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Abstract

In this short overview I present some recent results about the KAM theory for multidimensional partial differential equations (PDEs) trying to avoid technicalities. In particular I will not state a precise KAM theorem but I will focus on the dynamical consequences for the PDEs: the existence and the stability (or not) of quasi periodic in time solutions. Concretely, I present the complete study of the nonlinear beam equation on the d-dimensional torus recently obtained in collaboration with H. Eliasson and S. Kuksin. When $d \geq 2$ we are able to construct explicit examples where the quasi periodic solutions are linearly unstable, a new feature in Hamiltonian PDEs that could complement recent results in weak turbulence theory.

Contents

1. Introduction.

In this introduction I would like to convince non-specialists that a KAM theorem for a nonlinear PDE is something useful in the sense that it gives valuable informations on the dynamics of the PDE. Then I would like to explain why it is not so surprising that it works: roughly speaking, a KAM result for the nonlinear PDE

is a perturbative version of a simple fact for the associated linear PDE. I will finish the introduction with a short and non exhaustive presentation of the literature concerning KAM results for multidimensional PDEs.

In sections 2 and 3, I present the results of [11, 12] as an example of "complete" KAM study of a nonlinear PDE. In section 4, I give some keys to understand the difficulties to overcome in a space-multidimensional context.

1.1. Why a KAM theorem for PDEs and why it works?

A PDE can be viewed as a dynamical system in an infinite dimensional phase space. For most of conservative PDEs used in physics this system is Hamiltonian with the energy playing the role of the Hamiltonian function.

In the world of finite dimensional Hamiltonian systems, the class of integrable (in the sense of Liouville) systems is naturally central because we are able to describe totally their dynamics: the phase space is foliated by invariant tori corresponding to quasi-periodic solutions (see [1] for a general presentation or the second chapter of [24] for a nice overview). Then there is a marvelous theorem, namely the KAM theorem, that says that under a non resonances assumption, the small Hamiltonian perturbations of an integrable system will still exhibit a lot of quasi periodic solutions. Actually "most" of the invariant tori persist after the perturbation.

In the world of PDEs, the linear PDEs often plays the role of integrable systems. Then we expect to show with KAM technics that, close to the origin (where the nonlinearity can be considered as a perturbation) the nonlinear PDE still exhibits invariant tori and thus quasi periodic solutions. This is exactly what we can do modulo a non resonance assumption and also restricting ourself to finite dimensional tori inside the infinite dimensional phase space[1].

Of course this is not enough to understand all the dynamics of the nonlinear PDE since we only concentrate on a finite dimensional part of the phase space. Nevertheless these quasi periodic solutions are observed experimentally. The most famous example is the numerical experiment by Fermi-Pasta-Ulam(see [15] or [16] for a recent review).

Now let us try to understand why such a nice theorem is true. Actually we assume all that we need to be sure that we remain close to the linear case where the KAM conclusion (existence of invariant tori) can be seen as the consequence of the non interaction between the linear modes (the Fourier modes in the case of the torus). More precisely, let us consider a PDE on a compact manifold[2] of the following form

$$i\, u_t = Au + f(u) \tag{1.1}$$

where A is a linear operator diagonalized in an orthonormal basis of the phase space $(\varphi_j)_{j \in \mathcal{J}}$ and $f(u)$ is a nonlinear term, i.e. $f(u) = O(u^2)$. The solutions of the linear equation (when $f = 0$) read

$$u(t,x) = \sum_{j \in \mathcal{J}} c_j e^{i\omega_j t} \varphi_j(x) \tag{1.2}$$

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[1]In particular we cannot reach almost periodic solutions that correspond to infinite dimensional tori.

[2]The compactness insures a discrete spectrum which is indispensable in these theory.

where ω_j denotes the eigenvalue of A associated to the eigenfunction φ_j. So each linear mode j rotates with the frequency ω_j without interacting with the others. The natural question is: what happens when we turn on the nonlinearity? Of course the linear modes are going to interact but we can try to limit the nonlinear effects. Actually we assume:

- Smallness: we search for small solution u in such a way the nonlinear term is a perturbation.

- Regularity: we assume that the nonlinearity $u \mapsto f(u)$ preserves the regularity.

- Non resonances: we assume that the linear frequencies are non resonant. The sense of non resonant will be explained later, for the moment let us say that essentially this means that the frequencies are not rationally dependent and that this makes difficult the interactions between linear modes.

These are the three essential hypotheses for a KAM result and the typical result is that "most of" the quasi periodic solutions constructed on finitely many modes still persist after we turn on the nonlinearity. "Most of" means that the result holds true for an asymptotically (with respect to the smallness of the solution) full measure set of parameters and we distinguish two cases:

- KAM with exterior parameters: the operator A depends on parameters, $A = A(\rho)$. In that case when you change the parameters you change the PDE (1.1) and the KAM theorem does not concern one PDE but a family of PDEs.

