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Michael Ruzhansky James Smith

Abstract

Global time estimates of $L^p - L^q$ norms of solutions to general strictly hyperbolic partial differential equations are considered. The case of special interest in this paper are equations exhibiting the dissipative behaviour. Results are applied to discuss time decay estimates for Fokker-Planck equations and for wave type equations with negative mass.

1. Introduction

The paper is devoted to the time decay of $L^p - L^q$ norms of solutions to constant coefficients strictly hyperbolic equations of general form. It is known that such estimates lead to Strichartz estimates which are a powerful technique when dealing with nonlinear problems.

We will assume that the principal part of the equation is strictly hyperbolic. The full equation may have variable multiplicities because of the lower order terms. One question of interest is to identify properties of such equations which determine the time decay rate of solutions. Another question of interest is what happens when there are multiple characteristic roots.

Equations of higher orders appear in many applications. In particular, they arise as dispersion equations for hyperbolic systems, for example in the study of the Fokker-Planck equation and Grad systems in nonequilibrium thermodynamics. Moreover, in approximations of solutions to the Fokker-Planck equation the order of the corresponding system tends to infinity. However, it turns out to still be possible to determine the decay rate of its solutions. The behaviour exhibited by these examples is similar to the behaviour of the dissipative wave equation in the sense that characteristic roots lie in the complex upper half plane and come to the origin as single roots and at isolated points. That is why in this paper we will concentrate on equations of such type in Theorem 2.2, although we will also present a more

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general Theorem 2.1. Results described here are formulated for scalar equations. However, they can be easily extended to systems. They also yield the well-posedness results for semilinear equations. Details of such analysis will appear elsewhere.

Second order equations.
The study of $L^p - L^q$ decay estimates, or *Strichartz estimates*, for linear evolution equations began in 1970 when Robert Strichartz published two papers, [Str70a] and [Str70b]. He proved that if $u = u(x,t)$ satisfies the Cauchy problem for the homogeneous linear wave equation

$$\begin{cases} \partial_t^2 u(x,t) - \Delta_x u(x,t) = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty)\,, \\ u(x,0) = \phi(x),\ \partial_t u(x,0) = \psi(x), & x \in \mathbb{R}^n\,, \end{cases} \tag{1.1}$$

where the initial data ϕ and ψ lie in suitable function spaces such as $C_0^\infty(\mathbb{R}^n)$, then the *a priori* estimate

$$\|(u(\cdot,t), u_t(\cdot,t), \nabla_x u(\cdot,t))\|_{L^q} \le C(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\nabla_x\phi, \psi)\|_{W_p^{N_p}} \tag{1.2}$$

holds when $n \ge 2$, $p^{-1}+q^{-1} = 1$, $1 < p \le 2$ and $N_p \ge n(p^{-1}-q^{-1})$. Here $W_p^{N_p}$ stands for the standard Sobolev space with N_p derivatives over L^p. Using this estimate, Strichartz proved global existence and uniqueness of solutions to the Cauchy problem for nonlinear wave equations with suitable ("small") initial data. This procedure of proving an *a priori* estimate for a linear equation and using it, together with local existence of a nonlinear equation, to prove global existence and uniqueness for a variety of nonlinear evolution equations is now standard; a systematic overview, with examples including the equations of elasticity, Schrödinger equations and heat equations, can be found, for example, in [Rac92].

There are two main approaches used in order to prove (1.2); firstly, one may write the solution to (1.1) using the d'Alembert ($n = 1$), Poisson ($n = 2$) or Kirchhoff ($n = 3$) formulae, and their generalisation to large n,

$$u(x,t) = \begin{cases} \dfrac{1}{\prod_{j=1}^{\frac{n-1}{2}}(2j-1)}\Big[\partial_t (t^{-1}\partial_t)^{\frac{n-3}{2}}\Big(t^{n-1}\fint_{\partial B_t(x)} \phi\, dS\Big) \\ \qquad\qquad + (t^{-1}\partial_t)^{\frac{n-3}{2}}\Big(t^{n-1}\fint_{\partial B_t(x)} \psi\, dS\Big)\Big] \quad (\text{odd } n \ge 3) \\ \dfrac{1}{\prod_{j=1}^{n/2} 2j}\Big[\partial_t (t^{-1}\partial_t)^{\frac{n-2}{2}}\Big(t^{n}\fint_{B_t(x)} \dfrac{\phi(y)}{\sqrt{t^2-|y-x|^2}}\, dy\Big) \\ \qquad\qquad + (t^{-1}\partial_t)^{\frac{n-2}{2}}\Big(t^{n}\fint_{B_t(x)} \dfrac{\psi(y)}{\sqrt{t^2-|y-x|^2}}\, dy\Big)\Big] \quad (\text{even } n)\,, \end{cases}$$

(for the derivation of these formulae see, for example, [Eva98]), as is done in [vW71] and [Rac92]. Alternatively, one may write the solution as a sum of Fourier integral operators:

$$u(x,t) = \mathcal{F}^{-1}\Big(\frac{e^{it|\xi|}+e^{-it|\xi|}}{2}\,\hat\phi(\xi) + \frac{e^{it|\xi|}-e^{-it|\xi|}}{2|\xi|}\,\hat\psi(\xi)\Big)\,.$$

This is done in [Str70a], [Bre75] and [Pec76], for example. Using one of these representations for the solution and techniques from either the theory of Fourier integral operators ([Pec76]), Bessel functions ([Str70a]) or standard analysis ([vW71]), the estimate (1.2) may be obtained.

Another problem of interest where an $L^p - L^q$ decay estimate for the linear equation is used to prove existence and uniqueness for the related nonlinear problem is the Cauchy problem for the Klein–Gordon equation. Precisely, if $u = u(x,t)$ satisfies the initial value problem

$$\begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + m^2 u(x,t) = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty)\,, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), & x \in \mathbb{R}^n\,, \end{cases} \tag{1.3}$$

where $\phi, \psi \in C_0^\infty(\mathbb{R}^n)$, say, and m is a constant (representing a *mass term*), then

$$\|(u(\cdot,t), u_t(\cdot,t), \nabla_x u(\cdot,t))\|_{L^q} \leq C(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|(\nabla_x\phi, \psi)\|_{W_p^{N_p}}\,, \tag{1.4}$$

where p, q, N_p are as before. Comparing (1.2) to (1.4), we see that the estimate for the solution to the Klein–Gordon equation decays more rapidly—there is an improvement in the exponent of the decay function of $-\frac{1}{2}(\frac{1}{p} - \frac{1}{q})$. The estimate is proved in [vW71], [Pec76] and [Hör97] in different ways, each suggesting reasons for this improvement: in [vW71], the function

$$v = v(x, x_{n+1}, t) := e^{-imx_{n+1}} u(x,t)\,, \quad x_{n+1} \in \mathbb{R}\,,$$

is defined; using (1.3), it is simple to show that v satisfies the wave equation in $\mathbb{R}^{n+1}$, and thus estimate (1.2) holds for v, yielding the desired estimate for u. This is elegant, but cannot easily be adapted to other situations due to the importance of the structures of the Klein–Gordon and wave equations for this proof. In [Pec76] and [Hör97], a representation of the solution via Fourier integral operators is used and the stationary phase method then applied in order to obtain estimate (1.4).

A third problem of interest for us is the Cauchy problem for the dissipative wave equation,

$$\begin{cases} u_{tt}(x,t) - \Delta_x u(x,t) + u_t(x,t) = 0\,, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), & x \in \mathbb{R}^n\,, \end{cases}$$

where $\psi, \phi \in C_0^\infty(\mathbb{R}^n)$. In this case,

$$\|\partial_t^r \partial_x^\alpha u(\cdot,t)\|_{L^q} \leq C(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-r-\frac{|\alpha|}{2}} \|(\phi, \nabla\psi)\|_{W_p^{N_p}}\,.$$

This is proved in [Mat76] with a view to showing well-posedness of related semilinear equations. Once again, this estimate (for the solution $u(x,t)$ itself) is better than that for the solution to the wave equation by $-\frac{1}{2}(\frac{1}{p} - \frac{1}{q})$; there is an even greater improvement for higher derivatives of the solution. As before, the proof of this may be done via a representation of the solution using the Fourier transform:

$$u(x,t) = \begin{cases} \mathcal{F}^{-1}\Bigg(\Bigg[\dfrac{e^{-t/2}\sinh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)}{\sqrt{1-4|\xi|^2}} + e^{-t/2}\cosh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)\Bigg]\hat{\phi}(\xi) \\ \qquad\qquad + \dfrac{2e^{-t/2}\sinh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)}{\sqrt{1-4|\xi|^2}}\hat{\psi}(\xi)\Bigg), \quad |\xi| \le 1/2, \\ \mathcal{F}^{-1}\Bigg(\Bigg[\dfrac{e^{-t/2}\sin\left(\frac{t}{2}\sqrt{4|\xi|^2-1}\right)}{\sqrt{4|\xi|^2-1}} + e^{-t/2}\cos\left(\frac{t}{2}\sqrt{4|\xi|^2-1}\right)\Bigg]\hat{\phi}(\xi) \\ \qquad\qquad + \dfrac{2e^{-t/2}\sin\left(\frac{t}{2}\sqrt{4|\xi|^2-1}\right)}{\sqrt{4|\xi|^2-1}}\hat{\psi}(\xi)\Bigg), \quad |\xi| > 1/2. \end{cases}$$

Matsumura divides the phase space into the regions where the solution has different properties and then uses standard techniques from analysis.

Problem.
It is, therefore, interesting to ask why the addition of lower order terms improves the rate of decay of the solution to the equation; furthermore, we would like to understand why the improvement in the decay is the same for both the addition of a mass term and for the addition of a dissipative term. In the proof of each of the estimates (see the papers cited above), the critical role is played by the characteristic roots of the equations. In fact, it is the difference in the behaviour of the characteristic roots of the Klein–Gordon equation and the dissipative wave equation which yield improvement over the decay rate for the wave equation.

The aim of this paper is to investigate this phenomenon for higher order hyperbolic equations and see how lower order terms affect the rate of decay compared to that for the homogeneous m^{th} order equation and the examples above. Equations of this type appear in many applications. In particular, they arise as dispersion equations for hyperbolic $m \times m$ systems. The order m may be large, as in, for example, Grad systems coming from nonequilibrium thermodynamics, where it corresponds to the number of moments under consideration. Moreover, in applications to Fokker-Planck equations describing the distribution of Brownian particles, the order m corresponds to the Galerkin approximation of solutions, so it is increasing to infinity. In all these cases equations become too large and involved to analyse explicitly, so we are led to study properties which determine the decay rate of $L^p - L^q$ estimates in the general form.

We will consider the Cauchy problem for m^{th} order constant coefficient linear strictly hyperbolic equation of the general form for $u = u(x,t)$:

$$\begin{cases} D_t^m u + \sum\limits_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum\limits_{l=0}^{m-1} \sum\limits_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, \quad t > 0, \\ D_t^l u(x,0) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1, \ x \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

where $P_j(\xi)$ is a constant coefficient homogeneous polynomial of order j, and the $c_{\alpha,r}$ are constants.

