

*Journées*

# **ÉQUATIONS AUX DÉRIVÉES PARTIELLES**

Évian-les-Bains, 5 juin–9 juin 2006

Elise Fouassier

**High frequency limit of Helmholtz equations: the case of a discontinuous index**

*J. É. D. P.* (2006), Exposé n° IV, 19 p.

<[http://jedp.cedram.org/item?id=JEDP\\_2006\\_\\_\\_\\_A4\\_0](http://jedp.cedram.org/item?id=JEDP_2006____A4_0)>

cedram

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

# High frequency limit of Helmholtz equations: the case of a discontinuous index

Elise Fouassier

## Abstract

In this text, we compute the high frequency limit of the Helmholtz equation with source term, in the case of a refraction index that is discontinuous along a sharp interface between two unbounded media. The asymptotic propagation of energy is studied using Wigner measures.

## 1. Introduction

In this talk, we are interested in the analysis of the high frequency limit of the following Helmholtz equation

$$-i\alpha_\varepsilon \varepsilon u^\varepsilon + \varepsilon^2 \Delta u^\varepsilon + n^2(x)u^\varepsilon = -f^\varepsilon(x) = \frac{-1}{\varepsilon^{\frac{d-1}{2}}} f\left(\frac{x}{\varepsilon}\right), \quad (1.1)$$

where the variable  $x$  belongs to  $\mathbb{R}^d$  for some  $d \geq 3$ .

We assume that the refraction index is given by

$$n^2(x) = \begin{cases} n_+^2(x) & \text{if } x_d \geq 1 \\ n_-^2(x) & \text{if } x_d < 1. \end{cases} \quad (1.2)$$

We also assume that there exists  $n_0 > 0$  such that  $n^2(x) \geq n_0^2$  for all  $x \in \mathbb{R}^d$ , which means that equation (1.1) is uniformly of “Helmholtz type”. Problem (1.1), (1.2) corresponds to a transmission problem across the flat interface  $\Gamma = \{x_d = 1\}$ . We assume that the jump at the interface  $\Gamma$  satisfies  $[n^2](x) = n_-^2(x) - n_+^2(x) > 0$  for all  $x \in \Gamma$ . This is the only interesting situation, as we explain below.

Equation (1.1) modelizes the propagation of a source wave in a medium with refraction index  $n^2(x)$ . There, the small positive parameter  $\varepsilon$  is related to the frequency  $\omega = \frac{1}{2\pi\varepsilon}$  of  $u^\varepsilon$ . In this text, we study the high frequency limit, *i.e.* the asymptotics  $\varepsilon \rightarrow 0$ .

The source term  $f^\varepsilon$  models a source signal concentrating close to the origin at the scale  $\varepsilon$ , the concentration profile  $f$  being a given function. Since  $\varepsilon$  is also the scale of the oscillations dictated by the Helmholtz operator  $\Delta + \frac{n^2(x)}{\varepsilon^2}$ , resonant interactions can occur between these oscillations and the oscillations due to the source  $f^\varepsilon$ .

Moreover, the interface induces a refraction phenomenon of the energy. As we will see later on, the energy concentrates along the rays of geometric optics. We choose here the jump of the index at the interface to be positive, which is the interesting case since those rays are attracted by the regions of high index.

These are the two phenomena that we aim at studying quantitatively in the asymptotics  $\varepsilon \rightarrow 0$ . We refer to Section 2 for the precise assumptions we need on the source  $f$ , together with the refraction index  $n^2$ .

We assume that the regularizing parameter  $\alpha_\varepsilon$  is positive, with  $\alpha_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . The positivity of  $\alpha_\varepsilon$  ensures the existence and uniqueness of a solution  $u^\varepsilon$  to the Helmholtz equation (1.1) in  $L^2(\mathbb{R}^d)$  for any  $\varepsilon > 0$ . In some sense, the sign of the term  $-i\alpha_\varepsilon \varepsilon u^\varepsilon$  prescribes a radiation condition at infinity for  $u^\varepsilon$ . One of the key difficulties in our problem is to follow this condition in the limiting process  $\varepsilon \rightarrow 0$ . We will discuss that point later on.

We study the high frequency limit in terms of Wigner measures (or semiclassical measures). This is a means of describing the propagation of quadratic quantities, like the local energy density  $|u^\varepsilon(x)|^2$ , as  $\varepsilon \rightarrow 0$ . The Wigner measure  $\mu(x, \xi)$  is the energy carried by rays at the point  $x$  with frequency  $\xi$ . These measures were introduced by E. Wigner [20] and then mathematically developed by P. Gérard [9], P.-L. Lions, T. Paul [15]. They are relevant when a typical length  $\varepsilon$  is prescribed. They have already proven to be an efficient tool in such problems ([2], [3], [10], [17]).

Let us now state our main results. First, we introduce various measures:  $\mu$ ,  $\mu_\pm$  denote the Wigner measures associated respectively with  $u^\varepsilon$  and with the restrictions  $u_\pm^\varepsilon$  of  $u^\varepsilon$  to each medium. These three measures are defined on  $T^*\mathbb{R}^d$ . Last, we prove that there exist two measures  $\mu^{\partial\pm}$  defined on  $T^*\Gamma$  that are, in some sense, the traces of  $\mu_\pm$  at the interface:

$$\mu_\pm = \mathbf{1}_{\{x_d \geq 1\}} \mu_\pm + \delta(x_d - 1) \otimes \delta(\xi_d) \otimes \mu^{\partial\pm}.$$

Our first result, that is valid for a general index of refraction, describes how the sharp interface induces a refraction phenomenon. Depending on the propagation direction, the energy density is either totally reflected, or partially reflected and partially transmitted according to Snell-Descartes's law. More precisely, we prove the following theorem.

**Theorem 1.** (*General case*)

*Assume there is dispersion at infinity of the rays of geometrical optics (which corresponds to geometrical hypotheses on the refraction index  $n$ , see (H2)-(H6) page 5).*

*Assume also:*

(a) *non-interference (no density comes from both sides at a same point of the interface,*

(b) *no energy is trapped in the interface ( $\mu^{\partial\pm} = 0$ ).*

*Then, the Wigner measure associated with ( $u^\varepsilon$ ) is given by*

$$\mu(x, \xi) = \int_{-\infty}^0 (S_t^* Q)(x, \xi) dt, \tag{1.3}$$

where  $S_t^*$  is the Snell-Descartes semi-group associated with the refraction index  $n$  and  $Q$  is given by

$$Q(x, \xi) = \frac{1}{2^{d+1}\pi^{d-1}} \delta(x) \delta(|\xi|^2 - n^2(0)) \left( |\hat{f}(\xi)|^2 + \hat{f}(\xi) \bar{q}(\xi) \right), \quad (1.4)$$

where  $q$  is an  $L^2$  density on the sphere  $\{|\xi|^2 = n^2(0)\}$ .

In this theorem, formula (1.3) means that  $\mu$  is the integral along all the rays of geometric optics, and up to infinite time, of the energy source  $Q$ . Outside the interface, the rays, more precisely the bicaracteristics, are given by

$$\begin{cases} \dot{X}(t) &= \Xi(t), & X(0) &= x \\ \dot{\Xi}(t) &= \frac{1}{2} \nabla_x n^2(X(t)), & \Xi(0) &= \xi, \end{cases}$$

The Snell-Descartes semigroup also contains the relations of reflexion/transmission at the interface. In this text, we will not give a precise definition of Snell's semi-group. We refer, for instance, to [13], [17], or [7].

The energy source  $Q$  comes from the resonant interaction between the source  $f^\varepsilon$  and the solution  $u^\varepsilon$ . In particular,  $Q$  is concentrated at the origin via the Dirac mass  $\delta(x)$  and on the resonant frequencies  $|\xi|^2 = n^2(0)$ . The value of the auxiliary function  $q$  is related to the radiation condition at infinity satisfied by the weak limit  $w$  of the rescaled sequence of solutions  $w^\varepsilon(x) = \varepsilon^{\frac{d-1}{2}} u^\varepsilon(\varepsilon x)$ . This limit  $w$  clearly satisfies the following Helmholtz equation with constant index

$$\Delta w + n^2(0)w = -f. \quad (1.5)$$

Unfortunately, the equation (1.5) does not identify  $w$  in a unique way. In the general case, we cannot identify  $w$  as *the* outgoing solution to this equation, i.e. we cannot identify  $q$ . Two difficulties arise: the treatment of the interface and the variability of the indices  $n_\pm(x)$ .

