Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Évian, 2 juin–6 juin 2008

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J. É. D. P. (2008), Exposé n^o VII, 12 p.

<http://jedp.cedram.org/item?id=JEDP_2008____A7_0>

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GDR 2434 (CNRS)

Scattering for the Beam equation

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In this paper, we aim at describing recent results about scattering for the nonlinear Beam equation. The Beam equation, sometimes also called fourth-order wave equation is defined as follows

$$\partial_{tt}u + \Delta^2 u + mu + \lambda |u|^{p-1}u = 0. \tag{0.1}$$

It is a formal fourth-order extension of the classical Klein-Gordon equation and appears in the study of elasticity (we refer to Love [12] for a comprehensive presentation), in the study of the motion of a suspension bridge in McKenna and Walter [14, 15], and also as a toy model for the propagation of water waves in Bretherton [2]. In this equation, u, the unknown is a real-valued function defined on space-time $\mathbb{R} \times \mathbb{R}^n$, the mass $m > 0$ and $\lambda > 0$ are parameters which can easily be normalized to 1. The nonlinearity is chosen to be pure-power-like for simplicity, and we restrict p to be smaller than the H^2-critical exponent, thus if $n \geq 5$, we require that $p < 1 + 8/(n-4)$. Besides its physical interest, this equation is also a toy model as a general dispersive equation, lacking the very special forms of the Schrödinger or wave equation. In these conditions, to get strong results, we need to rely mostly on dispersive properties of the equation.

In Levandosky and Strauss [10], the authors could find find an analogue of the Morawetz-estimate for the wave equation that applied to the Beam equation in high dimensions $n \geq 5$. This led them to conjecture that, as for the Klein-Gordon equation, scattering should hold true for the Beam equation (0.1), at least for $n \geq 5$, in the subcritical range $1+8/n < p < 1+8/(n-4)$. Subsequent work by Lin [11] and Miao [13] proved that, in this case, the local energy was integrable in time. Finally in Pausader [16], it was proved that this conjecture holds true. In the following, our main emphasis will be on the proof of this result. More precisely, we will prove the following (see below for definitions of strong solutions and scattering)

Theorem 0.1. *Let $n \geq 5$, $\lambda > 0$ and $1 + 8/n < p < 1 + 8/(n-4)$, then any nonlinear strong solution of* (0.1) *scatters.*

We start by presenting general features of the equation, in particular, we obtain good global in time decay estimates. Then, we prove scattering in two steps. First, we prove that the solution satisfies an almost finite propagation speed principle, and then we make use of the Morawetz estimate of Levandosky and Strauss to obtain decay of the solution in Lebesgues spaces. This is sufficient, thanks to our analysis of the linear propagator. In the end, we give some further results about the scattering operator, and give miscellaneous remarks.

1. Presentation of the scattering problem and Beam equation

1.1. Analysis of the linear equation

An important ingredient in the analysis of the nonlinear Beam equation (0.1) is the analysis of the associated linear equation, namely

$$\begin{aligned} &\partial_{tt}u + \Delta^2 u + u = h, \\ &(u(0), \partial_t u(0)) = (u_0, u_1) \end{aligned} \tag{1.1}$$

where the initial data belongs to the energy space: $(u_0, u_1) \in H^2 \times L^2 = \mathcal{E}$. Indeed the solutions of (1.1) will be our model both locally in time, and asymptotically, since we want to prove scattering.

Equation (1.1) can be treated in the setting of semigroups, since, letting $\mathcal{E}_b = H^1 \times H^{-1}$ and considering the unbounded operator on $\mathcal{E}_b$

$$(D(A), A) = (H^3 \times H^1, A(u,v) = (v, -(\Delta^2 u + u)),$$

then A is skew symmetric and (1.1) can be rewritten $\dot{x} = Ax$ for $x(t) = (u(t), u_t(t))$. Since A is skew-symmetric and commutes with derivative operators, the flow of A, e^{tA} is an isometry on $\mathcal{E}$. In the sequel, we write $\mathcal{W}(t) = e^{tA}$ for the free propagator. We also note the following sympletic form on $\mathcal{E}_b$

$$\Omega((u_0, u_1), (v_0, v_1)) = \int_{\mathbb{R}^n} (u_0 v_1 - u_1 v_0)\, dx. \tag{1.2}$$

Previous results about the decay of $\mathcal{W}(t)$ were obtained by Levandosky [9] using Fourier restriction methods. Here, we will separate the Frequency space in two regions corresponding to two different models for our equation. In the sequel, P_N, $P_{\leq N}$ and $P_{\geq N}$ refer to Littlewood-Paley projections.