- KAM with interior parameters: the constant c_j in (1.2) are the parameters. In that case when you change the parameters you change the initial condition.

Of course the second case is much more satisfactory but more subtle. The parameters allow to move the initial frequencies and it is easier to do it directly at the level of the linear operator.

The KAM theory also allows to decide on the linear stability of the quasi periodic solutions[3] that we construct. Although we always observed stability for 1-d PDE, we have recently (see [12]) constructed examples that exhibit unstable invariant tori (cf. the notion of whiskered tori [8]). Such tori that are invariant under the flow generated by the PDE, exhibit hyperbolic directions that could be used to escape from the torus. Such a perspective could complement the weak turbulence approach.

1.2. Short review of related literature

If the KAM theorem is now well documented for nonlinear Hamiltonian PDEs in 1-dimensional context (see [21, 22, 25] for a first overview) only few results exist for multidimensional PDEs.
Existence of quasi-periodic solutions of space-multidimensional PDE were first proved in [6] (see also [7]) but with a technic based on the Nash-Moser thorem that do not allow to analyse the linear stability of the obtained solutions. Some KAM-theorems for small-amplitude solutions of multidimensional beam equations (see (2.1) above)

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[3]I.e. the stability of the dynamical system obtained by linearizing the PDE around the solution.

with typical m were obtained in [17, 18]. Both works treat equations with a constant-coefficient nonlinearity $g(x, u) = g(u)$, which is significantly easier than the general case. The first complete KAM theorem for space-multidimensional PDE was obtained in [14]. Also see [2, 3].

The technics developed by Eliasson-Kuksin has been improved in [11, 12] to allow a KAM result without external parameters. In these two papers we prove the existence of small amplitude quasi-periodic solutions of the beam equation on the d-dimensional torus. We further investigate the stability of these solutions and give explicit examples where the solution is linearly unstable and thus exhibits hyperbolic features (a sort of whiskered torus). These results are presented in section 3.

NLS equations in the d-dimensional torus and without external parameters were considered in [28] and [26, 27], using the KAM-techniques of [6, 7] and [14], respectively. Their main disadvantage compare to the 1d theory (see [23]) is severe restrictions (a non-degeneracy condition) on the finite set of linear modes on which the quasi-periodic solutions are based. The notion of non-degeneracy is so complicated that it is even not easy to give examples of non-degenerate sets.

All these examples concern PDEs on the torus, essentially because in that case the corresponding linear PDE is diagonalized in the Fourier basis and the structure of the resonant sets remains almost the same. Recently I have considered (see [19]) two important examples that do not fit in this Fourier context: the Klein-Gordon equation on the sphere $\mathbb{S}^2$ and the quantum harmonic oscillator on $\mathbb{R}^2$. In both cases I use external parameters. For the existence of quasi-periodic solutions for NLW and NLS on compact Lie groups via a Nash Moser approach see [5, 4].

Remark 1.1. The KAM technics are also used to prove the reducibility of non homogeneous linear PDE. We consider the linear PDE

$$i\, u_t = Au + V(t, x)u$$

and the question is: Is this system reducible to a homogeneous system? And a related question: Is the flow bounded in Sobolev spaces? (see [13, 20, 9] for a first overview on this problem).

2. The nonlinear beam equation with external parameters

Consider the d dimensional beam equation on the torus[4]

$$u_{tt} + \Delta^2 u + mu + V \star u + \varepsilon g(x, u) = 0, \quad x \in \mathbb{T}^d \qquad (2.1)$$

where m is the mass, g is a real analytic function on $\mathbb{T}^d \times I$ and I is a neighborhood of the origin in $\mathbb{R}$. The convolution potential $V : \mathbb{T}^d \to \mathbb{R}$ plays the role of external parameter. It is supposed to be analytic with real Fourier coefficients $\hat{V}(a)$, $a \in \mathbb{Z}^d$, satisfying

$$|a|^4 + \hat{V}(a) + m > 0 \qquad \forall\, a \in \mathbb{Z}^d. \qquad (2.2)$$

Introducing $v = -\dot{u}$ and denoting $\Lambda = (\Delta^2 + m + V\star)^{1/2}$, we write eq. (2.1) as

$$\begin{cases} \dot{u} &= -v, \\ \dot{v} &= \Lambda^2 u + \varepsilon g(x, u). \end{cases}$$

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[4]Notice that in this section we prefer to bring out the small parameter ε in front of the nonlinearity but, of course, by a scaling of the equation we can make it disappear and then consider small-amplitude solutions as in section 3.

