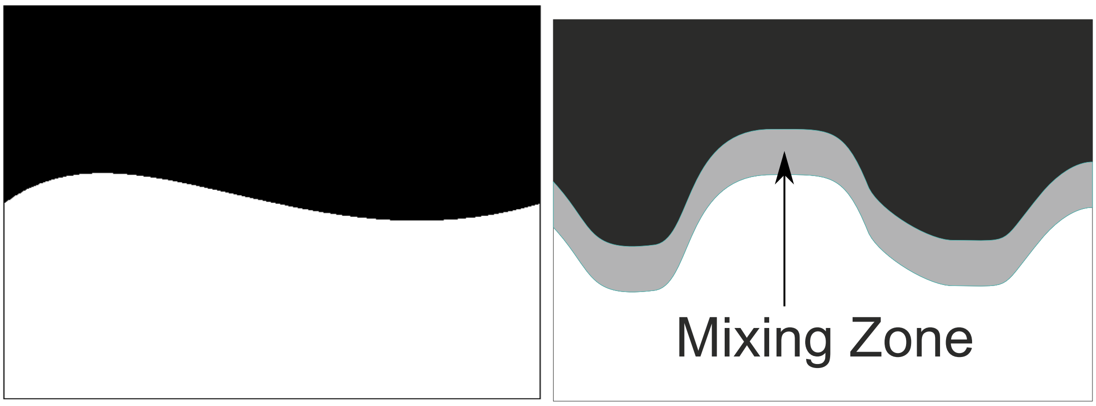
1. A mixing solution does not change the values that the density takes initially.

2. There is total mixing in the mixing zone.

An interesting question is which is the distribution of the density inside of the mixing zone $\Omega_{\text{mix}}(t)$. We think that the following holds:

There exist infinitely many mixing solutions such that the mixing is linearly degraded.

Hopefully a proof of it will arise in a forthcoming note. Roughly speaking, this assertion means that, there exist infinitely many solutions, such that the mean values of its density go from ρ^- to ρ^+ from the top to the bottom of the mixing zone in a linear way. Indeed, the mean values of the density of any of these solutions are given by the values of the same subsolution. In the next section we will present the concept of subsolution and will explain with some more details this issue.



(a) Initial data. (b) Mixing solution.

Figure 2.1: A Muskat type initial data in the fully unstable regime and a mixing solution

3. The subsolution

In [4] and [36] it has been shown that the existence of mixing solutions relies in the existence of a subsolution. A rigorous definition of subsolution can be found in [4]. Here we will consider that a subsolution is a 5−upla $(\rho, \mathbf{u}, \mathbf{m}) \in C\left((0,T) \times \mathbb{R}^2\right)$ satisfying (weakly)

$$\partial_t \rho + \nabla \cdot \mathbf{m} = 0, \tag{3.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{3.2}$$

$$\nabla^{\perp} \cdot \mathbf{u} = -\partial_{x_1} \rho \tag{3.3}$$

on $(0,T) \times \mathbb{R}^2$ and the inclusions

$$\left| \mathbf{m} - \rho \mathbf{u} + \frac{1}{2}\left(1-\rho^2\right)(0,1) \right| < \frac{1}{2}\left(1-\rho^2\right) \quad \text{on } \Omega_{\text{mix}}(t) \tag{3.4}$$

and

$$\rho = \mp 1, \quad \mathbf{m} = \mp \mathbf{u}, \quad \text{on } \Omega^{\pm}(t)$$

where $\Omega^{\pm}(t)$ and $\Omega_{\text{mix}}(t)$ satisfy the properties in the definition of mixing solutions. Here we will work with $\rho^{\pm} = \mp 1$ for sake of simplicity but any others values $\rho^+ < \rho^-$ could be considered.

The method of the proof of the fact that subsolutions yields solution is based on the adaptation of the method of convex integration for the incompressible Euler equation in the Tartar framework developed recently by De Lellis and Székelyhidi (see [3], [10], [18], [20], [21], [22], [23], [24], [38] and [37] for the incompressible Euler and for another equations [6], [7], [8], [2] and [33]).