Using the Fourier transform, we can give explicitly the solutions of (1.1) with $h = 0$. These are

$$u(t) = \cos(t\sqrt{1+\Delta^2})u_0 + \frac{\sin(t\sqrt{1+\Delta^2})}{\sqrt{1+\Delta^2}} u_1 \tag{1.3}$$

and we see that we need only understand the linear operator $e^{it\sqrt{1+\Delta^2}}$.

First, we investigate the high frequency regime, $|\xi| \to \infty$. The phase function, $\phi(\xi) = \sqrt{1+|\xi|^4}$ converges to the standard Schrödinger phase $\phi_s(\xi) = |\xi|^2$ on annuli $\{N \leq |\xi| \leq 2N\}$, as $N \to +\infty$. In this case, we get that, for $N \geq 1$, $P_N e^{it\sqrt{1+\Delta^2}}$ has the same dispersion property as $P_N e^{it\Delta}$, and hence enjoys the same Strichartz estimates.

Now, let us consider the low frequency case. As $|\xi| \to 0$, we have that

$$\phi(\xi) = 1 + \frac{|\xi|^4}{2} + o(|\xi|^4),$$

and in this case, factoring out a phase factor like e^{it}, we expect the same behavior as for a fourth-order Schrödinger equation.

Concretely, we can prove the following estimate, which is a refinement of the estimate in Pausader [16]. A couple (q, r) with $(q, r, n) \neq (2, \infty, 2)$ is called admissible if $2 \leq q, r \leq \infty$ and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}.$$

Proposition 1.1. *Let $(u_0, u_1) \in H^2 \times L^2$ and $h \in C(I, H^{-2})$. Then, the solution $v \in C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, L^2)$ of (1.1), satisfies the following Strichartz-type estimates*

$$\begin{aligned} \||\nabla|^{\frac{2}{q}} P_{\leq 1} u\|_{L^q(I, L^r))} &\lesssim \|P_{\leq 1} u_0\|_{L^2} + \|P_{\leq 1} u_1\|_{L^2} + \||\nabla|^{-\frac{2}{a}} P_{\leq 1} h\|_{L^{a'}(I, L^{b'})} \\ \|P_{\geq 1} \Delta u\|_{L^q(I, L^r)} &\lesssim \|P_{\geq 1} \Delta u_0\|_{L^2} + \|P_{\geq 1} u_1\|_{L^2} + \|P_{\geq 1} h\|_{L^{a'}(I, L^{b'})} \end{aligned} \tag{1.4}$$

where (q, r) and (a, b) are any admissible pairs.

Notice the additional derivative in the low-frequency norm, which is the price to pay to obtain global Strichartz estimates in the presence of the degeneracy at the 0 frequency[1]. The Strichartz estimates above come from the following decay rates for solutions of the homogeneous equation (1.1) with $h = 0$. Let $2 \leq p \leq \infty$, then

$$\begin{aligned} \|P_{\leq 1} |\nabla|^{n(1-\frac{2}{p})} u\|_{L^p} &\lesssim t^{-\frac{n}{2}(1-\frac{2}{p})} \left(\|P_{\leq 1} u_0\|_{L^{p'}} + \|P_{\leq 1} u_1\|_{L^{p'}}\right) \\ \|P_{\geq 1} u\|_{L^p} &\lesssim t^{-\frac{n}{2}(1-\frac{2}{p})} \left(\|P_{\geq 1} u_0\|_{L^{p'}} + \|(1+\Delta^2)^{-\frac{1}{2}} P_{\geq 1} u_1\|_{L^{p'}}\right). \end{aligned} \tag{1.5}$$

The crucial fact here is that the high frequencies decay much faster than the low frequencies[2]. In the following, we sometimes identify a linear homogeneous solution and its initial data.