Also, in the expression (1.3), the integral up to infinite time translates the radiation condition at infinity satisfied by the measure  $\mu$ . The follow-up of this condition in the limiting process is one the key difficulties in our study. Last, the assumption that no energy is trapped in the interface is linked both with the radiation condition at infinity satisfied by the trace of the Wigner measure  $\mu$  on the interface, and with the (absence of) energy carried by gliding rays at the interface.

In the particular case when the indices  $n_+$  and  $n_-$  are *constant*, a situation that we call the *homogeneous case* in the sequel, we prove that the previous assumptions are satisfied. The dispersion at infinity is obvious in that case since the rays are pieces of lines. The proofs of hypotheses (a)-(b) together with the identification of  $q$  in that case constitute our second main result.

**Theorem 2.** (*Homogeneous case*)

When the two indices  $n_+$  and  $n_-$  are constant, we have:

- (i) the non-interference hypothesis is satisfied,
- (ii)  $\mu^{\partial^\pm} = 0$ ,
- (iii)  $q = 0$  (i.e.  $w$  is the outgoing solution to the Helmholtz equation  $\Delta w + n_-^2 w = f$ ).

The combination of Theorem 1 and Theorem 2 gives a completely explicit expression for the Wigner measure  $\mu$  in the homogeneous case.

Our text is organized as follows. In Section 2, we first recall some definitions and we then give our main assumptions on the refraction index and the source profile  $f$ . In Section 3, we establish uniform bounds on the sequence  $(u^\varepsilon)$  and the sequence of Wigner transforms  $(W^\varepsilon(u^\varepsilon))$ , in order to ensure the existence of a Wigner measure associated with  $(u^\varepsilon)$ . In Section 4, we obtain the transport equations satisfied by Wigner measures outside the interface and up to the boundary. In Sections 5 and 6, we give the outline of the proofs of Theorem 1 and Theorem 2 respectively.

## 2. Notations and assumptions on the source and the refraction index

### 2.1. Semiclassical measures and Wigner transform

In this section, we recall some usual definitions and notations.

We use the following definition for the Fourier transform:

$$\hat{u}(\xi) = (\mathcal{F}_{x \rightarrow \xi} u(x))(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} u(x) dx.$$

The Weyl semiclassical operator  $a^w(x, \varepsilon D_x)$  (or  $Op_\varepsilon^w(a)$ ) is the continuous operator from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$  associated with the symbol  $a \in \mathcal{S}'(T^*\mathbb{R}^d)$  by the Weyl quantization rule

$$(a^w(x, \varepsilon D_x)u)(x) = \frac{1}{(2\pi\varepsilon)^d} \int_{\mathbb{R}_\xi^d} \int_{\mathbb{R}_y^d} e^{i(x-y) \cdot \xi / \varepsilon} a\left(\frac{x+y}{2}, \xi\right) f(y) d\xi dy.$$

For  $u, v \in \mathcal{S}(\mathbb{R}^d)$  and  $\varepsilon > 0$ , we define the Wigner transform

$$\begin{aligned} W^\varepsilon(u, v)(x, \xi) &= \mathcal{F}_{y \rightarrow \xi} \left( u\left(x + \frac{\varepsilon}{2}y\right) \bar{v}\left(x - \frac{\varepsilon}{2}y\right) \right), \\ W^\varepsilon(u) &= W^\varepsilon(u, u). \end{aligned}$$

We have the following formula: for  $u, v \in \mathcal{S}'(\mathbb{R}^d)$  and  $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ ,

$$\langle W^\varepsilon(u, v), a \rangle_{\mathcal{S}', \mathcal{S}} = \langle u, a^w(x, \varepsilon D_x)v \rangle_{\mathcal{S}', \mathcal{S}}, \quad (2.1)$$

where the duality brackets  $\langle \cdot, \cdot \rangle$  are semi-linear with respect to the first argument. This formula is also valid for  $u, v$  lying in other spaces as we will see in Section 3.

If  $(u^\varepsilon)$  is a bounded sequence in  $L^2(\mathbb{R}^d)$  (or some weighted  $L^2$  space as we will see in the sequel), it turns out that, up to extracting a subsequence, there exists a *Wigner measure* (or *semiclassical measure*)  $\mu$  associated with  $(u^\varepsilon)$ , *i.e.* a positive Radon measure on the phase space  $T^*\mathbb{R}^d = \mathbb{R}_x^d \times \mathbb{R}_\xi^d$  satisfying:

$$\forall a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d}), \lim_{\varepsilon \rightarrow 0} \langle u^\varepsilon, a^w(x, \varepsilon D_x)u^\varepsilon \rangle_{L^2} = \lim_{\varepsilon \rightarrow 0} \langle W^\varepsilon(u^\varepsilon), a \rangle = \int a(x, \xi) d\mu$$

### 2.2. Assumptions on the refraction index and the source

In the sequel, we denote  $x = (x', x_d)$  a point in  $\mathbb{R}^d$ .

In order to get uniform (in  $\varepsilon$ ) bounds on the sequence  $(u^\varepsilon)$ , we use the following

homogeneous Besov-like norms: for  $u, f \in L^2_{loc}$ ,

$$\begin{aligned}\|u\|_{\dot{B}^*} &= \sup_{R>0} \frac{1}{R} \int_{B(R)} |u|^2 dx, \\ \|f\|_{\dot{B}} &= \sum_{j \in \mathbb{Z}} \left( 2^{j+1} \int_{C(j)} |f|^2 dx \right)^{1/2},\end{aligned}$$

where  $B(R)$  denotes the ball of radius  $R$ , and  $C(j)$  the ring  $\{x \in \mathbb{R}^d / 2^j \leq |x| < 2^{j+1}\}$ .

These norms were introduced (in their inhomogeneous version) by Agmon and Hörmander [1], and they have been used recently by Perthame and Vega [18].

They satisfy the following duality relation

$$\left| \int u(x) f(x) dx \right| \leq \|u\|_{\dot{B}^*} \|f\|_{\dot{B}}.$$

We denote for  $x \in \mathbb{R}^d$ ,  $|x| = \sqrt{\sum_{j=1}^d x_j^2}$  and  $\langle x \rangle = (1 + |x|^2)^{1/2}$ .

We are now ready to state our assumptions. Our first (technical) assumption, borrowed from [2], concerns the regularizing parameter:

$$(H1) \quad \alpha_\varepsilon \geq \varepsilon^\gamma \text{ for some } \gamma > 0.$$

Next, we need assumptions on the refraction index that are mainly related to the dispersion at infinity of the rays of geometric optics. The following five are those made in [7] to obtain the estimates on  $u^\varepsilon$ .

$$(H2) \quad \text{There exists } c > 0 \text{ such that } [n^2](x) \geq c \text{ for all } x \in \Gamma.$$

$$(H3) \quad \text{There exists } n_0 > 0 \text{ such that } n \in L^\infty, n \geq n_0.$$

$$(H4)$$

$$2 \sum_{j \in \mathbb{Z}} \sup_{C(j)} \frac{(x \cdot \nabla n^2(x))_-}{n^2(x)} := \beta_1 < \infty.$$

$$(H5)$$

$$\sum_{j \in \mathbb{Z}} \sup_{C(j)} 2^{j+1} \frac{(\partial_d n^2(x))_+}{n^2(x)} := \beta_2 < \infty.$$

$$(H6) \quad \beta_1 + \beta_2 < 1.$$

Next, following Benamou *et al.* [2], in order to follow the radiation condition in the limiting process, we assume a stronger decay at infinity on the index:

$$(H7) \quad \langle x \rangle^{N_0} \nabla_x n_\pm^2 \in L^\infty \text{ for some } N_0 > 2.$$

$$(H8) \quad \nabla_x n_\pm^2 \text{ are locally Lipschitz on } \{x \in \mathbb{R}^d / x_d \geq 1\}.$$

