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Thierry De Pauw

On ∞-harmonic functions

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Contents

1. Limits of p-harmonic functions

Let $U \subset \mathbb{R}^m$ be a bounded open set with Lipschitz boundary, $g \in \operatorname{Lip}(\partial U)$, and $1 < p < \infty$. The following minimization problem admits a unique minimizer which we shall subsequently denote by u_p (the dependence upon U and g is not relevant for now):

$$(\mathscr{P}_p) \begin{cases} \text{minimize } \|\nabla u\|_p \\ \text{among } u \in W^{1,p}(U) \text{ with } \operatorname{trace} u = g\,. \end{cases}$$

It is well known that u_p has vanishing p-Laplacian, i.e.

$$\operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p) = 0 \tag{1}$$

weakly (in the sense of distributions or, equivalently, weakly in $W^{1,p}(U)$). The best interior regularity theory for u_p is $C^{1,\alpha}$ (with $\alpha = \alpha(m,p)$); it is shown to be optimal by some examples.

Questions dealt with in the present section are:

(A) Does the family u_p converge to some u as $p \to \infty$, and if yes in what sense?

(B) Does such a limiting function verify a partial differential equation?

(C) If yes, is that partial differential equation the Euler-Lagrange equation of some variational problem?

The author is a *chercheur qualifié* of the *Fonds National de la Recherche Scientifique*, Belgium.

We now proceed to briefly review the answers to these questions.

By the minimizing property of u_p and their common boundary value it is trivially checked that $\sup\{\|\nabla u_p\|_p : 1 < p < \infty\} < \infty$. On the other hand Morrey's inequality says that if $p > m$ then $W^{1,p}(U) \subset C^{0,1-m/p}(\overline{U})$ and, in fact,

$$|u(x) - u(y)| \leqslant C(m,p)\|\nabla u\|_p |x-y|^{1-m/p}$$

whenever $u \in W^{1,p}(U)$, $x, y \in U$, and, say, U is convex. A straightforward task consists in keeping track of the constant in the various potential estimates needed to prove the above inequality. Careful scrutinizing of such proofs shows that

$$\limsup_{p\to\infty} C(m,p) < \infty\,.$$

It therefore follows that the family $\{u_p : p > m\} \subset C(\overline{U})$ is equicontinuous. Since also these functions achieve the same boundary value it follows from Ascoli's Theorem that the same family is relatively compact. Hence there are $p_1 < p_2 < \dots$ and $u \in C(\overline{U})$ such that $p_j \to \infty$ as $j \to \infty$ and $u_{p_j} \to u$ uniformly as $j \to \infty$. (in fact such cluster point u is unique). This answers question (A).

With regard to question (B) we first develop the p-Laplacian $\triangle_p u$ of a function $u \in C^2(U)$:

$$\triangle_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\left(\frac{\langle \nabla u \cdot D^2 u, \nabla u\rangle}{|\nabla u|^2} + \frac{\triangle_2 u}{p-2}\right)\,.$$

Assume now that u_p and the limiting u are all of class C^2. Heuristically $\triangle_p u_p(x) = 0$ implies that (as $p \to \infty$) $\langle \nabla u(x)\cdot D^2 u(x), \nabla u(x)\rangle = 0$ (there is a dichotomy according to whether $\nabla u(x) = 0$ or not). We define

$$\triangle_\infty u = \langle \nabla u \cdot D^2 u, \nabla u\rangle$$

for $u \in C^2(U)$. The question becomes whether the limiting u is a solution of $\triangle_\infty u = 0$ in some weak sense. Notice that this equation is not in divergence form (or at least not readily so — in fact this question is the motivation of some recent work [6, section 4], [4]). Therefore it seems hopeless to seek for a weak formulation in the distributional sense since integration by parts is not available. Notwithstanding the concept of *viscosity solution* (see e.g. [3]) proves useful in this situation. One first shows that the u_p are solutions of $\triangle_p u_p = 0$ in the viscosity sense (this is based upon the comparison principle for weakly p-harmonic functions) and then one uses a standard technique of viscosity solutions to show that the limiting u solves $\triangle_\infty u = 0$ in the viscosity sense. This answers question (B).