1.2. Formal structure of the equation

In order to extend the local wellposedness results, we use the conservation laws derived from the formal structure of the equation. We define the energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|u_t|^2 + |\Delta u|^2 + m|u|^2 + \frac{2}{p+1} |u|^{p+1} \right) dx,$$

the momentum

$$\mathrm{Mom}(u) = \int_{\mathbb{R}^n} u_t(x) \nabla u(x) dx,$$

and the angular momentum (for $i \neq j$)

$$\Omega_{ij} = \int_{\mathbb{R}^n} \left(x_j u_t \partial_i u - x_i u_t \partial_j u\right) dx.$$

These are conserved for all strong solutions u of (0.1). For a complex-valued function, we also define the charge

$$\mathrm{Ch}(u) = \mathrm{Im} \int_{\mathbb{R}^n} u_t(t, x) \bar{u}(t, x) dx. \tag{1.6}$$

Note that the equation (0.1) is formally the symplectic flow associated with the Hamiltonian E and the symplectic form (1.2), the symplectic gradient of the Momentum is just the usual gradient (generator of the translations). the symplectic gradients of the angular momenti are the generators of rotations.

[1] This is due to the fourth-order dispersion. By contrast, local estimates can be deduced using the Schrödinger decomposition of the linear equation $\partial_{tt} + \Delta^2 = (\partial_t + i\Delta)(\partial_t - i\Delta)$.

[2] since to get decay of the solution for the low frequencies, one must use Sobolev's inequality, and lower p, hence lower the decay rate

Conservation of energy is almost always central in any result related to the beam equation, either through the global bounds that it gives in the defocusing case, as in Pausader [16], or through the fact that it gives a conserved quantity in certain proofs of blow-up, as in Hebey and Pausader [6]. Conservation of momentum is very important through the fact that momentum can be altered to give different Morawetz/Virial-type estimate as (2.1) below. We also refer to Pausader [17] for other such bounds. The conservation of Charge can be used to rule out scattering when the pure power-like nonlinearity is replaced by $f(u) = F'(u)$, where $F(x) < 0$ for some x, at least for complex-valued functions. We refer to Glassey [5] for a similar application in the Klein-Gordon case.

In order to use at best the conserved quantities, we work in the energy space $\mathcal{E} = H^2 \times L^2$. A strong solution is then a function $u \in \mathbb{E} = C(\mathbb{R}, H^2) \cap C^1(\mathbb{R}, L^2) \cap C^2(\mathbb{R}, H^{-2})$ that satisfies (0.1) in H^{-2} for all times. Another equivalent requirement is that $u \in \mathbb{E}$ satisfy the Duhamel formula

$$(u(t), u_t(t)) = \mathcal{W}(t)(u_0, u_1) - \int_0^t \mathcal{W}(t-s)(0, |u|^{p-1}u(s))ds \tag{1.7}$$

for all time t.

Using the Strichartz-type estimate and the conservation laws, one can prove that the Cauchy problem associated to (0.1), with initial data in $\mathcal{E}$ is locally well-posed in $\mathbb{E}$ and that all the strong solutions are global[3].

1.3. Scattering

Scattering is a quite accurate description of the dynamics of a nonlinear equation in the sense that is tells that the dynamics is in fact linear, up to waiting for a certain time. More precisely, we say that a solution u of (0.1) scatters if there exists a solution ω of the corresponding homogeneous linear equation (1.1) with $h = 0$ such that the difference of the two functions goes to 0 as t goes to $+\infty$. We say that scattering holds for the Beam equation if every strong solution scatters.

When this situation holds true, we can define the mapping $W : u \mapsto \omega$, and it is easy to see that this mapping can be inverted. We then define the scattering operator $S = W\mathcal{J}W^{-1}$, where $\mathcal{J}$ is the operator associated to the inversion of time. This operator synthesizes all the nonlinear dynamics in the sense that any nonlinear solution has the following behavior: it starts with a linear behavior similar to that of a linear solution ω, enjoys some nonlinear behavior, then gets back to a linear behavior similar to that of $S\omega$.

While such a neat description as scattering for all solutions should be restricted to very favorable cases, such as the defocusing case or situation in which the nonlinearity is prevented from becoming too strong, (as in "Payne-Sattinger" type potential well, see Payne and Sattinger [19] and Kenig and Merle [7], Duyckaerts, Holmer and Roudenko [4] for recent examples), it is conjectured that the "general" asymptotic behavior of a nonlinear solution is a decoupling between a scattering part that radiates away and a soliton-like part which enjoys special features.

