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# Uniqueness results for some PDEs

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**Abstract**

Existence of solutions to many kinds of PDEs can be proved by using a fixed point argument or an iterative argument in some Banach space. This usually yields uniqueness in the same Banach space where the fixed point is performed. We give here two methods to prove uniqueness in a more natural class. The first one is based on proving some estimates in a less regular space. The second one is based on a duality argument. In this paper, we present some results obtained in collaboration with Pierre-Louis Lions, with Kenji Nakanishi and with Fabrice Planchon.

## 1. Introduction

Consider the following system of equations

$$\begin{cases} u_t = Lu + N(u) \\ u(t=0) = u^0 \end{cases} \tag{1}$$

where $L$ is linear and $N(u)$ is a nonlinear term and the initial data $u_0$ is given in some $H^s$ space. We want to study the well-posedness of (1) in $H^s$, namely we want to prove the existence and uniqueness of a solution $u$ to (1) in the space $C([0,T);H^s)$. Here, the $H^s$ space can also be replaced by a Lebesgue space $L^p$, a Besov space or a more general Banach space... Under some conditions on $L$ and $N$ the existence can be proved by using a fixed point argument or and iterative argument in some Banach space $X$ such that $X \subset C([0,T);H^s)$. This, of course, yields a better existence result than the required one since it proves that the solution $u$ is actually in $X$ which is in general smaller than $C([0,T);H^s)$. However, the uniqueness is only proved in $X$ and in many examples it turns out to be not easy to prove that we have uniqueness of solutions in $C([0,T);H^s)$. In this review paper, we want to address this issue and give two general methods, one can use to try to prove uniqueness in the more natural class $C([0,T);H^s)$. The first one is based on proving estimates

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in a space which is less regular than $X$ but which contains $C([0,T);H^s)$ or more generally a space which contains the difference between two solutions $u$ and $v$ which are both in $C([0,T);H^s)$. More precisely, the space $X$ can be written as $X = X^s$ for some family of spaces $X^s$. Then, using that $u$ and $v$ are two solutions of (1) and that $u, v \in C([0,T);H^s)$, we can prove that $u - v \in X^{s'}$ for some $s' < s$. Then, we can prove some estimates in $X^{s'}$. In some examples the spaces $X^s$ and $X^{s'}$ only differ by the regularity index and in some other cases, $X^s$ and $X^{x'}$ are completely different.

The second method consists in proving an existence result for a dual problem and then using the solution of the dual problem as a test function. In other words, we replace the uniqueness problem by an existence problem. We will apply these two methods to different parabolic and hyperbolic examples.

In, the next section, we recall how we can prove existence for (1) by using a fixed point argument. In the other sections of this review paper, we study different examples. The author would like to thank Fabrice Planchon for providing many related references and giving some remarks about the manuscript.

## 2. Existence using a fixed point argument

We can prove the existence of solutions to (1) using a fixed point argument or an iterative scheme. Let $u_L$ be the solution of the linear problem

$$\begin{cases} \partial_t u_L = L u_L \\ u_L(t=0) = u^0. \end{cases} \tag{2}$$

To perform a fixed point argument, we have to find a Banach space $B_T^s$ such that $u_L \in B_T^s$ and

$$B_T^s \hookrightarrow C([0,T);H^s).$$

We consider the function $\Phi$ defined by

$$\begin{array}{cccc} \Phi & : & B_T^s & \to & B_T^s \\ & & v & \mapsto & u \end{array}$$

where $u$ solves the following equation

$$\begin{cases} \partial_t u = L u + N(u_L + v) \\ u(t=0) = 0. \end{cases} \tag{3}$$

We assume that $\Phi$ satisfies

$$\begin{aligned} ||\Phi(v)||_{B_T^s} &\leq C_T A(||u_L||_{B_T^s} + ||v||_{B_T^s}) \\ ||\Phi(v_1) - \Phi(v_2)||_{B_T^s} &\leq C_T A'(||u_L||_{B_T^s} + ||v_1||_{B_T^s} + ||v_2||_{B_T^s})(||v_1 - v_2||_{B_T^s}) \end{aligned}$$

where $C_T$ goes to 0 when $T$ goes to 0. Besides, $A(c), A'(c) \geq 0$ and are non-increasing and $A(c) + A'(c)$ goes to 0 when $c$ goes to 0.

Using the above hypotheses, it is easy to see that taking $T$ small enough, we can prove the local existence and uniqueness of a fixed point $v$ for the function $\Phi$. Moreover $u_L + v$ is a solution to the equation (1).

On the other hand if we also assume that $\sup_T C_T \leq C_\infty$ and that $A(c)/c \to \alpha < 1/C_\infty$ when $c$ goes to 0, then we can prove the global existence of a solution $w = u_L + v$ to (1) by using a fixed point argument in the Banach space $B^s_\infty$.

Going back to the uniqueness issue, we point out that in most of the interesting examples (specially the critical ones) the fixed point argument can not be performed directly in $C([0,T);H^s)$ and one has to work (sometimes very hard) to exhibit a space $B^s_T$ where we can perform the fixed point.

Instead of continuing in a very general framework, we will give some examples.