The obvious candidate for a variational problem that u would solve is the following:

$$(\mathscr{P}_\infty)\begin{cases}\text{minimize } \|\nabla u\|_\infty\\ \text{among } u \in W^{1,\infty}(U) \text{ with trace } u = g\,.\end{cases}$$

It turns out this problem is not well-posed as one can see from the following easy example when $m = 1$, $U = (0,1)\cup(1,2)$ and $g(0) = 0$, $g(1) = 1$, $g(2) = 1$. Every Lipschitz function u with $\operatorname{Lip} u = 1$ and verifying the required boundary values will solve $(\mathscr{P}_\infty)$, and there are plenty. The difficulty comes from the fact that $(\mathscr{P}_\infty)$ "doesn't localize". Specifically let $V \subset\subset U$ and write $(\mathscr{P}_{p,V})$ and $(\mathscr{P}_{\infty,V})$ the corresponding variational problems (i.e. with U replaced by V). If u_p solves $(\mathscr{P}_{p,U})$ then $u_p \restriction V$ solves $(\mathscr{P}_{p,V})$ (with respect to its own boundary values on ∂V) for every

$V \subset\subset U$ when $p \neq \infty$. For $p = \infty$ this is not anymore the case. The reason for this being of course that the energy $V \mapsto \|\nabla u\|_{p,V}^p$ is a measure for $p \neq \infty$ whereas for $p = \infty$ additivity is lost. The proper variational problem to be considered here is the following. We first define

$$\operatorname{Lip}(u, x) = \inf\{\operatorname{Lip}(u \restriction B(x, r)) : 0 < r < \operatorname{dist}(x, \partial U)\}.$$

1.1. **Definition.** We say that a function $u \in C(U)$ is *strongly absolutely minimizing in* U (and we write $u \in SAM(U)$) whenever it verifies the following condition. For every $V \subset\subset U$ one has

$$\sup\{\operatorname{Lip}(u, x) : x \in V\} \leqslant \sup\{\operatorname{Lip}(u', x) : x \in V\}$$

for every $u' \in C(\overline{V})$ with $u = u'$ on ∂V.

Then the limiting u is strongly absolutely minimizing in U (even though this is not immediate at all). This answers question (C)

2. Various equivalent formulations

We now turn to defining several concepts which turn out to be equivalent to that of being the limit of a sequence of p_j-harmonic functions, $p_j \to \infty$. We start with a variant of the property of being strongly absolutely minimizing.

2.1. **Definition.** We say that a function $u \in C(U)$ is *absolutely minimizing* (and we write $u \in AM(U)$) if the following holds. For every $V \subset\subset U$ one has

$$\|\nabla u\|_{\infty,V} \leqslant \|\nabla u'\|_{\infty,V} \tag{2}$$

whenever $u' \in C(\overline{V})$ and $u = u'$ on ∂V.

That $SAM(U) = AM(U)$ is a consequence of the next Remark.

2.2. *Remark. Let* $V \subset \mathbb{R}^m$ *be open and* $u \in C(\overline{V})$*. One has*

$$\|\nabla u\|_{\infty,V} = \sup\{\operatorname{Lip}(u, x) : x \in V\} \tag{3}$$

Let Γ denote the right member above and assume that $\Gamma < \infty$. It then follows from Stepanoff's Theorem ([7, 3.1.9]) that u is differentiable $\mathscr{L}^m$ almost everywhere in V. Accordingly we can associate with each $\gamma < \|\nabla u\|_{\infty,V}$ some $x \in V$ such that u is differentiable at x and $\gamma < |\nabla u(x)|$. Since also $|\nabla u(x)| \leqslant \operatorname{Lip}(u, x)$ we infer that $\gamma \leqslant \Gamma$ and the inequality $\|\nabla u\|_{\infty,V} \leqslant \Gamma$ follows from the arbitrariness of γ. In order to prove the reverse inequality we let $x \in V$ and choose $r > 0$ so that $B(x, r) \subset V$. Then

$$\operatorname{Lip}(u, x) \leqslant \operatorname{Lip}(u \restriction B(x, r)) = \|\nabla u\|_{\infty,B(x,r)} \leqslant \|\nabla u\|_{\infty,V}$$

where the equality holds because $B(x, r)$ is convex. The conclusion follows from the arbitrariness of x.