As we will see in Section 3, to get uniform bounds on  $u^\varepsilon$ , we assume that the source term satisfies

$$(H9) \quad \|f\|_{\dot{B}}, \|\nabla f\|_{\dot{B}} < \infty.$$

In order to compute the limit of the energy source, we make, as in [2], the stronger assumption

$$(H10) \quad \langle x \rangle^N f \in L^2(\mathbb{R}^d) \text{ for some } N > \frac{1}{2} + \frac{3\gamma}{\gamma+1}, \text{ and } \langle x \rangle^{N_1} \partial_{x_d} f \in L^2(\mathbb{R}^d) \text{ for some } N_1 > 1/2.$$

Finally, we assume

$$(H11) \quad f \in H^{\frac{1}{2}+s}(\mathbb{R}^d) \text{ for some } s > 0,$$

so that the traces of  $f$  on hyperplanes are well-defined in  $L^2(\mathbb{R}_{x'}^{d-1})$ , and

$$(H12) \quad \lim_{\varepsilon \rightarrow 0} \|f\left(\cdot, \frac{1}{\varepsilon}\right)\|_{L^2(\mathbb{R}_{x'}^{d-1})} = 0.$$

The last assumption can be rewritten as  $\|f^\varepsilon(\cdot, 1)\|_{L^2(\mathbb{R}_{x'}^{d-1})} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , so it means that no source density remains at the interface as  $\varepsilon \rightarrow 0$ .

Rather than keeping in mind these (weak) assumptions on the source  $f$ , one can think about  $f$  as a smooth and compactly supported function. It is not a key difficulty in our analysis.

Let us comment the assumptions we make on the index  $n$ . The conditions (H2) and (H5) are specific to the case with interface: they mainly ensure that the energy goes from one side of the interface to the other. The hypothesis (H4) together with (H3), ensures the dispersion at infinity of the rays of geometrical optics outside the interface. (H4) is a kind of a *virial assumption*. We would like to point out that we do not require that the index  $n$  goes to a constant at infinity.

### 3. Bounds on $u^\varepsilon$ , $W^\varepsilon(u^\varepsilon)$ , $W^\varepsilon(f^\varepsilon, u^\varepsilon)$

The first step in our study is to prove uniform bounds on the sequence of Wigner transforms ( $W^\varepsilon(u^\varepsilon)$ ), which will ensure the existence of a Wigner measure associated with the sequence of solutions ( $u^\varepsilon$ ) (up to extracting a subsequence). As in [2], we deduce these bounds from uniform homogeneous bounds on ( $u^\varepsilon$ ).

#### 3.1. Bounds on the solution to the Helmholtz equation

In this part, we give uniform bounds on the sequences ( $u^\varepsilon$ ) and ( $\varepsilon \nabla u^\varepsilon$ ) and their traces on the interface. This will allow us to define the various Wigner measures that appear in our problem. The following theorem is proved in [7] (using the multiplier method introduced by Perthame and Vega [18]):

**Theorem 3.1.** ([7]) *Under the hypothesis (H2)-(H7), the solution to the Helmholtz equation (1.1) satisfies*

$$\|\varepsilon \nabla u^\varepsilon\|_{\dot{B}^*}^2 + \|u^\varepsilon\|_{\dot{B}^*}^2 + \int_{\Gamma} |n^2| |u^\varepsilon|^2 dx' + \int_{\Gamma} |n^2| |\varepsilon \nabla u^\varepsilon|^2 dx' \leq C(\|f\|_{\dot{B}}^2 + \|\nabla f\|_{\dot{B}}^2), \quad (3.1)$$

where  $C$  does not depend on  $\varepsilon$ .

We draw a consequence of these bounds that will be useful for our purpose. First, we study the limit of the rescaled sequence defined by

$$w^\varepsilon(x) = \varepsilon^{\frac{d-1}{2}} u^\varepsilon(\varepsilon x)$$

that appears while computing the limit of the source term in the transport equation satisfied by the Wigner measure  $\mu$ . One can notice that, thanks to the homogeneity of the norm  $\dot{B}^*$ , we have the following scaling invariance

$$\begin{aligned} \|w^\varepsilon\|_{\dot{B}^*} &= \|u^\varepsilon\|_{\dot{B}^*}, \\ \|\nabla w^\varepsilon\|_{\dot{B}^*} &= \|\varepsilon \nabla u^\varepsilon\|_{\dot{B}^*}. \end{aligned}$$

**Proposition 3.2.** *We can extract from  $(w^\varepsilon)$  a subsequence which converges weak-\* in  $\dot{B}^*$  and strongly in  $L_{loc}^2(\mathbb{R}^d)$  to a solution  $w$  of*

$$\Delta w + n(0)^2 w = -f. \quad (3.2)$$

As a consequence, there exists a density  $q \in L^2(|\xi|^2 = n^2(0))$  such that

$$\hat{w}(\xi) = \hat{w}_0(\xi) + i \frac{\pi}{2} q(\xi) \delta(|\xi|^2 - n^2(0)), \quad (3.3)$$

where  $w_0$  is the outgoing solution to (3.2), given by

$$\hat{w}_0(\xi) = (|\xi|^2 - n^2(0) + i0)^{-1} \hat{f}(\xi) = \left( p.v. \left( \frac{1}{|\xi|^2 - n^2(0)} \right) + i \frac{\pi}{2} \delta(|\xi|^2 - n^2(0)) \right) \hat{f}(\xi).$$

**Remark** *In general, we cannot identify  $w$  as the outgoing solution to (3.2). This problem already appears in the case of a smooth index of refraction (i.e. without interface). It has been solved in that case only recently by two different approaches by Castella [4], and Wang, Zhang [19]. Here, we only prove that  $w = w_0$  in the homogeneous case (Theorem 2).*

*Proof.* The first part of point (i) can be easily deduced from Theorem 3.1 using Rellich's theorem. The formula (3.3) can be found in [1].  $\square$

### 3.2. Bounds on the Wigner transforms $W^\varepsilon(u^\varepsilon)$ and $W^\varepsilon(f^\varepsilon, u^\varepsilon)$

From Theorem 3.1, we now deduce bounds on the sequences of Wigner transforms  $(W^\varepsilon(u^\varepsilon))$  and  $(W^\varepsilon(f^\varepsilon, u^\varepsilon))$ . We obviously need uniform bounds on  $(W^\varepsilon(u^\varepsilon))$ . The study of the sequence  $(W^\varepsilon(f^\varepsilon, u^\varepsilon))$  is also necessary to handle the source term in the high frequency limit. Indeed,  $W^\varepsilon(u^\varepsilon)$  satisfies the following equation, where  $W^\varepsilon$  stands for  $W^\varepsilon(u^\varepsilon)$ :

$$\alpha_\varepsilon W^\varepsilon + \xi \cdot \nabla_x W^\varepsilon + Z^\varepsilon \star_\xi W^\varepsilon = \frac{i}{2\varepsilon} \mathcal{I}m W^\varepsilon(f^\varepsilon, u^\varepsilon) =: Q^\varepsilon \quad (3.4)$$



with  $Z^\varepsilon(x, \xi) = \frac{i}{2\varepsilon} \mathcal{F}_{y \rightarrow \xi} \left( n^2 \left( x + \frac{\varepsilon}{2} y \right) - n^2 \left( x - \frac{\varepsilon}{2} y \right) \right)$ .

The following two results are proved in [2]. The second point is due to the particular choice of the scaling of the source term in the Helmholtz equation (1.1).