Defining $\psi = \frac{1}{\sqrt{2}}(\Lambda^{1/2}u + i\Lambda^{-1/2}v)$ we get

$$\frac{1}{i}\dot{\psi} = \Lambda\psi + \varepsilon\frac{1}{\sqrt{2}}\Lambda^{-1/2}g\left(x, \Lambda^{-1/2}\left(\frac{\psi + \bar{\psi}}{\sqrt{2}}\right)\right).$$

Thus, if we endow the space $L2(\mathbb{T}^d, \mathbb{C}) = \{\psi(x) = \frac{1}{\sqrt{2}}(\tilde{u}(x) + i\tilde{v}(x))\}$ with the standard real symplectic structure, given by the two-form $id\psi \wedge d\bar{\psi} = d\tilde{u} \wedge d\tilde{v}$, then eq. (2.1) becomes a hamiltonian system

$$\dot{\psi} = i\frac{\partial H}{\partial\bar{\psi}} \tag{2.3}$$

with the Hamiltonian

$$H(\psi, \bar{\psi}) = \int_{\mathbb{T}^d}(\Lambda\psi)\bar{\psi}dx + \varepsilon\int_{\mathbb{T}^d}G\left(x, \Lambda^{-1/2}\left(\frac{\psi + \bar{\psi}}{\sqrt{2}}\right)\right)dx. \tag{2.4}$$

Here G is a primitive of g with respect to the variable u: $g = \partial_u G$. The linear operator Λ is diagonal in the complex Fourier basis
$\{\varphi_s(x) = (2\pi)^{-d/2}e^{is\cdot x}, \ s \in \mathbb{Z}^d\}$:

$$\Lambda\varphi_s = \lambda_s\varphi_s, \quad s \in \mathbb{Z}^d, \qquad \lambda_s = \sqrt{|s|^4 + m + \hat{V}_s}.$$

We decompose ψ and $\bar{\psi}$ in this basis as follows:

$$\psi = \sum_{s\in\mathbb{Z}^d}\xi_s\varphi_s, \quad \bar{\psi} = \sum_{s\in\mathbb{Z}^d}\eta_s\varphi_{-s}.$$

On the complex phase-space $\mathcal{P}_{\mathbb{C}} := \ell^2(\mathbb{Z}^d, \mathbb{C}) \times \ell^2(\mathbb{Z}^d, \mathbb{C})$, endowed with the symplectic form $i\sum_s d\xi_s \wedge d\eta_s$, we consider the (complex) Hamiltonian system

$$\begin{cases} \dot{\xi}_s = i\frac{\partial H}{\partial\eta_s} \\ \dot{\eta}_s = -i\frac{\partial H}{\partial\xi_s} \end{cases} \quad s \in \mathbb{Z}^d, \tag{2.5}$$

where the Hamiltonian H is given by $H = H_0 + P$ with

$$H_0 = \sum_{s\in\mathbb{Z}^d}\lambda_s\xi_s\eta_s, \quad P = \varepsilon\int_{\mathbb{T}^d}G\left(x, \sum_{s\in\mathbb{Z}^d}\frac{\xi_s\varphi_s(x) + \eta_s\varphi_{-s}(x)}{\sqrt{2\lambda_s}}\right)dx. \tag{2.6}$$

The beam equation (2.1) is then equivalent to the hamiltonian system (2.5), restricted to the real subspace

$$\mathcal{P}_{\mathbb{R}} := \{(\xi, \eta) \in \ell^2(\mathbb{Z}^d) \times \ell^2(\mathbb{Z}^d, \mathbb{C}) \mid \eta_s = \bar{\xi}_s, \ s \in \mathbb{Z}^d\}.$$

Let $\mathcal{A}$ be any subset of cardinality n in $\mathbb{Z}^d$. We denote $\mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A}$. Let us fix any n vector $I = \{I_a > 0, a \in \mathcal{A}\}$ with positive components. The n-dimensional torus in $\mathcal{P}_{\mathbb{R}}$

$$\begin{cases} \xi_a\eta_a = I_a, & a \in \mathcal{A} \\ \xi_s = \eta_s = 0, & s \in \mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A}, \end{cases}$$

is invariant for the unperturbed linear equation (2.5), where $H = H_0$. In a neighborhood of this torus in $\mathbb{R}^{2n} = \left((\xi_a, \bar{\xi}_a), a \in \mathcal{A}\right)$, introduce action-angle variables (r_a, θ_a):

$$\xi_a = \sqrt{(I_a + r_a)}e^{i\theta_a}, \qquad \eta_a = \sqrt{(I_a + r_a)}e^{-i\theta_a}. \tag{2.7}$$

The unperturbed Hamiltonian now becomes

$$H_0 = \sum_{a \in \mathcal{A}} \omega_a(\rho) r_a + \sum_{s \in \mathcal{L}} \lambda_s(\xi_s \eta_s)$$

with

$$\omega_a = \sqrt{|a|^4 + \rho_a + m}, \quad a \in \mathcal{A}; \qquad \lambda_s = \sqrt{|s|^2 + m}, \quad s \in \mathcal{L}.$$