Very briefly, the version of convex integration used initially by De Lellis and Székelyhidi understands a nonlinear PDE, $F(\rho, u) = 0$ as a combination of a linear system $L(\rho, u, q) = 0$ and a pointwise constraint $(\rho, u, q) \in K$ where K is a convenient set of states and q is an artificial new variable. Then L gives rises to a wave cone Λ and the geometry of the Λ hull of K, K^{Λ}, rules whether the convex integration method will yield solutions. An h-principle holds in this context: if for a given initial data there exists an evolution which belongs to K^{Λ}, called a subsolution, then one find infinitely many weak solutions.

For the case of the IPM system, in [15], the authors initiated this analysis and used a version of the convex integration method which avoids the computation of Λ hulls based on T4 configurations, key in other applications of convex integration, e.g to the (lack of) regularity of elliptic systems [31, 30, 29]. Keeping the discussion imprecise, their criteria amounts to say that $(0, 0)$ must be in the convex hull of $\Lambda \cap K$ in a stable way. Shvydkoy extended this approach to a general family of active scalars, where the velocity is an even singular integral operator, in [34]. Recently, in [28], Isett and Vicol using more subtle versions of convex integration show the existence of weak solution for IPM with $C^{\alpha}-$regularity. All of these solutions, increase the modulus of the density. We remark that the mixing solutions do not change the values of the density.

Székelyhidi refined the result of [15] in [36] computing explicitly the Λ-hull for the case of IPM. Notice that this increase the number of subsolutions (and thus the solutions available).

Next we will explain how to construct a subsolution for IPM. First of all we construct the mixing zone. In order to do it we will use a pseudo-interface

$$\Gamma_p(t) = \{\mathbf{x} \in \mathbb{R}^2 \,:\, \mathbf{x} = (x,\, f(x,t)),\, x \in \mathbb{R}\},$$

where $f(x,t)$ is a function to be determined below and with $f(x, 0) = f_0(x)$. For sake of simplicity it is assumed that $\Gamma_p(t)$ can be written as the graph of a function, but more general situations could be considered. The mixing zone is then defined by

$$\Omega_{\text{mix}}(t) = \left\{\mathbf{x} \in \mathbb{R}^2 \,:\, \mathbf{x} = (x, \lambda + f(x,t)),\, |\lambda| < \varepsilon,\, x \in \mathbb{R}\right\},$$

where $\varepsilon = ct$ and $c \in (0, 2)$. We will call to the constant c the speed of propagation of the mixing zone and measure the rate of growth of the mixing. The domains $\Omega^+(t)$ and $\Omega^-(t)$ will be the regions below and above $\Omega_{\text{mix}}(t)$ respectively.

The second step is to define the density and the velocity. We will use (x, λ)-coordinates in the mixing zone to do it

$$\mathbf{x}(x, \lambda, t) = (x, \lambda + f(x,t)).$$

The density will be given by

$$\rho(\mathbf{x}, t) = \begin{cases} +1 & \mathbf{x} \in \Omega^-(t) \\ \rho^{\sharp}(x, \lambda, t) = \frac{\lambda}{\varepsilon} & x \in \mathbb{R}, \quad |\lambda| < \varepsilon \\ -1 & \mathbf{x} \in \Omega^+(t) \end{cases} \tag{3.5}$$

where $\rho^{\sharp}(x, \lambda, t) \equiv \rho(\mathbf{x}(x, \lambda, t), t)$ for $x \in \mathbb{R}$ and $|\lambda| < \varepsilon$ and therefore $\mathbf{x}(x, \lambda, t) \in \Omega_{\text{mix}}(t)$.

Once we have the density, we construct the velocity by using Biot-Savart law, i.e.,

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(\mathbf{x} - \mathbf{y})^{\perp}}{|\mathbf{x} - \mathbf{y}|^2} \left(-\partial_{y_1} \rho(\mathbf{y}, t)\right) d\mathbf{y},$$

which in the mixing zone yields

$$\mathbf{u}^{\sharp}(x, \lambda, t) = \frac{1}{2\pi\varepsilon} \int_{-\infty}^{\infty} \int_{-\varepsilon}^{\varepsilon} \frac{(-(\lambda - \lambda' + \Delta f), x - y)}{(x-y)^2 + (\lambda - \lambda' + \Delta f)^2} \partial_y f(y, t) d\lambda' dy, \tag{3.6}$$

where $\mathbf{u}^{\sharp}(x, \lambda, t) = \mathbf{u}(\mathbf{x}(x, \lambda, t), t)$ and $\Delta f = f(x,t) - f(y,t)$.