We seek *a priori* estimates for the solution to this problem of the type

$$\|D_x^\alpha D_t^r u(\cdot,t)\|_{L^q} \le K(t) \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p-l}}\,, \tag{1.6}$$

where $1 \le p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $N_p = N_p(\alpha, r)$ is a constant depending on p, α and r, and $K(t)$ is a function to be determined.

Homogeneous equations.
The case where the operator in (1.5) is homogeneous has been studied extensively and provides many interesting relations to the geometric properties of characteristics. In this case we have

$$\begin{cases} L(D_x, D_t)u = 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty) \\ D_t^l u(x,0) = f_l(x), \quad l = 0,\dots,m-1,\ x \in \mathbb{R}^n\,, \end{cases} \tag{1.7}$$

where L is a homogeneous m^{th} order constant coefficient strictly hyperbolic differential operator; the symbol of L may be written in the form

$$L(\tau,\xi) = (\tau - \varphi_1(\xi))\dots(\tau - \varphi_m(\xi)), \text{ with } \varphi_1(\xi) > \dots > \varphi_m(\xi) \quad (\xi \ne 0).$$

In a series of papers, [Sug94], [Sug96] and [Sug98], Sugimoto showed how the geometric properties of the characteristic roots $\varphi_1(\xi),\dots,\varphi_m(\xi)$ affect the $L^p - L^q$ estimate. To understand this, let us summarise the method of approach.

Firstly, the solution can be written as the sum of Fourier multipliers:

$$u(x,t) = \sum_{l=0}^{m-1} [E_l(t) f_l](x), \quad \text{where } E_l(t) = \sum_{k=1}^{m} \mathcal{F}^{-1} e^{it\varphi_k(\xi)} a_{k,l}(\xi)\mathcal{F}$$

and $a_{k,l}(\xi)$ is homogeneous of order $-l$. Now, the problem of finding an $L^p - L^q$ decay estimate for the solution is reduced to showing that operators of the form

$$M_r(D) := \mathcal{F}^{-1} e^{i\varphi(\xi)} |\xi|^{-r} \chi(\xi)\mathcal{F}\,,$$

where $\varphi(\xi) \in C^\omega(\mathbb{R}^n \setminus \{0\})$ is homogeneous of order 1 and $\chi \in C^\infty(\mathbb{R}^n)$ is equal to 1 for large ξ and zero near the origin, are $L^p - L^q$ bounded for suitably large $r \ge l$. In particular, this means that, for such r,

$$\|E_l(1) f\|_{L^q} \le C \|f\|_{W_p^{r-l}}\,.$$

Indeed, it may be assumed, without loss of generality, that $t = 1$ since for $t > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$, we easily have

$$[E_l(t) f](x) = t^l [E_l(1) f(t\cdot)](t^{-1}x)\,.$$

Using this identity gives

$$\begin{aligned} \|E_l(t) f\|_{L^q}^q &= t^{lq} \|[E_l(1) f_t](t^{-1}\cdot)\|_{L^q}^q = t^{lq} \int_{\mathbb{R}^n} |[E_l(1) f_t](t^{-1}x)|^q \, dx \\ &\overset{(x=tx')}{=} t^{lq} \int_{\mathbb{R}^n} t^n |[E_l(1) f_t](x')|^q \, dx' = t^{lq+n} \|E_l(1) f_t\|_{L^q}^q\,. \end{aligned}$$

Then, noting that a simple change of variables yields

$$\|f_t\|_{W_p^k}^p \leq Ct^{kp-n}\|f\|_{W_p^k}^p ,$$

we have,

$$\|E_l(t)f\|_{L^q} \leq Ct^{l+\frac{n}{q}}\|f_t\|_{W_p^{r-l}} \leq Ct^{r-n(\frac{1}{p}-\frac{1}{q})}\|f\|_{W_p^{r-l}} ;$$

hence,

$$\|u(\cdot,t)\|_{L^q} \leq Ct^{r-n(\frac{1}{p}-\frac{1}{q})}\sum_{l=0}^{m-1}\|f_l\|_{W_p^{r-l}} .$$

It has long been known that the values of r for which $M_r(D)$ is $L^p - L^q$ bounded depends on the geometry of the level set

$$\Sigma_\varphi = \{\xi \in \mathbb{R}^n \setminus \{0\} : \varphi(\xi) = 1\} .$$

In [Lit73], [Bre75] it is shown that if the Gaussian curvature of Σ_φ is never zero then $M_r(D)$ is $L^p - L^q$ bounded when $r \geq \frac{n+1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$. This is extended in [Bre77], where it is proven that $M_r(D)$ is $L^p - L^q$ bounded provided $r \geq \frac{2n-\rho}{2}\left(\frac{1}{p}-\frac{1}{q}\right)$, where $\rho = \min_{\xi\neq 0} \operatorname{rank} \operatorname{Hess} \varphi(\xi)$.

Sugimoto extended this further in [Sug94], where he showed that if Σ_φ is convex then $M_r(D)$ is $L^p - L^q$ bounded when $r \geq \left(n - \frac{n-1}{\gamma(\Sigma)}\right)\left(\frac{1}{p}-\frac{1}{q}\right)$; here,

$$\gamma(\Sigma) := \sup_{\sigma\in\Sigma}\sup_{P}\gamma(\Sigma;\sigma,P) , \quad \Sigma \subset \mathbb{R}^n \text{ a hypersurface} ,$$

where P is a plane containing the normal to Σ at σ and $\gamma(\Sigma;\sigma,P)$ denotes the order of the contact between the line $T_\sigma \cap P$, T_σ is the tangent plane at σ, and the curve $\Sigma \cap P$.

In order to apply this result to the solution of (1.7), it is necessary to find a condition under which the level sets of the characteristic roots are convex. The following notion is the one that is sufficient. Let $L = L(D_x, D_t)$ be a homogeneous m^{th} order constant coefficient partial differential operator. It is said to satisfy the *convexity condition* if the Hessian, $\operatorname{Hess}\varphi_k(\xi)$, corresponding to each of its characteristic roots $\varphi_1(\xi), \ldots, \varphi_m(\xi)$ is semi-definite for $\xi \neq 0$.

It can be shown that if an operator L does satisfy this convexity condition, then the above results can be applied to the solution and thus an estimate of the form (1.6) holds with

$$K(t) = (1+t)^{-\frac{n-1}{\gamma}\left(\frac{1}{p}-\frac{1}{q}\right)} ,$$

where $\gamma = \max_{1\leq k\leq m}\gamma(\Sigma_{\phi_k})$. We also have $\gamma \leq m$.

Finally, in the case when this convexity condition does not hold, it was shown in [Sug96] and [Sug98] that, in general, $M_r(D)$ is $L^p - L^q$ bounded when $r \geq \left(n - \frac{1}{\gamma_0(\Sigma)}\right)\left(\frac{1}{p}-\frac{1}{q}\right)$, where

$$\gamma_0(\Sigma) := \sup_{\sigma\in\Sigma}\inf_{P}\gamma(\Sigma;\sigma,P) \leq \gamma(\Sigma).$$

For $n = 2$, $\gamma_0(\Sigma) = \gamma(\Sigma)$, so, the convexity condition may be lifted in that case. However, in [Sug96], examples are given when $n \geq 3$, $p = 1, 2$ where this lower bound for r is the best possible and, thus, the convexity condition is necessary for the above estimate. It turns out that the case $n \geq 3$, $1 < p < 2$ is more interesting and is studied in greater depth in [Sug98], where microlocal geometric properties must be looked at in order to obtain an optimal result. It can be noted that in $L^p - L^p$ estimates other geometric properties of phase function and wave fronts become important, see the survey [Ruzh00] for more details.

Two remarks are worth making; firstly, the convexity condition result recovers the Strichartz decay estimate for the wave equation, since that clearly satisfies such a condition, Secondly, the convexity condition is an important restriction on the geometry of the characteristic roots that affects the $L^p - L^q$ decay rate; hence, in the case of an m^{th} order operator with lower order terms we must expect some geometrical conditions on the characteristic roots to obtain decay.

2. Main Results

In this paper we will present conditions under which we can obtain $L^p - L^q$ decay estimates for the general m^{th} order linear, constant coefficient, strictly hyperbolic Cauchy problem

$$\begin{cases} P(D_t, D_x) \equiv D_t^m u + \sum_{j=1}^{m} P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, \quad t > 0, \\ D_t^l u(x,0) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1, \ x \in \mathbb{R}^n \,. \end{cases} \tag{2.1}$$

As usual, the strict hyperbolicity means that the principal symbol of the operator $P(D_t, D_x)$ is strictly hyperbolic, i.e. has real roots, distinct for $\xi \neq 0$. However, characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$ of the full symbol may have any multiplicities. Since we are interested in the question of how do lower order terms influence time decay rates, we do not want to worry about the well-posedness of the Cauchy problem and therefore assume that the principal part of the operator is strictly hyperbolic. Our main Theorem 2.1 states how different behaviour of the characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$ affect the rate of decay that can be obtained. For now we will assume that symbols of $P_j(D_x)$ are homogeneous polynomials of order j with constant coefficients. However, results extend to the case when P_j are pseudo-differential operators of order j and when lower order terms are pseudo-differential in D_x, provided the statement of Lemma 3.4 holds. This case is essential when considering hyperbolic systems and their dispersion equations.

It is natural to impose the condition:

$$\operatorname{Im} \tau_k(\xi) \geq 0 \quad \text{for } k = 1, \dots, m \ \text{ and for all } \ \xi \in \mathbb{R}^n; \tag{2.2}$$

this is equivalent to requiring the characteristic polynomial of the operator to be stable at all points $\xi \in \mathbb{R}^n$, and thus cannot be lifted, since we can not expect any time decay if this condition fails. Coefficients of equation (2.1) are allowed to be complex as long as condition (2.2) holds. Of course, because of the strict hyperbolicity, coefficients of the principal part are real.

Also, it is sensible to divide the considerations of how characteristic roots behave into two parts: their behaviour for large values of $|\xi|$ and for bounded values of $|\xi|$. These two cases are then subdivided further; in particular the following are the key properties to consider:

- multiplicities of roots (this only occurs in the case of bounded $|\xi|$);
- whether roots lie on the real axis or are separated from it;
- behaviour as $|\xi| \to \infty$ (only in the case of large $|\xi|$);
- how roots meet the real axis (if they do);
- properties of the Hessian of the root, $\operatorname{Hess}\tau_k(\xi)$;
- a convexity-type condition, as in the case of homogeneous roots.

Some definitions will be needed for the main theorem. Given a smooth function $\tau : \mathbb{R}^n \to \mathbb{R}$ and $\lambda \in \mathbb{R}$, set

$$\Sigma_\lambda \equiv \Sigma_\lambda(\tau) := \{\xi \in \mathbb{R}^n : \tau(\xi) = \lambda\} \, .$$

In the case where $\tau(\xi)$ is homogeneous of order one, write $\Sigma_\tau := \Sigma_1(\tau)$—for such τ, we then have $\Sigma_\lambda(\tau) = \lambda\Sigma_\tau$. Also, a smooth function $\tau : \mathbb{R}^n \to \mathbb{R}$ will be said to satisfy the *convexity condition* if Σ_λ is convex for each $\lambda \in \mathbb{R}$. Note that the empty set is considered to be convex. Finally, we will use notation in the introduction for the *maximal orders of contact* of a hypersurface, $\gamma(\Sigma)$ and $\gamma_0(\Sigma)$. We note that if $p(\xi)$ is a polynomial of order m and $\Sigma = \{\xi \in \mathbb{R}^n : p(\xi) = 0\}$ is compact then $\gamma_0(\Sigma) \leq \gamma(\Sigma) \leq m$; this is useful when applying the result below to hyperbolic differential equations and is proved in [Sug96].

