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Résumé

Nous montrons deux résultats d'approximation de dimension finie et une propriété "nonsqueezing" symplectique pour le flot Korteweg-de Vries (KdV) sur le cercle $\mathbb{T}$. Le résultat nonsqueezing dépend des résultats d'approximation mentionnés et du théorème nonsqueezing de Gromov en dimension finie. Contrairement aux travaux de Kuksin [22] qui a lancé l'étude de résultats nonsqueezing pour des systèmes hamiltoniens de dimension infinie, l'argument nonsqueezing ici ne construit pas de capacité de façon directe. De cette manière, nos résultats sont semblables à ceux obtenus pour le flot NLS par Bourgain [3]. Cependant, une difficulté majeure ici est le manque d'estimations de lissage qui nous permettraient d'approximer facilement le flot KdV de dimension infinie par un flot hamiltonien de dimension finie. Pour contourner ce problème, nous inversons la transformation de Miura et travaillons au niveau de l'équation KdV modifiée (mKdV), pour laquelle une estimation de lissage peut être obtenue.

Abstract

We prove two finite dimensional approximation results and a symplectic non-squeezing property for the Korteweg-de Vries (KdV) flow on the circle $\mathbb{T}$. The nonsqueezing result relies on the aforementioned approximations and the finite-dimensional nonsqueezing theorem of Gromov [14]. Unlike the work of Kuksin [22] which initiated the investigation of non-squeezing results for infinite dimensional Hamiltonian systems, the nonsqueezing argument here does not construct a capacity directly. In this way our results are similar to those obtained for the NLS flow by Bourgain [3]. A major difficulty here though is the lack of any sort of smoothing estimate which would allow us to easily approximate the infinite dimensional KdV flow by a finite-dimensional Hamiltonian flow. To resolve this problem we invert the Miura transform and work on the level of the modified KdV (mKdV) equation, for which smoothing estimates can be established.

1. Introduction

The material contained in this note is taken from a paper [10] by the same authors that is concerned with the symplectic behavior of the Korteweg-de Vries (KdV) flow

$$u_t + u_{xxx} = 6uu_x; \quad u(0,x) = u_0(x) \tag{1.1}$$

on the circle $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$, where $u(t,x)$ is real-valued. In particular we investigate how the flows may (or may not) be accurately approximated by certain finite-dimensional models, and then use such an approximation to conclude a symplectic non-squeezing property. All the results presented here are proved in In order to describe the symplectic space involved, and state the result precisely, we need to set notation and recall some previous results describing the well-posedness of the initial value problem (1.1).

On the circle we have the spatial Fourier transform

$$\widehat{u}(k) := \frac{1}{2\pi} \int_0^{2\pi} u(x) \exp(-ikx) \; dx \tag{1.2}$$

for all $k \in \mathbb{Z}$, and the spatial Sobolev spaces

$$\|u\|_{H_x^s} := (2\pi)^{1/2} \| \langle k \rangle^s \widehat{u} \|_{l_k^2}$$

for $s \in \mathbb{R}$, where $\langle k \rangle := (1 + |k|^2)^{1/2}$. These are natural spaces for analyzing the KdV flow.

Let P_0 denote the mean operator

$$P_0 u := \frac{1}{2\pi} \int_0^{2\pi} u$$

or equivalently

$$\widehat{P_0 u}(k) = \chi_{k=0} \widehat{u}(k).$$

The KdV flow is mean-preserving, and it will be convenient to work in the case when u has mean zero[1]. Accordingly we define the mean-zero periodic Sobolev spaces H_0^s by

$$H_0^s := \{u \in H_x^s : P_0 u = 0\}$$

endowed with the same norm as H_x^s.

Recent work on the local and global well-posedness theory in H_0^s for (1.1) is basic to our results here. For example, the geometric conclusions from finite-dimensional Hamiltonian dynamics which we ultimately need for our nonsqueezing result can only be applied in the setting of rather rough solutions to the initial value problem (1.1). We now pause to summarize some of the analytical techniques that have been developed for the study of such rough solutions, and the resulting regularity theory (see e.g. [1], [19], [6], and [8], [9]).

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[1]One can easily pass from the mean zero case to the general mean case by a Galilean transformation $u(t,x) \rightarrow u(t, x - P_0(u)t) - P_0(u)$.

1.1. Summary of local and global well-posedness theory

If the initial datum u_0 for (1.1) is smooth, then there is a global smooth solution[2] $u(t)$ (see e.g. [27]). We can thus define the non-linear flow map $S_{KdV}(t)$ on $C^\infty(\mathbb{T})$ by $S_{KdV}(t)u_0 := u(t)$. In particular this map is densely defined on every Sobolev space H_0^s.