**Proposition 3.3.** *Assume that the sequence  $(u^\varepsilon)$  is bounded in  $\dot{B}^*$ , and  $f \in L^2_N(\mathbb{R}^d)$  with  $N > \frac{1}{2}$ .*

(i) *For any  $\lambda > 0$ , the sequence of Wigner transforms  $(W^\varepsilon(u^\varepsilon))$  is bounded in the Banach space  $X_\lambda^*$  below and, extracting a subsequence, converges weak-\* to a nonnegative, locally bounded measure  $\mu$ .*

*The Banach space  $X_\lambda^*$  is defined as the dual space of the set  $X_\lambda$  of functions  $\hat{\varphi}(x, \xi)$  such that  $\varphi(x, y) := \mathcal{F}_{\xi \rightarrow y}(\hat{\varphi}(x, \xi))$  satisfies*

$$\int_{\mathbb{R}^d} \sup_{x \in \mathbb{R}^d} (1 + |x| + |y|)^{1+\lambda} |\varphi(x, y)| dy < \infty. \quad (3.5)$$

(ii) *If we denote  $f^\varepsilon(x) = \frac{1}{\varepsilon^{\frac{d-1}{2}}} f\left(\frac{x}{\varepsilon}\right)$ , the sequence  $(W^\varepsilon(f^\varepsilon, u^\varepsilon))$  is bounded in  $\mathcal{S}'(T^*\mathbb{R}^d)$  and for all  $\psi \in \mathcal{S}(T^*\mathbb{R}^d)$ , we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \langle W^\varepsilon(f^\varepsilon, u^\varepsilon), \psi \rangle_{\mathcal{S}', \mathcal{S}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \bar{w}(\xi) \hat{f}(\xi) \psi(0, \xi) d\xi, \quad (3.6)$$

where  $w$  is defined in Proposition 3.2.

## 4. Transport equations on the Wigner measures

The next step in our study is the derivation of the transport equations satisfied by the various Wigner measures that appear in our problem. These equations are of two different types. The first one is the transport equation satisfied by the Wigner measure  $\mu$  in the *interior* of each medium, it is deduced from the case with a smooth index of refraction studied in [2]. The other two equations concern the Wigner measures associated with the restrictions of  $(u^\varepsilon)$  to each side of the interface *up to the boundary*: the presence of the interface induces some extra source terms in these equations that involve the Wigner measures associated with the traces of  $u^\varepsilon$  and  $\varepsilon \partial_d u^\varepsilon$  on the interface.

As we have already noted for the Helmholtz equation, the kinetic transport equation (of Liouville type) satisfied by the Wigner measure  $\mu$  must be complemented by a radiation condition at infinity to determine a unique solution.

### 4.1. Notations

Throughout our study, we shall use the following notations:

For a function  $\varphi$  defined on  $\mathbb{R}^d \times \mathbb{R}^k$  for some  $k \geq 0$ , we denote  $\varphi_{\Gamma}$  the trace of  $\varphi$  on  $\Gamma \times \mathbb{R}^k$ .

For all  $x \in \mathbb{R}^d$ ,  $\xi' \in \mathbb{R}^{d-1}$ , we denote  $\omega_\pm(x, \xi') = n_\pm^2(x) - |\xi'|^2$ .

We denote  $u_\pm^\varepsilon = \mathbf{1}_{\{x_d \gtrless 1\}} u^\varepsilon$  the restrictions of  $u^\varepsilon$  in each medium, defined on  $\mathbb{R}^d$ . Next, the sequences  $(u_\pm^\varepsilon)$  are bounded in  $\dot{B}^*(\mathbb{R}^d)$ . Thus, we can associate with them two Wigner measures  $\mu_-$  and  $\mu_+$  on  $T^*\mathbb{R}^d$  as defined in Theorem 3.3.

Since the sequences of traces  $(u_{\Gamma}^\varepsilon)$  and  $((\varepsilon \partial_{x_d} u^\varepsilon)_{\Gamma})$  are bounded in  $L^2(\Gamma)$ , we can

also associate with the sequence  $(u_{|\Gamma}^\varepsilon, (\varepsilon \partial_{x_d} u^\varepsilon)_{|\Gamma})$  a matrix valued Wigner measure

$$\begin{pmatrix} \nu & \bar{\nu}^J \\ \nu^J & \dot{\nu} \end{pmatrix}.$$

## 4.2. Behavior of the Wigner measure in the interior of each medium

In the interior of each medium, the refraction index is smooth. The behavior of the Wigner measure in that case is studied in Benamou *et al* [2]. We recall their result in Theorem 4.2. Actually, they proved the analogous of Theorem 4.2 with a weaker radiation condition at infinity. The condition we state here can be easily deduced from the one they proved together with the following localization property. It is well-known without source term, and it is still valid here thanks to the particular scaling of  $f^\varepsilon$ .

**Proposition 4.1.**

$$\text{supp}(\mathbf{1}_{T^*(\mathbb{R}^d \setminus \Gamma)} \mu) \subset \{(x, \xi) \in T^*\mathbb{R}^d / |\xi|^2 = n(x)^2\}.$$

**Theorem 4.2.** *Under the assumptions (H1)-(H10), the measure  $\mu$  satisfies the following transport equation as a distribution in  $\mathcal{D}'(T^*(\mathbb{R}^d \setminus \Gamma))$ :*

$$\xi \cdot \nabla_x \mu + \frac{1}{2} \nabla_x n^2(x) \cdot \nabla_\xi \mu = Q(x, \xi) \quad \text{in } T^*(\mathbb{R}^d \setminus \Gamma), \quad (4.1)$$

where  $Q(x, \xi) = \frac{1}{2^{d+1}\pi^{d-1}} \delta(x) \delta(|\xi|^2 - n^2(0)) \hat{f}(\xi) (\bar{f}(\xi) + \bar{q}(\xi))$ , and  $q \in L^2(\xi^2 = n^2(0))$  is given in Proposition 3.2.

Moreover,  $\mu$  satisfies the following outgoing condition at infinity:

$$\mu(x, \xi) \rightarrow 0 \quad \text{when } |x| \rightarrow \infty \quad \text{with } x \cdot \xi < 0 \quad \text{and } x_d \neq 1.$$

## 4.3. Study up to the boundary

The above result does not say anything about the Wigner measure  $\mu$  close to the boundary  $\Gamma$ , where refraction occurs. We write the transport equations up to the boundary using only test functions that are polynomials with respect to the  $\xi_d$ -variable, which corresponds to tangential test operators. These operators, that act as differential operators in the  $d$ -th variable, will be adapted to the treatment of the interface (by integration by parts).

Using these test operators, we then study the propagation of the Wigner measure up to the boundary. More precisely, since the behaviour at the boundary depends on the side from which the rays come, we study separately the measures associated with the restrictions of  $(u^\varepsilon)$  to each medium,  $\mu_\pm$ .

First, as for the Wigner measure  $\mu$ , we have the following localization property for the measures associated with the restrictions  $(u_\pm^\varepsilon)$ .

**Proposition 4.3.**

$$(i) \quad \text{supp}(\mu_\pm) \subset \{|\xi|^2 = n_\pm^2(x)\}.$$

(ii)  $\mu = \mu_+ + \mu_-$ .

*Proof.* Point (ii) is consequence of point (i) together with the orthogonality property on Wigner measures.  $\square$

The following property specifies what happens at the boundary.

**Proposition 4.4.** *For all  $\varphi_0, \varphi_1$  in  $\hat{\mathcal{C}}_c^\infty(\mathbb{R}_x^d \times \mathbb{R}_{\xi'}^{d-1})$ , we have*

$$\begin{aligned} - \left\langle \mu_\pm, \left( \xi \cdot \nabla_x + \frac{1}{2} \nabla_x n_\pm^2 \cdot \nabla_\xi \right) (\varphi_0 + \varphi_1 \xi_d) \right\rangle &= \langle Q_\pm, \varphi_0 + \varphi_1 \xi_d \rangle \\ &\pm \frac{1}{2} \langle \nu, (|\xi'|^2 - n_\pm^2) \varphi_{1|\Gamma} \rangle_{T^*\Gamma} \pm \langle \mathcal{R}e \nu^J, \varphi_{0|\Gamma} \rangle_{T^*\Gamma} \pm \frac{1}{2} \langle \dot{\nu}, \varphi_{1|\Gamma} \rangle_{T^*\Gamma} \end{aligned} \quad (4.2)$$

where  $Q_+ = 0$ , and  $Q_- = \frac{1}{2^{d+1}\pi^{d-1}} \delta(x) \delta(|\xi|^2 - n^2(0)) (|\hat{f}(\xi)|^2 + \hat{f}(\xi) \bar{q}(\xi))$ ,  $q$  being given in Proposition 3.2.