Now we may state the main theorem:

Theorem 2.1. *Suppose $u = u(x,t)$ satisfies the m^{th} order linear, constant coefficient, strictly hyperbolic Cauchy problem* (2.1). *Denote the characteristic roots of the operator by $\tau_1(\xi), \dots, \tau_m(\xi)$ and assume that* (2.2) *holds.*

We introduce two functions, $K^{(l)}(t)$ and $K^{(b)}(t)$, which take values as follows:

1. *Consider the behaviour of each characteristic root, $\tau_k(\xi)$, in the region $|\xi| \geq N$, where N is some large number. The following table gives values for the function $K_k^{(l)}(t)$ corresponding to possible properties of $\tau_k(\xi)$; if $\tau_k(\xi)$ satisfies more than one, then take $K_k^{(l)}(t)$ to be function that decays the slowest as $t \to \infty$.*

Location of $\tau_k(\xi)$	Additional Property	$K_k^{(l)}(t)$
away from real axis		$e^{-\delta t}$, some $\delta > 0$
on real axis	$\det \mathrm{Hess}\, \tau_k(\xi) \neq 0$	$(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$
	$\mathrm{rank}\, \mathrm{Hess}\, \tau_k(\xi) = n-1$	$(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$
	convexity condition, γ	$(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$
	no convexity condition, γ_0	$(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$
asymptotic to real axis	$\det \mathrm{Hess}\, \tau_k(\xi) \neq 0$	$(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$
	$\mathrm{rank}\, \mathrm{Hess}\, \tau_k(\xi) = n-1$	$(1+t)^{-\frac{n-1}{2}(\frac{1}{p}-\frac{1}{q})}$
	no convexity condition, γ_0	$(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$

Then take $K^{(l)}(t) = \max_{k=1\dots,m} K_k^{(l)}(t)$.

1. *Consider the behaviour of the characteristic roots in the bounded region* $|\xi| \leq N$; *again, take* $K^{(b)}(t)$ *to be the maximum (slowest decaying) function for which there are roots satisfying the conditions in the following table*:

Location of Root(s)	Properties	$K^{(\mathrm{b})}(t)$
away from axis	no multiplicities	$e^{-\delta t}$, some $\delta > 0$
	L roots coinciding	$(1+t)^{L-1}e^{-\delta t}$
on axis, no multiplicities	$\det \mathrm{Hess}\, \tau_k(\xi) \neq 0$	$(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$
	convexity condition, γ	$(1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$
	no convexity condition, γ_0	$(1+t)^{-\frac{1}{\gamma_0}(\frac{1}{p}-\frac{1}{q})}$
meeting axis with finite order s	L roots coincide on set of codimension ℓ	$(1+t)^{L-1-\frac{\ell}{s}(\frac{1}{p}-\frac{1}{q})}$

Then, with $K(t) = \max\left(K^{(b)}(t), K^{(l)}(t)\right)$, *the following estimate holds*:

$$\|D_x^\alpha D_t^r u(\cdot,t)\|_{L^q} \leq K(t) \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_p-l}},$$

where $1 < p \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, *and* $N_p = N_p(\alpha, r)$ *is a constant depending on* p, α *and* r.

Let us make a number of remarks on how to understand this theorem. Since the decay rate does depend on the behaviour of characteristic roots at different points, we single out properties which determine this decay rate. Since the same characteristic root, say τ_k, may exhibit different properties at different points, we look at the corresponding rates $K^{(\mathrm{b})}(t), K^{(\mathrm{l})}(t)$ under each possible condition and then take the slowest one for the final answer. It also means that if we microlocalise in a region where only one of these properties holds, we can get the decay rate straight from the table for the corresponding solution. In some cases, especially when roots do not lie on the axis for large ξ, the result may be extended to $p = 1$.

In Part I of the statement, it can be shown by the perturbation arguments that only three cases are possible for large ξ, namely, the characteristic root may be uniformly separated from the real axis, it may lie on the axis, or it may converge to the real axis at infinity. If, for example, the root lies on the axis and, in addition, it satisfies the convexity condition with index γ, we get the corresponding decay rate $K^{(1)}(t) = (1+t)^{-\frac{n-1}{\gamma}(\frac{1}{p}-\frac{1}{q})}$. Indices γ and γ_0 in the tables are defined as the maximum of the corresponding indices $\gamma(\Sigma_\lambda)$ and $\gamma(\Sigma_\lambda)$, where $\Sigma_\lambda = \{\xi : \tau_k(\xi) = \lambda\}$, over all k and over all λ, for which ξ lies in the corresponding zone.

The statement in Part II is more involved since we may have multiple roots intersecting on rather irregular sets. The number L of coinciding roots corresponds to the number of roots which actually contribute to the loss of regularity. For example, operator $(\partial_t^2 - \Delta)(\partial_t^2 - 2\Delta)$ would have $L = 2$ for both pairs of roots intersecting at the origin. Meeting the axis with finite order s means that we have the estimate

$$\operatorname{dist}(\xi, Z_k)^s \leq c|\operatorname{Im} \tau_k(\xi)| \tag{2.3}$$

for all the intersecting roots, where $Z_k = \{\xi : \operatorname{Im} \tau_k(\xi) = 0\}$. In Part II of Theorem 2.1, the condition that L roots meet the axis with finite order s on a set of codimension ℓ means that all these estimates hold and that there is a (regular) set Z of codimension ℓ such that $Z_k \subset Z$ for all corresponding k. In Theorem 2.2 we will discuss the special case of a single root τ_k meeting the axis at a point ξ_0 with order s, which means that $\operatorname{Im} \tau_k(\xi_0) = 0$ and that we have the estimate $|\xi - \xi_0|^s \leq c|\operatorname{Im} \tau_k(\xi)|$. In fact, under certain conditions an improvement in this part of the estimates is possible, see Theorem 2.2 and Remark 2.3.

In addition to the theorem, if we have L multiple roots which coincide on the real axis on a set S of codimension ℓ, we have an estimate

$$|u(t,x)| \leq C(1+t)^{L-1-\ell} \sum_{l=0}^{m-1} \|f_l\|_{L^1}, \tag{2.4}$$

if we cut off the Fourier transform of the Cauchy data to the ϵ-neighbourhood S^ϵ of S with $\epsilon = 1/t$. Here we may relax the definition of the intersection above and say that if L roots coincide on a set S, then they coincide on a set of codimension ℓ if the measure of the ϵ-neighborhood S^ϵ of S satisfies $|S^\epsilon| \leq C\epsilon^\ell$ for small $\epsilon > 0$; here $S^\epsilon = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, S) \leq \epsilon\}$. The estimate (2.4) follows from the procedure described below of the resolution of multiple roots. We can then combine this with the remaining cases outside of this neighborhood, where it is possible to establish decay by different arguments. In particular, this is the case of homogeneous equations with roots intersecting at the origin. However, one sometimes needs to introduce special norms to handle L^2-estimates around the multiplicities. Details of this will appear elsewhere. Finally, in the case of a simple root we may set $L = 1$, and $\ell = n$, if it meets the axis at a point.

Theorem 2.1 allows a microlocalisation and estimates for the corresponding oscillatory integrals. In fact, Theorem 2.1 follows from its microlocal version in regions where characteristic roots are simple. In regions with multiple roots one requires additional arguments resolving the singularities caused by multiple roots followed by estimates for relevant pieces of the solution. A microlocal version of the theorem leads to better estimates since in Theorem 2.1 we take the slowest among all microlocal decay rates. Such a microlocal version and the full proof of Theorem 2.1 will appear elsewhere.

For our applications in Section 4, we only need this result in the special case where characteristic roots meet the real axis with finite order; therefore, we shall state and outline the prove the theorem in this special case.

Theorem 2.2. *Consider the m^{th} order strictly hyperbolic Cauchy problem* (2.1) *for operator $P(D_t, D_x)$, with initial data $f_j \in L^p \cap W_2^{[\frac{n}{2}]+1+|\alpha|-j+r}$ for $j = 0, \ldots, m-1$, where $1 \le p \le 2$ and $2 \le q \le \infty$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that the characteristic roots $\tau_1(\xi), \ldots, \tau_m(\xi)$ of $P(\tau, \xi) = 0$ satisfy* (2.2) *and the following conditions:*

1. *there is some $\epsilon > 0$ such that for all $k \in \{1, \ldots, m\}$ we have*
$$\liminf_{|\xi|\to\infty} \operatorname{Im} \tau_k(\xi) \ge \varepsilon\,;$$
2. *for each $\xi_0 \in \mathbb{R}^n$ there is at most one k for which $\operatorname{Im} \tau_k(\xi_0) = 0$ and there exists a constant $c > 0$ such that*
$$|\xi - \xi_0|^s \le c|\operatorname{Im} \tau_k(\xi)|,$$
for ξ in some neighbourhood of ξ_0.

Then the solution $u = u(x,t)$ to Cauchy problem (2.1) *satisfies the estimate*
$$\|D_t^r D_x^\alpha u(\cdot,t)\|_{L^q} \le C_{\alpha,r}(1+t)^{-\frac{n}{s}(\frac{1}{p}-\frac{1}{q})} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{\frac{2-p}{p}\left([\frac{n}{2}]+1\right)+|\alpha|+r-j}}.$$

Essentially, this theorem is a special case of Theorem 2.1, where we get the exponential decay from Part I, exponential decay from multiple roots away from the real axis in Part II, as well as the last line of the table in Part II with $L = 1$ and $l = n$, since we have only a single root coming to the axis. The main problem here is the possible appearance of multiple roots in the complex upper half plane. If several roots meet on the axis, the decay is then given in Part II of Theorem 2.1, where we observe the appearance of the extra power t^{L-1} compared to Theorem 2.2. If roots come to the axis on a set of other codimension ℓ, the order should change according to Theorem 2.1. If conditions of Theorem 2.2 hold only with ξ_0 we will call the polynomial $P(\tau, \xi)$ strongly stable. Such polynomials will be discussed in more detail in Section 4.

Remark 2.3: The order of time decay in Theorem 2.2 may be improved in the following cases. If $\operatorname{Im} \tau_k(\xi_0) = 0$ in (H2) implies that $\xi_0 = 0$, then we actually get
$$\left\|D_t^r D_x^\alpha u(\cdot,t)\right\|_{L^q(\mathbb{R}^n_x)} \le C(1+t)^{-\frac{n+|\alpha|}{s}\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{\frac{2-p}{p}\left([\frac{n}{2}]+1\right)+|\alpha|+r-j}}.$$

Now, assume that for all ξ_0 in (H2) we also have the estimate
$$c_0|\xi - \xi_0|^s \le c|\operatorname{Im} \tau_k(\xi)| \le c_1|\xi - \xi_0|^{s_1}. \tag{2.5}$$

with some constants $c_0, c_1 > 0$.