2.3. **Definition.** We say that $u \in C(U)$ *enjoys comparison with cones from above* (and we write $u \in CCA(U)$) whenever the following condition holds. For every $V \subset\subset U$, every $z \in \mathbb{R}^m \setminus V$ and every $a \in \mathbb{R}$ one has

$$\sup\{u(x) - a|x - z| : x \in V\} = \sup\{u(x) - a|x - z| : x \in \partial V\}.$$

We say u *enjoys comparison with cones from below* (and we write $u \in CCB(U)$) if $-u \in CCA(U)$. Finally if $u \in CC(U) := CCA(U) \cap CCB(U)$ then we say that u *enjoys comparison with cones.*

We now sketch a proof that $AM(U) \subset CC(U)$. Since $u \in AM(U)$ implies $-u \in AM(U)$ it suffices to show that $AM(U) \subset CCA(U)$. Let $V \subset\subset U$, $z \in \mathbb{R}^m \setminus V$ and $a \in \mathbb{R}$. Put

$$W = V \cap \{x : u(x) - a|x - z| > \sup\{u(y) - a|y - z| : y \in \partial V\}\} .$$

We claim that $W = \emptyset$ for otherwise the "cone function"

$$u'(x) = a|x - z| + \sup\{u(y) - a|y - z| : y \in \partial V\}$$

would have the following property: $u = u'$ on ∂W. It then remains to check that this implies that $u = u'$ in W, a contradiction (see Theorem 5.4(B)). One way to understand this is to check that $u' \in AM(\mathbb{R}^m \setminus \{z\})$ (hence $u' \in AM(W)$) and that there is uniqueness for the Dirichlet problem for absolutely minimizing functions.

We now define the *upper slope* of $u \in C(U)$ at $x \in U$ and $0 < r < \operatorname{dist}(x, \partial U)$ as follows

$$\operatorname{slope}^*(u, x, r) = \sup\left\{\frac{u(y) - u(x)}{r} : y \in \partial B(x, r)\right\} .$$

The *lower slope* is defined similarly

$$\operatorname{slope}_*(u, x, r) = \inf\left\{\frac{u(y) - u(x)}{r} : y \in \partial B(x, r)\right\} .$$

2.4. **Definition.** We say that $u \in C(U)$ *has the monotonicity of slope property* (and we write $u \in MSP(U)$) whenever the following holds. For every $z \in U$ the function

$$(0, \operatorname{dist}(z, \partial U)) \to \mathbb{R} : r \mapsto \operatorname{slope}^*(u, z, r)$$

is nondecreasing, and the function

$$(0, \operatorname{dist}(z, \partial U)) \to \mathbb{R} : r \mapsto \operatorname{slope}_*(u, z, r)$$

is nonincreasing.

In order to establish that $CC(U) \subset MSP(U)$ it suffices to notice that, for fixed $z \in U$, the "cone function"

$$u'(x) = u(x) + |x - z| \sup\left\{\frac{u(y) - u(z)}{r} : y \in \partial B(z, r)\right\}$$

bounds u from above on the boundary of $B(z, r) \setminus \{z\}$.

A nice and simple argument in [1, section 4.4] shows that if $u \in MSP(U)$ and $u \in C^2(U)$ then $\triangle_\infty u = 0$. Since in general functions belonging to any of the classes defined above are not C^2 (see next section) we need a weaker concept of a solution of the equation $\triangle_\infty u = 0$.

2.5. **Definition.** We say a function $u \in C(U)$ is a *viscosity solution* of the equation $\triangle_\infty u = 0$ if the following condition holds. If $\varphi \in C^2(U)$, $x \in U$ and $u - \varphi$ has a local maximum (resp. local minimum) at x then $\triangle_\infty \varphi(x) \geqslant 0$ (resp. $\triangle_\infty(x) \leqslant 0$).