Using conservation of Energy and the isometric property of $\mathcal{W}$ on $\mathcal{E}$, we observe that when $p > 1 + 4/n$, all solutions of (0.1) scatter weakly in the sense that there

[3] One also sees that they are the unique functions $u \in L^\infty(\mathbb{R}, H^2) \cap W^{1,\infty}(\mathbb{R}, L^2)$ satisfying (0.1) in the sense of distributions.

exists $(u_0^+, u_1^+) \in \mathcal{E}$ such that

$$(u(t), u_t(t)) - \mathcal{W}(t)(u_0^+, u_1^+) \rightharpoonup (0,0) \tag{1.8}$$

in $\mathcal{E}$ as $t \to +\infty$. Besides, this scattering state is formally given by

$$(u_0^+, u_1^+) = (u_0, u_1) - \int_0^\infty \mathcal{W}(-s)(0, |u|^{p-1}u(s))ds \tag{1.9}$$

and subsequent work concentrate on giving a sense to formula (1.9). A sufficient condition is that u has finite space time norm in appropriate Lebesgues spaces. For that, a useful criterion in the subcritical case is the following:

If u decays to 0 in some Lebesgues space norm, and $1+8/n < p < 1+8/(n-4)$, then u belongs to the appropriate space-time norm and hence scatters.

Consequently, our remaining analysis aims at proving such a decay.

2. Nonlinear Bounds

To go beyond global existence and study asymptotic results, one needs to find bounds on the total contribution of the nonlinearity over time. A prefered candidate to do the job is a Morawetz-type inequality. Levandosky and Strauss [10] have found the following analogue of the Morawetz estimate for the Klein-Gordon equation:

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{|u(t,x)|^{p+1}}{|x|} dxdt \lesssim \|u_t\|_{L^\infty L^2} \|\nabla u\|_{L^\infty L^2} \tag{2.1}$$

which is valid in dimensions $n \geq 5$ for any u a strong solution of (0.1). This gives very powerful control on the solution close to the origin. However, it gives poor control far away from the origin (observe in particular that the integral over the space-time region $\{|x| \geq t \log^2 t\}$ is trivially bounded by the energy).

The question now is how to use the control given by (2.1). A first case is when u is restricted to be spherically symmetrical. In this case, everything "happens" close to the origin, and using the Morawetz estimate one easily shows that u decays in L^{p+1}_x and hence scatters. However, in general, it is possible that all the "nonlinear behaviors" happen far away from the origin, or that the solution presents some kind of drift. For example, in the focusing case $\lambda < 0$, such traveling waves exist for all $1 < p < 1+8/(n-4)$ as shown in Levandosky [8]. In the defocusing case, however, these exact solutions cannot exist.

In order to prevent a solution concentrated around the origin to move too far away quickly, and thereby make the Morawetz-type estimate useless, we first prove that the solution satisfies an approximate finite speed propagation principle.

2.1. Almost finite speed of propagation

The main result of this step is to show that the solution has to stay in some (space-time) cone, up to some error. However, the smaller the error, the larger the cone. In this sense, we say that the solution satisfies a kind of almost-finite speed of propagation principle.

In order to achieve our goal, we use an observation by Tao [20] that global solutions of bounded energy to highly dispersive subcritical equations remain bounded in frequency.

Proposition 2.1. *Let $n \geq 5$, and let u be a strong solution of* (0.1)*, then there exists a solution ω of the linear equation* (1.1) *and $\eta > 0$ such that the following holds*

$$\begin{aligned} &u(t) = \omega(t) + v(t), \\ &v(t) \rightharpoonup 0 \textit{ in } H^2 \textit{ as } t \to +\infty \textit{ and} \\ &E(P_{\geq N} v, P_{\geq N} v_t) \lesssim_u o(1) + N^{-\eta}, \end{aligned}$$

where $o(1) \to 0$ as $t \to +\infty$, independently of N. In particular, for any $\epsilon > 0$, there exists N such that

$$\|P_{\geq N} u(t)\|_{H^2} \leq \epsilon \tag{2.2}$$

for all times $t \geq 0$.