We denoted $\hat{V}_a = \rho_a$ for $a \in \mathcal{A}$, the external parameters, and for simplicity of notation have chosen $\hat{V}_s = 0$ for $s \in \mathcal{L}$. In particular

$$V = V(\rho) = (2\pi)^{-d/2} \sum_{a \in \mathcal{A}} \rho_a e^{ia \cdot x}.$$

Let us denote

$$U_{I,m}(r,\theta;\xi,\eta)(x) = \sum_{a \in \mathcal{A}} \frac{\sqrt{(I_a + r_a)} \left(e^{i\theta_a} \varphi_a + e^{-i\theta_a} \varphi_{-a} \right)}{\sqrt{2\lambda_a}} + \sum_{s \in \mathcal{L}} \frac{\xi_s \varphi_s + \eta_s \varphi_{-s}}{\sqrt{2\lambda_s}}. \quad (2.8)$$

Let us set $u_0(\theta, x) = U_{I,m}(0, \theta; 0, 0)(x)$. Then for any $I \in \mathbb{R}_+^n$ and $\theta_0 \in \mathbb{T}^d$ the function $(t, x) \mapsto u_0(\theta_0 + t\omega, x)$ is a solution of (2.1) with $\varepsilon = 0$ and is quasi-periodic of quasi-period $\omega(\rho)$. We have proved in [11] that for most external parameter ρ this quasi-periodic solution persists (but is sightly deformed) when we turn on the nonlinearity :

Theorem 2.1. *Fix $m \geq 0$ and $I \in \mathbb{R}_+^n$. For ε sufficiently small there is a Borel subset $\mathcal{D} \subset [0,1]^n$, $\mathrm{meas}([0,1]^n \setminus \mathcal{D}) \leq C\varepsilon^\alpha$, such that for $\rho \in \mathcal{D}$ there is a function $u_1(\theta, x)$, analytic in $\theta \in \mathbb{T}^n$ and smooth in $x \in \mathbb{T}^d$, satisfying*

$$\sup_{\theta \in \mathbb{T}^n} \|u_1(\theta, \cdot) - u_0(\theta, \cdot)\|_{H^s} \leq \gamma_s \varepsilon,$$

for any[5] $s \geq 0$ and there is a C^1-mapping $\omega' : \mathcal{D} \to \mathbb{R}^n$, $\|\omega' - \omega\|_{C^1(\mathcal{D})} \leq \beta\varepsilon$, such that for any $\rho \in \mathcal{D}$ the function $u(t, x) = u_1(\theta + t\omega'(\rho), x)$ is a linearly stable solution of the beam equation

$$u_{tt} + \Delta^2 u + mu + V(\rho) \star u + \varepsilon g(x, u) = 0, \quad x \in \mathbb{T}^d.$$

The positive constants α and β depend only on d, n and s, while C also depends on g and γ_s also depends on s.

Remark 2.2. The one dimensional case ($d = 1$) is essentially a corollary of [21]. The d-dimensional case was considered in [17] but only in the case where g does not depend on x.

3. The nonlinear beam equation without external parameters

We now consider equation (2.1) with $V = 0$ and $\varepsilon = 1$ but we conserve the mass m as a unique external parameter. Further we assume that

$$g(x, u) = 4u^3 + O(u^4) = 4u^3 + \partial_u \tilde{G}(x, u).$$

---

[5]Here $\| \cdot \|_{H^s}$ denotes the standard Sobolev norm of order s.

With the same notations as in section 2 we then get $H = H_0 + P_4 + R_5$ where

$$H_0 = \sum_{s \in \mathbb{Z}^d} \lambda_s \xi_s \eta_s,$$

$$P_4 = (2\pi)^{-d} \sum_{(i,j,k,\ell) \in \mathcal{J}} \frac{(\xi_i + \eta_{-i})(\xi_j + \eta_{-j})(\xi_k + \eta_{-k})(\xi_\ell + \eta_{-\ell})}{4\sqrt{\lambda_i \lambda_j \lambda_k \lambda_\ell}}, \qquad (3.1)$$

$$R_5 = \varepsilon \int_{\mathbb{T}^d} \tilde{G}\left(x, \sum_{s \in \mathbb{Z}^d} \frac{\xi_s \varphi_s(x) + \eta_s \varphi_{-s}(x)}{\sqrt{2\lambda_s}}\right) dx$$

where $\mathcal{J}$ denotes the zero momentum set:

$$\mathcal{J} := \{(i,j,k,\ell) \in \mathbb{Z}^d \mid i + j + k + \ell = 0\}.$$

In the previous section H was considered as a perturbation of the quadratic Hamiltonian H_0. But when $V = 0$ (and $d \neq 1$) this Hamiltonian is too resonant: $\lambda_a = \lambda_b$ when $|a| = |b|$. So in this section we will consider H as a perturbation of the quartic Hamiltonian $H_0 + P_4$ and we will use the first part of the nonlinearity to destroy the exact resonances of the linear part of the PDE. This nice idea was first used in [23] for the 1d-NLS. Concretely we will use the initial actions as internal parameters and we set

$$I_a = \rho_a^2, \quad a \in \mathcal{A}. \qquad (3.2)$$

To avoid internal resonances (i.e. resonances inside the torus) we need to restrict our analysis to admissible sets $\mathcal{A}$:

Definition 3.1. *A finite set $\mathcal{A} \in \mathbb{Z}^d$ is called admissible if*

$$j, k \in \mathcal{A}, \; j \neq k \Rightarrow |j| \neq |k|.$$

We still denote $\mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A}$ and set

$$\mathcal{L}_f = \{s \in \mathcal{L} \mid \exists \, a \in \mathcal{A} \text{ such that } |a| = |s|\} \qquad (3.3)$$

which is a finite subset of $\mathcal{L}$. In particular,

$$\mathcal{L}_f = -(\mathcal{A} \setminus \{0\}) \quad \text{if} \quad d = 1 \quad \text{and } \mathcal{A} \text{ is admissible.} \qquad (3.4)$$

For each $a \in \mathcal{L}_f$ there exists a unique element of $\mathcal{A}$, denoted $\ell(a)$, satisfying

$$|a| = |\ell(a)|.$$

Further we define two subsets of $\mathcal{L}_f \times \mathcal{L}_f$:

$$(\mathcal{L}_f \times \mathcal{L}_f)_+ := \{(a,b) \in \mathcal{L}_f \times \mathcal{L}_f \mid \ell(a) + \ell(b) = a + b\} \qquad (3.5)$$

$$(\mathcal{L}_f \times \mathcal{L}_f)_- = \{(a,b) \in \mathcal{L}_f \times \mathcal{L}_f \mid a \neq b \text{ and } \ell(a) - \ell(b) = a - b\}. \qquad (3.6)$$

If $d = 1$, then in view of (3.4) we have $\ell(a) = -a$ and the sets $(\mathcal{L}_f \times \mathcal{L}_f)_\pm$ are empty. For $d \geq 2$, in general, both of them are non-trivial. Obviously

$$(\mathcal{L}_f \times \mathcal{L}_f)_+ \cap (\mathcal{L}_f \times \mathcal{L}_f)_- = \emptyset. \qquad (3.7)$$

Finally we introduce $K(\rho)$, a finite symmetric complex matrix, acting on the space

$$Y^f := \text{span}\left\{(\xi_s, \eta_s), s \in \mathcal{L}_f\right\} \ni \zeta^f$$

such that the corresponding quadratic form is

$$
\begin{aligned}
\langle K(\rho)\zeta^f, \zeta^f \rangle = 3(2\pi)^{-d} \Big(& \sum_{\ell \in \mathcal{A},\ a \in \mathcal{L}_f} \frac{(3\delta_{\ell,|a|} - 2)}{\lambda_\ell \lambda_a} \rho_\ell^2 \xi_a \eta_a \\
+ & \sum_{(a,b) \in (\mathcal{L}_f \times \mathcal{L}_f)_+} \frac{\rho_{\ell(a)} \rho_{\ell(b)}}{\lambda_a \lambda_b} (\eta_a \eta_b + \xi_a \xi_b) + 2 \sum_{(a,b) \in (\mathcal{L}_f \times \mathcal{L}_f)_-} \frac{\rho_{\ell(a)} \rho_{\ell(b)}}{\lambda_a \lambda_b} \xi_a \eta_b \Big).
\end{aligned}
\tag{3.8}
$$

The quadratic form $\langle K(\rho)\zeta^f, \zeta^f \rangle$ is essentially the resonant part of P_4 reduced to Y^f. It appears as a remaining term of P_4 after a Birkhoff normal form procedure and some other symplectic transformations (see [12] and section 4.2).

Let us set $u_{I,m}(\theta, x) = U_{I,m}(0, \theta; 0, 0)(x)$ where $U_{I,m}$ is defined in (2.8). As in the previous section, $(t, x) \mapsto u_{I,m}(\theta_0 + t\omega, x)$ is a quasi-periodic solution of the linear beam equation (i.e. (2.1) with $\varepsilon = 0$ and $V = 0$). Our main theorem analyses persistence of this solution in (2.1) with $\varepsilon = 1$ for typical small vectors I and generic m:

Theorem 3.2. *Assume that the nonlinearity $g(x, u) = 4u^3 + O(u^4)$ is analytic, and that the set $\mathcal{A}$ (card $\mathcal{A} = n$) is admissible. Then there exists a zero-measure Borel set $\mathcal{C} \subset [1, 2]$ and a Borel function $\nu_0 : [1, 2] \to \mathbb{R}$, strictly positive outside $\mathcal{C}$, such that for $m \notin \mathcal{C}$ and $0 < \nu \le \nu_0(m)$*