If $s \geq -1/2$, then the equation (1.1) is globally well-posed in H_0^s. In other words, the flow map $S_{KdV}(t)$ is uniformly continuous (indeed, it is analytic) on H_0^s for times t restricted to a compact interval $[-T, T]$, and for such s we have bounds of the form

$$\sup_{|t| \leq T} \|S_{KdV}(t)u_0\|_{H_0^s} \leq C(s, T, \|u_0\|_{H_0^s}), \tag{1.3}$$

(see [19], [8], [9]). For $s < -1/2$ the flow map $S_{KdV}(t)$ is no longer uniformly continuous [6] (see also [20]) or analytic [4], so from the point of view which requires a uniformly continous flow in time, the Sobolev space $H_0^{-1/2}$ is the endpoint space for the KdV flow. Coincidentally, this space is also a natural phase space for which KdV becomes a Hamiltonian flow; we will have more to say about this at the end of the introduction. Note however that if one asks only that the flow be continuous in time, then global well-posedness for (1.1) has been established for all $s \geq -1$ in [17] using inverse scattering methods. Combining mapping properties of the Miura Transform and the result in [28], local well-posedness of (1.1) in H_0^s with a (not uniformly) continuous flow map holds for $-5/8 < s < -1/2$.

1.2. Low frequency approximation of KdV

The KdV flow (1.1) is, formally at least, a Hamiltonian flow on an infinite-dimensional space. In order to rigorously prove in this context results from symplectic geometry, in principle one could proceed in two very different ways. The first is by introducing in the infinite-dimensional setting the geometric tools developed in the finite one, like holomorphic curves for example. The second is by approximating the given infinite-dimensional flow by a finite-dimensional one. So far this second procedure has been the preferred approach (see [22], [3] and [10]), although there are no results at this point indicating that the first one could not be successful.

When we started thinking about proving for the KdV flow the analogue of the finite dimensional non-squeezing theorem of Gromov [14] recalled in Theorem 1.6, we had in mind the proof that Bourgain gives in [3] in order to prove a non-squeezing theorem for the $1D$ cubic nonlinear Schrödinger flow. But we immediately faced a serious obstruction: the finite dimensional approximation for the KdV flow analogue to the one used by Bourgain was not a good approximation in this case. Let's now make this statement precise. To approximate the KdV flow by a finite-dimensional model we proceed in the most obvious way: we restrict the infinitely many Fourier

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[2]This result can also be obtained by inverse scattering methods, since the KdV equation is completely integrable. However, our methods here do not use inverse scattering techniques, although the special algebraic structure of KdV (in particular, the Miura transform [25]) is certainly exploited.

modes to the ones supported on $[-N, N]$, for some large fixed N and we study the KdV flow

$$u_t + u_{xxx} = P_{\leq N}(6uu_x); \quad u(0) = u_0, \qquad (1.4)$$

where $P_{\leq N}$ is the Fourier projection to frequencies $\leq N$:

$$\widehat{P_{\leq N}u}(k) = \chi_{|k| \leq N}\widehat{u}(k).$$

Denote the flow map associated to (1.4) by $S_{P_{\leq N}KdV}(t)$. This flow has several advantageous properties; for instance, $S_{P_{\leq N}KdV}(t)$ is a symplectomorphism on the space $P_{\leq N}H_0^{-1/2}$, associated with a natural symplectic structure (see next subsection). Since $P_{\leq N}H_0^{-1/2}$ is a finite dimensional space, it is easy to see (e.g. using L^2 norm conservation and Picard iteration) that this flow $S_{P_{\leq N}KdV}$ is globally smooth and well-defined. As mentioned above in [3], the NLS flow $iu_t + u_{xx} = |u|^2u$ was similarly truncated, and it was shown that the truncated flow was a good approximation to the original (infinite dimensional) flow. Unfortunately, the same result does not apply for KdV:

Theorem 1.1. *Let $k_0 \in \mathbb{Z}^*$, $T > 0$, $A > 0$. Then for any $N \gg C(A, T, k_0)$ there exists initial data u_0 with $\|u_0\|_{H_0^{-1/2}} \leq A$ and $\text{supp}\,(\widehat{u_0}) \subset \{|k| \leq N\}$ such that*

$$|(\widehat{S_{KdV}(T)u_0})(k_0) - (\widehat{S_{P_{\leq N}KdV}(T)u_0})(k_0)| \geq c(T, A, k_0) \qquad (1.5)$$

for some $c(T, A, k_0) > 0$.

In other words, $S_{P_{\leq N}KdV}$ does not converge to S_{KdV} even in a weak topology.

Basically, the problem is that the multiplier $\chi_{[-N,N]}$ corresponding to $P_{\leq N}$ is very rough, and this creates significant deviations between S_{KdV} and $S_{P_{\leq N}KdV}$ near the Fourier modes $k = \pm N$. In cubic equations such as mKdV (see (1.8) below) or the cubic nonlinear Schrödinger equation, these deviations would stay near the high frequencies $\pm N$, but in the quadratic KdV equation these deviations create significant fluctuations near the frequency origin, eventually leading to failure of weak convergence in (1.5).