In order to prove Proposition 2.1, we use the fact that formula (1.7) shows that u can be decomposed in two different terms with very different features. First, we have the contribution of a linear solution. This has finite space-time norm (so that all the "nonlinear behavior" is quite localized), but does not enjoy any gain in regularity. The other term now comes from the contribution of the nonlinearity. It is less explicit, but has better regularization properties. The first term can be manipulated easily. To deal with the second, we wish to isolate the contribution of the nonlinear term. However, the scattering formula (1.9) tells us that this nonlinear contribution should also create some linear evolution. To get around that problem, we isolate the nonlinear term "when t tends to infinity", using (1.8) to find what the asymptotic linear term should be. This way, we can produce a formula similar to (1.7):

$$(u(t), u_t(t)) = \mathcal{W}(t)(u_0^+, u_1^+) + \int_t^{\to\infty} \mathcal{W}(t-s)(0, |u|^{p-1}u(s))ds, \tag{2.3}$$

where the integral is understood as the weak limit of the integrals over finite segment $[t, T]$ as $T \to +\infty$. Using also the classical Duhamel formula (1.7) to compensate for the poor definition of the integral in (2.3), we get two expressions for u that involve the contribution of the nonlinear term on disjoint time interval. In order to find better regularity in the nonlinear term, we project both formula (1.7) and (2.3), on the high frequencies, take the scalar product and we express $(v(t), v_t(t)) = (u(t), u_t(t)) - \mathcal{W}(t)(u_0^+, u_1^+)$ as a sum of different terms, the worst of which is the "Double Duhamel term" which after some integration by parts, reads as follows:

$$\mathcal{I}(t) = \int_{s=0}^{t} \int_{s'=t}^{\to\infty} \langle \mathcal{W}(s'-s)(0, P_{\geq N}|u|^{p-1}u(s)), (0, P_{\geq N}|u|^{p-1}u(s'))\rangle ds ds'$$

To estimate this term, we split it in two parts.

First, we consider the contribution of times s, s' which are far away from the given time t. For these times, in particular, $|s' - s| >> 1$, and we can use the fact that the high frequency part of the linear propagator $\mathcal{W}$ has a very fast decay, as shown in (1.5). In fact, in dimensions $n \geq 5$, the decay of the high frequencies is so fast that one can make the integrand in $\mathcal{I}$ integrable around infinity. Thus, the contribution of $\{(s, s') : 0 \leq s \leq t - N^\eta \leq t + N^\eta \leq s'\}$ can be made arbitrarily small provided N is taken sufficiently large.

Then, it remains to deal with the contribution of the times (s, s') which are not too far away from the time t we consider. In this region, we can no longer take advantage of the dispersion. Besides, this region is by no means small (it is of size N^η, N large).

However, we consider only the high frequencies, and we have a subcritical equation (0.1) with a bound on a critical norm, namely the Energy. The idea is then to push the solution to higher and higher frequencies, which will "select" $\dot{H}^2$ behavior, and thus should damp the effect of the nonlinearity. In fact one can prove the following estimate:

$$\|P_{\geq N}u\|_{L^{a'}(I,L^{b'})} \lesssim_{E(u)} N^{-\kappa}(1+|I|)^{\frac{1}{a'}} \tag{2.4}$$

which enables us to get a small contribution of the nonlinearity in the suitable norm for long time interval.

As a remark, we note that this procedure is quite general, and Proposition 2.1 holds true in many cases when a solution of a highly-dispersive equation has bounded H^2-norm uniformly in time (we refer to Tao [20] for a previous use of that strategy).

Proposition 2.1 goes a long way towards the resolution of the scattering conjecture, as it identifies the scattering element, and derives some properties of the error term v. In particular, our goal is to prove that v decays to 0 strongly in H^2.

Now, the next step is to use the fact that linear solutions bounded in frequency almost move at a finite speed, or in our case, the corresponding property of the linear propagator, together with Proposition 2.1 to obtain an almost-finite propagation speed principle for the nonlinear solution, in order to be able to use the Morawetz estimate (2.1). More precisely, we claim that

Proposition 2.2. *Let u be a solution of (0.1) such that (2.2) holds true. If $\varepsilon > 0$ is sufficiently small, then there exists $R > 0$ such that*

$$\int_{\{|x|\geq R(1+t)\}} |u(t,x)|^{p+1}dx \lesssim_{E(u)} \varepsilon^{\kappa} \tag{2.5}$$

for all times $t \geq 0$, where $\kappa > 0$ is independent of u, ε.