## 3. The Navier-Stokes equation

The Navier-Stokes system describes the motion of a viscous incompressible fluid. When the fluid fills a regular domain $\Omega$ (which can be bounded or unbounded) of $\mathbb{R}^N$, the system is written on $(0,T) \times \Omega$

$$\partial_t u + u.\nabla u - \nu \Delta u = -\nabla p \tag{4}$$
$$\mathrm{div}(u) = 0 \tag{5}$$

together with initial and boundary conditions

$$u(0) = u^0 \tag{6}$$
$$u = 0 \text{ on } \partial\Omega \tag{7}$$

where $u(t,x) \in \mathbb{R}^N$ is the velocity, $p(t,x) \in \mathbb{R}$ is the pressure and $t \in (0,T)$, $x \in \Omega$. Besides the result of J. Leray [15] who proved that if $u^0 \in L^2(\Omega)$ then there exists a global weak solution to (4) and (5) in $L^\infty(0,T;L^2) \cap L^2(0,T;H^1_0)$, we have the following existence of mild-solution due to T. Kato [11] for the case $\Omega = \mathbb{R}^N$

**Theorem 3.1** *If $u_0 \in L^N(\mathbb{R}^N)$, then there exists a unique maximal time $T^*$ and a unique solution of the Navier-Stokes system in*

$$C([0,T^*);L^N(\mathbb{R}^N)) \cap \{u/t^{\frac{1}{4}}u(t) \in C([0,T^*[;L^{2N}(\mathbb{R}^N))\}. \tag{8}$$

The fact that $u$ belongs to $\{u/t^{\frac{1}{4}}u(t) \in C([0,T^*[;L^{2N})\}$ is essential for the application of the fixed point argument in the proof by Kato. For $N = 3$, the uniqueness in $C([0,T^*);L^3(\mathbb{R}^3))$ was proved by G. Furioli, P-G. LemariÈ-Rieusset and E. Terraneo [7], [8] by proving some estimates on the difference between two solutions in the space $B^{\frac{1}{2}}_{2,\infty}$. In collaboration with P.-L. Lions [16, 17], we gave a more general proof of uniqueness which also applies to the case of domains with boundaries. We have

**Theorem 3.2** *If $u^1$ and $u^2$ are two solutions of the Navier-Stokes system (4) and (5), with the same initial condition $u^0$ in $C([0,T);L^N(\Omega))$ with $N \geq 3$, then $u^1 = u^2$.*

**Theorem 3.3** *If $u^1$ and $u^2$ are two solutions of the Navier-Stokes system (4) and (5), with the same initial condition $u^0$ in $L^\infty([0,T);L^N(\Omega))$ with $N \geq 4$, then $u^1 = u^2$.*

The proof of the first theorem 3.2 is based on proving existence to the following dual problem. Let $\Phi_i$, with $1 \le i \le N$, be a solution in $\Omega \times (0,T)$ of the following backward Stokes system

$$-\partial_t \Phi_i - u^1_j \partial_j \Phi_i - \nu \Delta \Phi_i - u^2_j \partial_i \Phi_j + \partial_i \Psi = F_i \tag{9}$$

$$\mathrm{div}(\Phi) = 0 \quad \Phi = 0 \text{ on } \partial\Omega \quad \Phi(T) = 0$$

where the summation is performed over $j$ and where $F \in C_0^\infty(\Omega \times (0,T))$. Then if we multiply formally the following equation satisfied by $u^1 - u^2$

$$\partial_t(u^1 - u^2) - \nu\Delta(u^1 - u^2) + u^1.\nabla(u^1 - u^2) + (u^1 - u^2).\nabla u^2 = -\nabla(p^1 - p^2) \tag{10}$$

by $\Phi$, we get after some computations

$$\int_0^T \int_\Omega (u^1 - u^2).F = 0. \tag{11}$$

Since (11) holds for all $F \in C_0^\infty(\Omega \times (0,T))$, we deduce that $u^1 = u^2$. To justify this computation, we show the existence of $\Phi$ by showing some a priori estimates and then using the regularity of the Stokes operator we deduce that $\Phi$ is regular enough and decays fast enough at infinity to make the computations rigorous. By doing so, we have replaced a proof of uniqueness by an existence proof of some solution satisfying enough regularity and decay.

To prove theorem 3.3, we argue as follows. Since, we know that there exists a solution in $C([0,T);L^N(\Omega))$ we can assume, with out loss of generality, that $u^2 \in C([0,T);L^N(\Omega))$. Then, multiplying at least formally (10) by $v = u_1 - u_2$, we get

$$\begin{aligned} & \frac{1}{2}\partial_t ||v||^2_{L^2} + \nu ||\nabla v||^2_{L^2} \le \int_\Omega |v| \, |\nabla v| \, |u^2| \\ \le \; & C \, \varepsilon ||\nabla v||_{L^2} ||v||_{L^{2N/(N+2)}} + C_\varepsilon ||v||_{L^2} ||\nabla v||_{L^2} \\ \le \; & 2C \, \varepsilon ||\nabla v||^2_{L^2} + C ||v||^2_{L^2} \end{aligned} \tag{12}$$

where we have decomposed $u^2$ in the following form

$$u^2 = v^2 + w^2 \text{ with } ||v^2||_{C([0,T);L^N(\Omega))} \le \varepsilon \text{ and } ||w^2||_{L^\infty_{x,t}} \le C_\varepsilon.$$

Then, choosing $\varepsilon$ small enough and applying a Gronwall lemma, we deduce that $v = 0$. To make this computation, we have to prove that $u^1 - u^2 \in L^\infty(L^2) \cap L^2(H^1)$ since $N \ge 4$. This can be proved by using the regularity of the Stokes operator (see [17] for the details).