We are now ready to state the somewhat surprising list of equivalence. It is remarkable for instance that the "very weak" monotonicity of slope property is equivalent to being absolutely minimizing. A complete proof of this result can be found in the survey [1]. In the last section of this note we prove "by hand" that the function $u(x) = |x|$, $x \in \mathbb{R}^m$, is strongly absolutely minimizing in $U = \mathbb{R}^m \setminus \{0\}$ — since clearly $\triangle_\infty u = 0$ in U in the classical sense, this can be considered a particular case of (5) $\Rightarrow$ (1).

2.6. **Theorem.** *Let $U \subset \mathbb{R}^m$ be open and $u \in C(U)$. The following conditions are equivalent.*

1. *u is strongly absolutely minimizing;*
2. *u is absolutely minimizing;*
3. *u enjoys comparison with cones;*
4. *u has the monotonicity of slope property;*
5. *u is a viscosity solution of the equation $\triangle_\infty u = 0$.*

From now on we will say that $u \in C(U)$ is ∞-*harmonic* whenever it verifies the equivalent conditions stated above.

3. Examples

One readily checks that for $u \in C^2(U)$ one has $\triangle_\infty u = \frac{1}{2}\langle \nabla u, \nabla |\nabla u|^2 \rangle$. This implies for instance that $u(x) = |x|$ is solution of the equation $\triangle_\infty u = 0$ in $\mathbb{R}^m \setminus \{0\}$. More generally every $u \in C^1(U)$ which solves the eikonal equation $|\nabla u| = 1$ is a viscosity solution of $\triangle_\infty u = 0$ in U. In particular if $C \subset \mathbb{R}^m$ is a closed convex set then $u(x) = \operatorname{dist}(x, C)$ is absolutely minimizing in $U = \mathbb{R}^m \setminus C$. Letting $m = 2$ and C be a line segment we see that absolutely minimizing functions need not have better regularity than $C^{1,1}$. In fact if $u \in C^1(U)$ solves the eikonal equation then $u \in C^{1,1}(U)$.

The function $u(x_1, x_2) = x_1^{4/3} - x_2^{4/3}$ is a viscosity solution of $\triangle_\infty u = 0$ in $\mathbb{R}^2$, see [1, Example 4.12]. In particular one cannot hope for better regularity than $C^{1,1/3}$ when u is absolutely minimizing. This example also shows that there is no maximum principle for $|\nabla u|$ whenever u is ∞-harmonic. One can perhaps trace this back to the fact that no equation (or partial differential inequality) holds for ∇u; derivatives of viscosity solutions do not verify the corresponding differentiated equation in general. However the Harnack inequality holds for $|\nabla u|$ when u is a C^4 ∞-harmonic function (see [5] and also [9]).

4. Regularity

The following Harnack and Caccioppoli type inequalities are consequences of the monotonicity of slope formula. The proofs are taken from [1].

4.1. **Theorem.** *Assume that $U \subset \mathbb{R}^m$ is open, $u \in MSP(U)$, $a \in U$, $r > 0$, $B(a,r) \subset U$ and $0 < \lambda < 1/3$. Then the following hold.*

(A) *(Harnack inequality) If moreover $u \leqslant 0$ on $B(a,r)$ then*

$$\max_{B(a,\lambda r)} u \leqslant \left(\frac{1 - 3\lambda}{1 - \lambda} \right) \min_{B(a,\lambda r)} u \,.$$

(B) *(Caccioppoli inequality)*

$$\operatorname{Lip}(u, B(a, \lambda r)) \leqslant \left(\frac{1}{1 - 3\lambda} \right) \frac{1}{r} \left(\max_{B(a,r)} u - \max_{B(a,\lambda r)} u \right) .$$