This means that u is essentially concentrated in a cone of slope R. If u were a solution of the wave equation, we would have Proposition 2.2 with a cone of slope 1. Here the slope depends on the error ε.

In order to prove Proposition 2.2, we first remark that, using our assumption (2.2), it suffices to consider the low frequency component of u, and then that, taking R sufficiently large, estimate (2.5) is easily proven for the linear solution with the same initial data as u (i.e. for the linear component of u in (1.7)). Now, in order to extend it to the nonlinear solution, we consider α and θ satisfying

$$\frac{2n}{n+4} - \delta < \alpha < \frac{2n}{n+4}\ ,\ 0 < \theta < 1 \text{ and } \frac{1}{\alpha p} = \frac{1-\theta}{2} + \frac{\theta}{\alpha'} \tag{2.6}$$

for some small $\delta > 0$, and the set

$$I(C) = \left\{ t \geq 0; \int_{\{|x|\geq R(1+t)\}} |u(t,x)|^{p\alpha} \leq C\varepsilon^{p\alpha} \right\}.$$

Our goal is then to prove that for C sufficiently large, uniformly in ε, there exists $R > 0$ such that $I(C)$ contains a neighborhood of 0 and that $I(2C) \subset I(C)$.

In order to prove this, we let $S_t = \{|x| \geq R(1+t)\}$, and we split u into several pieces, using (1.7)

$$\begin{aligned}
\mathbb{1}_{S_t} u(t) &= \mathbb{1}_{S_t} P_{\geq N} u(t) + \mathbb{1}_{S_t} P_{\leq N} \pi_1 \mathcal{W}(t)(u_0, u_1) \\
&\quad - \mathbb{1}_{S_t} P_{\leq N} \int_{\min(0,t-\delta)}^{t} \pi_1 \mathcal{W}(t-s)(0, |u|^{p-1}u(s))ds \\
&\quad - \mathbb{1}_{S_t} P_{\leq N} \int_{t=0}^{t-\delta} \pi_1 \mathcal{W}(t-s)(0, \mathbb{1}_{S^c_{t-s}} |u|^{p-1}u(s))ds \\
&\quad - \mathbb{1}_{S_t} P_{\leq N} \int_{t=0}^{t-\delta} \pi_1 \mathcal{W}(t-s)(0, \mathbb{1}_{S_{t-s}} |u|^{p-1}u(s))ds \\
&= r_1 + r_2 + r_3 + r_4 + r_5
\end{aligned}$$

The first and second terms are small provided that R is large enough, as said before. The third one is small, provided $\delta > 0$ is sufficiently small, depending on $\varepsilon, E(u)$. As these are the only terms if $t \leq \delta$, we get that $I(C)$ contains a neighborhood of 0. Now, the difficulty is to bound the contribution of the nonlinearity over long periods of time.

First, we consider the fourth term. This terms corresponds to the contributions of the points close to the origin (i.e. inside the cone) at previous times to the dynamics of the points far away from the origin at time t (i.e. outside the cone). However, this interaction is only possible through the action of a linear propagator bounded in frequency, which thus only enables a finite speed (here of order $2N$) of propagation. Consequently, this contribution is small (by "non-stationary phase" arguments) if we choose the slope of the cone sufficiently large (here $R \gtrsim N^{1+\delta}$ will do).

Second, we need to control the contribution of the last term: the points previously outside the cone that interact with the points now outside the cone. Of course, that interaction is not banned by the "boundedness in frequency" hypothesis. However, in this situation, the term that interact are of size ε, and interact in a nonlinear way, so the interaction is of size $\varepsilon^p << \varepsilon$. More precisely, we first discard the contribution from $t-1$ to $t-\delta$, which is easily seen to smaller than $\varepsilon^{p-\delta}$, and then, using the fact that the sum of the last two integrals is bounded in L^2 and (2.6), we write, for $t \in I(C)$,