If $\operatorname{Im}\tau_k(\xi_0) = 0$ in (H2) implies that $\operatorname{Re}\tau_k(\xi_0) = 0$, then we actually get

$$\left\| D_t^r D_x^\alpha u(\cdot,t) \right\|_{L^q(\mathbb{R}^n_x)} \le C(1+t)^{-\left(\frac{n+rs_1}{s}\right)\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{\frac{2-p}{p}\left(\left[\frac{n}{2}\right]+1\right)+|\alpha|+r-j}}.$$

And finally, assume that for all ξ_0 such that $\operatorname{Im}\tau_k(\xi_0) = 0$ in (H2), we also have $\xi_0 = 0$ and $\operatorname{Re}\tau_k(\xi_0) = 0$. Then we actually get

$$\left\| D_t^r D_x^\alpha u(\cdot,t) \right\|_{L^q(\mathbb{R}^n_x)} \le C(1+t)^{-\frac{n+|\alpha|+rs_1}{s}\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{j=0}^{m-1} \|f_j\|_{W_p^{\frac{2-p}{p}\left(\left[\frac{n}{2}\right]+1\right)+|\alpha|+r-j}}.$$

The proof of this is based on Remark 3.8. In particular, these estimates cover the case of dissipative wave equation and applications in Section 4. A similar remark can be made for Theorem 2.1, where we also get the corresponding improvements.

3. Outline of the proof

Here we will outline the proof of Theorem 2.2. For large frequencies we have simple roots separated from the real axis so we can expect exponential decay in time there. For small frequencies, while separated from the real axis, we may have multiple roots, which may intersect on a rather irregular set. We will cut off around this set and show that we can get additional polynomial growth in time dependent on the "dimension" of this set, which is matched against exponential decay. Technically we have to establish a number of additional estimates on the solution in this case since the usual solution representation blows up around points of multiplicity. Finally, we can show the polynomial decay in time when characteristic roots approach the real axis.

3.1. Some properties of hyperbolic polynomials

Here we will describe some useful properties of hyperbolic polynomials. Let $L = L(D_x, D_t)$ be a linear m^{th} order constant coefficient partial differential operator. Then each of the characteristic roots of L, denoted $\tau_1(\xi), \ldots, \tau_m(\xi)$, is continuous in $\mathbb{R}^n$; furthermore, for each $k = 1, \ldots, m$, the characteristic root $\tau_k(\xi)$ is analytic in

$$\{\xi \in \mathbb{R}^n : \tau_k(\xi) \neq \tau_l(\xi)\, \forall\, l \neq k\}\ .$$

Let now $L = L(D_x, D_t)$ be a linear m^{th} order constant coefficient strictly hyperbolic partial differential operator. Then there exists a constant N such that, the characteristic roots $\tau_1(\xi), \ldots, \tau_m(\xi)$ of L are pairwise distinct for $|\xi| \ge N$. We also have the following symbolic properties of characteristic roots:

Proposition 3.1. *Let $L = L(D_x, D_t)$ be a linear m^{th} order constant coefficient hyperbolic partial differential operator with characteristic roots $\tau_1(\xi)$, ..., $\tau_m(\xi)$; then*

1. *for each $k = 1, \dots, m$, there exists a constant $C > 0$ such that*

$$|\tau_k(\xi)| \le C(1 + |\xi|) \quad \text{for all } \xi \in \mathbb{R}^n \,.$$

Suppose that the maximum order of the lower order terms is $0 \le K \le m-1$. Furthermore, assume that L is strictly *hyperbolic, and denote the roots of the principal part $L_m(\xi, \tau)$ by $\varphi_1(\xi), \dots, \varphi_m(\xi)$. Then we have the following*:

1. *For each $\tau_k(\xi)$, $k = 1, \dots, m$, there exists a corresponding root of the principal symbol $\varphi_k(\xi)$ (possibly after reordering) such that*

$$|\tau_k(\xi) - \varphi_k(\xi)| \le C(1 + |\xi|)^{K+1-m} \quad \text{for all } \xi \in \mathbb{R}^n \,.$$

2. *There exists $N > 0$ such that, for each characteristic root of L and for each multi-index α, we can find constants $C = C_{k,\alpha} > 0$ such that*

$$\left|\partial_\xi^\alpha \tau_k(\xi)\right| \le C|\xi|^{1-|\alpha|} \quad \text{for all } |\xi| \ge N \,,$$

3. *There exists $N > 0$ such that, for each $\tau_k(\xi)$ a corresponding root of the principal symbol $\varphi_k(\xi)$ can be found (possibly after reordering) which satisfies, for each multi-index α and $k = 1, \dots, m$,*

$$\left|\partial_\xi^\alpha \tau_k(\xi) - \partial_\xi^\alpha \varphi_k(\xi)\right| \le C|\xi|^{K+1-m-|\alpha|} \quad \text{for all } |\xi| \ge N$$

for each multi-index α and $k = 1, \dots, m$.

3.2. Representation of the solution

Recall that we begin with the Cauchy problem with solution $u = u(x,t)$:

$$\begin{cases} D_t^m u + \sum_{j=1}^m P_j(D_x) D_t^{m-j} u + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} D_x^\alpha D_t^r u = 0, & t > 0, \\ D_t^l u(x,0) = f_l(x) \in C_0^\infty(\mathbb{R}^n), \quad l = 0, \dots, m-1, \ x \in \mathbb{R}^n \,, \end{cases} \tag{3.1}$$

where symbol $P_j(\xi)$ of $P_j(D)$ is a constant coefficient homogeneous polynomial of order j, and the $c_{\alpha,r}$ are constants.

Applying the partial Fourier transform with respect to x yields an ordinary differential equation for $\hat{u} = \hat{u}(\xi, t) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x,t)\, dx$:

$$D_t^m \hat{u} + \sum_{j=1}^m P_j(\xi) D_t^{m-j} \hat{u} + \sum_{l=0}^{m-1} \sum_{|\alpha|+r=l} c_{\alpha,r} \xi^\alpha D_t^r \hat{u} = 0 \,, \tag{3.2a}$$

$$D_t^l \hat{u}(\xi, 0) = \hat{f}_l(\xi), \quad l = 0, \dots, m-1, \tag{3.2b}$$

where $(\xi, t) \in \mathbb{R}^n \times [0, \infty)$. Let $E_j = E_j(\xi, t)$, $j = 0, \dots, m-1$, be the solutions to (3.2a) with initial data

$$D_t^l E_j(\xi, 0) = \begin{cases} 1 & \text{if } l = j, \\ 0 & \text{if } l \ne j. \end{cases} \tag{3.2c}$$

Then the solution u of (3.1) can be written in the form

$$u(x,t) = \sum_{j=0}^{m-1} (\mathcal{F}^{-1} E_j \mathcal{F} f_j)(x,t), \tag{3.3}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the partial Fourier transform with respect to x and its inverse respectively.

Now, as (3.2a), (3.2c) is the Cauchy problem for a linear ordinary differential equation, we can write, denoting the characteristic roots of (3.1) by $\tau_1(\xi), \ldots, \tau_m(\xi)$,

$$E_j(\xi,t) = \sum_{k=1}^{m} A_j^k(\xi,t) e^{i\tau_k(\xi)t}$$

where $A_j^k(\xi,t)$ are polynomials in t whose coefficients depend on ξ. Moreover, for each $k = 1, \ldots, m$ and $j = 0 \ldots, m-1$, the $A_j^k(\xi,t)$ are independent of t at points of the (open) set $\{\xi \in \mathbb{R}^n : \tau_k(\xi) \neq \tau_l(\xi) \, \forall \, l \neq k\}$; when this is the case, we write $A_j^k(\xi,t) \equiv A_j^k(\xi)$. For $A_j^k(\xi)$, we have the following properties:

Lemma 3.2. *Suppose $\xi \in S_k := \{\xi \in \mathbb{R}^n : \tau_k(\xi) \neq \tau_l(\xi) \, \forall \, l \neq k\}$; then we have the following formula*:

$$A_j^k(\xi) = \frac{(-1)^j \displaystyle\sum^k_{1 \le s_1 < \cdots < s_{m-j-1} \le m} \prod_{q=1}^{m-j-1} \tau_{s_q}(\xi)}{\displaystyle\prod_{l=1, l \neq k}^{m} (\tau_l(\xi) - \tau_k(\xi))}, \tag{3.4}$$

where $\sum^k$ means sum over the range indicated excluding k. Furthermore, we have, for each $j = 0, \ldots, m-1$ and $k = 1, \ldots, m$,

1. *$A_j^k(\xi)$ is smooth in S_k;*

2. *$A_j^k(\xi) = O(|\xi|^{-j})$ as $|\xi| \to \infty$.*

Proof. The representation (3.4) follows from Cramer's rule (and is done explicitly in [Kli67]): $A_j^k(\xi) = \frac{\det V_j^k}{\det V}$, where $V := \left(\tau_i^{l-1}(\xi)\right)_{i,l=1}^{m}$ is the Vandermonde matrix and V_j^k is the matrix obtained by taking V and replacing the k^{th} column by $(\underbrace{0 \ \ldots \ 0}_{j} \ 1 \ 0 \ \ldots \ 0)^{\mathrm{T}}$.

Smoothness of $A_j^k(\xi)$ in S_k is obvious and the asymptotic behaviour is a consequence of 1 of Proposition 3.1 since (3.4) holds for all $|\xi| > N$. □

In view of Lemma 3.2, choose $N_1 > 0$ so that the $\tau_k(\xi)$, $k = 1, \ldots, n$, are distinct for $|\xi| > N_1$. Also, choose $N_2 > 0$ so that all points at which any of the roots, $\tau_k(\xi)$, meet the real axis—i.e. points $\xi \in \mathbb{R}^n$ such that, for all $\varepsilon > 0$, there exist $\xi_1, \xi_2 \in B_\varepsilon(\xi)$ with $\operatorname{Im} \tau_k(\xi_1) = 0$ and $\operatorname{Im} \tau_k(\xi_2) \neq 0$—lie in $B_{N_2}(0)$. Set $N = \max(N_1, N_2)$.