Since the obstruction above was generated by a sharp truncation of the frequencies, the next obvious step was to look at a smooth truncation. Let $b(k)$ be the restriction to the integers of a real even bump function adapted to $[-N, N]$ which equals 1 on $[-N/2, N/2]$, and consider the evolution

$$u_t + u_{xxx} = B(6uu_x); \quad u(0) = u_0 \qquad (1.6)$$

where

$$\widehat{Bu}(k) = b(k)\widehat{u}(k).$$

Let S_{BKdV} denote the flow map associated to (1.6). Observe that this is a finite-dimensional flow on the space $P_{\leq N}H_0^s$. Unfortunately, S_{BKdV} is not a symplectomorphism, but we will explain in (1.26) below how by conjugating a flow of the form (1.6) with a simple multiplier operator we will arrive at our desired finite dimensional symplectomorphism on $P_{\leq N}H^{-\frac{1}{2}}(\mathbb{T})$ that well-approximates the full KdV flow at low frequencies. This desired symplectomorphism is labelled $S_{KdV}^{(N)}(t)$ in (1.26) below[3], and once the aforementioned approximation properties are established, the

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[3]The equation which defines this flow can be find in [10].

nonsqueezing result will follow almost immediately after quoting the finite dimensional nonsqueezing result of Gromov [14].

The first step in the argument is to show we can approximate S_{KdV} by S_{BKdV} in the strong H_0^s topology:

Theorem 1.2. *Fix $s \geq -1/2$, $T > 0$, and $N \gg 1$. Let $u_0 \in H_0^s$ have Fourier transform supported in the range $|k| \leq N$. Then*

$$\sup_{|t| \leq T} \|P_{\leq N^{1/2}}(S_{BKdV}u_0(t) - S_{KdV}(t)u_0)\|_{H_0^s} \leq N^{-\sigma}C(s, T, \|u_0\|_{H_0^s})$$

for some $\sigma = \sigma(s) > 0$.

In particular, we can accurately model the KdV evolution for band-limited initial data by a finite-dimensional flow, at least for frequencies $|k| \leq N^{1/2}$.

Theorem 1.2 can be viewed as a statement that one can (smoothly) truncate the KdV evolution at the high frequencies without causing serious disruption to the low frequencies, in spite of the obstruction posed by Theorem 1.1. Our second main result is in a similar vein:

Theorem 1.3. *Fix $s \geq -1/2$, $T > 0$, $N \geq 1$. Let $u_0, \tilde{u}_0 \in H_0^s$ be such that $P_{\leq 2N}u_0 = P_{\leq 2N}\tilde{u}_0$ (i.e. u_0 and $\tilde{u}_0$ agree at low frequencies). Then we have,*

$$\sup_{|t| \leq T} \|P_{\leq N}(S_{KdV}(t)\tilde{u}_0 - S_{KdV}(t)u_0)\|_{H_0^s} \leq N^{-\sigma}C(s, T, \|u_0\|_{H_0^s}, \|\tilde{u}_0\|_{H_0^s})$$

for some $\sigma = \sigma(s) > 0$.

The point of Theorem 1.3 is that changes to the initial data at frequencies $\geq 2N$ do not significantly affect the solution at frequencies $\leq N$, as measured in the strong H_0^s topology. This is in stark contrast to the negative result in Theorem 1.1. The point is that there is some delicate cancellative structure in the KdV equation which permits the decoupling of high and low frequencies, and this structure is destroyed by projecting the KdV equation crudely using (1.4).

To prove Theorem 1.2 and Theorem 1.3, we shall need to exploit the subtle cancellation mentioned in the previous paragraph in order to avoid the obstructions arising from Theorem 1.1. We do not know how to do this working directly with the KdV flow. Rather, we are able to prove estimates which explicitly account for this subtle structure in KdV by using the *Miura transform* $u = \mathbf{M}v$, defined by

$$u = \mathbf{M}v := v_x + v^2 - P_0(v^2). \tag{1.7}$$

As discovered in [25], this transform allows us to conjugate the KdV flow to the *modified Korteweg-de Vries* (mKdV) flow

$$v_t + v_{xxx} = F(v); \quad v(x, 0) = v_0(x) \tag{1.8}$$

where the non-linearity $F(v)$ is given by

$$F(v) := 6(v^2 - P_0(v^2))v_x. \tag{1.9}$$

The modified KdV equation has slightly better smoothing properties than the ordinary KdV equation, and in addition the process of inverting the Miura transform adds one degree of regularity (from $H_0^{-1/2}$ to $H_0^{1/2}$). In particular, the types of counterexamples arising in Theorem 1.1 do not appear in the mKdV setting, and by proving a slightly more refined trilinear estimate than those found in e.g. [9] we are able to prove the above two theorems by passing to the mKdV setting using the Miura transform. Of course, in order to close the argument we will need some efficient estimates on the invertibility of the Miura transform; we set up these estimates (which may be of independent interest) in [10].

1.3. Application to symplectic non-squeezing

We can apply the above approximation results to study the symplectic behavior of KdV in a natural phase space $H_0^{-1/2}(\mathbb{T})$. Before doing so, we recall some context and results from previous works. We are following here especially the exposition from [16, 23].