The third and last step is to define the vector field $\mathbf{m}$. In $\Omega^{\pm}(t)$ we need to impose

$$\mathbf{m}(\mathbf{x},t) = \mp\mathbf{u}(\mathbf{x},t) \quad \mathbf{x}\in\Omega^{\pm}(t).$$

and in the mixing zone we will take

$$\mathbf{m} = \rho\mathbf{u} - \left(\gamma + \frac{1}{2}\right)\left(1-\rho^2\right)(0,1) \tag{3.7}$$

where γ will be chosen later. Since, by construction, ρ, $\mathbf{u}$ and $\mathbf{m}$ are continuous on $\mathbb{R}^2$ for $t>0$, in order to find a weak solution of the system (3.1), (3.2) and (3.3) we have to solve

$$\nabla\cdot\mathbf{m} = -\partial_t\rho \quad \text{on } \Omega_{\text{mix}}(t), \tag{3.8}$$

with ρ, $\mathbf{u}$ and $\mathbf{m}$ as in (3.5), (3.6) and (3.7) (notice here that because of the continuity there is no any term coming from the boundary of $\Omega_{\text{mix}}(t)$).

Passing to $(x,\,\lambda)-$coordinates inside of the mixing zone, some computations yields the following expression for $\gamma^{\sharp}(x,\lambda,t)=\gamma(\mathbf{x}(s,\lambda,t),t)$, from (3.8)

$$\gamma^{\sharp}(x,\lambda,t) = \frac{c-1}{2} + \frac{1}{1-(\rho^{\sharp})^2}\frac{1}{\varepsilon}\int_{-\varepsilon}^{\lambda}\left(\mathbf{u}^{\sharp}(x,\lambda',t)\cdot(-\partial_x f(x,t),\,1) - \partial_t f(x,t)\right)d\lambda'\,. \tag{3.9}$$

In addition the inclusion (3.4) reads

$$|\gamma| \le \frac{1}{2}\,. \tag{3.10}$$

Then, since we would like, both $\gamma(x,\lambda,t)\in L^{\infty}$ (recall $\mathbf{m}$ must be continuous in $\mathbb{R}^2$) and the second term in (3.9) small to satisfy (3.10) we need to impose

$$\partial_t f(x,t) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\mathbf{u}^{\sharp}(x,\lambda',t)\cdot(-\partial_x f(x,t),1)d\lambda' \tag{3.11}$$

which is an equation for our pseudo-interface.

This choice of $f(x,t)$ allow to prove (see [4]) that

$$|\gamma| \le \frac{|c-1|}{2} + o(1),$$

when $t\to 0$, which implies (3.10) for small enough time t fixed $c\in(0,2)$.

This concludes the construction of a subsolution for IPM (but we have to show existence of solutions for (3.11)).

Coming back to the distribution of the density inside of the mixing zone. The following should hold: there exist infinitely many solutions $(\rho,\mathbf{u})$ such that

$$\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}\frac{R^{1/2}}{2}\int_{\lambda-\frac{1}{R^{1/2}}}^{\lambda+\frac{1}{R^{1/2}}}\rho^{\sharp}(x,\lambda',t)d\lambda' dx = \frac{\lambda}{\varepsilon} \quad \forall t\in(0,T) \text{ and } |\lambda|<\varepsilon,$$

where $\rho^{\sharp}(x,\lambda,t)=\rho(\mathbf{x}(x,\lambda,t),t)$. Thus, the mixing goes from +1 to -1 linearly from the top to the bottom of the mixing zone, for all of these solutions (they are the same to a macroscopic level).

An alternative proof of the existence of mixing solutions has been recently published in [25]. In this paper the authors used a piecewise constant density for the subsolution.

4. The equation for the pseudo-interface

Then, in order to show the existence of a subsolution for the IPM system we just have to solve the equation (3.11).

Here we just will explain the main behaviour of this equation by using a toy model and present the main theorems in [4] which give rise to the existence of solutions.