$$\begin{aligned}
\|\mathbb{1}_{S_t} u(t)\|_{L^{p\alpha}} &\leq 4c\varepsilon + \|r_4 + r_5\|_{L^2}^{1-\theta} \|r_4 + r_5\|_{L^{\alpha'}}^{\theta} \\
&\leq 4c\varepsilon + c' \left(\varepsilon^p + \int_0^{t-1} \|\pi_1 P_{\geq 1} \mathcal{W}(t-s)(0, |u|^{p-1}u(s))\|_{L^{\alpha'}} ds \right)^{\theta} \qquad (2.7) \\
&\leq 4c\varepsilon + c' \left(\varepsilon^p + \sup_{s \in [0,t-1]} \|\mathbb{1}_{S_s} u(s)\|_{L^{p\alpha}}^p \right)^{\theta}
\end{aligned}$$

where c, c' are independent of t, C, ε. Using (2.6) and (2.7), we see that for C sufficiently large, we get that $I(2C) \subset I(C)$, and thus u remains concentrated in the corresponding cone in $L^{p\alpha}$. By conservation of the energy, Hölder's inequality then gives (2.5).

2.2. Morawetz estimate and space-time bound

Using the Morawetz estimate (2.1) and Proposition 2.2, we can find an arbitrary long interval I on which the solution has arbitrarily small nonlinear behavior as follows: Given $\epsilon > 0$ and $L > 1$, we get R such that (2.5) holds true with ϵ/L

instead of ε. Independently, in any given cone, we can find intervals of length $2L$ such that the nonlinear term is small. Indeed, letting

$$a_k = \int_{2kL}^{2(k+1)L} \int_{\{|x| \le R(1+t)\}} |u(t,x)|^{p+1} dx,$$

the Morawetz estimate (2.1) ensures that

$$\sum_{k \ge 0} \frac{a_k}{2R((k+2)L)} < \infty,$$

and for that series to converge, we must have terms a_k which are arbitrarily small. Thus, there exists an interval $I = [a, a+2L]$ such that

$$\int_I \int_{\mathbb{R}^n} |u(t,x)|^{p+1} dx dt \le \epsilon.$$

Thanks to the subcriticality of the nonlinearity, we can improve this smallness in $L^{p+1}(I \times \mathbb{R}^n)$ to a smallness in $L^\infty(I, L^{p+1})$ as follows. Let $M = \|u\|_{L^\infty(I, L^{p+1})}$ and let $t_0 \in I$ be a point where the maximum is attained (recall that $u \in C(I, L^{p+1})$). Then, using the bound on the H^2-norm given by conservation of energy, we see that there exists $\kappa > 0$ and $N \lesssim_{E(u)} M^{-\kappa}$ such that $\|P_{\le N} u(t_0)\|_{L^{p+1}} \ge M/2$. Besides, still by conservation of energy, we also get that $\|\partial_t P_{\le N} u\|_{L^2} \le 2E(u)$, hence

$$\|P_{\le N} u(t_0) - P_{\le N} u(s)\|_{L^2} \lesssim_{E(u)} (t_0 - s)^{\frac{1}{2}}.$$

Since $\|P_{\le N} u(t_0) - P_{\le N} u(s)\|_{L^\infty} \lesssim_{E(u)} N^{\frac{n}{2}}$, we get by Hölder's inequality that for all s such that $|t_0 - s| \lesssim_{E(u)} M^{(p+1)c}$ for $c > 0$ independent of M, there holds that $\|P_{\le N} u(s)\|_{L^{p+1}} \ge M/2$. Since the Littlewood-Paley projection is bounded, we get, integrating on I, that $\|u\|_{L^{p+1}(I \times \mathbb{R}^n)} \gtrsim_{E(u)} M^{1+c}$, hence, smallness in $L^{p+1}(I \times \mathbb{R}^n)$ implies smallness in $L^\infty(I, L^{p+1})$.