Definition 1.4. *Consider a pair $(\mathbb{H}, \omega)$ where ω is a symplectic form[4] on the Hilbert space $\mathbb{H}$. We say $(\mathbb{H}, \omega)$ is the symplectic phase space of a PDE with Hamiltonian $H[u(t)]$ if the PDE can be written in the form,*

$$\dot{u}(t) = J\nabla H[u(t)]. \tag{1.10}$$

Here J is an almost complex structure[5] on $\mathbb{H}$, which is compatible with the Hilbert space inner product $\langle \cdot, \cdot \rangle$. That is, for all $u, v \in \mathbb{H}$,

$$\omega(u, v) = \langle Ju, v \rangle. \tag{1.11}$$

The notation ∇ in (1.10) denotes the usual gradient with respect to the Hilbert space inner product,

$$\langle v, \nabla H[u] \rangle \equiv dH[u](v) \tag{1.12}$$

$$\equiv \frac{d}{d\epsilon}\Big|_{\epsilon=0} H[u + \epsilon v]. \tag{1.13}$$

One easily checks that an equivalent way to write the PDE corresponding to the Hamiltonian $H[u(t)]$ in $(\mathbb{H}, \omega)$ is

$$\dot{u}(t) = \nabla_\omega H[u(t)] \tag{1.14}$$

where the symplectic gradient $\nabla_\omega H[u]$ is defined in analogy with (1.12),

$$\omega(v, \nabla_\omega H[u]) = dH[u](v). \tag{1.15}$$

For example, on the Hilbert space $H_0^{-\frac{1}{2}}(\mathbb{T})$, we can define the symplectic form

$$\omega_{-\frac{1}{2}}(u, v) := \int_{\mathbb{T}} u(x)\partial_x^{-1} v(x)\ dx \tag{1.16}$$

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[4]That is, a nondegenerate, antisymmetric form $\omega : \mathbb{H} \times \mathbb{H} \to \mathbf{C}$. We identify in the usual way $\mathbb{H}$ and it's tangent space $T_x\mathbb{H}$ for each $x \in \mathbb{H}$.

[5]That is, a bounded, anti-selfadjoint operator with $J^2 = -(\text{identity})$.

where $\partial_x^{-1} : H_0^{-1/2}(\mathbb{T}) \to H_0^{1/2}(\mathbb{T})$ is the inverse to the differential operator ∂_x defined via the Fourier transform by

$$\widehat{\partial_x^{-1} f}(k) := \frac{1}{ik}\widehat{f}(k).$$

The KdV flow (1.1) is then *formally* the Hamiltonian equation in $(H_0^{-1/2}(\mathbb{T}), \omega_{-\frac{1}{2}})$ corresponding to the (densely defined) Hamiltonian

$$H[u] := \int_{\mathbb{T}} \frac{1}{2}u_x^2 + u^3 dx. \tag{1.17}$$

Indeed, working formally[6] we have for any $v \in H_0^{-\frac{1}{2}}(\mathbb{T})$,

$$\begin{aligned}
\frac{d}{d\epsilon}\Big|_{\epsilon=0} H[u + \epsilon v] &= \int_{\mathbb{T}} u_x v_x + 3u^2 v \, dx \\
&= \int_{\mathbb{T}} (-u_{xx} + 3u^2)v \, dx \\
&= \int_{\mathbb{T}} \partial_x^{-1}(-u_{xxx} + 6uu_x)v \, dx \\
&= -\int_{\mathbb{T}} (-u_{xxx} + 6uu_x)\partial_x^{-1} v \, dx \\
&= \omega_{-\frac{1}{2}}(u_{xxx} - 6uu_x, v) \\
&= \omega_{-\frac{1}{2}}(v, -u_{xxx} + 6uu_x).
\end{aligned}$$

Comparing (1.14)-(1.15) with (1.1), we see KdV is indeed the Hamiltonian PDE corresponding to $H[u]$ on the infinite dimensional symplectic space $(H_0^{-\frac{1}{2}}, \omega_{-\frac{1}{2}})$. In particular, the flow maps $S_{KdV}(t)$ are, formally, symplectomorphisms on $H_0^{-1/2}(\mathbb{T})$.

That the KdV flow arises as a Hamiltonian flow from a symplectic structure as described above was discovered by Gardner and Zakharov-Faddeev (see [13, 32]). A second structure was given by Magri [24] using $\int u^2 dx$ as Hamiltonian, but it is not as convenient as the first structure for our strategy to prove nonsqueezing. Roughly speaking, it seems the symplectic form in this second structure could possibly be used to establish a nonsqueezing property - in the $H^{-\frac{3}{2}}$ topology - of a finite dimensional analog of (1.1). However, since the well-posedness theory, and the accompanying estimates, for the full KdV flow do not presently exist at such rough norms, we do not see how we could approximate the full KdV flow in a space as rough as $H^{-\frac{3}{2}}$ with a finite dimensional flow. The first structure described above allows us to adopt this strategy in the space $H_0^{-\frac{1}{2}}$, within which we do have well-posedness. (See below for references for this approach to proving nonsqueezing for PDE. See e.g [26, 12] for more details and history of the various symplectic structures for KdV.)

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[6]By the word 'formally', we mean here that no attempt is made to justify various differentiations or integration by parts. Later, when we localize the space $H_0^{-\frac{1}{2}}$ and Hamiltonian in frequency and write down the corresponding equations, the reader can carry out the analogous computation where the justification of the necessary calculus will be evident.