1) we can find a Borel set $\mathcal{D}_m \subset [\nu, 2\nu]^n$ asymptotically of full measure as $\nu \to 0$, i.e. satisfying $\mathrm{meas}([\nu, 2\nu]^n \setminus \mathcal{D}_m) \le C(m)\nu^{n+\alpha}$ with $\alpha > 0$, a function $v : \mathbb{T}^n \times \mathbb{T}^d \times \mathcal{D}_m \to \mathbb{R}$, analytic in θ and smooth in $x \in \mathbb{T}^d$, satisfying

$$
\sup_{\theta \in \mathbb{T}^d, I \in \mathcal{D}_m} \| v(\theta, \cdot; I) - u_{I,m}(\theta, \cdot) \|_{H^s(\mathbb{T}^d)} \le C(m, s)\nu^\beta, \qquad \beta > 0,
$$

for each $s \ge 0$, and a mapping $\omega' = \omega'_m : [0, \nu]^n \to \mathbb{R}^n$, $\|\omega' - \omega\|_{C^1} \le C(m)\nu^\beta$, such that, for any $I \in \mathcal{D}_m$, the function $u(t, x) = v(\theta + t\omega'(I), x; I)$ is a solution of the beam equation

$$
u_{tt} + \Delta^2 u + mu + g(x, u) = 0, \quad x \in \mathbb{T}^d.
$$

2) This solution is linearly stable if and only if the Hamiltonian operator[6] iJK, explicitly constructed in terms of the set $\mathcal{A}$ (see (3.8)) is stable[7]. This is always the case if $d = 1$, while for $d \ge 2$ for some choices of the set $\mathcal{A}$ the solution is linearly unstable.

For instance, in [12] we verify that

$$
\mathcal{A} = \{(0, 1), (1, -1)\}
$$

is admissible and that the corresponding matrix K is always unstable when $\rho_{(0,1)} = \rho_{(1,-1)}$. As a consequence the quasi-periodic solutions constructed on the two nodes of $\mathcal{A}$ are linearly unstable.

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[6]To the Hamiltonian function $\frac{1}{2}\langle K(\rho)\zeta^f, \zeta^f \rangle$ we associate the Hamiltonian operator iJK where J is the block diagonal matrix $J = \mathrm{diag} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

[7]Which means that all its eigenvalues have a negative real part.

4. Some details on resonances and on resonant normal forms

4.1. Resonances

The KAM proof is based on an iterative procedure that requires to solve a so-called homological equation at each step. Roughly speaking, it consists in inverting an infinite dimensional matrix whose eigenvalues are the so-called small divisors:

$$\tilde{\omega} \cdot k \quad k \in \mathbb{Z}^{\mathcal{A}},$$

$$\tilde{\omega} \cdot k + \tilde{\lambda}_a \quad k \in \mathbb{Z}^{\mathcal{A}}, \ a \in \mathcal{L},$$

$$\tilde{\omega} \cdot k + \tilde{\lambda}_a \pm \tilde{\lambda}_b \quad k \in \mathbb{Z}^{\mathcal{A}}, \ a, b \in \mathcal{L}$$

where $\tilde{\omega} = \tilde{\omega}(\rho)$ and $\tilde{\lambda}_a = \tilde{\lambda}_a(\rho)$ are small perturbations (changing at each KAM step) of the original frequencies $\omega(\rho) = (\sqrt{|a|^2 + \rho_a})_{a \in \mathcal{A}}$ and $\lambda_a = |a|^2$, $a \in \mathcal{L}$ (cf. section 2).

Ideally we would like to bound away from zero all these small divisors. In particular, this leads to infinitely many non resonances conditions of the type

$$|\tilde{\omega} \cdot k + \tilde{\lambda}_a - \tilde{\lambda}_b| > \frac{\kappa}{|k|^\alpha}, \quad k \in \mathbb{Z}^{\mathcal{A}}, \ a, b \in \mathcal{L}$$

for some parameters $\kappa > 0$ and $\alpha > 0$. Of course we have to exclude the case $k = 0$, $a = b$ for which the small divisor is identically zero and this is precisely the reason why the external frequencies $\tilde{\lambda}_a$, $a \in \mathcal{L}$, move at each step.

When $d = 1$ we have $|\lambda_a - \lambda_b| \geq 2|a|$ for $|b| \neq |a|$. Therefore for each fixed k there are only finitely many non resonances conditions and you can expect to satisfy them for a large set of parameters ρ.

Now when $d \geq 2$, the frequencies λ_a, $a \in \mathcal{L}$, are not sufficiently separated and you really have to manage infinitely many non resonances conditions[8] for each k. It is not possible to control so many small divisors. Part of the solution consists in decomposing $\mathcal{L}$ in blocks $[a] := \{c \in \mathcal{L} \mid |c| = |a|,\}$ and to solve the homological equation according to this clustering. Then we only have to control the small divisors

$$|\tilde{\omega} \cdot k + \tilde{\lambda}_a - \tilde{\lambda}_b| \quad \text{for } k \in \mathbb{Z}^{\mathcal{A}}, \ a, b \in \mathcal{L}, \ |a| \neq |b|$$

which is more reasonable.