## 5. Outline of the proof of the refraction result for two homogeneous media

In this text, we only give details about the proof of Theorem 1 *in the case of two homogeneous media*. Indeed, the strategy of proof is exactly the same as in the general case but the geometry of the rays is easy to treat in that particular case (the rays are pieces of lines). Moreover, in this special case, we get a completely explicit formula for the Wigner measure associated with  $(u^\varepsilon)$ , in particular because we can identify the various radiation conditions at infinity that are necessary to entirely determine the Wigner measure  $\mu$ . We refer to [8] for details about the general case.

In this section, we assume that  $n_+$  and  $n_-$  are two constants with  $n_- > n_+ > 0$ . Now we state our main result in the case of two homogeneous media.

**Theorem 5.1.** *Assume (H1) and (H9)-(H12). Let  $u^\varepsilon$  be the solution to the Helmholtz equation (1.1). Assume that the refraction indices  $n_+$  and  $n_-$  are constant, with  $n_- > n_+$ . Then, the Wigner measure associated with  $(u^\varepsilon)$  is given by*

$$\begin{aligned} \mu(x, \xi) &= \\ &\mathbf{1}_{\{x_d < 1, \xi_d \geq 0\}} \int_{-\infty}^0 Q(x + t\xi, \xi) dt \\ &+ \mathbf{1}_{\{x_d \leq 1, -\sqrt{[n^2]} \leq \xi_d < 0\}} \left( \int_{\frac{1-x_d}{\xi_d}}^0 Q(x + t\xi, \xi) dt + \int_{-\infty}^{\frac{1-x_d}{\xi_d}} Q(\tilde{x} + t\tilde{\xi}, \tilde{\xi}) dt \right) \\ &+ \mathbf{1}_{\{x_d \leq 1, \xi_d < -\sqrt{[n^2]}\}} \left( \int_{\frac{1-x_d}{\xi_d}}^0 Q(x + t\xi, \xi) dt + \int_{-\infty}^{\frac{1-x_d}{\xi_d}} \alpha^R(\xi') Q(\tilde{x} + t\tilde{\xi}, \tilde{\xi}) dt \right) \\ &+ \mathbf{1}_{\{x_d \geq 1, \xi_d > 0\}} \left( \int_{\frac{1-x_d}{\xi_d}}^0 Q(x + t\xi, \xi) dt + \int_{-\infty}^{\frac{1-x_d}{\xi_d}} \alpha^T(\xi') Q(\tilde{x} + t\tilde{\xi}, \tilde{\xi}) dt \right), \end{aligned}$$

where

$$\begin{aligned}
Q(x, \xi) &= \frac{1}{2^{d+1}\pi^{d-1}} \delta(x) \delta(\xi^2 - n_-^2) |\hat{f}(\xi)|^2, \\
\check{\xi} &= (\xi', -\xi_d), \quad \check{x} = (x', 2 - x_d), \\
\tilde{\xi} &= \left( \xi', \operatorname{sgn}(\xi_d) \sqrt{\xi_d^2 + [n^2]} \right), \quad \tilde{x} = \left( x', 1 + (x_d - 1) \frac{\tilde{\xi}_d}{|\xi_d|} \right),
\end{aligned}$$

and the coefficients of partial reflection and partial transmission are

$$\alpha^R(\xi') = \left| \frac{2\sqrt{\omega_-(\xi')}}{\sqrt{\omega_+(\xi')} + \sqrt{\omega_-(\xi')}} \right|^2, \quad \alpha^T(\xi') = \left| \frac{\sqrt{\omega_+(\xi')} - \sqrt{\omega_-(\xi')}}{\sqrt{\omega_+(\xi')} + \sqrt{\omega_-(\xi')}} \right|^2.$$

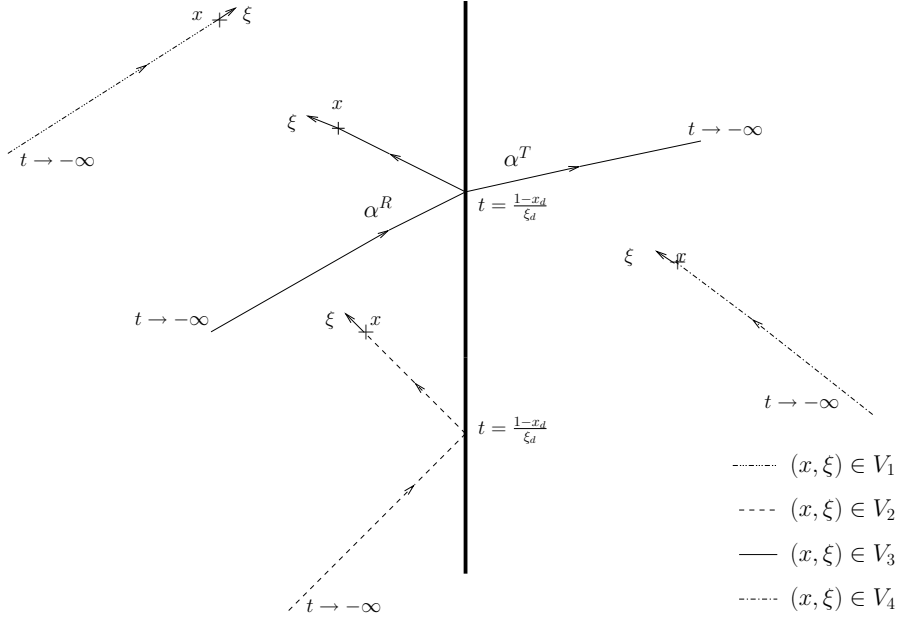


Figure 1: *Rays of geometrical optics in the homogeneous case.*

Before going further, let us comment Theorem 5.1 with the help of Figure 1 (where the regions  $V_j$ ,  $j = 1, \dots, 4$  are defined in Section 5.2). In order to compute the value of  $\mu$  at the point  $(x, \xi)$ , we first use the transport equation (4.1) to obtain the relation between  $\mu(x, \xi)$  and the value of  $\mu$  along the bicaracteristics  $(x + t\xi, \xi)$  until the time when this curve reaches the interface:

$$\mu(x, \xi) = \mu(x + t\xi, \xi) + \int_t^0 Q(x + s\xi, \xi) ds.$$

The first part of  $\mu$  in Theorem 5.1, i.e. when  $(x, \xi) \in V_1$  in Figure 1, corresponds to points  $(x, \xi)$  on the left side of the interface such that the bicaracteristics passing through  $(x, \xi)$  at  $t = 0$  does not reach the interface for  $t \in (-\infty, 0)$ . The value of  $\mu$  at such points is obtained using the radiation condition at infinity. The second part of  $\mu$ , i.e. when  $(x, \xi) \in V_2$  in Figure 1, corresponds to points  $(x, \xi)$  on the left side of the interface such that the bicaracteristics passing through  $(x, \xi)$  at  $t = 0$  reaches

the interface at a point where the ray is totally reflected (at time  $(1 - x_d)/\xi_d$ ). Finally, the third and fourth parts of  $\mu$  correspond to the two parts of the ray drawn for  $(x, \xi) \in V_3$  in Figure 1. For such points, the energy is partially reflected and partially transmitted at the interface.

We give the outline of the proof of Theorem 5.1 in the subsequent sections 5.1 and 5.2. We first define the boundary measures (related to the traces of the measures  $\mu_\pm$  at the boundary). Then, we obtain the propagation relations at the boundary (total reflexion and refraction) using the transport equations up to the boundary obtained in the previous section. Finally, we get the Wigner measure  $\mu$  by solving the transport equation satisfied by  $\mu$  and using both the radiation condition at infinity and the propagation relations at the boundary.

## 5.1. Boundary measures

In this section, we introduce the boundary measures related to the trace of  $\mu$  on the interface and we give relations between these measures and the semiclassical measures  $\nu$ ,  $\dot{\nu}$  and  $\nu^J$  associated with the traces of  $u^\varepsilon$  and its derivative  $\varepsilon \partial_d u^\varepsilon$  on the interface. This task is performed using the transport equations on  $\mu_\pm$  up to the boundary (4.2).