For any $u_* \in H_0^{-1/2}(\mathbb{T})$, $r > 0$, $k_0 \in \mathbb{Z}^*$, and $z \in \mathbf{C}$, we consider the infinite-dimensional ball

$$\mathbf{B}^\infty(u_*; r) := \{u \in H_0^{-1/2}(\mathbb{T}) : \|u - u_*\|_{H_0^{-1/2}} \leq r\}$$

and the infinite-dimensional cylinder

$$\mathbf{C}_{k_0}^\infty(z; r) := \{u \in H_0^{-1/2}(\mathbb{T}) : |k_0|^{-1/2}|\widehat{u}(k_0) - z| \leq r\}.$$

The final result of this paper is the following symplectic non-squeezing theorem,

Theorem 1.5. *Let $0 < r < R$, $u_* \in H_0^{-1/2}(\mathbb{T})$, $k_0 \in \mathbb{Z}^*$, $z \in \mathbf{C}$, and $T > 0$. Then*

$$S_{KdV}(T)(\mathbf{B}^\infty(u_*; R)) \not\subseteq \mathbf{C}_{k_0}^\infty(z; r).$$

In other words, there exists a global $H_0^{-1/2}(\mathbb{T})$ solution u to (1.1) such that

$$\|u(0) - u_*\|_{H_0^{-1/2}} \leq R$$

and

$$|k_0|^{-1/2}|\widehat{u(T)}(k_0) - z| > r.$$

Note that no smallness conditions are imposed on u_*, R, z, or T.

Roughly speaking, this Theorem asserts that the KdV flow cannot squash a large ball into a thin cylinder. Notice that the balls and cylinders can be arbitrarily far away from the origin, and the time T can also be arbitrary. Note though that this result is interesting even for $u_* = 0, z = 0$ and smooth initial data u_0, as it tells us that the flow cannot at any time uniformly squeeze the ball $B^\infty(0, R)$ even at a fixed frequency k_0. By Theorem (1.5), the well-posedness theory for KdV reviewed above, and density considerations, we know that for any $T, r < R$, there will be some initial data $u_0 \in B^\infty(0; R)$ for which[7] $|\widehat{u}(k_0, T)| > |k_0|^{\frac{1}{2}}r$. (See [5], page 96 for the same discussion in the context of a nonlinear Klein-Gordon equation.) A second immediate application of Theorem 1.5 to smooth solutions was highlighted in a different context already in [22], namely that such smooth solutions of (1.1) cannot uniformly approach some asymptotic state: for any neighborhood $B^\infty(u_0; R)$ of the initial data in $H^{-\frac{1}{2}}(\mathbb{T})$ and for any time t, the diameter of the set $S_{KdV}(t)(B^\infty(u_0; R))$ cannot be less than R.

The motivation for Theorem 1.5, and an important component of its proof, is the finite-dimensional nonsqueezing theorem of Gromov [14] (see also subsequent extensions in [15], [16]). The extension to the infinite-dimensional setting provided by a nonlinear PDE seems nontrivial. The program was initiated by Kuksin [22], [23] for certain equations where the nonlinear flow is a compact perturbation of the linear flow. That the KdV equation doesn't meet this requirement can be seen by

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[7]We are using here the statement of the Theorem only in the case $u_* = 0, z = 0$. Of course one gets a similar conclusion to the one we draw here, but with different weights and a different initial data set, by simply using the L^2 conservation and time reversability properties of the flow. That is, for any $R > r$, there is data $\tilde{u}_0 \in \{\|f\|_{L^2(\mathbb{T})} \leq R\}$ such that the evolution $\tilde{u}$ of this data satisfies $|\widehat{\tilde{u}}(k_0, T)| > r$.

an argument involving simple computations similar to those supporting Theorem 1.1: fix $\sigma \ll 1$ and for each integer $N \geq 1$ consider initial data,

$$u_{0,N}(x) := \sigma N^{\frac{1}{2}} \cos(Nx).$$

Clearly the set $\{u_{0,N} : N = 1, 2, \ldots\}$ is bounded in $H_0^{-\frac{1}{2}}$. However, when one computes the second iterate[8] $u_N^{[2]}$ one sees that it differs from the linear evolution of $\widehat{u_N^{[0]}}$ at frequency $k = N$ in that,

$$\widehat{u_N^{[2]}}(N, t) - \widehat{u_N^{[0]}}(N, t) \sim N^{\frac{1}{2}} \sigma^3 e^{iN^3 t}. \tag{1.18}$$

By the local well-posedness theory we know, assuming σ is sufficiently small compared to t, that the difference between the second iterate and the actual nonlinear evolution $u_N(t)$ of the data $u_{0,N}$ satisfies,

$$\|u_N(t) - u_N^{[2]}(t)\|_{H_0^{-\frac{1}{2}}(\mathbb{T})} \lesssim \sigma^4. \tag{1.19}$$

Together, (1.18) and (1.19) show that if $\{N_k\}$ is a sequence of integers relatively prime to one-another[9], then

$$\widehat{u_{N_k}}(N_l, t) - \widehat{u_{N_k}^{[0]}}(N_l, t) \sim \delta_{k,l} \cdot \sigma^3 \cdot N_k^{\frac{1}{2}} e^{iN_k^3 t}.$$

Hence the set $\{u_{N_k}(t) - u_{N_k}^{[0]}(t)\}$ has no limit point in $H_0^{-\frac{1}{2}}(\mathbb{T})$.