## 4. Maxwell-Dirac

To write the Maxwell-Dirac equation, we introduce the following notations

$$\partial = (\partial_0, \ldots, \partial_3) = (\partial_t, \nabla), \quad D_\alpha = \partial_\alpha + iA_\alpha, \tag{13}$$

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \tag{14}$$

where $(t,x)$ denotes the space-time coordinates, $A = (A_0, \dots, A_3) = (A_0, \boldsymbol{A})$ denotes the electromagnetic potential, $D$ denotes the covariant derivative, and $F$ denotes the electromagnetic field. $F$ is decomposed into the electric field $E = (F_{10}, F_{20}, F_{30}) = -(F_{01}, F_{02}, F_{03})$ and the magnetic field $B = (F_{23}, F_{31}, F_{12}) = -(F_{32}, F_{13}, F_{21})$. In other words, they are given by

$$E = \nabla A_0 - \dot{\boldsymbol{A}}, \qquad B = \nabla \times \boldsymbol{A}. \tag{15}$$

The existence of $A$ satisfying these relations is equivalent to the following equations for $E$ and $B$:

$$\nabla \cdot B = 0, \quad \dot{B} + \nabla \times E = 0. \tag{16}$$

In the sequel, we employ the convention of tacit summation over coupled upper and lower indices, where Greek letters run from 0 to 3 while Latin letters run from 1 to 3. We denote $X^\alpha = g^{\alpha\beta} X_\beta$ with $g^{\alpha\beta} = diag(-1,1,1,1)$ for any tensor $X$. We consider the following Maxwell-Dirac system which describes the evolution of the wave function of a self-interacting relativistic electron:

$$\begin{cases} \partial_\alpha F^{\alpha\beta} = J^\beta = \langle \gamma^0 u, \gamma^\beta u \rangle, \\ i\gamma^\alpha D_\alpha u = mu, \end{cases} \tag{17}$$

where $u \in \mathbb{C}^4$ denotes the spinor field coupled with $F$, $m \geq 0$ is a constant, $J_0 = -|u|^2$ is the charge density, $\boldsymbol{J} = (J_1, J_2, J_3)$ is the electric current given by $J_k = \langle \gamma^0 u, \gamma^k u \rangle$ for $1 \leq k \leq 3$, and $\langle a, b \rangle$ denotes the real part of the inner product, namely $\langle a, b \rangle = \Re(a\bar{b})$. The Dirac matrices are given by

$$\gamma^0 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0_2 & \sigma_k \\ -\sigma_k & 0_2 \end{pmatrix}, \tag{18}$$

where $0_2$ is the null $2 \times 2$ matrix, $I_2$ is the $2 \times 2$ identity and the Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{19}$$

This system has a conserved total charge $\int J_0 dx = -\int |u|^2 dx$ and the conserved energy given by

$$\begin{aligned} \mathcal{E} &= \int 2\langle iD_0 u, u \rangle + |E|^2 + |B|^2 dx \\ &= \int 2\langle \gamma^j D_j u, i\gamma^0 u \rangle + 2m\langle u, \gamma^0 u \rangle + |E|^2 + |B|^2 dx, \end{aligned} \tag{20}$$

which does not have a definite sign. The Maxwell-Dirac in the Coulomb gauge $(\nabla \cdot \boldsymbol{A} = 0)$ can be written as

$$\begin{cases} i\gamma^\alpha \partial_\alpha u - mu = \gamma^\alpha A_\alpha u, \\ -\partial_t^2 \boldsymbol{A} + \Delta \boldsymbol{A} = \mathcal{P}\boldsymbol{J}, \\ \Delta A_0 = -|u|^2, \quad \nabla \cdot \boldsymbol{A} = 0, \end{cases} \tag{21}$$

where $\mathcal{P}$ denotes the projection on divergence-free vectors which is given by $\mathcal{P}\boldsymbol{J} = \boldsymbol{J} - \nabla\Delta^{-1}(\nabla \cdot \boldsymbol{J})$. We complement (21) with the following initial data

$$\begin{aligned} &(u(0), \boldsymbol{A}(0), \dot{\boldsymbol{A}}(0)) \in H^{1/2}(\mathbb{R}^3; \mathbb{C}^4) \times (\dot{H}^1 \times L^2)(\mathbb{R}^3; \mathbb{R}^3), \\ &\nabla \cdot \boldsymbol{A}(0) = \nabla \cdot \dot{\boldsymbol{A}}(0) = 0, \end{aligned} \tag{22}$$

In collaboration with Kenji Nakanishi [18] we proved

**Theorem 4.1** (with Kenji Nakanishi) *Given any initial data as in (22), there exists a unique solution $(u, A)$ to the Maxwell-Dirac in the Coulomb gauge (21) on some time interval $(-T, T)$ satisfying*

$$(u, A_0, \boldsymbol{A}, \dot{\boldsymbol{A}}) \in C(-T, T; H^{1/2} \times C_0 \times \dot{H}^1 \times L^2). \tag{23}$$