On the other hand we have to face the problem that the size of the block $[a]$ is growing with $|a|$. As a consequence you loose regularity each time you solve a homological equation (essentially because the norm of the inverse of a matrix is directly related to its size). Of course this is not acceptable for an infinite induction. The very nice idea developed in [14] consists in considering a sub-clustering constructed as the equivalence classes of the equivalence relation on $\mathbb{Z}^d$ generated by the pre-equivalence relation

$$a \sim b \Longleftrightarrow \begin{cases} |a| = |b| \\ |a - b| \leq \Delta \end{cases}$$

---

[8]In fact this is not true at first step since $\lambda_a = \lambda_b = |a|^2$ for $|a| = |b|$ and the set $\{\lambda_a - \lambda_b \mid a, b \in \mathcal{L}, \ |b| \neq |a|\}$ is included in $\mathbb{Z}$. Nevertheless at each KAM step, the external frequencies λ_a, $a \in \mathcal{L}$ will move a little bit and then the new set $\{\tilde{\lambda}_a - \tilde{\lambda}_b \mid a, b \in \mathcal{L}, \ |b| \neq |a|\}$ will not be discrete anymore.

Let $[a]_\Delta$ denote the equivalence class of a. The crucial fact (proved in [14]) is that the blocks are finite with a maximal "diameter"

$$\max_{[a]_\Delta = [b]_\Delta} |a - b| \leq C_d \Delta^{\frac{(d+1)!}{2}}$$

depending only on Δ. With such clustering, you do not lose regularity when solving the homological equation. Further, working in a phase space of analytic functions u or equivalently, exponentially decreasing Fourier coefficients, it turns out that the homological equation is "almost" block diagonal relatively to this clustering. Then you increase the parameter Δ at each step of the KAM iteration.

4.2. Resonant normal form

When you do not allow external parameters (as in section 3), H_0 is resonant in the sense that some of the linear frequencies are rationally dependent. For instance $\omega_{\ell(a)} = \lambda_a$ for $a \in \mathcal{L}_f$ (see notations in section 3) and thus the internal mode $\ell(a)$ is resonant with the external mode a. In other words the small divisor $\omega \cdot k + \lambda_a$ vanishes when[9] $k_j = -\delta(j, \ell(a))$. As explained in section 3 we want to use P_4, the quartic part of the Hamiltonian, to destroy these exact resonances.
We first perform a Birkhoff normal form procedure that allows us to kill all the non resonant monomials in P_4: we construct a symplectic change of variable τ, close to the identity, such that

$$(H_0 + P_4) \circ \tau = H_0 + Z_4 + R$$

where R is a remainder term that can be considered as a perturbation,

$$Z_4 = (2\pi)^{-d} \sum_{(i,j,k,\ell) \in \mathcal{R}} \frac{\xi_i \xi_j \eta_k \eta_\ell}{4\sqrt{\lambda_i \lambda_j \lambda_k \lambda_\ell}}$$

is the effective resonant part and

$$\mathcal{R} = \left\{ (i, j, k, \ell) \in (\mathbb{Z}^d)^4 \mid i + j = k + \ell \text{ and } \{|i|, |j|\} = \{|k|, |\ell|\} \right\}$$

is the resonant set. The quartic polynomial Z_4 still contains a lot of monomials. Some of them are very bad (in the sense that they depend on the internal angles $(\theta_\ell)_{\ell \in \mathcal{A}}$ and thus are not integrable) like $\xi_{\ell(a)} \xi_{\ell(b)} \eta_a \eta_b$ for $(a, b) \in (\mathcal{L}_f \times \mathcal{L}_f)_+$ or $\xi_{\ell(a)} \xi_b \eta_a \eta_{\ell(b)}$ for $(a, b) \in (\mathcal{L}_f \times \mathcal{L}_f)_-$. We use some rotations on the internal angles to reduce them and to build the quadratic form $K(\rho)$ (see [12]). There are also very nice term in Z_4, especially $\xi_\ell \xi_a \eta_\ell \eta_a$ for $\ell \in \mathcal{A}$ and $a \in \mathcal{L}$ which, using (2.7) and (3.2), reads $(I_\ell + r_\ell)\xi_a \eta_a = (\rho_\ell^2 + r_\ell)\xi_a \eta_a$. Then forgetting for a moment all the "bad" terms, these nice terms of Z_4 will be added to the term $\lambda_a \xi_a \eta_a$ in H_0 to form a new external frequency

$$\Lambda_a(\rho) = \lambda_a + 6(2\pi)^{-d} \sum_{\ell \in \mathcal{A}} \frac{\rho_\ell^2}{\lambda_\ell \lambda_a}.$$

So in this simplified model, the frequencies explicitly depend on the internal parameters and we can try to apply a KAM result with parameters.