In [22] Kuksin first defines the concept of *capacity* then he proves that the *capacity* is preserved by the infinite dimensional flow maps that he considers. A corollary of this result is then non-sqeezing theorems. It is important to mention that also in this work a finite dimensional approximation is considered. The nonsqueezing results of Kuksin were extended to certain stronger nonlinearities by Bourgain [3, 5] - for instance [3] treats the the cubic non-linear Schrödinger flow on $L^2(\mathbb{T})$. In these works, the full solution map is shown to be well-approximated by a finite dimensional flow constructed by cutting the solution off to frequencies $|k| \leq N$ for some large N. The nonsqueezing results in [3, 5] follow then from a direct application of Gromov's finite dimensional nonsqueezing result to this approximate flow.

As mentioned above, the argument we follow here for the KdV flow is similar[10] to the work in [3, 5], but seems to require a bit more care. The complication seems to us to be somehow rooted in the counterexample of Theorem 1.1, which clearly exhibits that a sharp cut-off is not appropriate in constructing the approximating flow, but which seems also to be subtly related to the fact that the estimates necessary to approximate the full KdV flow by a more gradually truncated flow are unavailable to us when we work directly with the KdV equation. We have already sketched how

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[8]See [10] for the notation used here, and if necessary for what we hope is a sufficiently detailed discussion to allow the reader to reproduce the elementary computations we quote here.

[9]Note (for example by examining the iterates and using well-posedness) that $\widehat{u}_N(t)$ is supported only at frequencies which are integer multiples of N.

[10]We do not know wether one could define also for the KdV and 1D cubic Schrödinger equations the concept of *capacity* and prove that the respective flows preserve it.

we will deal with this difficulty (that is, by passing to the modified KdV equation) in the discussion which followed Theorem 1.3 above.

We now provide some details of the previous paragraph's sketch, in particular we indicate the difficulties that arise when one tries to repeat the argument in [3, 5].

Let $N \geq 1$ be an integer. By simply restricting the form $\omega_{-\frac{1}{2}}$, the space $(P_{\leq N} H_0^{-1/2}(\mathbb{T}), \omega_{-\frac{1}{2}})$ is a $2N$-dimensional real symplectic space and hence by general arguments (see e.g. Proposition 1 in [16]) is symplectomorphic to the standard space $(\mathbb{R}^{2N}, \omega_0)$. We will make explicit use of such an equivalence below: any $u \in P_{\leq N} H_0^{-1/2}(\mathbb{T})$ is determined completely by

$$
\begin{aligned}
(\text{Re}(\widehat{u}(1)), \ldots, &\text{Re}(\widehat{u}(N)), \text{Im}(\widehat{u}(1)), \ldots, \text{Im}(\widehat{u}(N))) \\
&\equiv (e_1(u), \ldots, e_n(u), f_1(u), \ldots, f_N(u)) \in \mathbb{R}^{2N}.
\end{aligned}
\tag{1.20}
$$

In terms of the coordinates (1.20) the form $\omega_{-\frac{1}{2}}$ defined in (1.16) can be written using the Plancherel theorem as,

$$
\begin{aligned}
\omega_{-\frac{1}{2}}(u, v) &= \sum_{\substack{k = -N \\ k \neq 0}}^{N} \widehat{u}(-k) \frac{1}{ik} \widehat{v}(k) \\
&= \sum_{k=1}^{N} \frac{1}{ik} \left(\widehat{u}(-k)\widehat{v}(k) - \widehat{u}(k)\widehat{v}(-k) \right) \\
&= \sum_{k=1}^{N} \frac{2}{k} (\text{Im}(\widehat{v}(k)\overline{\widehat{u}(k)})) \\
&= \sum_{k=1}^{N} \frac{2}{k} (e_k(u) \cdot f_k(v) - e_k(v) \cdot f_k(u)).
\end{aligned}
$$

Write Γ for the $N \times N$ matrix $\Gamma \equiv \text{diag}(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \ldots \frac{1}{\sqrt{N}})$, $\Lambda \equiv \text{diag}(\Gamma, \Gamma)$, and $u = (\vec{e}(u), \vec{f}(u)) \in \mathbb{R}^{2N}$ for the coordinates in $P_{\leq N} H_0^{-1/2}(\mathbb{T})$, we summarize the discussion above by saying,

$$
\omega_{-\frac{1}{2}}(u, v) = \omega_0(\Lambda(\vec{e}(u), \vec{f}(u)), \Lambda(\vec{e}(v), \vec{f}(v))),
\tag{1.21}
$$

where as before we've written ω_0 for the standard symplectic form on $\mathbb{R}^{2N}$. In other words,

$$
\Lambda : (P_{\leq N} H_0^{-1/2}(\mathbb{T}), \omega_{-\frac{1}{2}}) \to (R^{2N}, \omega_0)
$$

is a symplectomorphism.