*where $C_0$ is the completion of $C_0^\infty$ with respect to the $L^\infty$ norm.*

The existence part improves Bournaveas' result [1], where a strictly greater regularity was necessary, namely $C(H^{1/2+\varepsilon} \times H^{1+\varepsilon})$. To prove the existence, we have to use a fixed point argument in an $X^{s,b}$ which we are going to define. For $s, b \in \mathbb{R}$ and an interval $I \subset \mathbb{R}$, we define

$$\|u\|_{X_\pm^{s,b}(I)} := \inf_{u(t)=U(\pm t)v(t) \text{on} I} \|v\|_{H_t^b(\mathbb{R}; H_x^s)}, \tag{24}$$

where $H^s$ denotes the inhomogeneous Sobolev space. Schematically, we have

$$X_\pm^{s,b} = U(\pm t) H_t^b H_x^s. \tag{25}$$

We also denote $X^{s,b} = X_+^{s,b} + X_-^{s,b}$. We remark that those spaces do not change if we consider the Klein-Gordon propagator $U_m(\pm t) = e^{\pm it\sqrt{m^2 - \Delta}}$ instead of $U(\pm t) = U_0(\pm t)$, since $U_m(-t)U_0(t)$ is a uniformly bounded operator on any $H_t^b(H_x^s)$ and $H_t^b(\dot{H}_x^s)$. Then, it is obvious that we have

$$\|U_m(\pm t)\varphi\|_{X_\pm^{s,b}(I)} \lesssim \|\varphi\|_{H^s} \tag{26}$$

for any $m, s, b, I$. We estimate the solution $(u, A)$ in the following spaces : $u \in X^{1/2,b}$, $\dot{u} \in X^{-1/2,b}$, $\boldsymbol{A} \in \dot{X}^{1,b}$ and $\dot{\boldsymbol{A}} \in X^{0,b}$ for some $b > 1/2$. By the standard multiplication estimate in the Besov spaces and the Hölder inequality, we obtain the following estimates. Notice that $\dot{B}_{2,1}^{3/2} \subset C_0$ by the Sobolev embedding.

$$\begin{aligned} &\|A_0\|_{L^\infty(\dot{B}_{2,1}^{3/2})} \lesssim \||u|^2\|_{L^\infty(\dot{B}_{2,1}^{-1/2})} \lesssim \|u\|_{L^\infty H^{1/2}}^2, \\ &\|\dot{A}_0\|_{\dot{B}_{2,\infty}^{1/2}} \lesssim \|u\dot{u}\|_{\dot{B}_{2,\infty}^{-3/2}} \lesssim \|u\|_{L^\infty H^{1/2}} \|\dot{u}\|_{L^\infty H^{-1/2}}, \\ &\|A^\alpha A_\alpha u\|_{L^\infty H^{-1/2}} \lesssim (\|\boldsymbol{A}\|_{L^\infty \dot{H}^1} + \|A_0\|_{L^\infty \dot{B}_{2,1}^{3/2}})^2 \|u\|_{L^\infty H^{1/2}}, \\ &\|A_0 \dot{u}\|_{L^\infty H^{-1/2}} \lesssim \|A_0\|_{L^\infty \dot{B}_{2,1}^{3/2}} \|\dot{u}\|_{L^\infty H^{-1/2}}, \\ &\|\Phi \boldsymbol{J}\|_{L^2 L^2} \lesssim \|u\|_{L^4 L^4}^2. \end{aligned} \tag{27}$$

Using null-form structure, we get

$$\|\boldsymbol{A} \cdot \nabla u\|_{L^2 H^{-1/2}} \lesssim \|\boldsymbol{A}\|_{\dot{X}^{1,b}} \|u\|_{X^{1/2,b}}, \tag{28}$$

since we have $\boldsymbol{A}\cdot\nabla u = \partial_i(\partial_i\Delta^{-1}A_j)\partial_j u - \partial_j(\partial_i\Delta^{-1}A_j)\partial_i u$ by virtue of $\nabla\cdot\boldsymbol{A} = 0$. Using the embedding $L^2L^{4/3}(I) \subset \dot{X}_{\pm}^{-1/2,-1/4-\varepsilon}(I)$, we get

$$\begin{aligned} \|(\partial\boldsymbol{A})u\|_{X^{-1/2,-1/4-\varepsilon}} &\lesssim \|(\partial\boldsymbol{A})u\|_{L^4L^{4/3}} \lesssim \|\partial\boldsymbol{A}\|_{L^\infty L^2}\|u\|_{L^4_{t,x}}, \\ \|(\partial A_0)u\|_{X^{-1/2,-1/4-\varepsilon}} &\lesssim \|(\partial A_0)u\|_{L^4L^{4/3}} \lesssim \|\partial A_0\|_{\dot{B}^{1/2}_{2,\infty}}\|u\|_{L^\infty_t L^2_x\cap L^4_{t,x}}. \end{aligned} \tag{29}$$

Thus we can estimate $\Box u$ in $X^{-1/2,-1/4-\varepsilon}$ and $\Box\boldsymbol{A}$ in $L^2_{t,x}$. Then, using the above estimates as well as a classical iteration argument, we deduce the existence of a unique local solution for the Maxwell-Dirac system satisfying $u \in X^{1/2,b}$, $\dot{u} \in X^{-1/2,b}$, $A_0 \in C(C_0)$, $\boldsymbol{A} \in \dot{X}^{1,b}$ and $\dot{\boldsymbol{A}} \in X^{0,b}$ for some $3/4 > b > 1/2$.