Proof. Let $x, y \in B(a, \lambda r)$. We abbreviate $d(x) = \operatorname{dist}(x, \partial B(a,r))$ and we notice that

$$d(x) \geqslant (1-\lambda) r\,, \tag{4}$$

and

$$|y - x| \leqslant 2\lambda r\,. \tag{5}$$

As $\lambda < 1/3$ we see that $2\lambda r < (1-\lambda)r$. Consider $\rho \in \mathbb{R}$ such that

$$|y - x| < \rho < d(x)\,.$$

Since $y \in B(x,\rho) \subset U$ we infer from the monotonicity formula that

$$\begin{aligned} \frac{u(y)-u(x)}{|y-x|} &\leqslant \operatorname{slope}^*(u, x, |y-x|) \\ &\leqslant \operatorname{slope}^*(u, x, \rho) \\ &= \frac{1}{\rho} \max\{u(\xi) - u(x) : \xi \in \partial B(x,\rho)\} \\ &\leqslant -\frac{u(x)}{\rho}\,, \end{aligned}$$

where the last inequality follows from the assumption that $u \leqslant 0$ on $B(a,r)$ and the inclusion $\partial B(x,\rho) \subset B(a,r)$. Letting $\rho \uparrow d(x)$ we obtain

$$u(y) - u(x) \leqslant -u(x) \frac{|y-x|}{d(x)} \tag{6}$$

and, equivalently,

$$u(y) \leqslant u(x) \left(1 - \frac{|y-x|}{d(x)} \right)\,. \tag{7}$$

We infer from (4) and (5) that

$$\frac{|y-x|}{d(x)} \leqslant \frac{2\lambda}{1-\lambda}$$

and in turn from (6), since $u(x) \leqslant 0$, that

$$u(y) - u(x) \leqslant -u(x) \frac{2\lambda}{1-\lambda}\,,$$

i.e.,

$$u(y) \leqslant u(x) \left(\frac{1-3\lambda}{1-\lambda} \right)\,.$$

Conclusion (A) readily follows from the arbitrariness of $x, y \in B(a, \lambda r)$.

In order to prove (B) we start by assuming that $u \leqslant 0$ on $B(a,r)$. Given $x, y \in B(a, \lambda r)$ we infer from (6) that

$$\frac{u(y)-u(x)}{|y-x|} \leqslant -u(x)\frac{1}{d(x)}$$

which, according to (4) and the inequality $u(x) \leqslant 0$ is bounded by

$$\begin{aligned} &\leqslant -u(x)\frac{1}{(1-\lambda)r} \\ &\leqslant -\min\{u(\xi) : \xi \in B(a,\lambda r)\}\frac{1}{(1-\lambda r)} \end{aligned}$$

and, by (C),

$$\leqslant -\left(\frac{1-\lambda}{1-3\lambda}\right)\max\{u(\xi) : \xi \in B(a,\lambda r)\}\frac{1}{(1-\lambda)r}\,.$$

As $x, y \in B(a, \lambda r)$ are arbitrary we obtain

$$\operatorname{Lip}(u, B(a\lambda r)) \leqslant \left(\frac{1}{1-3\lambda}\right)\frac{1}{r}\left(-\max_{B(a,\lambda r)} u\right)\,. \tag{8}$$

In case u is not necessarily nonpositive on $B(a,r)$ we apply (8) to $\hat{u} = u - \max_{B(a,r)} u$. This finishes the proof of (B). □

We infer from conclusion (B) that if u has the monotonicity of slope property then u is Lipschitzian. Whether u is C^1 or not remains an open question at the time of this writing except in case $m = 2$. The following is from [8].

4.2. **Theorem** (O. Savin)**.** *Let $U \subset \mathbb{R}^2$ be open and let $u : U \to \mathbb{R}$ be ∞-harmonic. It then follows that $u \in C^1(U)$.*

Regarding the general case there is some weaker information available. Given an set $E \subset \mathbb{R}^m$, $x \in U$ and $r > 0$ we abbreviate

$$E_{x,r} = \frac{E-x}{r}\,.$$

4.3. **Definition.** Let $U \subset \mathbb{R}^m$ be open, $x \in U$ and $r > 0$. We define the difference quotient operator $D_{x,r} : C(U) \to C(U_{x,r})$ by the formula

$$D_{x,r}(u) : U_{x,r} \to \mathbb{R} : h \mapsto \frac{u(x+rh)-u(x)}{r}\,,$$

$u \in C(U)$.