### Existence and notations

Since outside the interface,  $\mu$  is a solution to the transport equation

$$\xi \cdot \nabla_x \mu = Q(x, \xi) = \frac{1}{2^{d+1} \pi^{d-1}} \delta(x) \delta(|\xi|^2 - n_-^2) |\hat{f}(\xi)|^2,$$

we deduce

$$\mu \in \mathcal{C} \left( \mathbb{R}_{x_d}, \mathcal{D}' \left( \mathbb{R}_{x'}^{d-1} \times (\mathbb{R}_\xi^d \setminus \{\xi_d = 0\}) \right) \right).$$

For this reason, we can define, in  $\{\xi_d \neq 0\}$ , the traces

$$\mu_\pm^0 = \mu|_{x_d=1^\pm}. \quad (5.1)$$

These measures inherit the positivity of  $\mu$  and they satisfy the jump formula:

$$\partial_{x_d}(\mathbf{1}_{x_d \geq 1} \mu) = \mathbf{1}_{x_d \geq 1} \partial_{x_d} \mu \pm \delta(x_d - 1) \otimes \mu_\pm^0.$$

Since we have the localization property  $\text{supp}(\mu_\pm) \subset \{|\xi|^2 = n_\pm^2\}$ , there exist four nonnegative measures  $\mu_\pm^{out}, \mu_\pm^{in}$  (see Figure 2) such that

$$\mu_+^0 = \delta(\xi_d + \sqrt{\omega_+}) \otimes \mu_+^{in} + \delta(\xi_d - \sqrt{\omega_+}) \otimes \mu_+^{out}, \quad (5.2)$$

$$\mu_-^0 = \delta(\xi_d - \sqrt{\omega_-}) \otimes \mu_-^{in} + \delta(\xi_d + \sqrt{\omega_-}) \otimes \mu_-^{out}, \quad (5.3)$$

where  $\omega_\pm$  has been defined in Section 4.1,  $\omega_\pm = n_\pm^2 - \xi'^2$ .

Our goal is now to find relations between  $\mu_-^{in}, \mu_+^{in}, \mu_-^{out}$  and  $\mu_+^{out}$  that translate the transmission/reflection phenomena at the interface.

First, let us introduce the following last measures.

**Lemma 5.2.** *There exist two nonnegative measures  $\mu^{\partial\pm}$  on  $T^*\Gamma$  with support in the set  $\{|\xi'|^2 = n_\pm^2\}$  such that*

$$\mu_\pm = \mathbf{1}_{\{x_d \geq 1\}} \mu_\pm + \delta(x_d - 1) \otimes \delta(\xi_d) \otimes \mu^{\partial\pm}.$$

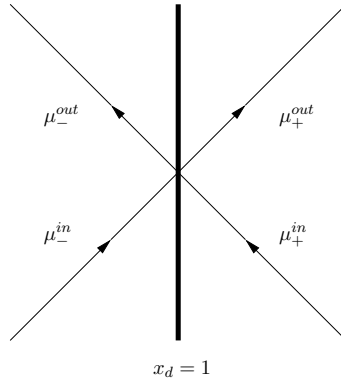


Figure 2: *Boundary measures*

**Remark** *This means that the density at the interface ( $x_d = 1$ ) can be only carried by the gliding rays  $\xi_d = 0$ . In the particular case we are studying, these rays don't "come from" one medium since  $\xi$  is constant along a ray. Hence, we will have to study separately the density inside the interface.*

Next, we get the relations that we are looking for, depending on the regions of  $T^*\Gamma$ . They are obtained by "taking the trace" of the transport equations in Proposition 4.4 (choosing appropriate test functions).

**Lemma 5.3.** *For  $\xi_d \neq 0$ , in the set  $\{\omega_{\pm} > 0\}$ , we have*

$$(i) \pm \mathcal{R}e \nu^J = \sqrt{\omega_{\pm}} (\mu_{\pm}^{out} - \mu_{\pm}^{in}),$$

$$(ii) \frac{1}{2} \dot{\nu} = \omega_{\pm} (\mu_{\pm}^{out} + \mu_{\pm}^{in} - \frac{1}{2} \nu).$$

**Lemma 5.4.**

$$(i) \dot{\nu} = 0 \text{ on } \{\omega_+ = 0\}.$$

$$(ii) \mathcal{R}e(\nu^J) = 0 \text{ on } \{\omega_+ \leq 0\}.$$

## 5.2. Reflexion/transmission at the interface

In this section, we end the proof of our main theorem in the case of two homogeneous media. We prove it by solving Cauchy problems with respect to the  $x_d$  variable. These problems are of two types: in the regions where the rays of geometrical optics do not reach the interface when  $t \rightarrow -\infty$ , we solve Cauchy problems with boundary conditions at infinity in space; in the other regions, we solve Cauchy problems with initial data at  $x_d = 1$ .

We use the following partition of phase space

$$\begin{aligned}
T^*\mathbb{R}^d &= \{x_d < 1, \xi_d \geq 0\} \cup \{x_d \leq 1, -\sqrt{[n^2]} \leq \xi_d < 0\} \\
&\cup \{x_d \leq 1, \xi_d < -\sqrt{[n^2]}\} \cup \{x_d > 1, \xi_d \leq 0\} \\
&\cup \{x_d \geq 1, \xi_d > 0\} \cup \{x_d = 1, \xi_d = 0\} \\
&= V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6.
\end{aligned}$$

The value of  $\mu$  in the first five regions will be obtained by solving the transport equation (4.1) on each region  $V_j$  ( $j = 1, \dots, 5$ ), using the radiation condition at infinity and Lemmas 5.3 and 5.4 to get the values at the boundary. At variance, the value of  $\mu$  in  $V_6$  cannot be obtained using a transport equation since no ray coming from one media reaches the interface with  $\xi_d = 0$  (the rays are given by  $(x + t\xi, \xi)$  in the homogeneous case). Thus, we have to study directly  $\mu^{\partial\pm}$ . The following proposition implies that  $\mu = 0$  in the region  $V_6$ .

In the first region  $V_1$ ,  $\mu$  is the solution to  $\xi \cdot \nabla_x \mu = Q$  with the outgoing condition at infinity  $\mu(x, \xi) \rightarrow 0$  as  $|x| \rightarrow \infty$  with  $x \cdot \xi < 0$  (it is a consequence of the radiation condition). On the other hand, if  $(x, \xi) \in V_1$ , then for all  $t < 0$ ,  $(x + t\xi, \xi) \in V_1$ . We deduce

$$\mu(x, \xi) = \mu(x + t\xi, \xi) + \int_t^0 Q(x + s\xi, \xi) ds.$$

Taking the limit  $t \rightarrow -\infty$ , we obtain the value of  $\mu$  in  $V_1$ :

$$\mu(x, \xi) = \int_{-\infty}^0 Q(x + s\xi, \xi) ds.$$

As a consequence,

$$\delta(\xi_d - \sqrt{\omega_-}) \otimes \mu_-^{in}(x', \xi') = \int_{-\infty}^0 Q(x' + s\xi', 1 + s\xi_d, \xi) dt.$$

In particular, the measure  $\mu_-^{in}$  is known.