To understand equation (3.11) it is convenient looking first to Muskat equation (1.6). If we assume that the interface $\Gamma(t)$ can be parameterize as the graph of a function, i.e., $\Gamma(t)=\{\mathbf{x}\in\mathbb{R}^2\,:\,\mathbf{x}=(x,\,f(x,t)),\,x\in\mathbb{R}\}$, we find from (1.6) the following equation

$$\partial_t f(x,t) = \frac{\rho^+-\rho^-}{2\pi}\int_{\mathbb{R}}\frac{y}{y^2+(f(x,t)-f(x-y,t))^2}(\partial_x f(x,t)-\partial_x f(x-y,t))dy.$$

Linearizing this equation around the flat solution $f = 0$ we find

$$\partial_t f(x,t) = -\frac{\rho^+ - \rho^-}{2}\Lambda f(x,t). \tag{4.1}$$

The operator Λ is square root of the Laplacian, $\Lambda = (-\Delta)^{\frac{1}{2}}$, with Fourier symbol $\widehat{\Lambda f}(\xi,t) = |\xi|\hat{f}(\xi,t)$ and representations

$$\Lambda f(x,t) = \frac{1}{\pi}P.V.\int_{\mathbb{R}} \frac{\partial_x f(x-y,t)}{y}dy = \frac{1}{\pi}P.V.\int_{\mathbb{R}} \frac{f(x)-f(x-y)}{y^2}dy.$$

Then, by looking to the Fourier side, we can solve (4.1) to find

$$\hat{f}(\xi,t) = e^{-(\rho^+-\rho^-)t|\xi|}\hat{f}_0(\xi,t).$$

From, this expression, it is easy to see that the solutions strongly depend on the sign of $\rho^+ - \rho^-$. If the denser part of the fluid is below, then $\rho^+ > \rho^-$ and the problem is well-posed in Sobolev spaces among many other good properties. But if the denser part of the fluid is above the problem is ill-posed.

It happens that the local existence theory for equation (1.6) behaves in a similar way that its linear version. The reason way of this phenomenon is the following: by taking four derivatives (this is not sharp but enough for this explanation) in (1.6) we find that

$$\partial_t F(\alpha,t) = -\sigma(\alpha,t)\Lambda F + a(\alpha,t)\partial_\alpha F(\alpha,t) + G(\alpha,t),$$

where $F(\alpha,t) = \partial_\alpha^4 z(\alpha,t)$, $a(\alpha,t)$ and $G(\alpha,t)$ satisfy that

$$||\partial_\alpha a||_{L^\infty} \leq ||\mathbf{z} - (\alpha,0)||_{H^4}, \quad ||G||_{H^4} \leq ||\mathbf{z} - (\alpha,0)||_{H^4},$$

and the Rayleigh-Taylor function $\sigma(\alpha,t)$ is given by

$$\sigma(\alpha,t) = (\rho^+ - \rho^-)\frac{\partial_\alpha z_1(\alpha,t)}{|\partial_\alpha \mathbf{z}(\alpha,t)|^2}.$$

Therefore G is a low order term, $a(\alpha,t)\partial_\alpha F(\alpha,t)$ is a transport term with a smooth velocity and the main term in the equation is $-\sigma(\alpha,t)\Lambda F$. From this splitting is easy to see that, if $\sigma(\alpha,t) > 0$ the local in time behaviour of the equation is nice and quite the opposite if $\sigma(\alpha,t)$ is negative in some point.

For the equation for our pseudo-interface we have that a time $t = 0$

$$\partial_t f(x,t)|_{t=0} = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{y^2 + (f(x,0) - f(x-y,0))^2}(\partial_x f(x,0) - \partial_x f(x-y,0)dy,$$

which is the same expression we have for the Muskat equation at time $t = 0$. This is a problem because we are loosing a derivative through a term of the form $\Lambda f(x,0)$ (with the bad sign). However for any $t > 0$ the density is not a step function any more but a Lipschitz function and the velocity becomes nicer and the equation (3.11) does not lose any derivative through a Λ-operator.