4.4. **Definition.** Let $U \subset \mathbb{R}^m$ be open, $x \in U$ and $u \in C(U)$. A function $v : \mathbb{R}^m \to \mathbb{R}$ is called a *derived function of u at x* if there are positive real numbers $r_1, r_2, \ldots$ such that $r_j \to 0$ as $j \to \infty$ and

$$\sup\left\{\left|D_{x,r_j}(u)(h) - v(h)\right| : h \in K\right\} \to 0 \text{ as } j \to \infty$$

for every compact $K \subset \mathbb{R}^m$. The collection of derived functions of u at x will be denoted by $\operatorname{Der}(u,x)$.

A diagonal argument based on Ascoli's Theorem shows that $\mathrm{Der}(u,x) \neq \emptyset$ whenever $\mathrm{Lip}(u,x) < \infty$. In fact u is differentiable at x if and only if $\mathrm{Der}(u,x)$ is a singleton and its element is a linear function (the derivative of u at x). It may occur that $\mathrm{Der}(u,x)$ is singletonic but u is not differentiable at x ($u(x) = |x|$, $x = 0$). It can also happen that every $v \in \mathrm{Der}(u,x)$ is linear but u is not differentiable at x as the following example shows.

4.5. **Example** (D. Preiss)**.** Let $m = 1$ and $u(x) = x \sin(\log|\log|x||)$, $x \in \mathbb{R} \setminus \{0\}$, $u(0) = 0$. It is an easy matter to check that u is Lipschitzian when restricted to some neighborhood of 0 and that $\mathrm{Lip}(u,0) = 1$. Furthermore, given $h > 0$ one sees that

$$D_{0,r_j}(u)(h) = h \sin(\log|\log r_j + \log h|)$$

whenever $r_j > 0$, $j = 1, 2, \ldots$. Given $-1 \leqslant t \leqslant 1$ and choosing properly the sequence $r_j \to 0$ as $j \to \infty$ one can readily achieve

$$\sup\left\{\left|D_{0,r_j}(u)(h) - th\right| : h \in K\right\} \to 0 \text{ as } j \to \infty$$

for every compact $K \subset \mathbb{R}$. In other words $\mathrm{Der}(u,0)$ consists in the linear functions $v_t(h) = th$, $h \in \mathbb{R}$, corresponding to each $-1 \leqslant t \leqslant 1$.

For a proof of the following consult [2].

4.6. **Theorem** (M.G. Crandall and L.C. Evans)**.** *Let $U \subset \mathbb{R}^m$ be open, $u : U \to \mathbb{R}$ and $x \in U$. If u has the monotonicity of slope property then every $v \in \mathrm{Der}(u,x)$ is linear and* $\mathrm{Lip}\, v = \mathrm{Lip}(u,x)$.

5. A calibration for cone functions

The proof presented in this last section is perhaps original. We will show that the function $u(x) = |x|$, $x \in \mathbb{R}^m$, is strongly absolutely minimizing in $\mathbb{R}^m \setminus \{0\}$. In order to do this we need a series of (trivial) preliminary remarks.

5.1. *Remark. Let $V \subset \mathbb{R}^m$ be open, $u' \in C(V)$ and assume that*

$$\Gamma = \sup\{\mathrm{Lip}(u',x) : x \in V\} < \infty\,.$$

Then for every $a, b \in V$ if $[\![a,b]\!] \subset V$ then

$$|u'(b) - u'(a)| \leqslant \Gamma|b-a|\,.$$

Given $\varepsilon > 0$ we associate with each $x \in [\![a,b]\!]$ some $r(x) > 0$ such that $|u'(x+h) - u'(x)| \leqslant (\Gamma+\varepsilon)|h|$ whenever $h \in B(0,r(x))$. The compactness of $[\![a,b]\!]$ implies the existence of $x_1, \ldots, x_\kappa \in [\![a,b]\!]$ such that $[\![a,b]\!] \subset \cup\{B(x_k, r(x_k)) : k = 1, \ldots, \kappa\}$. One readily infers that $|u'(b) - u'(a)| \leqslant (\Gamma+\varepsilon)|b-a|$. The conclusion follows from the arbitrariness of $\varepsilon > 0$.