Now, we remark that if the "reaction has ended" (in the sense that u has little nonlinear presence for a long time interval), then it cannot spontaneously ignite again in the absence of a perturbation by the linear part. Indeed for any time $t \ge a+L$, using (1.7), we see that $u(t)$ is the sum of a linear solution with same initial data, the contribution of the nonlinearity for times before a, and the contribution of the nonlinearity between a and t,

$$\begin{aligned}(u(t), u_t(t)) =& \mathcal{W}(t)(u_0, u_1) - \mathcal{W}(t-a) \int_0^a \mathcal{W}(a-s)(0, |u|^{p-1} u(s)) ds \\ &- \int_a^t \mathcal{W}(t-s)(0, |u|^{p-1} u(s)) ds.\end{aligned}$$

But, by the decay of linear homogeneous solutions, if L is large enough, the first term is small. Similarly, the second term is also a linear homogeneous solution, and has therefore uniformly bounded H^2-norm. Using also the decay in L^∞ of linear homogeneous solutions (1.5), we get that if L is sufficiently large, this term is small. Finally, we run a nonlinear argument similar to the one in Proposition 2.1 to prove that if the nonlinearity is smaller than ϵ, and if the contribution of the other terms are small, then the nonlinear term remains small.

2.3. More about the scattering operator

Once it is shown to exist, the scattering operator, whose formula is given by (1.9), almost directly comes with some regularity properties that can be derived from its functional equation:

$$(u(t), u_t(t)) = \mathcal{W}(t)(u_0^+, u_1^+) + \int_t^{+\infty} \mathcal{W}(t-s)(0, |u|^{p-1}u(s))ds.$$

The results presented here are from Pausader and Strauss [18]. Using an implicit function theorem, one can get the following regularity property:

Proposition 2.3. *Let $n \geq 5$, $1 + 8/n < p < 1 + 8/(n-4)$, then any nonlinear solution u of* (0.1) *scatters to a linear homogeneous solution ω of* (1.1)*. Besides, the mapping, $u(0) \mapsto \omega(0)$ and S are $C^{\lfloor p \rfloor}$-diffeomorphisms of $\mathcal{E}$ commuting with translation and satisfying $2E(u) = \|\partial_t \omega(0)\|_{L^2}^2 + \|(1-\Delta)\omega(0)\|_{L^2}^2$.*

Besides, still using the functional property, we get a second order expansion of the scattering operator around $(0,0)$,

$$S(u, v) = (u, v) - \lambda \int_{-\infty}^{+\infty} \mathcal{W}(-s)(0, |\omega|^{p-1}\omega(s))ds + O(u, v)^{2p-1}$$

where $\omega(s)$ solves (1.1) with initial data (u, v). Using the second term in the expansion of the scattering operator, one can find that it determines uniquely the nonlinearity (i.e. it gives λ and p in (0.1)). More generally, if we replace $\lambda|u|^{p-1}u$ by an arbitrary function $f(u)$ satisfying suitable conditions so that scattering still holds, then the scattering operator determines the Taylor expansion of f at 0.

In case the nonlinearity is analytic, the scattering operator is analytic, and one can give an explicit expansion of S around $(0,0)$. Of course, in this case the scattering operator completely determines the nonlinearity. We refer to Carles and Gallagher [3] for other results in the case of wave and Schrödinger equations.

3. Miscellaneous remarks

Remark 3.1. *We have proved that for any u solution of* (0.1)*, we have that*

$$\int_{\mathbb{R}\times\mathbb{R}^n} |u(t,x)|^{p+1} dx dt < \infty. \tag{3.1}$$

Then, using an analysis similar to the one in Bahouri and Gerard [1] *(but in the easier context of a subcritical equation), we can prove that the bound in* (3.1) *depends on u only through the energy $E(u)$.*

Remark 3.2. *In a similar way, in dimensions $n \leq 4$, we can use a contradiction argument similar to the one in Kenig and Merle* [7] *to prove the scattering conjecture when $p > 1 + 8/n$. However, the argument in this case does not give such a good description of the behavior of solutions. We refer to Pausader* [17]*.*

Remark 3.3. *One could wonder about the significance of the exponents in theorem 0.1. They correspond to the L^2 and H^2-critical exponents, which in turn correspond to the quantities for which we know an a priori bound, thanks to the conservation laws. However, we note that, for small initial data in $\dot{H}^{-1} \cap H^2 \times \dot{H}^{-1} \cap L^2$, one has scattering even for a smaller power $1 + 8/(n+2) \leq p \leq 1 + 8/n$, at least in dimensions $n \geq 6$. On the other hand, there is a limit as to how much one can* *lower the exponent, since if $p < 1 + 2/n$, using* (1.2)*, it can be shown that any smooth function whose Fourier support is away from* 0 *does not lie in the image of the scattering operator (if such an operator exists).*

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