In the same spirit, by taking five derivatives we can split the equation (3.11) into three terms

$$\partial_t F = K_{\partial_x f(x,t)} \otimes F_x + a(x,t)\partial_x F + G \tag{4.2}$$

where

$$K_{\partial_x f(x,t)} \otimes F_x \equiv \frac{1}{2\pi\varepsilon}\int_{-\infty}^{\infty}\int_{-\varepsilon}^{\varepsilon} \frac{(-(\lambda - \lambda' + \partial_x f(x,t)), x-y)}{(x-y)^2 + (\lambda - \lambda' + \partial_x f(x,t))^2}\partial_y F(y,t)d\lambda' dy,$$

and $F = \partial_x^5 f$, and $a(x,t)$ and $G(x,t)$ can be considered harmless functions (a precise definition of "harmless" can be found in [4]).

To sketch the behavior of the first term on the right hand side of (4.2) we will analyze the following toy model

$$\partial_t f(x,t) = \mathcal{L}f(x,t) \tag{4.3}$$

where

$$\widehat{\mathcal{L}f}(\xi) = \frac{|\xi|}{1+t|\xi|}\hat{f}(\xi).$$

For this equation it is easy to see that a $t = 0$ we lose a derivative through a Λ-operator, i.e.,

$$\partial_t f(x,t)|_{t=0} = \Lambda f(x,0)$$

but for any $t > 0$ we do not lose any derivative any more. In fact, (4.3) can be integrated to obtain explicit solutions given by

$$f(x,t) = f_0(x) + t\Lambda f_0(x)$$

Therefore there exist solutions for the toy model and they lose a derivative with respect to the initial data. This is effect of the lost of a derivative a time $t = 0$.

For equation 3.11 there are no many hopes to find explicit solutions and a different strategy need to be applied. For example we could think about how to use energy estimates to show the existence of solution. For the toy model we could proceed in the following way. Since

$$\frac{d}{dt}\frac{\hat{f}(\xi,t)}{1+t|\xi|} = 0$$

we have that

$$\frac{d}{dt}\left|\left|D^{-1}f\right|\right|_{H^{k+1}} = 0$$

where $\widehat{D^{-1}} = \frac{1}{1+t|\xi|}$. This energy estimate allows us to prove local existence of solution in H^k for an initial data in H^{k+1}. Next we will explain how to apply this idea to our pseudo-interface. First we look to equation (4.2) and we consider just the first term on the right hand side, i.e.,

$$\partial_t F = K_{\partial_x f(x,t)} \otimes \partial_x F.$$

Second we freeze the term $\partial_x f(x,t)$ to obtain the equation

$$\partial_t F = K_A \otimes \partial_x F,$$

with A =cte.

For this last equation it can be proven that

$$\frac{d}{dt}\left|\left|\tilde{\mathfrak{m}}D^{-1}F\right|\right|_{L^2} = 0$$

where

$$\widehat{\tilde{\mathfrak{m}}F}(\xi,t) = \tilde{m}(\xi,t)\hat{F}(\xi,t)$$

with

$$\tilde{m}(\xi,t) = e^{-H(t|\xi|,A)+\log(1+t|\xi|)} \tag{4.4}$$

and

$$H(t|\xi|, A(x,t)) = \int_0^{t|\xi|} \frac{1}{\tau}\left\{1 + \frac{1}{4\pi\tau}\left(e^{-4\pi\tau\sigma}\left(\cos(4\pi\tau\sigma A) - A\sin(4\pi\tau\sigma A)\right) - 1\right)\right\} d\tau.$$

Here it is remarkable that

$$\frac{1}{C} \leq ||\tilde{m}||_{L^\infty} \leq C \tag{4.5}$$

and because of that we can think of $\tilde{m}D^{-1}$ just like D^{-1}.

Finally we put all this ingredients together. In order to do it we define the pseudo-differential operator

$$\tilde{\mathfrak{m}}F(x,t) = \int_{\mathbb{R}} e^{2\pi i\xi x}\tilde{m}(\xi,x,t)\hat{F}(\xi,t)d\xi$$

where $\tilde{m}(\xi,x,t)$ is like in (4.4) but replacing A by $\partial_x f(x,t)$ and we compute

$$\frac{d}{dt}\left|\left|\tilde{\mathfrak{m}}D^{-1}F\right|\right|_{L^2}$$

by using (4.2). Of course, this derivative is not zero anymore, but fortunately it does not lose any derivative at any time and it is possible to close a suitable estimate to show local existence for (3.11).