5.2. *Remark. Let $U \subset \mathbb{R}^m$ be open, $u \in \mathrm{Lip}(U)$, $u' \in C(U)$, $V \subset\subset U$ and assume that $u = u'$ on ∂V and that*

$$\Gamma = \sup\{\mathrm{Lip}(u',x) : x \in V\} < \infty\,.$$

Then the function u'' defined in U by

$$u''(x) = \begin{cases} u'(x) & \textit{if } x \in V \\ u(x) & \textit{if } x \notin V\,, \end{cases}$$

verifies $\operatorname{Lip}(u'', x) \leqslant \max\{\Gamma, \operatorname{Lip} u\}$ *for every* $x \in U$. It is of course sufficient to check it for $x \in \partial V$. Let $y \in U$ be such that $[\![x, y]\!] \subset U$. If $y \notin V$ then clearly $|u''(y) - u''(x)| = |u(y) - u(x)| \leqslant (\operatorname{Lip} u)|y - x|$. If $y \in V$ then we denote by z the point from $\partial V \cap [\![y, x]\!]$ closest to y. It then follows from Remark 5.1 that

$$\begin{aligned}|u''(y) - u''(x)| &\leqslant |u''(y) - u''(z)| + |u''(z) - u''(x)| \\ &= |u'(y) - u'(z)| + |u(z) - u(x)| \\ &\leqslant \Gamma|y - z| + (\operatorname{Lip} u)|z - x| \\ &\leqslant \max\{\Gamma, \operatorname{Lip} u\}|y - x| \,.\end{aligned}$$

The next remark is about a kind of Gauss-Green formula. Its interest stems from the fact that open sets V with $\mathscr{L}^m(\partial V) > 0$ are allowed.

5.3. *Remark. Assume that*

(A) $U \subset \mathbb{R}^m$ *is open,* $v \in C^1(U, \mathbb{R}^m)$*,* $\operatorname{div} v = 0$*;*

(B) $f \in C(U)$ *and* $\sup\{\operatorname{Lip}(f, x) : x \in U\} < \infty$*;*

(C) $V \subset\subset U$ *and* $f = 0$ *on* ∂V.

Then $\int_V \langle \nabla f, v\rangle d\mathscr{L}^m = 0$. We start by defining a function $g : U \to \mathbb{R}$ by the formula

$$g(x) = \begin{cases} f(x) & \text{if } x \in V \\ 0 & \text{if } x \notin V \,. \end{cases}$$

It follows from Remark 5.2 that $\operatorname{Lip}(g, x) \leqslant \Gamma$ for every $x \in U$, whence also g is differentiable $\mathscr{L}^m$ almost everywhere in U according to Rademacher's Theorem. Next we choose a decreasing sequence of open sets $V_j \subset\subset U$, $j = 1, 2, \ldots$, such that ∂V_j is an $m-1$ dimensional submanifold of class 1 and $\overline{V} = \cap_{j=1}^\infty V_j$. Applying the Gauss-Green Theorem we obtain

$$\begin{aligned}\int_{V_j} \langle \nabla g, v\rangle d\mathscr{L}^m &= \int_{V_j} \operatorname{div}(gv) d\mathscr{L}^m - \int_{V_j} g \operatorname{div} v d\mathscr{L}^m \\ &= \int_{\partial V_j} g\langle v, n_{V_j}\rangle d\mathscr{H}^{m-1} \\ &= 0 \,,\end{aligned}$$

$j = 1, 2, \ldots$. According to the bounded convergence Theorem we infer that

$$\begin{aligned}0 &= \int_{\overline{V}} \langle \nabla g, v\rangle d\mathscr{L}^m \\ &= \int_V \langle \nabla f, v\rangle d\mathscr{L}^m + \int_{\partial V} \langle \nabla g, v\rangle d\mathscr{L}^m \,.\end{aligned}$$

Whence it remains only to show that $\int_{\partial V} \langle \nabla g, v\rangle d\mathscr{L}^m = 0$. In case $\mathscr{L}^m(\partial V) = 0$ this is obvious. Otherwise we claim that if $x \in \partial V$ is an $\mathscr{L}^m$ density point of ∂V and g is differentiable at x then $\nabla g(x) = 0$. This is because $g = 0$ on ∂V, whence the approximate gradient $\operatorname{ap} \nabla g(x) = 0$, and since g is differentiable at x one infers $\nabla g(x) = \operatorname{ap} \nabla g(x) = 0$. Since $\mathscr{L}^m$ almost all points $x \in \partial V$ enjoy both properties according to the Lebesgue density Theorem, the conclusion follows at once.