Let $\chi(\xi) = \chi_N(\xi) \in C_0^\infty(\mathbb{R}^n)$, $0 \le \chi(\xi) \le 1$, be a cut-off function that is identically 1 for $|\xi| < N$ and identically zero for $|\xi| > 2N$. Then (3.3) can be rewritten as:

$$u(x,t) = \sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j \chi \mathcal{F} f_j)(x,t) + \sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j (1-\chi) \mathcal{F} f_j)(x,t)\,. \tag{3.5}$$

3.3. Large $|\xi|$

The second term of (3.5) is the most straightforward to study: by the choice of N,

$$E_j(\xi,t)(1-\chi)(\xi) = \sum_{k=1}^{m} A_j^k(\xi)(1-\chi)(\xi) e^{i\tau_k(\xi)t}\,;$$

therefore, since each summand is smooth in $\mathbb{R}^n$,

$$\sum_{j=0}^{m-1} \mathcal{F}^{-1}(E_j(1-\chi)\mathcal{F} f_j)(x,t)$$
$$= \frac{1}{(2\pi)^n} \sum_{j=0}^{m-1} \sum_{k=1}^{m} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + \tau_k(\xi)t)} A_j^k(\xi)(1-\chi)(\xi) \hat{f}_j(\xi)\, d\xi\,.$$

Note that, unlike in the case of homogeneous strictly hyperbolic equations we may not assume that $t = 1$. Each of these integrals may be studied separately. Indeed, we have the following result:

Proposition 3.3. *Let $\tau : U \to \mathbb{C}$ be a smooth function, $U \subset \mathbb{R}^n$ open, and $a_j \in \mathcal{S}^{-j}_{1,0}(U)$. Assume:*

1. *there exists $\delta > 0$ such that $\operatorname{Im}\tau(\xi) \ge \delta$ for all $\xi \in U$;*

2. *$|\tau(\xi)| \le C(1+|\xi|)$ for all $\xi \in U$.*

Then,

$$\left\| \int_U e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi) \xi^\alpha \tau(\xi)^r \hat{f}(\xi)\, d\xi \right\|_{L^\infty(\mathbb{R}^n_x)} \le C e^{-\delta t} \|f\|_{W_1^{N_0+|\alpha|+r-j}}$$

and

$$\left\| \int_U e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi) \xi^\alpha \tau(\xi)^r \hat{f}(\xi)\, d\xi \right\|_{L^2(\mathbb{R}^n_x)} \le C e^{-\delta t} \|f\|_{W_2^{|\alpha|+r-j}}$$

for all $t > 0$, $N_0 > n$, multi-indices α, $r \in \mathbb{R}$ and $f \in C_0^\infty(U)$.

So, for all $t > 0$,

$$\left\| D_t^r D_x^\alpha \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi) \hat{f}(\xi)\, dx \Big) \right\|_{L^\infty} \le C e^{-\delta t} \|f\|_{W_1^{N_1+|\alpha|+r-j}}\,,$$
$$\left\| D_t^r D_x^\alpha \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi) \hat{f}(\xi)\, dx \Big) \right\|_{L^2} \le C e^{-\delta t} \|f\|_{W_2^{|\alpha|+r-j}}\,,$$

where $N_1 > n$, $r \geq 0$, α multi-index; interpolating then gives,

$$\left\| D_t^r D_x^\alpha \Big(\int_{\mathbb{R}^n} e^{i(x\cdot\xi+\tau(\xi)t)} a_j(\xi)\hat{f}(\xi)\, dx \Big) \right\|_{L^q} \leq C e^{-\delta t} \|f\|_{W_p^{N_p+|\alpha|+r-j}} ,$$

where $p^{-1} + q^{-1} = 1$, $1 \leq p \leq 2$, $N_p \geq n\Big(\frac{1}{p} - \frac{1}{q}\Big)$, $r \geq 0$, α a multi-index and $f \in C_0^\infty(\mathbb{R}^n)$. Thus, in this case we have exponential decay of the solution.

3.4. Bounded $|\xi|$

Let us now consider the terms of the first sum in (3.5), the case of low frequencies,

$$\mathcal{F}^{-1}(E_j \chi \mathcal{F} f)(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \Big(\sum_{k=1}^m e^{i\tau_k(\xi)t} A_j^k(\xi,t) \Big) \chi(\xi) \hat{f}(\xi)\, d\xi \,. \tag{3.6}$$

Unlike in the case above, here the characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$ are not necessarily distinct at all points in the support of the integrand (which is contained in the ball of radius $2N$ about the origin); in particular, this means that the $A_j^k(\xi,t)$ genuinely depend on t and we have no simple formula valid for them in the whole region.

For this reason, we begin by systematically separating neighbourhoods of points where roots meet—referred to henceforth as multiplicities—from the rest of the region, and then considering the two cases separately.

First, we need to understand in what type of sets the roots $\tau_k(\xi)$ can intersect:

Lemma 3.4. *The complement of the set of multiplicities of a linear strictly hyperbolic constant coefficient partial differential operator $L(D_x, D_t)$,*

$$S := \{\xi \in \mathbb{R}^n : \tau_j(\xi) \neq \tau_k(\xi) \text{ for all } j \neq k\} ,$$

is dense in $\mathbb{R}^n$.

Proof. First note

$$S = \{\xi \in \mathbb{R}^n : \Delta_L(\xi) \neq 0\} ,$$

where Δ_L is the discriminant of $L(\xi,\tau)$. Now, by Sylvester's Formula (see, for example, [GKZ94]), Δ_L is a polynomial in the coefficients of $L(\xi,\tau)$, which are themselves polynomials in ξ. Hence, Δ_L is a polynomial in ξ; as it is not identically zero (for large $|\xi|$, the characteristic roots are distinct, and hence it is non-zero at such points), it cannot be zero on an open set, and hence its complement is dense in $\mathbb{R}^n$. □

Corollary 3.5. *Let $L(\xi,\tau)$ be a linear strictly hyperbolic constant coefficient partial differential operator with characteristic roots $\tau_1(\xi), \dots, \tau_m(\xi)$. Suppose $\mathcal{M}_{kl} \subset \mathbb{R}^n$ is a set such that $\tau_k(\xi) = \tau_l(\xi)$, for some $k \neq l$, for all $\xi \in \mathcal{M}_{kl}$. For $\varepsilon > 0$, define*

$$\mathcal{M}_{kl}^\varepsilon := \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}_{kl}) \leq \varepsilon\} ;$$

denote the minimal $\nu \in \mathbb{N}$ such that $\operatorname{meas}(\mathcal{M}_{kl}^\varepsilon) \leq C\varepsilon^\nu$ *for all sufficiently small $\varepsilon > 0$ by* $\operatorname{codim} \mathcal{M}_{kl}$. *Then* $\operatorname{codim} \mathcal{M}_{kl} \geq 1$.

Proof. Follows straight from Lemma 3.4: the fact that $\mathcal{M}_{kl}$ has non-empty interior ensures that its ε-neighbourhood is bounded by $C\varepsilon$ in at least one dimension for all small $\varepsilon > 0$. ☐

With this in mind, we subdivide integral (3.6): suppose L roots meet on a set $\mathcal{M}$ with $\operatorname{codim} \mathcal{M} = \ell$; without loss of generality, assume the coinciding roots are $\tau_1(\xi), \dots, \tau_L(\xi)$. By continuity, there exists an $\varepsilon > 0$ such that only characteristic roots coinciding with $\tau_k(\xi)$, $k \in \{1, \dots, L\}$, in $\mathcal{M}^\varepsilon$ are $\tau_1(\xi), \dots, \tau_L(\xi)$. Furthermore, we may assume that $\partial\mathcal{M}^\varepsilon \in C^1$: for each $\varepsilon > 0$ there exists a set S_ε with C^1 boundary such that $\mathcal{M}^\varepsilon \subset S_\varepsilon$ and $\operatorname{meas}(\mathcal{M}^\varepsilon) \to \operatorname{meas}(S_\varepsilon)$ as $\varepsilon \to 0$. Then:

1. Let $\chi_{\mathcal{M},\varepsilon} \in C^\infty(\mathbb{R}^n)$ be a smooth function identically 1 on $\mathcal{M}^\varepsilon$ and identically zero outside $\mathcal{M}^{2\varepsilon}$; now consider the subdivision of (3.6):

$$\int_{B_{2N}(0)} e^{ix\cdot\xi} E_j(\xi,t)\hat{f}(\xi)\,d\xi = \int_{B_{2N}(0)} e^{ix\cdot\xi} E_j(\xi,t)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi + \int_{B_{2N}(0)} e^{ix\cdot\xi} E_j(\xi,t)(1-\chi_{\mathcal{M},\varepsilon})(\xi)\hat{f}(\xi)\,d\xi\,;$$

   for the second integral, simply repeat the above procedure around any root multiplicities in $B_{2N}(0) \setminus \mathcal{M}^\varepsilon$.

2. For the first integral, the case where the integrand is supported on $\mathcal{M}^\varepsilon$, split off the coinciding roots from the others:

$$\begin{aligned}\int_{B_{2N}(0)} & e^{ix\cdot\xi} E_j(\xi,t)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi \\ &= \int_{B_{2N}(0)} e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi \\ &\quad + \int_{B_{2N}(0)} e^{ix\cdot\xi}\Big(\sum_{k=L+1}^{m} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi. \qquad (3.7)\end{aligned}$$

3. For the first integral, we use techniques discussed in Section 3.5 below to estimate it.

4. For the second there are two possibilities: firstly, two or more of the roots $\tau_{L+1}(\xi), \dots, \tau_m(\xi)$ coincide in $\mathcal{M}^{2\varepsilon}$—in this case, repeat the procedure above for this integral. Alternatively, these roots are all distinct in $\mathcal{M}^{2\varepsilon}$—in this case, it suffices to study each integral separately as the $A_k^j(\xi,t)$ are independent of t, and thus the expression (3.4) is valid and we can write

$$\begin{aligned}\int_{B_{2N}(0)} & e^{ix\cdot\xi}\Big(\sum_{k=L+1}^{m} e^{i\tau_k(\xi)t}A_j^k(\xi,t)\Big)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi \\ &= \sum_{k=L+1}^{m}\int_{B_{2N}(0)} e^{i[x\cdot\xi+\tau_k(\xi)t]}A_j^k(\xi)\chi_{\mathcal{M},\varepsilon}(\xi)\hat{f}(\xi)\,d\xi\,;\end{aligned}$$

note that in this case we may use that the region is bounded to ensure the continuous functions are also bounded.

Continue this procedure until all multiplicities are accounted for in this way.

3.4.1. Roots separated from the real axis

The case where characteristic roots are separated from the real axis is similar to that for large $|\xi|$. Let us assume that $\tau_k(\xi)$ has no multiplicities in the set Ω; now, a result similar to Proposition 3.3 holds for general integrals of this form, and thus

$$\left\| D_t^r D_x^\alpha \Big(\int_\Omega e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\hat{f}(\xi)\,dx \Big) \right\|_{L^q} \leq C e^{-\delta t} \|f\|_{L^p} \,,$$

where $p^{-1}+q^{-1}=1$, $1\leq p\leq 2$, $N_p \geq n\Big(\frac{1}{p}-\frac{1}{q}\Big)$, $r\geq 0$, α a multi-index, $f\in C_0^\infty(\mathbb{R}^n)$, $\delta>0$ is a constant such that $\operatorname{Im}\tau(\xi)\geq\delta$ for all $\xi\in\Omega$ and $C\equiv C_{\Omega,r,\alpha,p}>0$. So, in this case we have also have exponential decay of the solution.

3.4.2. Roots meeting the real axis with finite order

In the case of bounded $|\xi|$, we must also consider the situation where the phase function $\tau(\xi)$ meets the real axis. Suppose $\xi_0\in\Omega$ is such a point, i.e. $\operatorname{Im}\tau(\xi_0)=0$, while in each punctured ball around ξ_0, $B'_\varepsilon(\xi_0)\subset\Omega$, $\varepsilon>0$, there exists $\xi\in B'_\varepsilon(\xi_0)$ so that $\operatorname{Im}\tau(\xi)>0$. Then, we claim that ξ_0 is a root of $\operatorname{Im}\tau(\xi)$ of finite order s: indeed, if ξ_0 were a zero of $\operatorname{Im}\tau(\xi)$ of infinite order, then, by the analyticity of $\operatorname{Im}\tau(\xi)$ at ξ_0 (which follows straight from the analyticity of $\tau(\xi)$ at ξ_0) it would be identically zero in a neighbourhood of ξ_0, contradicting the assumption.