Following [3], our goal is to find a flow which satisfies three conditions: it should be finite dimensional - that is, map $P_{\leq N} H^{-\frac{1}{2}}(\mathbb{T})$ into itself; it should be a symplectic map for each t; and it should well-approximate the full flow $S_{KdV}(t)$ in a sense that we will make rigorous momentarily. For now, we write $S_{\text{Good!}}^{(N)}(t)$ for this yet to be

determined flow.

$$
\begin{array}{ccc}
(P_{\leq N} H_0^{-\frac{1}{2}}, \omega_{-\frac{1}{2}}) & \xrightarrow{\ \Lambda\ } & (\mathbb{R}^{2N}, \omega_0) \\[4pt]
{\scriptstyle S^{(N)}_{\mathrm{Good!}}(t)}\Big\downarrow & & \\[4pt]
(P_{\leq N} H_0^{-\frac{1}{2}}, \omega_{-\frac{1}{2}}) & \xrightarrow[\ \Lambda\ ]{} & (\mathbb{R}^{2N}, \omega_0)
\end{array}
\tag{1.22}
$$

Note then that the map,

$$
\Lambda \circ S^{(N)}_{\mathrm{Good!}}(t) \circ \Lambda^{-1} : (\mathbb{R}^{2N}, \omega_0) \longrightarrow (\mathbb{R}^{2N}, \omega_0)
\tag{1.23}
$$

is likewise a symplectomorphism to which we can apply the finite dimensional theory of symplectic capacity (see [14], and e.g. [16]). One defines, for any $\vec{x}_* \in \mathbb{R}^{2N}, u^{(N)}_* \in P_{\leq N} H_0^{-1/2}(\mathbb{T})$, $r > 0$, $0 < |k_0| \leq N$, and $z \in \mathbb{C}$, the finite-dimensional balls in $P_{\leq N} H_0^{-1/2}(\mathbb{T}), R^{2N}$, respectively, by the notation,

$$
\mathbf{B}^N(u^{(N)}_*; r) := \{ u^{(N)} \in P_{\leq N} H_0^{-1/2}(\mathbb{T}) : \| u^{(N)} - u^{(N)}_* \|_{H_0^{-1/2}} \leq r \}
\tag{1.24}
$$

$$
B(\vec{x}_*, r) := \{ \vec{x} \in \mathbb{R}^{2N} : |\vec{x} - \vec{x}_*| \leq r \}.
\tag{1.25}
$$

and the finite-dimensional cylinders in the same spaces by,

$$
\mathbf{C}^N_{k_0}(z; r) := \{ u^{(N)} \in P_{\leq N} H_0^{-1/2}(\mathbb{T}) : |k_0|^{-1/2} |\widehat{u^{(N)}}(k_0) - z| \leq r \}
$$

$$
\mathbf{C}_{k_0}(z; r) := \{ (\vec{e}, \vec{f}) \in \mathbb{R}^{2N} : |(e_{k_0} + \sqrt{-1} f_{k_0}) - z| \leq r) \}.
$$

>From [14], (see also e.g. Theorem 1, Page 55 in the exposition [16]) we have the finite-dimensional analogue of Theorem 1.5:

Theorem 1.6 ([14]). *Assume that for some* $R, r \geq 0, z \in \mathbb{C}, 0 \leq k_0 \leq N, \vec{x}_* \in \mathbb{R}^{2N}$ *there is a symplectomorphism* ϕ *defined on* $B(\vec{x}_*, R) \subset (\mathbb{R}^{2N}, \omega_0)$ *so that*

$$
\phi(B(\vec{x}_*, R)) \subset C_{k_0}(z; r).
$$

Then necessarily $r \geq R$.

We apply this theorem to the symplectomorphism $\Lambda \circ S^{(N)}_{\mathrm{Good!}} \circ \Lambda^{-1}$ defined in (1.23) above to conclude,

Theorem 1.7. *Let* $N \geq 1$, $0 < r < R$, $u^{(N)}_* \in P_{\leq N} H_0^{-1/2}(\mathbb{T})$, $0 < |k_0| \leq N$, $z \in \mathbb{C}$, *and* $T > 0$. *Let* $S^{(N)}_{Good!}(T) : P_{\leq N} H_0^{-1/2}(\mathbb{T}) \to P_{\leq N} H_0^{-1/2}(\mathbb{T})$ *be any symplectomorphism. Then*

$$
S^{(N)}_{Good!}(T)(\mathbf{B}^N(u^{(N)}_*; R)) \not\subseteq \mathbf{C}^N_{k_0}(z; r).
$$

To deduce Theorem 1.5 from Theorem 1.7, one would like to let $N \to \infty$ and show that the flow $S^{(N)}_{\mathrm{Good!}}(T)$ converged to $S_{KdV}(T)$ in some weak sense. More precisely, one would need,

Condition 1.8. *Let $k_0 \in \mathbb{Z}^*$, $T > 0$, $A > 0$, $0 < \varepsilon \ll 1$. Then there exists an $N_0 = N_0(k_0, T, \varepsilon, A) > |k_0|$ such that*

$$|k_0|^{-1/2}|\widehat{S_{KdV}(T)}u_0(k_0) - \widehat{S_{Good!}^{(N)}(T)}u_0(k_0)| \ll \varepsilon$$

for all $N \geq N_0$ and all $u_0 \in \mathbf{B}^N(0, A)$.