Now, we compute  $\mu$  in the region  $V_2$ . We consider the following part of the interface:  $\{\omega_- > 0\} \cap \{\omega_+ \leq 0\}$ , which corresponds to  $0 < \omega_- \leq [n^2]$ . On this set, since  $\mathcal{R}e \nu^J = 0$ , from Propositions 5.3 and 5.4, we get  $\mu_-^{out} = \mu_-^{in}$ . Hence, for  $-\sqrt{[n^2]} \leq \xi_d < 0$ , we recover

$$\begin{aligned}
\delta(\xi_d + \sqrt{\omega_-}) \otimes \mu_-^{out} &= \delta(\check{\xi}_d - \sqrt{\omega_-}) \otimes \mu_-^{in}(x', \check{\xi}') \\
&= \int_{-\infty}^0 Q(x' + s\check{\xi}', 1 + s\check{\xi}_d, \check{\xi}) ds, \tag{5.4}
\end{aligned}$$

where  $\check{\xi} = (\xi', -\xi_d)$ . Hence, we are left with the following Cauchy problem in the  $x_d$  variable with initial data (5.4) at the interface  $x_d = 1$ : For  $(x, \xi) \in V_2$ ,

$$\begin{cases} \partial_{x_d} \mu + \xi_d^{-1} \xi' \cdot \nabla_{x'} \mu = \xi_d^{-1} Q, & x_d < 1, \\ \mu|_{\{x_d=1\}}(x', \xi) = \int_{-\infty}^0 Q(x' + s\check{\xi}', 1 + s\check{\xi}_d, \check{\xi}) ds. \end{cases}$$

This problem is explicitly solvable. For  $(x, \xi) \in V_2$ , we obtain

$$\begin{aligned}
\mu(x, \xi) &= \mu|_{\{x_d=1\}}(x' + \frac{1-x_d}{\xi_d}\xi', \xi) - \int_{x_d}^1 Q(x' + \frac{s-x_d}{\xi_d}\xi', s, \xi) \frac{ds}{\xi_d} \\
&= \int_{-\infty}^0 Q(x' + (\frac{1-x_d}{\xi_d} + s)\check{\xi}', 1 + s\check{\xi}_d, \check{\xi}) ds \\
&\quad - \int_{x_d}^1 Q(x' + \frac{s-x_d}{\xi_d}\xi', s, \xi) \frac{ds}{\xi_d} \\
&= \int_{-\infty}^{\frac{1-x_d}{\xi_d}} Q(x' + t\check{\xi}', 2-x_d + t\check{\xi}_d, \check{\xi}) ds \\
&\quad + \int_{\frac{1-x_d}{\xi_d}}^0 Q(x' + t\xi', x_d + t\xi_d, \xi) ds
\end{aligned}$$

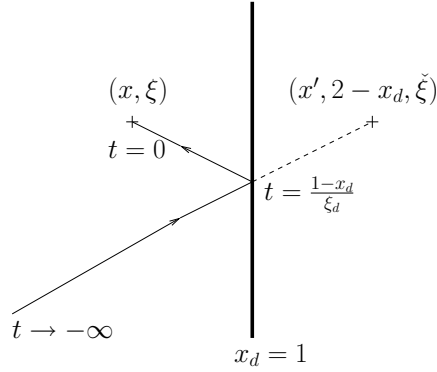


Figure 3: *Total reflection*

**Remark** One can notice that  $\frac{1-x_d}{\xi_d}$  is the time at which the bicharacteristics reaches the interface. The point  $(x', 2-x_d)$  is the symmetric of  $x$  with respect to the interface (see Figure 3).

Next, we consider the part  $\{\omega_- > 0\} \cap \{\omega_+ > 0\}$  of the interface. In the region  $V_4$ ,  $\mu$  satisfies the equation  $\xi \cdot \nabla_x \mu = 0$  with the outgoing radiation condition at infinity  $\mu(x, \xi) \rightarrow 0$  as  $|x| \rightarrow \infty$ ,  $x \cdot \xi < 0$ . Hence,  $\mu = 0$  in this region and

$$\mu_+^{in} = 0.$$

In the next lemma, we write the relations between the other three measures  $\mu_-^{in}$ ,  $\mu_-^{out}$ ,  $\mu_+^{out}$ . These relations translate the refraction phenomenon. The proof of this result is borrowed from [16].

**Lemma 5.5.** *If  $\mu_+^{in} = 0$ , then  $\mu_+^{out} = \alpha^T \mu_-^{in}$  and  $\mu_-^{out} = \alpha^R \mu_-^{in}$ .*

*Proof.* Using Lemma 5.3, and the fact that  $\mu_+^{in}$ , we get

$$\dot{\nu} + \omega_+ \nu = 2\omega_+ \mu_+^{out} = 2\sqrt{\omega_+} \mathcal{R}e \nu^J.$$



But the matrix measure  $\begin{pmatrix} \nu & \bar{\nu}^J \\ \nu^J & \dot{\nu} \end{pmatrix}$  is hermitian so that

$$|\nu^J| \leq (\nu)^{\frac{1}{2}}(\dot{\nu})^{\frac{1}{2}}.$$

Hence, we recover

$$2(\omega_+\nu)^{\frac{1}{2}}(\dot{\nu})^{\frac{1}{2}} \leq \dot{\nu} + \omega_+\nu = 2\sqrt{\omega_+} \operatorname{Re} \nu^J \leq 2(\omega_+\nu)^{\frac{1}{2}}(\dot{\nu})^{\frac{1}{2}}$$

and

$$\dot{\nu} = \omega_+\nu \quad \text{in } \{\omega_- > 0\} \cap \{\omega_+ > 0\}.$$

Thus, we now have five equations (the equation above and the four equations in Lemma 5.3) involving the six unknown measures  $\nu$ ,  $\dot{\nu}$ ,  $\nu^J$ ,  $\mu_+^{out}$ ,  $\mu_+^{in}$  and  $\mu_-^{in}$ . After some calculations, we deduce

$$\begin{cases} \mu_+^{out} &= \alpha^T \mu_-^{in}, \\ \mu_+^{in} &= \alpha^R \mu_-^{in}, \end{cases}$$

where the coefficients  $\alpha^R$  and  $\alpha^T$  are defined in Theorem 5.1.  $\square$

Using this lemma, we can now determine  $\mu$  in the remaining regions  $V_3$  and  $V_5$  by solving Cauchy problems with initial data at  $x_d = 1$ .

In the region  $V_3$ , making the same calculations as in the second region, we obtain the reflected part (with a partial reflexion coefficient):

$$\mu(x, \xi) = \int_{\frac{1-x_d}{\xi_d}}^0 Q(x' + t\xi', x_d + t\xi_d, \xi) ds + \int_{-\infty}^{\frac{1-x_d}{\xi_d}} \alpha^R(\xi) Q(x' + t\check{\xi}', 2 - x_d + t\check{\xi}_d, \check{\xi}) ds.$$

In the region  $V_5$ , we have

$$\mu(x, \xi) = \int_{\frac{1-x_d}{\xi_d}}^0 Q(x + t\xi, \xi) dt + \int_{-\infty}^0 \alpha^T(\xi) Q\left(x' + \frac{1-x_d}{\xi_d}\xi' + t\xi', 1 + t\check{\xi}_d, \check{\xi}\right) dt,$$

so

$$\mu(x, \xi) = \int_0^{\frac{1-x_d}{\xi_d}} Q(x + s\xi, \xi) ds + \int_{-\infty}^{\frac{1-x_d}{\xi_d}} \alpha^T(\xi) Q\left(x' + s\xi', 1 + (x_d - 1)\frac{\check{\xi}_d}{\xi_d} + s\check{\xi}_d, \check{\xi}\right) ds.$$

This ends the proof of Theorem 5.1.  $\square$

### 5.3. Remarks

We would like to end this section by making some comments on the assumptions (a) and (b) in Theorem 1.

With the notations we have now introduced, hypothesis (a) is equivalent to the fact that  $\mu_+^{in}$  and  $\mu_-^{in}$  are mutually singular. We have just proved that in the case of two homogeneous media,  $\mu_+^{in} = 0$ , so that the hypothesis of non interference is satisfied. This is due to the fact that the source is situated on one side of the interface and that the rays only reach once the interface. In the general case, this hypothesis may not be satisfied: for instance, because of rays coming from the left, transmitted and that may go back to the interface from the right side.

Let us now comment on hypothesis (b). The energy trapped in the interface can only be supported by rays such that  $\xi_d = 0$ . In the general case, there exist glancing rays that come from the two media, and thus could carry energy up to the interface.

Hence, in order to prove that the non-trapping hypothesis is satisfied, one should try to prove a radiation condition satisfied by the measure *inside the interface*.

## 6. Outline of the proof of Theorem 2

As we proved in the previous section, the measure  $\mu_+^{in}$  vanishes in the case of two homogeneous media, hence point (i) is proved.

In order to prove (ii) and (iii), we use the explicit formula that is available in this particular case for the resolvent of the Helmholtz operator. Here we only detail the proof of point (ii).