In condition (H2) of Theorem 2.2 we actually have that there exist constants $c_0,c_1>0$ such that, for all ξ sufficiently close to ξ_0,

$$c_0|\xi-\xi_0|^s \leq |\operatorname{Im}\tau(\xi)| \leq c_1|\xi-\xi_0|^2\,.$$

Indeed, the Taylor expansion of $\operatorname{Im}\tau(\xi)$ around ξ_0,

$$\operatorname{Im}\tau(\xi)=\sum_{i=1}^n \partial_{\xi_i}\operatorname{Im}\tau(\xi_0)(\xi_i-(\xi_0)_i)+O(|\xi-\xi_0|^2)\,,$$

is valid for $\xi\in B_\varepsilon(\xi_0)\subset\Omega$ for some small $\varepsilon>0$. Now, if $\xi\in B_\varepsilon(\xi_0)$, then $-\xi+2\xi_0\in B_\varepsilon(\xi_0)$ also. However,

$$\operatorname{Im}\tau(-\xi+2\xi_0)=-\sum_{i=1}^n \partial_{\xi_i}\operatorname{Im}\tau(\xi_0)(\xi_i-(\xi_0)_i)+O(|\xi-\xi_0|^2)\,;$$

thus, for $\varepsilon>0$ chosen small enough, this means that either $\operatorname{Im}\tau(\xi)\leq 0$ or $\operatorname{Im}\tau(-\xi+2\xi_0)\leq 0$—contradicting the hypothesis that $\operatorname{Im}\tau(\xi)\geq 0$ for all $\xi\in\Omega$; hence, $\partial_{\xi_i}\operatorname{Im}\tau(\xi_0)=0$ for each $i=1,\ldots,n$. In conclusion, $\operatorname{Im}\tau(\xi)=O(|\xi-\xi_0|^2)$ for all $\xi\in B_\varepsilon(\xi_0)$.

Now, we need the following result, which is based in the calculation of the L^p-L^q decay estimate for the dissipative wave equation in [Mat76], but is here extended to a more general situation so that it can be used on a wider class of equations:

Proposition 3.6. *Let $\phi : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open, be a continuous function and suppose $\xi_0 \in U$ such that $\phi(\xi_0) = 0$ and that $\phi(\xi) > 0$ in a punctured open neighbourhood of ξ_0, denoted by $V \setminus \{\xi_0\}$. Furthermore, assume that, for some $s > 0$, there exists a constant $c_0 > 0$ such that, for all $\xi \in V$,*

$$\phi(\xi) \geq c_0 |\xi - \xi_0|^s .$$

Then, for any function $a(\xi)$ that is bounded and compactly supported in U, and for all $t \geq 0$, $f \in C_0^\infty(\mathbb{R}^n)$, and $r \in \mathbb{R}$,

$$\int_V e^{-\phi(\xi)t} |\xi - \xi_0|^r |a(\xi)| |\hat{f}(\xi)| \, d\xi \leq C(1+t)^{-(n+r)/s} \|f\|_{L^1} , \tag{3.8}$$

and

$$\left\| e^{-\phi(\xi)t} |\xi - \xi_0|^r a(\xi) \hat{f}(\xi) \right\|_{L^2(V)} \leq C(1+t)^{-r/s} \|f\|_{L^2} . \tag{3.9}$$

Proof. First, we give a straightforward result that is useful in proving each of the estimates:

Lemma 3.7. *For each $\rho, M \geq 0$ and $\varsigma, c > 0$ there exists $C \equiv C_{\rho,\varsigma,M,c} \geq 0$ such that, for all $t \geq 0$,*

$$\int_0^M x^\rho e^{-cx^\varsigma t} \, dx \leq C(1+t)^{-(\rho+1)/\varsigma} \text{ and } \sup_{0 \leq x \leq M} x^\rho e^{-cx^\varsigma t} \leq C(1+t)^{-\rho/\varsigma} .$$

Proof. For $0 \leq t \leq 1$, each is clearly bounded: the first by $\frac{M^{\rho+1}}{\rho+1}$ and the second by M^ρ. For $t > 1$, set $y = xt^{1/\varsigma}$; with this substitution, the first becomes

$$\int_0^{Mt^{1/\varsigma}} y^\rho t^{-\rho/\varsigma} e^{-cy^\varsigma} t^{-1/\varsigma} \, dy \leq t^{-(\rho+1)/\varsigma} \int_0^\infty y^\rho e^{-cy^\varsigma} \, dy ,$$

while the second becomes

$$\sup_{0 \leq y \leq Mt^{1/\varsigma}} y^\rho t^{-\rho/\varsigma} e^{-cy^\varsigma} \leq t^{-\rho/\varsigma} \sup_{y \geq 0} y^\rho e^{-cy^\varsigma} ;$$

that the right-hand side of each is then bounded follows from standard results. $\square$

Returning to the proof of (3.8), as $a(\xi)$ is bounded in U by assumption, we have

$$\int_V e^{-\phi(\xi)t} |\xi - \xi_0|^r |a(\xi)| |\hat{f}(\xi)| \, d\xi \leq C \int_{V'} e^{-\phi(\xi)t} |\xi - \xi_0|^r |\hat{f}(\xi)| \, d\xi ,$$

where $V' = V \cap \operatorname{supp} a$; this, in turn, can be estimated in the following manner using the hypothesis on $\phi(\xi)$ and Hölder's inequality:

$$\begin{aligned} \int_{V'} e^{-\phi(\xi)t} |\xi - \xi_0|^r |\hat{f}(\xi)| \, d\xi &\leq C \int_{V'} e^{-c_0 |\xi - \xi_0|^s t} |\xi - \xi_0|^r |\hat{f}(\xi)| \, d\xi \\ &\leq C \int_{V'} e^{-c_0 |\xi - \xi_0|^s t} |\xi - \xi_0|^r \, d\xi \|\hat{f}\|_{L^\infty(V')} . \end{aligned}$$

Then, transforming to polar coordinates and using the Hausdorff–Young inequality, we find that, for some $\varepsilon > 0$ (chosen so that $V' \subset B_\varepsilon(\xi_0)$, possible since $a(\xi)$ is compactly supported),

$$\int_{V'} e^{-c_0|\xi-\xi_0|^s t}|\xi-\xi_0|^r \, d\xi \|\hat{f}\|_{L^\infty(V')} \\ \leq C \int_{S^{n-1}} \int_0^\varepsilon |\eta|^{r+n-1} e^{-c_0|\eta|^s t} \, d|\eta| d\omega \|f\|_{L^1(\mathbb{R}^n)} ,$$

Finally, by the first part of Lemma 3.7, we find

$$\int_V e^{-\phi(\xi)t}|\xi-\xi_0|^r |a(\xi)||\hat{f}(\xi)| \, d\xi \leq C \int_0^\varepsilon y^{r+n-1} e^{-c_0 y^s t} \, dy \|f\|_{L^1(\mathbb{R}^n)} \\ \leq C(1+t)^{-(n+r)/s}\|f\|_{L^1} .$$

This completes the proof of the first part.

Now let us look at the second part. By the second part of Lemma 3.7,

$$\left\| e^{-\phi(\xi)t}|\xi-\xi_0|^r a(\xi)\hat{f}(\xi) \right\|^2_{L^2(V)} \leq \int_{V'} e^{-2c_0|\xi-\xi_0|^s t}|\xi-\xi_0|^{2r}|\hat{f}(\xi)|^2 \, d\xi \\ \leq C(1+t)^{-2r/s} \int_{V'} e^{-c_0|\xi-\xi_0|^s t}|\hat{f}(\xi)|^2 \, d\xi .$$

The Hölder inequality implies that

$$\int_{V'} e^{-c_0|\xi-\xi_0|^s t}|\hat{f}(\xi)|^2 \, d\xi \leq \sup_{V'} \left| e^{-c_0|\xi-\xi_0|^s t} \right| \|\hat{f}\|^2_{L^2(V')} \leq C\|f\|^2_{L^2} ,$$

and together these give the required estimate (3.9). □

So, using this proposition, we have, for all $t > 0$, and sufficiently small $\varepsilon > 0$,

$$\left\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi_0)} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\hat{f}(\xi) \, d\xi \right\|_{L^\infty(\mathbb{R}^n_x)} \\ \leq \int_{B_\varepsilon(\xi_0)} e^{-\operatorname{Im}\tau(\xi)t}|a(\xi)||\tau(\xi)|^r|\xi|^\alpha|\hat{f}(\xi)| \, d\xi \leq C(1+t)^{-n/s}\|f\|_{L^1} ,$$

and, using the Plancherel Theorem,

$$\left\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi_0)} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\hat{f}(\xi) \, d\xi \right\|_{L^2(\mathbb{R}^n_x)} \\ = C\left\| e^{i\tau(\xi)t}\tau(\xi)^r \xi^\alpha a(\xi)\hat{f}(\xi) \right\|_{L^2(B_\varepsilon(\xi_0))} \leq C\|f\|_{L^2} ;$$

here we have used that $|\xi|^{|\alpha|}|\tau(\xi)|^r \leq C$ for $\xi \in V'$ for $r \in \mathbb{N}$, α a multi-index.

Thus, for all $t > 0$,

$$\left\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi_0)} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\hat{f}(\xi) \, d\xi \right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n}{s}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p} ,$$

where $1 \leq p \leq 2$, $p^{-1}+q^{-1} = 1$.

Remark 3.8: If $\xi_0 = 0$, then Proposition 3.6 further tells us that

$$\left\| D_t^r D_x^\alpha \int_{B_\varepsilon(0)} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\hat{f}(\xi)\, d\xi \right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\frac{n+|\alpha|}{s}\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p}\,.$$

If $\operatorname{Re}\tau(\xi_0) = 0$, then under condition (2.5) we have $|\tau(\xi)| \leq |\operatorname{Im}\tau(\xi)| \leq c_1|\xi-\xi_0|^{s_1}$ for ξ near ξ_0, and so we get

$$\left\| D_t^r D_x^\alpha \int_{B_\varepsilon(\xi_0)} e^{i(x\cdot\xi+\tau(\xi)t)} a(\xi)\hat{f}(\xi)\, d\xi \right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{-\left(\frac{n+rs_1}{s}\right)\left(\frac{1}{p}-\frac{1}{q}\right)} \|f\|_{L^p}\,.$$

If both assumptions hold, we get the improvement from both cases, which is the estimate by $C(1+t)^{-\left(\frac{n+|\alpha|+rs_1}{s}\right)\left(\frac{1}{p}-\frac{1}{q}\right)}$.

3.5. Estimates for bounded $|\xi|$ around multiplicities

Finally, let us turn to finding estimates for the first term of (3.7), which we may write in the form

$$\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi,t)\Big)\chi(\xi)\hat{f}(\xi)\, d\xi\,,$$

where the characteristic roots $\tau_1(\xi),\ldots,\tau_L(\xi)$ coincide on a set $\mathcal{M}\subset\Omega$ of codimension ℓ (in the sense of Corollary 3.5), $\Omega\subset\mathbb{R}^n$ is a bounded open set and $\chi\in C_0^\infty(\Omega)$.