For the uniqueness, we have to prove some similar estimate but at a lower level of regularity. We take two solutions $(u, A)$ and $(u_w, A_w)$ on some time interval $I = (-T, T)$ in the energy space. Without loss of generality, we can assume that $(u, A)$ is the solution constructed above and hence $u \in X^{1/2,b}$, $\dot{u} \in X^{-1/2,b}$, $\boldsymbol{A} \in \dot{X}^{1,b}$ and $\dot{\boldsymbol{A}} \in X^{0,b}$ for some $b > 1/2$. We denote $u' = u - u_w$ and $A' = A - A_w$. Using that $(u_w, A_w)$ solves the Maxwell-Dirac system, we can prove that $(u', \boldsymbol{A}')$ is in the following spaces: $u' \in X^{0,b}$, $\dot{u}' \in X^{-1,b}$, $\boldsymbol{A}' \in \dot{X}^{1/2,b}$ and $\dot{\boldsymbol{A}}' \in \dot{X}^{-1/2,b}$. Notice that these are exactly the same spaces where we have proved the existence with 1/2 derivative less. We introduce the Klein-Gordon propagator and the Dirac propagator given respectively by

$$K(t) := \Omega^{-1}\sin\Omega t, \quad D(t) := \dot{K}(t) + (\gamma^j\partial_j - im)\gamma^0 K(t), \tag{30}$$

where $\Omega = \sqrt{m^2 - \Delta}$. Hence, $u$ satisfies

$$u(t) = D(t)u(0) - i\int_0^t D(t-s)\gamma^0\gamma^\alpha(A_\alpha u)(s)ds \tag{31}$$

and $u'$ satisfies

$$u'(t) = -i\int_0^t D(t-s)\gamma^0\gamma^\alpha(A'_\alpha u + A'_\alpha u' + A_\alpha u')(s)ds. \tag{32}$$

Using that $(u, A)$ is a "good solution", we can estimate all the terms appearing in (32). We refer to [18] for the details. Hence, we get

$$\|u'\|_{X^{0,b}} + \|\boldsymbol{A}'\|_{\dot{X}^{1/2,b}} \lesssim C(|I|^{1/4} + |I|^{1-b})(\|u'\|_{X^{0,b}} + \|\boldsymbol{A}'\|_{\dot{X}^{1/2,b}})$$

where $C$ depends here on the two solutions $(u, A)$ and $(u_w, A_w)$. Taking $I$ to be small enough, we deduce that $(u, A) = (u_w, A_w)$ and we can iterate the argument on the whole interval $(-T, T)$.

**Remark 4.2** *In [18], we prove a similar result for the Maxwell-Klein-Gordon (MKG) system*

$$\begin{cases} \partial_\alpha F^{\alpha\beta} = J^\beta = -\Im(u\overline{D^\beta u}) = \langle iu, D^\beta u\rangle, \\ D_\alpha D^\alpha u = m^2 u, \end{cases} \tag{33}$$

*where $u$ is a complex scalar field. We refer the reader to [18] for the proof. We recall that, the existence and uniqueness with additional restrictions have been obtained by Klainerman and Machedon in [13] for MKG in the energy space $C(H^1)$. For MKG, unconditional uniqueness was also proved by Zhou in [30] who proved the uniqueness for a simplified model which carries only the null quadratic terms of MKG.*

## 5. Nonlinear wave equations

We consider the following quintic wave equation in three space dimensions,

$$\begin{cases} \Box\phi + \phi^5 = 0, \\ \phi(t=0) = \phi_0, \qquad \phi_t(t=0) = \phi_1, \end{cases} \tag{34}$$

Existence of global solutions in the energy space $C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2)$ is due to Shatah and Struwe [27] (see also [9, 26]). Uniqueness was proved only under an additional space-time integrability assumption of Strichartz type, which is a crucial ingredient to the proof of the existence result. This condition is for instance an $L^5(L^{10})$ bound. Another condition can also be found in [29]. To state our result, let us recall that a smooth solution of the wave equation (34) satisfies the following energy identity on each backward cone : Let $(t_0, x_0)$ be the vertex of such a backward cone $K$, $K = \{|x - x_0| = t_0 - t\}$ and $e(u) = |\partial u|^2/2 + u^6/6$ be the energy density (here and thereafter $\partial$ denotes the full space-time gradient). Then we have for all $s \leq t \leq t_0$

$$\int_{B(x_0,t_0-t)} e(u(t,x))dx + \frac{1}{\sqrt{2}}\int_s^t \int_{\partial B(x_0,t_0-\tau)} \frac{|\partial_K u(\tau)|^2}{2} + \frac{u(\tau)^6}{6} \quad d\sigma d\tau \\ \leq \int_{B(x_0,t_0-s)} e(u(s,x))dx, \tag{35}$$