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[9] Here $\delta(j, k)$ denotes the Kronecker symbol.

References

[1] V.I. Arnold, Mathematical methods in classical mechanics; 3d edition, Springer-Verlag, Berlin, 2006.

[2] M. Berti, P. Bolle, Sobolev quasi periodic solutions of multidimensional wave equations with a multiplicative potential, *Nonlinearity* **25** (2012), 2579-2613.

[3] M. Berti, P. Bolle, Quasi-periodic solutions with Sobolev regularity of NLS on T^d with a multiplicative potential, *J. Eur. Math. Soc.* **15** (2013), 229-286.

[4] M. Berti, L. Corsi, M. Procesi, An Abstract Nash-Moser Theorem and Quasi-Periodic Solutions for NLW and NLS on Compact Lie Groups and Homogeneous Manifolds, *Comm. Math. Phys* (2014).

[5] M. Berti, M. Procesi, Nonlinear wave and Schrödinger equations on compact Lie groups and Homogeneous spaces, *Duke Math. J.* **159**, 479-538 (2011).

[6] J. Bourgain, Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equation, *Ann. Math.* **148** (1998), 363-439.

[7] J. Bourgain, Green's function estimates for lattice Schrödinger operators and applications, *Annals of Mathematical Studies*, Princeton, 2004.

[8] R. De La Llave, C. E. Wayne, Whiskered and low dimensional tori in nearly integrable Hamiltonian systems, *MPEJ* **10** (2004), paper 5.

[9] J.-M. Delort, Growth of Sobolev norms for solutions of time dependent Schrödinger operators with harmonic oscillator potential, *Comm. Partial Differential Equations* **39** (2014), 1Ð33.

[10] L. H. Eliasson, Almost reducibility of linear quasi-periodic systems, in *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, 679-705, Proc. Sympos. Pure Math., 69, Amer. Math. Soc., Providence, RI, 2001.

[11] L. H. Eliasson, B. Grébert and S. B. Kuksin, KAM for the nonlinear beam equation 2: a normal form theorem, *preprint*.

[12] L. H. Eliasson, B. Grébert and S. B. Kuksin, KAM for the nonlinear beam equation 1: small-amplitude solutions, *preprint*.

[13] L. H. Eliasson and S. B. Kuksin, On reducibility of Schrödinger equations with quasiperiodic in time potentials, *Comm. Math. Phys.* **286** (2009), 125–135.

[14] L. H. Eliasson and S. B. Kuksin, KAM for the nonlinear Schrödinger equation, *Ann. Math* **172** (2010), 371-435.

[15] E. Fermi, J. R. Pasta and S. M. Ulam, Studies of nonlinear problems. Collected works of E. Fermi, vol.2. Chicago University Press, Chicago, 1965.

[16] G. Gallavotti (editor). The Fermi-Pasta-Ulam problem *Lectures Notes in Physics* **728** Springer-Verlag, Berlin, 2008

[17] J. Geng and J. You, A KAM theorem for Hamiltonian partial differential equations in higher dimensional spaces, *Comm. Math. Phys.*, **262** (2006), 343–372.

[18] J. Geng and J. You, KAM tori for higher dimensional beam equations with constant potentials, *Nonlinearity*, **19** (2006), 2405–2423.

[19] B. Grébert, KAM for KG on $\mathbb{S}^2$ and for the quantum harmonic oscillator on $\mathbb{R}^2$, *preprint* arXiv:1410.8084.

[20] B. Grébert and L. Thomann, KAM for the Quantum Harmonic Oscillator, *Comm. Math. Phys.*, **307** (2011), 383–427.

[21] S. B. Kuksin, Hamiltonian perturbations of infinite-dimensional linear systems with an imaginary spectrum, *Funct. Anal. Appl.*, **21** (1987), 192–205.

[22] S. B. Kuksin, Nearly integrable infinite-dimensional Hamiltonian systems, *Lecture Notes in Mathematics, 1556,* Springer-Verlag, Berlin, 1993.

[23] S. B. Kuksin and J. Pöschel, Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation, *Ann. Math.* **143** (1996), 149–179.

[24] T. Kappeler and J. Pöschel, KAM & KdV, Springer-Verlag, Berlin, 2003.

[25] J. Pöschel, A KAM-theorem for some nonlinear partial differential equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) 23 (1996), no. 1, 119–148.

[26] C. Procesi and M. Procesi, A normal form of the nonlinear Schrödinger equation with analytic non–linearities, *Comm. Math. Phys* **312** (2012), 501-557.

[27] C. Procesi and M. Procesi, A KAM algorithm for the resonant nonlinear Schrödinger equation, *preprint 2013*.

[28] W.-M. Wang, Nonlinear Schrödinger equations on the torus, *Monograph, 156 pp*, 2014 (submitted).

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