Unlike in the case away from multiplicities of characteristic roots, we have no explicit representation for the coefficients $A_j^k(\xi,t)$, which in turn means we cannot split this into L separate integrals. To overcome this, we first show, in Section 3.5.1, that a useful representation for the above integral exists that allows us to use techniques from earlier. Using this alternative representation, it is a simple matter to find estimates in the case where the image of the set $\mathcal{M}$ is separated from the real axis. The argument may be extended to the case when it arises on the real axis as a result of all the roots meeting the axis with finite order. Such argument is more elaborate but not necessary for Theorem 2.2.

3.5.1. Resolution of multiple roots

In this section, we find estimates for

$$\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi,t)\,,$$

where $\tau_1(\xi),\ldots,\tau_L(\xi)$ coincide on a set $\mathcal{M}$ of codimension ℓ. For simplicity, first consider the simplest case, $L=2$ and $\mathcal{M}=\{\xi_0\}$; the general case works in a more involved but similar way. So, assume

$$\tau_1(\xi_0)=\tau_2(\xi_0) \text{ and } \tau_k(\xi_0)\neq\tau_1(\xi_0) \text{ for } k=3,\ldots,m\,;$$

by continuity, there exists a ball of radius $\varepsilon>0$ about ξ_0, $B_\varepsilon(\xi_0)$, in which the only root which coincides with $\tau_1(\xi)$ is $\tau_2(\xi)$. Then:

Lemma 3.9. *For all $t \geq 0$ and $\xi \in B_\varepsilon(\xi_0)$,*

$$\Big|\sum_{k=1}^{2} e^{i\tau_k(\xi)t} A_j^k(\xi,t)\Big| \leq C(1+t)e^{-\min(\operatorname{Im}\tau_1(\xi),\operatorname{Im}\tau_2(\xi))t}\,, \tag{3.10}$$

where the minimum is taken over $\xi \in B_\varepsilon(\xi_0)$.

Proof. First, note that in the set

$$S := \{\xi \in \mathbb{R}^n : \tau_1(\xi) \neq \tau_k(\xi)\ \forall k = 2,\ldots,m \text{ and } \tau_2(\xi) \neq \tau_l(\xi)\ \forall l = 3,\ldots,m\}$$

the formula (3.4) is valid for $A_j^1(\xi)$ and $A_j^2(\xi)$. Now, recall that $E_j(\xi,t) = \sum_{k=1}^m e^{i\tau_k(\xi)t}A_j^k(\xi,t)$ is the solution to the Cauchy problem (3.2a), (3.2c), and thus is continuous; therefore, for all $\eta \in \mathbb{R}^n$ such that $\tau_1(\eta) \neq \tau_k(\eta)$ and $\tau_2(\eta) \neq \tau_k(\eta)$ for $k = 3,\ldots,m$ (but allow $\tau_1(\eta) = \tau_2(\eta)$), we have

$$\sum_{k=1}^{2} e^{i\tau_k(\eta)t} A_j^k(t,\eta) = \lim_{\xi\to\eta}\left(e^{i\tau_1(\xi)t}A_j^1(\xi) + e^{i\tau_2(\xi)t}A_j^2(\xi)\right),$$

provided ξ varies in the set S (thus, ensuring $e^{i\tau_1(\xi)t}A_j^1(\xi) + e^{i\tau_2(\xi)t}A_j^2(\xi)$ is well-defined). Hence, to obtain (3.10) for all $\xi \in B_\varepsilon(\xi_0)$, it suffices to show

$$\left|e^{i\tau_1(\xi)t}A_j^1(\xi) + e^{i\tau_2(\xi)t}A_j^2(\xi)\right| \leq Cte^{-\min(\operatorname{Im}\tau_1(\xi),\operatorname{Im}\tau_2(\xi))t}$$

for all $t \geq 0$, $\xi \in B'_\varepsilon(\xi_0) = B_\varepsilon(\xi_0) \setminus \{\xi_0\}$.

Now, for all $\xi \in B'_\varepsilon(\xi_0)$, $t \geq 0$,

$$\begin{aligned} e^{i\tau_1(\xi)t}A_j^1(\xi) &+ e^{i\tau_2(\xi)t}A_j^2(\xi) \\ &= \sinh[(\tau_1(\xi)-\tau_2(\xi))t](e^{i\tau_2(\xi)t}A_j^1(\xi) - e^{i\tau_1(\xi)t}A_j^2(\xi)) \\ &\qquad + \cosh[(\tau_1(\xi)-\tau_2(\xi))t](e^{i\tau_2(\xi)t}A_j^1(\xi) + e^{i\tau_1(\xi)t}A_j^2(\xi))\,. \end{aligned} \tag{3.11}$$

Furthermore, we have the following estimates for all $\xi \in B'_\varepsilon(\xi_0)$, $t \geq 0$:

$$\begin{aligned} &\left|\sinh[(\tau_1(\xi)-\tau_2(\xi))t](A_j^1(\xi)e^{i\tau_2(\xi)t} - A_j^2(\xi)e^{i\tau_1(\xi)t})\right| \\ &\qquad \leq Ct[|e^{i\tau_2(\xi)t}| + |e^{i\tau_1(\xi)t}|] \leq Cte^{-\min(\operatorname{Im}\tau_1(\xi),\operatorname{Im}\tau_2(\xi))t}\,, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &|\cosh[(\tau_1(\xi)-\tau_2(\xi))t](A_j^1(\xi)e^{i\tau_2(\xi)t} + A_j^2(\xi)e^{i\tau_1(\xi)t})| \\ &\qquad \leq Cte^{-\min(\operatorname{Im}\tau_1(\xi),\operatorname{Im}\tau_2(\xi))t}\,. \end{aligned} \tag{3.13}$$

The proof of the first is simple: just note that

$$\frac{\sinh[(\tau_1(\xi)-\tau_2(\xi))t]}{(\tau_1(\xi)-\tau_2(\xi))} \to t \ \text{ as } \ (\tau_1(\xi)-\tau_2(\xi)) \to 0\,,$$

or, equivalently, as $\xi \to \xi_0$ through S, and $A_j^k(\xi)(\tau_1(\xi)-\tau_2(\xi))$ is continuous in $B_\varepsilon(\xi_0)$ for $k = 1,2$. The proof of the second is more technical and uses the explicit representation (3.4) for the $A_j^k(\xi)$ at points away from multiplicities of $\tau_k(\xi)$; otherwise it is similar and we omit it here.

Combining (3.11), (3.12) and (3.13) we have (3.10), which completes the proof of the lemma.

□

Suppose now that the characteristic roots $\tau_1(\xi), \dots, \tau_L(\xi)$, $2 \leq L \leq m$, coincide on a set $\mathcal{M}$ of codimension ℓ, and that $\tau_1(\xi) \neq \tau_k(\xi)$ for all $\xi \in \mathcal{M}$ when $k = L+1, \dots, m$. By continuity, we may take $\varepsilon > 0$ so that the set $\mathcal{M}^\varepsilon = \{\xi \in \mathbb{R}^n : \operatorname{dist}(\xi, \mathcal{M}) \leq \varepsilon\}$ contains no points η at which $\tau_1(\eta), \dots, \tau_L(\eta) = \tau_k(\eta)$ for $k = L+1, \dots, m$. With this notation, we can extend Lemma 3.9 to the general situation:

Lemma 3.10. *For all $t \geq 0$ and $\xi \in \mathcal{M}^\varepsilon$,*

$$\Big|\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t)\Big| \leq C(1+t)^{L-1} e^{-t \min_{k=1,\dots,L} \operatorname{Im} \tau_k(\xi)},$$

where the minimum is taken over $\xi \in B_\varepsilon(\xi_0)$.

Note that this estimate does not depend on the codimension of $\mathcal{M}$ nor its geometric structure.

3.5.2. Phase function separated from the real axis

We now turn back to finding $L^p - L^q$ estimates for

$$\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t)\Big)\chi(\xi)\hat{f}(\xi)\, d\xi,$$

when $\tau_1(\xi), \dots, \tau_L(\xi)$ coincide on a set $\mathcal{M}$ of codimension ℓ; choose $\varepsilon > 0$ so that these roots do not intersect with any of the roots $\tau_{L+1}(\xi), \dots, \tau_m(\xi)$ in $\mathcal{M}^\varepsilon$.

Here we can assume that there exists $\delta > 0$ such that $\operatorname{Im}\tau_k(\xi) \geq \delta$ for all $\xi \in \mathcal{M}^\varepsilon$—so, $\min_k \operatorname{Im}\tau_k(\xi) \geq \delta$. For this, we use the same approach as in Section 3.4.1, but using Lemma 3.10 to estimate the sum. Firstly, the $L^1 - L^\infty$ estimate:

$$\begin{aligned}
&\Big\|D_t^r D_x^\alpha\Big(\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t)\Big)\chi(\xi)\hat{f}(\xi)\, dx\Big)\Big\|_{L^\infty(\mathbb{R}^n_x)} \\
&\qquad = \Big\|\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t)\tau_k(\xi)^r\Big)\xi^\alpha\chi(\xi)\hat{f}(\xi)\, dx\Big\|_{L^\infty(\mathbb{R}^n_x)} \\
&\qquad \leq \max_k \sup_\Omega |\tau_k(\xi)|^r \int_\Omega \Big|\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t)\Big| |\xi|^{|\alpha|} |\hat{f}(\xi)|\, dx \\
&\qquad \leq C(1+t)^{L-1} e^{-\delta t}\|\hat{f}\|_{L^\infty(\Omega)} \leq C(1+t)^{L-1} e^{-\delta t}\|f\|_{L^1}.
\end{aligned}$$

Similarly, the $L^2 - L^2$ estimate:

$$\begin{aligned}
&\Big\|D_t^r D_x^\alpha\Big(\int_\Omega e^{ix\cdot\xi}\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t)\Big)\chi(\xi)\hat{f}(\xi)\, dx\Big)\Big\|_{L^2(\mathbb{R}^n_x)} \\
&\qquad = \Big\|\Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi, t)\tau_k(\xi)^r\Big)\xi^\alpha\chi(\xi)\hat{f}(\xi)\Big\|_{L^2(\Omega)} \\
&\qquad \leq C(1+t)^{L-1} e^{-\delta t}\|\hat{f}\|_{L^2(\Omega)} \leq C(1+t)^{L-1} e^{-\delta t}\|f\|_{L^2}.
\end{aligned}$$

Then,

$$\left\| D_t^r D_x^\alpha \Big(\int_\Omega e^{ix\cdot\xi} \Big(\sum_{k=1}^{L} e^{i\tau_k(\xi)t} A_j^k(\xi,t) \Big) \chi(\xi) \hat{f}(\xi)\, dx \Big) \right\|_{L^q(\mathbb{R}^n_x)} \leq C(1+t)^{L-1} e^{-\delta t} \|f\|_{L^p} ,$$

where $p^{-1} + q^{-1} = 1$, $1 \leq p \leq 2$. Once again, we have exponential decay.

4. Applications

In this section we will briefly consider applications of Theorems 2.1 and 2.2 to Fokker-Planck equations and wave equations with dissipation and negative mass. There are further applications to Grad systems linearised near equilibrium points where Theorems 2.2 and 4.1 immediately yield the corresponding decay rates. The size of these systems depends on the number of moments and dimension of the space. Some examples of these systems and their stability has been analysed in [VR03].