where $\partial_K$ denotes the derivatives tangent to the backward cone $K$ and the inequality is actually an equality. The second term on the left-hand side is usually referred to as the (outgoing) flux through the cone $K$. Moreover, if we consider the forward cone $K_1$ of vertex $(t_0 - 2s, x_0)$, namely $K_1 = \{|x - x_0| = t - (t_0 - 2s)\}$, then $K \cap K_1 = \{(t, x) \; |t = t_0 - s, \; |x - x_0| = s\}$ and

$$\frac{1}{\sqrt{2}}\int_{t_0-2s}^{t_0-s} \int_{\partial B(x_0,\tau-(t_0-2s))} \frac{|\partial_{K_1} u(\tau)|^2}{2} + \frac{u(\tau)^6}{6} \quad d\sigma d\tau \\ + \frac{1}{\sqrt{2}}\int_{t_0-2s}^{t_0-s} \int_{\partial B(x_0,t_0-\tau)} \frac{|\partial_K u(\tau)|^2}{2} + \frac{u(\tau)^6}{6} \quad d\sigma d\tau \\ \leq \int_{B(x_0,2s)} e(u(t_0 - 2s, x))dx. \tag{36}$$

and the equality holds. The first term in the left hand side is usually referred to as the (incoming) flux through the cone $K_1$. In collaboration with Fabrice Planchon [19], we prove the following result

**Theorem 5.1** *(with Fabrice Planchon) Let $u$ be a weak solution to (34) which satisfies (36) and (35). Then this solution is unique among all weak solutions satisfying (36) and (35).*

To prove the theorem, let us take $u$ and $v$, two solution of (34) satisfying (36) and (35). Taking $\phi$ to be an admissible test function $\phi \in C_0^\infty([0, \infty), \mathbb{R}^3)$, we have

$$\int (u - v)(\Box\phi + V\phi) = 0. \tag{37}$$

where $V = u^4 + 4u^3 v + 6u^2 v^2 + 4uv^3 + v^4$. We intend to solve the following (dual) problem: let $F \in C_0^\infty((0,T) \times \mathbb{R}^3)$ and $\phi$ be the solution of the following backward wave equation

$$\begin{cases} \Box\phi + V\phi = F, \\ \phi(T) = \partial_t \phi(T) = 0, \end{cases} \tag{38}$$

for some $T > 0$ which will be chosen small enough. Let $K = \{(t,x) | \ |x - x_0| = t_0 - t, \ t_0 \le t \le T\}$. Then, the solution of (38) is given by

$$\phi(t_0, x_0) = \int_K \frac{F(z) - V\phi(z)}{|z - z_0|} d\sigma(z), \tag{39}$$

with $z = (t,x)$ and where $\sigma$ is the surface measure on $K$. Then, we proceed as Jörgens [10, 25], and get the following a priori estimate

$$\|\phi\|_{L^\infty((0,T)\times\mathbb{R}^3)} \le C(F) + \|\phi\|_{L^\infty((0,T)\times\mathbb{R}^3)} \sup_{z_0} \int_K \frac{|V|}{|z - z_0|} d\sigma(z), \tag{40}$$

and as $|V| \lesssim u^4 + v^4$, we use

$$\int_K \frac{|u|^4}{|z - z_0|} d\sigma(z) \le C \int_K |\partial_K u|^2 + |u|^6$$

which is controlled using (36) by the initial energy in the ball $B(x_0, T)$. By choosing $T$ small enough this can be made uniformly small and hence, using the a priori estimate (40), we can prove the existence of a solution to (38) in $L^\infty_{t,x}$. At this level of regularity $\phi$ can not be used as a test function in (37). However, using a regularizing procedure (see [19]), we can prove that $\int (u - v)F = 0$ and hence $u = v$.

**Remark 5.2** *In four space dimensions, a related problem is the cubic wave equation*

$$\begin{cases} \Box\phi + \phi^3 = 0, \\ \phi(t = 0) = \phi_0, \qquad \phi_t(t = 0) = \phi_1, \end{cases} \tag{41}$$

*For this equation F. Planchon [24] proved the uniqueness of solutions in $C(\mathbb{R}; H^1) \cap C^1(\mathbb{R}; L^2)$ without any extra assumption and regardless of the sign in front of $\phi^3$. The proof uses the end-point Strichartz estimate. In the defocusing case, one has again weak finite energy solutions; by taking advantage of finite speed of propagation, the proof can be adapted to obtain uniqueness of weak solutions satisfying only forward in time local energy inequalities.*

## 6. Wave maps

Let $(N, g)$ be a complete Riemannian manifold of dimension $k$ without boundary. We denote $(x^\alpha)$, $0 \le \alpha \le d$ the canonical coordinate system of $\mathbb{R} \times \mathbb{R}^d$ where $t = x^0$ denotes the time variable. Moreover, we denote $\partial_\alpha = \partial/\partial x^\alpha$ and use the the Minkowski metric on $\mathbb{R} \times \mathbb{R}^d$ to raise and lower indices. In particular, $\partial^0 = -\partial_0$ and $\partial^\alpha = \partial_\alpha$ for $1 \le \alpha \le d$. The wave map equation from $\mathbb{R} \times \mathbb{R}^d$ into $N$, reads