The following "calibration argument" may be original.

5.4. **Theorem.** *Let* $z \in \mathbb{R}^m$*,* $a, b \in \mathbb{R}$ *and define* $u(x) = b + a|x - z|$*,* $x \in \mathbb{R}^m$.

(A) *The function u is strongly absolutely minimizing in $\mathbb{R}^m \setminus \{z\}$.*

(B) *If $V \subset\subset \mathbb{R}^m \setminus \{z\}$, $u' \in C(\overline{V})$ is strongly absolutely minimizing in V and $u' = u$ on ∂V then $u' = u$ in V.*

Proof. There is of course no restriction to assume that $z = 0$ and $a = b = 0$, i.e. $u(x) = |x|$, $x \in \mathbb{R}^m$. We define $v(x) = x|x|^{-m}$, $x \in \mathbb{R}^m \setminus \{0\}$. We notice that, in $\mathbb{R}^m \setminus \{0\}$, v is C^1 smooth, div $v = 0$ and $\langle \nabla u, v \rangle = |v|$. Let $V \subset\subset \mathbb{R}^m \setminus \{0\}$ be nonempty and $u' \in C(\mathbb{R}^m \setminus \{0\})$ be such that $u' = u$ on ∂V. Since $\text{Lip}(u, x) = 1$ for every $x \in \mathbb{R}^m \setminus \{0\}$ we need only to show that

$$1 \leqslant \sup\{\text{Lip}(u', x) : x \in V\}\,. \tag{9}$$

Denote by Γ the right member of the above inequality and, avoiding a triviality, assume that $\Gamma < \infty$ (so that in particular u' is locally Lipschitzian in V, whence also differentiable $\mathscr{L}^m$ almost everywhere in V according to Rademacher's Theorem and, in view of Remark 2.2, $\|\nabla u'\|_{\infty,V} < \infty$). We define $u'' : U \to \mathbb{R}$ as follows.

$$u''(x) = \begin{cases} u'(x) & \text{if } x \in V \\ u(x) & \text{if } x \notin V\,. \end{cases}$$

According to Remark 5.2 we see that Remark 5.3 applies to $f = u'' - u$ whence

$$\int_V \langle \nabla u - \nabla u', v \rangle d\mathscr{L}^m = 0\,.$$

In turn,

$$\begin{aligned} \int_V |v| d\mathscr{L}^m &= \int_V \langle \nabla u, v \rangle d\mathscr{L}^m \\ &= \int_V \langle \nabla u', v \rangle d\mathscr{L}^m \\ &\leqslant \|\nabla u'\|_{\infty,V} \int_V |v| d\mathscr{L}^m\,. \end{aligned} \tag{10}$$

As $\int_V |v| d\mathscr{L}^m > 0$, we infer from Remark 2.2 that (9) holds. This proves (A).

Assume V and u' are as in (B). The minimizing property of u' together with Remark 2.2 implies that $\|\nabla u'\|_{\infty,U} \leqslant 1$. Plugging this inequality in (10) yields

$$\int_V \langle \nabla u', v \rangle d\mathscr{L}^m = \int_V \langle \nabla u, v \rangle d\mathscr{L}^m = \int_V |v| d\mathscr{L}^m\,.$$

It immediately follows that $\langle \nabla u', v \rangle = |v|$, $\mathscr{L}^m$ almost everywhere in V. Therefore $\nabla u' = v|v|^{-1} = \nabla u$, $\mathscr{L}^m$ almost everywhere in V. Since Lip $u' < \infty$ we infer from the Fundamental Theorem of Calculus that $u'(x) - u(0) = u(x)$, $x \in \mathbb{R}^m$. As $u' = u$ on ∂V we conclude that $u' = u$. □

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Université catholique de Louvain
Département de mathématiques
Chemin du cyclotron, 2
B-1348 Louvain-la-Neuve
Belgique
thierry.depauw@uclouvain.be