The main lemmas in order to show that estimate are the following

Lemma 1. *Let $F(\cdot,t)$ a distribution such that, its Fourier transform $\hat{F}(\cdot,t)$ is a function and*

$$\left|\left|\frac{1}{1+t|\cdot|}\hat{F}(\cdot,t)\right|\right|_{L^2(\mathbb{R})}<\infty$$

for every $t\in[0,T]$, for some $T>0$. Then

$$||\mathfrak{c}_1 F(\cdot,t)||_{L^2}\leq\langle A\rangle\left|\left|\mathcal{D}^{-1}F(\cdot,t)\right|\right|_{L^2}$$

where

$$\mathfrak{c}_1 F(x,t)\equiv\int_{\mathbb{R}}e^{2\pi i\xi x}m(\xi,x,t)\left(\hat{K}_{A(x)}(\xi)\widehat{\partial_x F}(\xi,t)-\mathcal{F}\left[K_{A(x)}\otimes\partial_x F\right](\xi,t)\right)d\xi.$$

This lemma allows to deal with the first term on the right hand side of (4.2).

Lemma 2. *Let $F(\cdot,t)$ a distribution such that, its Fourier transform $\hat{F}(\cdot,t)$ is a function and*

$$\left|\left|\frac{1}{1+t|\cdot|}\hat{F}(\cdot,t)\right|\right|_{L^2(\mathbb{R})}<\infty$$

for every $t\in[0,T]$, for some $T>0$. Then

$$||\mathfrak{c}_2 F(x,t)||_{L^2(dx)}\leq\langle A\rangle||a||_{H^3}\left|\left|\mathcal{D}^{-1}F(x,t)\right|\right|_{L^2(dx)}$$

where

$$\mathfrak{c}_2 F(x,t)=\int_{\mathbb{R}}e^{2\pi i x\xi}m(\xi,x,t)\left(a(x,t)\widehat{\partial_x F}-\mathcal{F}[a\partial_x F](\xi,t)\right)d\xi.$$

This lemma allows to deal with the transport term on the right hand side of (4.2).

Lemma 3. *Let $F(\cdot,t)\in L^2(\mathbb{R})$ for every $t\in[0,T]$ for some $T>0$. Then the pseudo-differential operator*

$$\tilde{\mathfrak{m}}F(x)\equiv\int_{\mathbb{R}}e^{2\pi i x\xi}\tilde{m}(\xi,x,t)\hat{F}(\xi)d\xi$$

satisfies

$$||\tilde{\mathfrak{m}}F(x)||_{L^2}\geq\langle A\rangle(1-\langle A\rangle t)||F||_{L^2}.$$

This lemma shows coercivity for $\tilde{\mathfrak{m}}$ (an analogous fact to (4.5)).

In these three lemmas $\langle A\rangle$ is a constant that depend on $||A||_{H^3}$.

The proofs of these lemmas can be found in [4]. In them some ideas from [27] are used. In this last paper one can find the L^2-boundedness of the operator $\tilde{\mathfrak{m}}F$.

5. Numerical analysis

In this section we present a numerical analysis of the equation (3.11) in which the formation of fingers in the pseudo-interface can be observed. In order to avoid problems with the numerical integration at the infinity we will work with periodic boundary conditions in the horizontal variable. Although the well-posedness has been proven in [4] for a asymptotically flat interfaces, is reasonable to think of a similar proof can be provided to the periodic in the horizontal variable case. In this paper we will assumed that this last fact holds. Then, our mixing zone will be given by

$$\Omega_{\text{mix}}(t)=\{\mathbf{x}\in\mathbb{R}^2\,:\,\mathbf{x}=(x,\,\lambda+f(x,t)),\quad|\lambda|<ct,\,x\in\mathbb{R}\}$$

with $f(x+2\pi,t)=f(x,t)$ for all $x\in\mathbb{R}$.