$$\begin{cases} D_\alpha \partial^\alpha u &= 0, \\ u(x,0) &= u_0(x), \\ \partial_t u(x,0) &= u_1(x) \qquad x \in \mathbb{R}^n, \quad t \ge 0. \end{cases} \tag{42}$$

where $D_\alpha$ is the pull-back of the covariant derivative on the target Riemannian manifold $N$. Low regularity solutions to (42) are usually constructed via fixed point methods. Hence, while one is ultimately seeking solutions which are continuous evolutions of the data, that is $(u, \partial_t u) \in C_t(\dot{H}^s) \times C_t(\dot{H}^{s-1})$, the necessary requirements to set up a fixed point lead to a smaller Banach space. For example, this translates into additional space-time integrability conditions, like $u \in L^p_t(L^q_x)$ for suitable $p, q$. Our aim is to remove these assumptions which are incorporated in the uniqueness and existence statement given by, say, Picard's theorem. Note that in the wave map situation, one does not construct a solution directly by iteration, at least when working at the critical regularity. Nevertheless, in order to obtain a priori estimates, one is led to add similar requirement ($\partial u \in L^2_t(L^{2n}_x)$ for example in [28, 22]).

From now on, we generically denote $(\nabla u, \partial_t u)$ as $\partial u$, so that any statement regarding $u$ and $\partial_t u$ can be summarized into one, like $\partial u \in C_t(\dot{H}^{s-1})$.

**Theorem 6.1** *Let $u$ be a solution to (42) on $[0, T^*)$, with $d \geq 4$. Then $u$ is the unique solution of (42) in the class*

$$\partial u \in C_t(\dot{H}^{\frac{d}{2}-1}).$$

**Remark 6.2** *The same result holds for $n = 3$ if we take $\partial u \in C_t(\dot{H}^{\frac{1}{2}+\varepsilon})$, $\varepsilon > 0$. In fact, both schemes of proof from [28, 22] work in that framework, modulo technicalities related to the low regularity.*

Let us explain the idea behind the proof of theorem 6.1. Using the finite speed of propagation, we can assume that we have small data. Using a Gauge transform (see [20] for the details), we can reduce the uniqueness problem to the uniqueness for the following model equation

$$\left\{ \begin{array}{rl} \Box q = & A \cdot \nabla q + q \nabla \cdot A + A^2 q + q^3 \\ \Delta A = & \nabla(A^2) + \nabla(q^2). \end{array} \right. \tag{43}$$

To prove existence for (43), one can perform a fixed point argument in the class $E = C_t(\dot{H}^1) \cap L^2(\dot{B}^{1/6,2}_6)$ for $q$, and $F = C_t(\dot{H}^1) \cap L^1(\dot{B}^{1,1}_4)$ for $A$ ([22]). To prove Uniqueness, we consider $\delta = q - q_w$ the difference between two solutions, and $\alpha = A - A_w$ the difference between the vectors, and set $(q, A)$ to be "the good solution" obtained by the fixed point argument. The equation for $(\delta, \alpha)$ is

$$\left\{ \begin{array}{rcl} \Box \delta & \equiv & A \cdot \nabla \delta + \alpha \nabla (q - \delta) + \delta \nabla A + (q - \delta) \nabla \alpha \\ & & + q\alpha(2A - \alpha) + \delta(A - \alpha)^2 + \delta(q^2 + q\delta + \delta^2) \\ \Delta \alpha & \equiv & \nabla(2A\alpha - \alpha^2) + \nabla(2q\delta - \delta^2). \end{array} \right. \tag{44}$$

Then, we prove an estimate for $(\delta, \alpha)$ in $X \times Z$ where $X = C_t(\dot{H}^{\frac{1}{6}}) \cap L^2_t(\dot{B}^{-\frac{2}{3},2}_6)$. and $Z = L^2_t(\dot{B}^{1,2}_{12/7}) \hookrightarrow L^2 L^3$. The first step is actually to prove that $(\delta, \alpha) \in X \times Z$ which is a consequence of the small energy bound and the fact that $(q, A)$ and $(q_w, A_w)$ are solutions of (43). The last step is to prove some estimates on the different nonlinear terms by studying the interactions between low, medium and high frequencies. Finally, we get

$$\|\delta\|_X + \|\alpha\|_Z \lesssim \varepsilon_0 \|\alpha\|_Z + \varepsilon_0 \|\delta\|_X + \|\delta\|_X \|q\|_{L^2 \dot{B}^{1/6}_6}.$$

for some small $\varepsilon_0$. Hence, $(\delta, \alpha) = (0, 0)$.

## 7. Schrödinger equation

We consider the following semilinear Schrodinger equation

$$\begin{cases} iu_t + \Delta u = \pm |u|^2 u \\ u(t=0) = u^0 \ \in H^s(\mathbb{R}^3). \end{cases} \tag{45}$$

For $s \geq 1/2$, Cazenave and Weissler [3] proved the existence of a solution in $C(H^s)$. The uniqueness was only known in a more restrictive class. For $s \geq 1$, Kato [12] proved the uniqueness in $C(H^s)$. In [6], Furioli and Terraneo extended this result to the case $s > 1/2$. The proof is again based on estimating the difference between two solutions in some less regular space.

## References


[1] N. Bournaveas, *Local existence for the Maxwell-Dirac equations in three space dimensions.* Comm. Partial Differential Equations 21 (1996), no. 5-6, 693–720.