Intuitively, since the source  $f^\varepsilon$  is situated outside the interface, and since the glancing rays ( $\xi_d = 0$ ) do not come from any of the two media in the homogeneous case ( $\xi$  is constant along the bicaracteristics), the source cannot carry energy inside the interface ( indeed,  $\mu^{\partial\pm}$  has support in  $\xi_d = 0$ ). However, we cannot show that  $\mu^{\partial\pm} = 0$  by local methods. The Helmholtz equation takes into account the rays *up to infinite time*. We need global information, that is the reason why we use the explicit formula for the resolvent.

Using this explicit expression, our study rests on (non-)stationary phase method with singularity. Indeed, we already know that  $\mu$  has support in the set  $\{\xi^2 = n(x)^2\}$ . Hence, if we denote  $\xi = (\xi', \xi_d) \in \mathbb{R}^d$ , the roots

$$\omega_\pm^\varepsilon(\xi') = \sqrt{\xi'^2 - n_\pm^2 \pm i\alpha_\varepsilon\varepsilon}$$

of the equations  $\xi_d^2 = n_\pm^2 - \xi'^2(-i\alpha_\varepsilon\varepsilon)$  naturally appear both in the phase and in the test functions. Typically, we need to estimate terms of the following type:

$$\frac{1}{\varepsilon^{\frac{3d+1}{2}}} \int \frac{1}{\omega_-^\varepsilon(\xi')} e^{i\frac{(x'-y')\cdot\xi'}{\varepsilon} + i\frac{y'\cdot\xi'}{\varepsilon} - \frac{\omega_-^\varepsilon(\xi')}{\varepsilon}} A(x', y', \zeta', \xi') dy' d\xi' d\zeta', \quad (6.1)$$

where  $x'$  is bounded, and the amplitude  $A$  is supported near  $\zeta'^2 = n_-^2$ . First, for  $|\xi'|$  far from  $n_-$ , the phase is non-stationary with respect to the  $y'$ -variable. Hence, we are left with the case when  $|\xi'|$  is close to  $n_-$ , *i.e.* close to the singularity of  $\omega_-^\varepsilon$ . After a change of variable, the term we have to estimate is of the form

$$\frac{1}{\varepsilon^{\frac{3d+1}{2}}} \int \frac{1}{\sqrt{t + i\alpha_\varepsilon\varepsilon}} e^{i\frac{\sqrt{t+n_-^2}}{\varepsilon} - \frac{\sqrt{t+i\alpha_\varepsilon\varepsilon}}{\varepsilon}} B(t) dt, \quad (6.2)$$

where  $B$  is supported close to  $t = 0$ .

Now, in order to treat the singularity of  $\sqrt{t + i\alpha_\varepsilon\varepsilon}$  when  $\varepsilon \rightarrow 0$  in (6.2), the key ingredient is a contour deformation in the complex plane, together with the use of almost-analytic extensions: there exists a extension  $\tilde{B}$  of  $B$  to the complex plane (compactly supported in  $\mathbb{C}$  if  $B$  is compactly supported in  $\mathbb{R}$ ) such that, for all  $N$ ,

$$\left| \frac{\partial}{\partial \bar{z}} \tilde{B}(z) \right| \leq C_N |\operatorname{Im} z|^N. \quad (6.3)$$

Using this extension and the Green-Riemann formula, we decompose the previous integral into the sum of an integral of  $\tilde{B}$  over  $\{\operatorname{Im} z = \beta\}$  ( $\beta > 0$  fixed) and of an integral of  $\partial\tilde{B}/\partial\bar{z}$  over  $\{\alpha_\varepsilon\varepsilon \leq \operatorname{Im} z \leq \beta\}$ . The first part can be estimated using the usual non-stationary phase theorem, since the root  $\sqrt{t + i\beta}$  is not singular anymore. To bound the second part, we separate the domains  $|\operatorname{Im} z| \leq \varepsilon^\delta$  and  $|\operatorname{Im} z| \geq \varepsilon^\delta$ . For  $|\operatorname{Im} z| \leq \varepsilon^\delta$ , we use the property (6.3) of almost-analytic extensions. For  $|\operatorname{Im} z| \geq$

$\varepsilon^\delta$ , we use the fact that  $\sqrt{z}$  is bounded from below by  $\varepsilon^{\delta/2}$ , so that each integration by part gives a power of  $\varepsilon^{1-\delta/2}$ . All these estimates allow us *in fine* to prove that the integral (6.2) is  $O(\varepsilon^\infty)$ .

## References

- [1] S. AGMON, L. HÖRMANDER, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Analyse Math., **30** (1976), 1-38.
- [2] J.D. BENAMOU, F. CASTELLA, T. KATSAOUNIS, B. PERTHAME, *High frequency limit of the Helmholtz equation*, Rev. Mat. Iberoamericana **18** (2002), no. 1, 187–209.
- [3] F. CASTELLA, B. PERTHAME, O. RUNBORG, *High frequency limit of the Helmholtz equation. Source on a general manifold*, Comm. P.D.E **3-4** (2002), 607-651.
- [4] F. CASTELLA, *The radiation condition at infinity for the high frequency Helmholtz equation with source term: a wave packet approach*, J. Funct. Anal. **223** (2005), no.1, 204-257.
- [5] J. DEREZIŃSKI, C. GÉRARD, *Scattering theory of classical and quantum N-particle systems*, Texts and Monographs in Physics, Springer, Berlin, 1997.
- [6] M. DIMASSI, J. SJÖSTRAND, *Spectral asymptotics in the semiclassical limit*, London Mathematical Society Lecture Notes Series, vol. 268, Cambridge University Press, Cambridge, 1999.
- [7] E. FOUASSIER, *Morrey-Campanato estimates for Helmholtz equations with two unbounded media*, Proc. Roy. Soc. Edinburg Sect. A **135** (2005), no.4, 767-776.
- [8] E. FOUASSIER, *High frequency analysis of Helmholtz equations: refraction by sharp interfaces*, to appear in Journal de Mathématiques Pures et Appliquées.
- [9] P. GÉRARD, *Mesures semi-classiques et ondes de Bloch*, In Séminaire Equations aux dérivées partielles 1988-1989, exp XVI, Ecole Polytechnique, Palaiseau (1988).
- [10] P. GÉRARD, E. LEICHTNAM, *Ergodic properties of eigenfunctions for the Dirichlet problem*, Duke Math. J., **71** (1993), 559-607.
- [11] P. GÉRARD, P.A. MARKOWITCH, N.J. MAUSER, F. POUPAUD, *Homogenization limits and Wigner transforms*, Comm. pure and Appl. Math., **50** (1997), 321-357.
- [12] C. GÉRARD, A. MARTINEZ, *Principe d'absorption limite pour des opérateurs de Schrödinger à longue portée*, C. R. Acad. Sci. Paris, Ser. I math, Vol 195, **3**, 121-123 (1988).
- [13] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators I and III*, Springer-Verlag.

- [14] L. HÖRMANDER, Lecture Notes at the Nordic Summer School of mathematics (1968).
- [15] P.-L. LIONS, T. PAUL, *Sur les mesures de Wigner*, Revista Matemática Iberoamericana, **9 (3)** (1993), 553-618.
- [16] L. MILLER, *Propagation d'ondes semi-classiques à travers une interface et mesures 2-microlocales*, Doctorat de l'École Polytechnique, Palaiseau (1996).
- [17] L. MILLER, *Refraction of high-frequency waves density by sharp interfaces and semiclassical measures at the boundary*, J. Math. Pures Appl. (9) **79** (2000), 227–269.
- [18] B. PERTHAME, L. VEGA, *Morrey-campanato estimates for the Helmholtz equation*, J. Funct. Anal. **164(2)** (1999), 340-355.
- [19] X.P. WANG, P. ZHANG, *High frequency limit of the Helmholtz equation with variable index of refraction*, Preprint (2004)
- [20] E. WIGNER, *On the quantum correction for thermodynamic equilibrium*, Phys. Rev., **40** (1932)

UMPA, ENS LYON, 46 ALLÉE D'ITALIE, 69364 LYON CEDEX 7, FRANCE  
 ET IRMAR, UNIVERSITÉ DE RENNES 1, CAMPUS DE BEAULIEU, 35042 RENNES  
 CEDEX, FRANCE  
 elise.fouassier@umpa.ens-lyon.fr