In order to obtain equation (3.11) in this new setting, we notice that the stream function ψ satisfying $\Delta\psi(\mathbf{x},t)=-\partial_{x_1}\rho(\mathbf{x},t)$ can now be written as follows (see [14]):

$$\psi(\mathbf{x},t)=\frac{1}{4\pi}\int_{\mathbb{T}\times\mathbb{R}}\log\left(\cosh(x_2-y_2)-\cos(x_1-y_1)\right)\left(-\partial_{y_1}\rho(\mathbf{y},t)\right)d\mathbf{y},$$

and then

$$\mathbf{u}(\mathbf{x},t)=\frac{1}{4\pi}\int_{\mathbb{T}\times\mathbb{R}}\frac{(-\sinh(x_2-y_2),\sin(x_1-y_1))}{\cosh(x_2-y_2)-\cos(x_1-y_1)}\left(-\partial_{y_1}\rho(\mathbf{y},t)\right)d\mathbf{y}.$$

Changing to (x,λ)-coordinates

$$\mathbf{x}(x,\lambda,t)=(x,\lambda+f(x,t)),$$

and using that

$$-\partial_{x_1}\rho(\mathbf{x}(x,\lambda,t),t)=\partial_x f(x,t)\frac{1}{\varepsilon}$$

with $\varepsilon=ct$, we obtain that

$$\mathbf{u}(\mathbf{x},t)=\frac{1}{4\pi}\int_{\mathbb{T}}\int_{-\varepsilon}^{\varepsilon}\frac{(-\sinh(x_2-\lambda'-f(x',t)),\sin(x_1-x'))}{\cosh(x_2-\lambda-f(x',t))-\cos(x_1-x')}\partial_{x'}f(x',t)dx'd\lambda'.$$

Thus, the identity

$$\begin{aligned}&\partial_{x'}\log\left(\cosh(x_2-\lambda-f(x',t))-\cos(x_1-x')\right)\\&=-\frac{\sinh(x_2-\lambda'-f(x',t))}{\cosh(x_2-\lambda-f(x',t))-\cos(x_1-x')}\partial_{x'}f(x',t)-\frac{\sin(x_1-x')}{\cosh(x_2-\lambda-f(x',t))-\cos(x_1-x')}\end{aligned}$$

yields

$$\mathbf{u}(\mathbf{x},t)=\frac{1}{4\pi\varepsilon}\int_{\mathbb{T}}\int_{-\varepsilon}^{\varepsilon}\frac{\sin(x_1-x')}{\cosh(x_2-\lambda'-f(x',t))-\cos(x_1-x')}(1,\partial_{x'}f(x',t))d\lambda'dx'.$$

The integration in λ' can be computed explicitly. We get

$$\begin{aligned}&\int_{-\varepsilon}^{\varepsilon}\frac{\sin(x_1-x')}{\cosh(x_2-\lambda'-f(x',t))-\cos(x_1-x')}d\lambda'\\&\qquad=2\arctan\left(\frac{e^{\varepsilon+f(x',t)-x_2}}{\sin(x-x')}\right)-2\arctan\left(\frac{e^{-\varepsilon+f(x',t)-x_2}}{\sin(x-x')}\right).\end{aligned}$$

By evaluating in $\mathbf{x}=\mathbf{x}(x,\lambda,t)$, we obtain that in the mixing zone

$$\mathbf{u}^\sharp(x,\lambda,t)=\frac{1}{2\pi\varepsilon}\int_{\mathbb{R}}(1,\partial_x f(x',t))\left(\arctan\left(\frac{e^{\varepsilon-\lambda-\Delta f}}{\sin(x-x')}\right)-\arctan\left(\frac{e^{-\varepsilon-\lambda-\Delta f}}{\sin(x-x')}\right)\right)dx'.$$

Since the equation for our pseudo-interface is given by

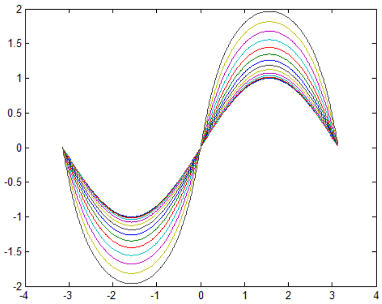
$$f_t(x,t)=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\mathbf{u}^\sharp(x,\lambda,t)\cdot(-\partial_x f(x,t),1)d\lambda$$