4.1. Fokker–Planck Equation

The classical Boltzmann equation for the particle distribution function $f = f(t, x, c)$, where $x, \mathbf{c} \in \mathbb{R}^n$, $n = 1, 2, 3$, is

$$(\partial_t + \mathbf{c} \cdot \nabla_x) f = S(f),$$

where $S(f)$ is the so-called integral of collisions. The important special case of this equation is the Fokker–Planck equation for the distribution of Brownian particles, when the integral of collisions is linear and is given by

$$S(f) = \nabla_{\mathbf{c}} \cdot (\mathbf{c} + \nabla_{\mathbf{c}}) f = \sum_{k=1}^{n} \partial_{c_k} (c_k + \partial_{c_k}) f.$$

In this case the kinetic Fokker-Planck equations takes the form

$$\left(\partial_t + \sum_{k=1}^{n} c_k \partial_{x_k} \right) f(t, x, c) = \sum_{k=1}^{n} \partial_{c_k} (c_k + \partial_{c_k}) f.$$

The Hermite-Grad method of dealing with Fokker-Planck equation consists in decomposing $f(t, x, \cdot)$ in the Hermite basis, i.e. writing

$$f(t, x, c) = \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} m_\alpha(x, t) \psi^\alpha(c),$$

where $\psi^\alpha(c) = (2\pi)^{-n/2} (-\partial_c)^\alpha \exp(-\frac{|c|^2}{2})$ are Hermite functions. They are derivatives of the Maxwell distribution ψ^0 which annihilates the integral of collisions and form a complete orthonormal basis in the weighted Hilbert space $L^2_w(\mathbb{R}^n)$ with weight $w = 1/\psi^0$. This decomposition yields the infinite system

$$\partial_t m_\beta(x, t) + \beta_k \partial_{x_k} m_{\beta - e_k}(x, t) + \partial_{x_k} m_{\beta + e_k}(x, t) + |\beta| m_\beta(x, t) = 0.$$

The Galerkin approximation f^N of the solution f is

$$f^N(t,x,c) = \sum_{0\le|\alpha|\le N} \frac{1}{\alpha!} m_\alpha(x,t)\psi^\alpha(c)\,,$$

with $m(x,t) = \{m_\beta(x,t) :\ 0 \le |\beta| \le N\}$ being the unknown function of coefficients. For $m(x,t)$ one obtains the following system of equations

$$D_t m(x,t) + \sum_j A_j D_{x_j} m(x,t) - iBm(x,t) = 0,$$

where B is a diagonal matrix, $B_{\alpha,\beta} = |\alpha|\delta_{\alpha,\beta}$, and the only non-zero elements of the matrix A_j are $a_j^{\alpha-e_j,\alpha} = \alpha_j$, $a_j^{\alpha+e_j,\alpha} = 1$. For details of these calculations see [VR04]. Hence, the dispersion equation for the system is

$$\begin{aligned} P(\tau,\xi) &\equiv \det(\tau I + \sum_j A_j\xi_j - iB) = 0 \\ P(\tau,0) &= \det(\tau I - iB) = \tau \prod_{j=1}^{N} (\tau - ji)^{\gamma_j}\,. \end{aligned} \tag{4.1}$$

Properties of this polynomial $P(\tau,\xi)$ have been extensively studied by Volevich and Radkevich in [VR04], who gave conditions and examples of situations when $\operatorname{Im}\tau_j(\xi) \ge 0$, for all $\xi \ne 0$. In our situation here we have to take additional care of possible multiple roots, as is done in Theorem 2.2.

Assume now that $P(\tau,\xi)$ is a stable polynomial, i.e. its roots $\tau(\xi)$ satisfy $\operatorname{Im}\tau(\xi) \ge 0$ and $\operatorname{Im}\tau(\xi) = 0$ imply $\xi = 0$. We will say that $P(\tau,\xi)$ is strongly stable if, moreover, its roots $\tau(\xi)$ satisfy $\operatorname{Im}\tau(\xi) \ge \epsilon > 0$ for large ξ. It follows that we satisfy the conditions of Theorem 2.2 and we have to determine the order with which the characteristic arrives at the origin. We have the following theorem about time decay of solutions to Cauchy problems for equations with strongly stable symbols.

Theorem 4.1. *Let a strongly stable polynomial $P(\tau,\xi)$ of order m have a strictly hyperbolic principal part and assume that $\partial_\tau P(0,0) \ne 0$. Let α be the multiindex of the smallest length $|\alpha|$ such that $\partial_\xi^\alpha P(0,0) \ne 0$. Let $u(x,t)$ be the solution of the Cauchy problem $P(D_t,D_x)u = 0, \partial_t^l u|_{t=0} = f_l$, $0 \le l \le m-1$. Let $1 \le p \le 2$, $2 \le q \le \infty$, and $1/p + 1/q = 1$. Then*

$$||u(\cdot,t)||_{L^q} \le C(1+t)^{-\frac{n}{|\alpha|}\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{l=0}^{m-1} ||f_l||_{W_p^{N_l}},$$

where $N_l = \frac{2-p}{p}([n/2]+1) - l$. Moreover, we have the estimate

$$\left\|\partial_t^r \partial_x^\beta u(\cdot,t)\right\|_{L^q(\mathbb{R}^n_x)} \le C(1+t)^{-\left(\frac{n+|\beta|}{|\alpha|}+r\right)\left(\frac{1}{p}-\frac{1}{q}\right)} \sum_{l=0}^{m-1} \|f_l\|_{W_p^{N_l+|\beta|+r}}\,.$$

Indeed, since the polynomial is strongly stable, the estimate in Theorem 4.1 follows from Theorem 2.2 and Remark 2.3. The improvement in the last estimate for derivatives comes from the fact that there is only one root τ such that $\tau(0) = 0$ and so the last statement of Remark 2.3 applies. In certain cases it can be shown that actually $|\alpha| = 2$, in which case we have the same decay as for dissipative wave equation.

Let us write the polynomial $P(\tau, \xi)$ from (4.1) in the form

$$P(\tau, \xi) = \sum_{j=0}^{M+1} (-i)^j P_j(\tau, \xi),$$

where P_j is a homogeneous polynomial of order $M+1-j$, and assume that $P(\tau, \xi)$ is strongly stable. In [VR04] it was shown that $P_{M+1}(\tau, \xi) \equiv 0$, $\gamma_M P_M(\tau, \xi) = M!\tau$ for some $\gamma_M > 0$, and $\gamma_{M-1} P_{M-1}(\tau, \xi) = M! \sum_{k=2}^{M+1} \frac{1}{k-1}\tau^2 - M!|\xi|^2$ for some $\gamma_{M-1} > 0$. It can be readily verified now that conditions of Theorem 4.1 hold with $|\alpha| = 2$, from which we get the estimate with $(1+t)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}$.

4.2. Wave type equations with (negative) mass and dissipation

Here we will show that we can still have time decay of solutions if we allow the negative mass but exclude certain low frequencies for Cauchy data. This is given in (4.2) below. Nonnegative but time dependent mass and dissipation with oscillations have been considered before. See, for example, [HR03] and references therein.

Let us consider second order equations of the following form

$$\begin{cases} \partial_t^2 u - c^2 \Delta u + \delta \partial_t u + \mu u = 0\,, \\ u(0, x) = 0,\ u_t(0, x) = g(x)\,. \end{cases}$$

Here δ is the dissipation and μ is the mass. For simplicity, the first Cauchy data is taken to be zero. The general case can be treated in the same way. Let us now apply Theorem 2.1 to the analysis of this equation. The associated characteristic polynomial is

$$\tau^2 - c^2|\xi|^2 - i\delta\tau - \mu = 0\,,$$

which has roots

$$\tau_{\pm}(\xi) = \frac{i\delta}{2} \pm \sqrt{c^2|\xi|^2 + \mu - \delta^2/4}\,.$$

Now, we have the following cases, which correspond to different cases of Theorem 2.1:

- $\delta = \mu = 0$. This is the wave equation.
- $\delta = 0$, $\mu > 0$. This is the Klein–Gordon equation.
- $\mu = 0$, $\delta > 0$. This is the dissipative wave equation.

- $\delta < 0$. In this case, $\operatorname{Im} \tau_-(\xi) \leq \frac{\delta}{2} < 0$ for all ξ, hence we cannot expect any decay in general.

- $\delta > 0$, $\mu > 0$. In this case the discriminant is always strictly greater than $-\delta^2/4$, and thus the roots always lie in the upper half plane and are separated from the real axis. So we have exponential decay.

  Here is the main case for us:

- $\delta \geq 0$, $\mu < 0$. In this case, note that $\operatorname{Im} \tau_-(\xi) \geq 0$ if and only if $c^2|\xi|^2 + \mu \geq 0$, i.e. the critical value is $|\xi| = \sqrt{|\mu|}/c$. Therefore, the answer depends on the Cauchy data g. In particular, if $\operatorname{supp} \hat{g}$ is contained in $c^2|\xi|^2 + \mu \geq 0$, then we may get decay of some type. More precisely, let $B(0, r)$ be the open ball with radius r centred at the origin. Then we have:

  - if g is such that $\operatorname{supp} \hat{g} \cap B(0, \frac{\sqrt{|\mu|}}{c}) \neq \varnothing$, then we have no decay;
  - if there is some $\epsilon > 0$ such that $\operatorname{supp} \hat{g} \subset \mathbb{R}^n \backslash B(0, \frac{\sqrt{|\mu|}}{c} + \varepsilon)$, then the roots are either separated from the real axis (if $\delta > 0$), and we get exponential decay, or lie on the real axis (if $\delta = 0$), and we get Klein–Gordon type behaviour (since the Hessian of τ is nonsingular).
  - if, for all g, $\operatorname{supp} \hat{g} \subset \mathbb{R}^n \setminus B(0, \frac{\sqrt{|\mu|}}{c}) = \left\{ |\xi| \geq \frac{\sqrt{|\mu|}}{c} \right\}$, then again we must consider $\delta = 0$ and $\delta > 0$ separately.
    If $\delta = 0$, then the roots lie completely on the real axis, and they meet on the sphere $|\xi| = \sqrt{|\mu|}/c$. It follows from (2.4) with $L = 2$ and $\ell = 1$ that, although the representation of solution as a sum of Fourier integrals breaks down at the sphere, the solution is still bounded in a $(1/t)$-neighbourhood of the sphere. In its complement we can get the decay.

    If $\delta > 0$, then the root τ_- comes to the real axis at $|\xi| = \frac{\sqrt{|\mu|}}{c}$, in which case we get the decay

    $$||u(\cdot, t)||_{L^q} \leq C(1+t)^{-\left(\frac{1}{p} - \frac{1}{q}\right)} ||g||_{L^p}. \tag{4.2}$$

    Indeed, in this case the order of the root τ_- at the axis is one, i.e. estimate (2.3) holds with $s = 1$. Here $1/p + 1/q = 1$ and $1 \leq p \leq 2$. Note also that compared to the case of no mass when $\ell = n$, the codimension of the set $\left\{ \xi \in \mathbb{R}^n : |\xi| = \frac{\sqrt{|\mu|}}{c} \right\}$ is $\ell = 1$. We can apply the last case of Part II of Theorem 2.1 with $L = 1$ and $s = \ell = 1$ which gives estimate (4.2).

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