[2] N. Bournaveas *Local existence of energy class solutions for the Dirac-Klein-Gordon equations.* Comm. Partial Differential Equations 24 (1999), no. 7-8, 1167–1193.

[3] T. Cazenave and F. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$.* Nonlinear Anal. 14 (1990), no. 10, 807–836.

[4] J.-Y. Chemin. Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. *Journal d'Analyse Mathématique*, 77(?):27–50, 1999.

[5] P.A.M. Dirac, *Principles of Quantum Mechanics*, Oxford University Press, 4th ed., London (1958)

[6] G. Furioli, and E. Terraneo. Besov spaces and unconditional well-posedness for the nonlinear Schrödinger equations in $\dot{H}^s(\mathbb{R}^n)$, to appear in *Commun. Contemp. Math.*, 2001

[7] G. Furioli, P.-G. Lemarié-Rieusset, and E. Terraneo. Sur l'unicité dans $L^3(\mathbf{R}^3)$ des solutions "mild" des équations de Navier-Stokes. *C. R. Acad. Sci. Paris Sér. I Math.*, 325(12):1253–1256, 1997.

[8] G. Furioli, P.-G. Lemarié-Rieusset, and E. Terraneo. Unicité dans $L^3(\mathbf{R}^3)$ et d'autres espaces fonctionnels limites pour Navier-Stokes. *Rev. Mat. Iberoamericana*, **3** (2000) 605-667.

[9] M. G. Grillakis, *Regularity and asymptotic behaviour of the wave equation with a critical nonlinearity*, Ann. of Math. (2), 132, 1990, 3, 485–509.

[10] K. Jörgens, Das Anfangswertproblem im Grossen f¸r eine Klasse nichtlinearer Wellengleichungen. (German) Math. Z. 77 1961 295–308.

[11] T. Kato, Strong $L^p$-solutions of the Navier-Stokes equation in $R^m$, with applications to weak solutions., Math. Z. 187 (1984), no. 4, 471–480.

[12] T. Kato. On nonlinear Schrödinger equations. II. $H^s$-solutions and unconditional well-posedness, *J. Anal. Math.*, 67, 1995, 281–306,

[13] S. Klainerman and M. Machedon, *On the Maxwell-Klein-Gordon equation with finite energy*, Duke Math. J. **74** (1994), no. 1, 19–44.

[14] S. Klainerman and M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. **46** (1993), no. 9, 1221–1268.

[15] J. Leray. Etude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures Appl.*, 12:1–82, 1933.

[16] P.-L. Lions and N. Masmoudi. Unicité des solutions faibles de Navier-Stokes dans $L^N(\Omega)$. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(5):491–496, 1998.

[17] P.-L. Lions and N. Masmoudi. Uniqueness of mild solutions of the Navier-Stokes system in $L^N(\Omega)$. *Comm. Partial Differential Equations.* 26 (2001), no. 11-12, 2211–2226.

[18] N. Masmoudi and K. Nakanishi, *Uniqueness of finite energy solutions for Maxwell-Dirac and Maxwell-Klein-Gordon equations*, to appear in Comm. Math. Physics, 2003.

[19] N. Masmoudi and F. Planchon, *On Uniqueness for the critical wave equation*, preprint, 2003.

[20] N. Masmoudi and F. Planchon, *On Uniqueness for wave maps*, preprint, 2003.

[21] S. Monniaux. Uniqueness of mild solutions of the Navier-Stokes equation and maximal $L^p$-regularity. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(8):663–668, 1999.

[22] Andrea Nahmod, Atanas Stefanov, and Karen Uhlenbeck. On the well-posedness of the Wave Map problem in high dimensions. preprint, 2001.

[23] B. Najman, *The nonrelativistic limit of the nonlinear Dirac equation*, Ann. Inst. Henri Poincaré, Anal. Non Lineaire **9** (1992) 3–12

[24] F. Planchon, On uniqueness for semilinear wave equations, to appear in *Math. Zeit.*, 2001

[25] J. Rauch, The $u^5$ Klein-Gordon equation. II. Anomalous singularities for semilinear wave equations. Nonlinear partial differential equations and their applications. CollËge de France Seminar, Vol. I (Paris, 1978/1979), pp. 335–364, Res. Notes in Math., 53, Pitman, Boston, Mass.-London, 1981.

[26] Jalal Shatah and Michael Struwe. Regularity results for nonlinear wave equations. *Ann. of Math. (2)*, 138(3):503–518, 1993.

[27] Jalal Shatah and Michael Struwe. Well-posedness in the energy space for semilinear wave equations with critical growth. *Internat. Math. Res. Notices*, (7):303ff., approx. 7 pp. (electronic), 1994.

[28] Jalal Shatah and Michael Struwe. The Cauchy problem for wave maps. *Int. Math. Res. Not.*, (11):555–571, 2002.

[29] M. Struwe. Uniqueness for critical nonlinear wave equations and wave maps via the energy inequality, *Comm. Pure Appl. Math.*, 52 n 9 , 1999, 1179-1188

[30] Y. Zhou *Uniqueness of generalized solutions to nonlinear wave equations.* Amer. J. Math. 122 (2000), no. 5, 939–965.

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