Once we find a finite dimensional symplectic flow $S_{\mathrm{Good!}}^{(N)}(t)$ for which Condition 1.8 holds, it is an easy matter to conclude Theorem 1.5. Indeed, let r, R, u_*, k_0, z, T be as in that Theorem, and choose $0 < \varepsilon < (R-r)/2$. The ball $\mathbf{B}^\infty(u_*; R)$ is contained in some ball $\mathbf{B}^\infty(0; A)$ centered at the origin. We choose $N \geq N_0(k_0, T, \varepsilon, A)$ so large that $\|u_* - P_{\leq N}u_*\|_{H_0^{-1/2}} \leq \varepsilon$. From Theorem 1.7 we can find initial data $u_0^{(N)} \in P_{\leq N}H^{-\frac{1}{2}}(\mathbb{T})$ satisfying $\|u_0^{(N)} - P_{\leq N}u_*\|_{H_0^{-1/2}} \leq R - \varepsilon$, and hence by the triangle inequality,

$$\|u_0^{(N)} - u_*\|_{H_0^{-1/2}} \leq R,$$

and so that at time T we have,

$$|k_0|^{-1/2}|\widehat{S_{\mathrm{Good!}}^{(N)}(T)}u_0^{(N)}(k_0) - z| > r + \varepsilon.$$

If we then apply Condition 1.8 and the triangle inequality we obtain Theorem 1.5 with $u_0 := u_0^{(N)}$,

$$|k_0|^{-1/2}|z - \widehat{S_{KdV}(T)}u_0^{(N)}(k_0)| \geq$$

$$|k_0|^{-1/2}\left||z - \widehat{S_{\mathrm{Good!}}^{(N)}(T)}u_0^{(N)}(k_0)| - |\widehat{S_{KdV}(T)}u_0^{(N)}(k_0) - \widehat{S_{\mathrm{Good!}}^{(N)}(T)}u_0^{(N)}(k_0)|\right|$$

$$> r + \epsilon - \epsilon \quad = r.$$

It remains to define the flow $S_{\mathrm{Good!}}^{(N)}(t)$. One might first try to follow Bourgain's treatment of several different Hamiltonion PDE, notably the cubic NLS flow on $L^2(\mathbb{T})$ (see [3], [5]). Note that the Hamiltonian $H[u]$ (1.17) is well-defined on $(P_{\leq N}H_0^{-1/2}(\mathbb{T}), \omega_{-\frac{1}{2}})$, and the equation giving the corresponding Hamiltonian flow on this space can be computed as before to be (1.4), which can be viewed either as a PDE or as a system of $2N$ ODE. The maps $S_{P_{\leq N}KdV}(t)$ are therefore symplectomorphisms, but from Theorem 1.1 we know that Condition 1.8 fails.

We proceed instead by using a flow of the form (1.6) as follows: Theorem 1.2 tells us that for any multiplier $\tilde{B}$ of the form described in (1.6), the finite dimensional flow $S_{\tilde{B}KdV}$ provides a good approximation to the low frequency behavior of S_{KdV}. However, the flows $S_{\tilde{B}KdV}$ are not symplectomorphisms, and hence cannot be candidates for our flow $S_{\mathrm{Good!}}^{(N)}(t)$ in the discussion above. Fortunately, there is a quick cure for this hiccup using the approximation given by Theorem 1.3 as follows: we will define a symplectic, finite dimensional flow $S_{KdV}^{(N)}(t)$ on $P_{\leq N}H_0^{-\frac{1}{2}}$ so that the following diagram commutes.

$$
\begin{array}{ccc}
u_0 \in P_{\leq N}H_0^{-\frac{1}{2}} & \xrightarrow{\ B\ } & Bu_0 \\
{\scriptstyle S_{KdV}^{(N)}(t)}\downarrow & & \downarrow{\scriptstyle S_{B^2KdV}(t)} \\
S_{KdV}^{(N)}(t)u_0 & \xrightarrow[\ B\ ]{} & w(t)
\end{array}
\qquad (1.26)
$$

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We write explicitly the PDE defining this flow in [10]. To finish the argument we have now to show that $S_{KdV}^{(N)}(t)$ well approximates $S_{KdV}(t)$ at frequency k_0, and hence qualifies as our choice of $S_{\text{Good!}}^{(N)}(t)$, we will simply spell out the following: Theorem 1.3 allows us to replace $S_{B^2 KdV}(t)$ on the right side of (1.26) with $S_{KdV}(t)$; and our choice $N \gg |k_0|$ allows us to ignore both the mappings on the top of (1.26) (again, by Theorem 1.3) and the bottom of (1.26) (by the definition of B, this is the identity at frequency k_0). All the details are given in [10].

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