we finally obtain

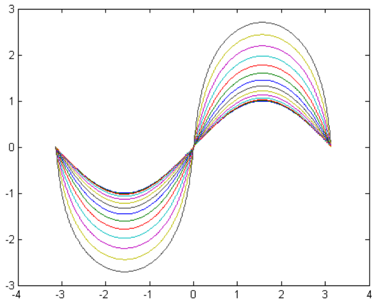
$$\partial_t f(x,t)=-\int_{\mathbb{T}}(\partial_x f(x,t)-\partial_x f(x',t))K_{ct}(x,x')dx' \tag{5.1}$$

where

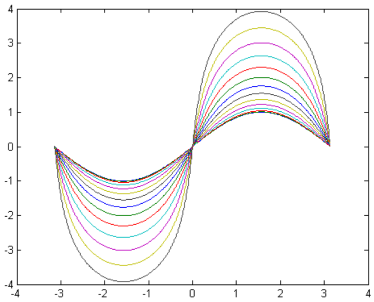
$$\begin{aligned}K_\varepsilon(x,x')=\frac{1}{4\pi\varepsilon}\int_{-1}^{1}\Bigg(&\arctan\left(\frac{e^{(\varepsilon(1-\lambda)-\Delta f)}-\cos(x-x')}{\sin(x-x')}\right)\\&-\arctan\left(\frac{e^{(-\varepsilon(1+\lambda)-\Delta f)}-\cos(x-x')}{\sin(x-x')}\right)\Bigg)d\lambda.\end{aligned}$$



(a) Speed of propagation $c = 0.5$.

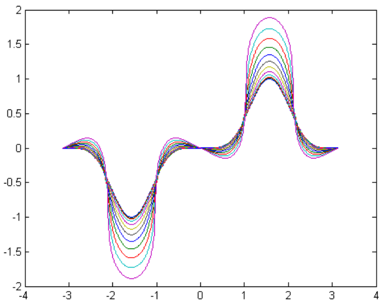


(b) Speed of propagation $c = 1$.

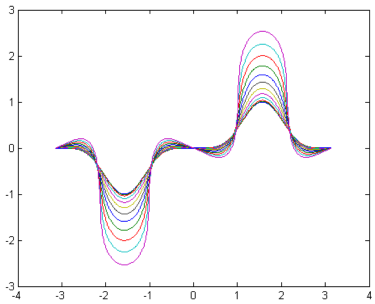


(c) Speed of propagation $c = 2$.

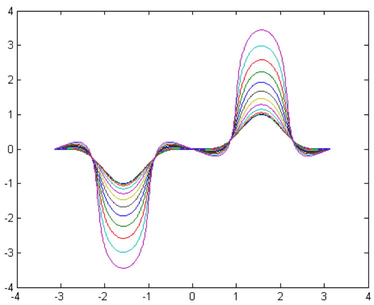
Figure 5.1: Initial data: $f_0(x) = \sin(x)$. Evolution from $t = 0$ to $t = 2.5$



(a) Speed of propagation $c = 0.5$.



(b) Speed of propagation $c = 1$.



(c) Speed of propagation $c = 2$.

Figure 5.2: Initial data: $f_0(x) = \sin^5(x)$. Evolution from $t = 0$ to $t = 2$

In both figures 5.1 and 5.2 we present numerical simulations of equation (5.1) with initial data $f_0(x) = \sin(x)$ and $f_0(x) = \sin^5(x)$ respectively, for different speeds of propagation c. As we can see in these draws the evolution is similar as predicted by the toy model (4.3), whose solutions are given by $f(x,t) = f_0(x) + t\Lambda f_0(x)$: the creation of a fingers which grow with time. Of course the dynamics is much more complicated and other phenomena also arise as for example a turning effect. Here we recall that given a solution for (5.1) in $[0,T]$ we only can guarantee the existence of a subsolution for IPM for a short time $T^* < T$.

In these simulations we have used a Runge-Kutta fourth-order method to carry out the integration on time. The discretization on space only causes problems a time $t = 0$. Because of that for the first step in the integration on time we have used the Euler method and we have taken into account that

$$\lim_{\varepsilon\to 0} K_\varepsilon(x,x') = \frac{1}{2\pi}\frac{\sin(x-x')}{\cosh(\Delta f) - \cos(x-x')}\,.$$

We compute the Principal Value of the integral removing the integration at the point $x = x'$. The derivatives have been performed by using fourth-order splines and the quadratures by using the trapezoidal rule. The codes have been run in MATLAB.

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