Alberto Enciso and Daniel Peralta-Salas

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*Tubes de vorticité étroits dans l'équation d'Euler
stationnaire*

Résumé

On expose quelques nouveaux résultats sur l'existence de solutions stationnaires à l'équation d'Euler sur $\mathbb{R}^3$ avec un ensemble de tubes de vorticité étroits (qui peuvent être noués et entrelacés) qu'on peut prescrire a priori.

Abstract

In this paper we outline some recent results concerning the existence of steady solutions to the Euler equation in $\mathbb{R}^3$ with a prescribed set of (possibly knotted and linked) thin vortex tubes.

1. Introduction

Let us consider the Euler equation in $\mathbb{R}^3$,

$$\partial_t u + (u \cdot \nabla)u = -\nabla P\,, \qquad \operatorname{div} u = 0\,,$$

which models the behavior of an inviscid incompressible fluid in space. In this equation the unknowns are the velocity field $u(x,t)$, which is a time-dependent vector field, and the pressure function $P(x,t)$.

In this paper we will be concerned with the existence of *steady* (that is, time-independent) solutions to the Euler equation in $\mathbb{R}^3$. As is well known, a vector field $u(x)$ is a steady solution of the Euler equation if and only if it satisfies the (quite unmanageable) system of PDEs

$$u \times \omega = \nabla B\,, \qquad \operatorname{div} u = 0\,, \tag{1.1}$$

where $\omega := \operatorname{curl} u$ is the *vorticity* and

$$B := P + \frac{1}{2}|u|^2$$

is the Bernoulli function. A priori, it is not easy to see for which choices of the function B we have solutions and which properties these solutions can exhibit.

Our goal is to review the recent construction of steady solutions to the Euler equation that describe nontrivial fluid structures. To be more precise, let us recall that a *vortex line* in a steady fluid is just an integral curve of the vorticity, that is, the solution $x(t)$ to the ODE

$$\dot{x} = \omega(x)$$

with some initial condition $x(0) = x_0$. A domain in $\mathbb{R}^3$ that is the union of vortex lines and whose boundary is an embedded torus is a (closed) *vortex tube*. Obviously the boundary of the tube is an invariant torus of the vorticity.

The main result that we will discuss in this note can be informally stated as follows (the precise statement is given in Theorem 3.1). To describe the kind of tubes we construct, let us introduce the notation

$$\mathcal{T}_\epsilon(\gamma) := \left\{ x \in \mathbb{R}^3 : \operatorname{dist}(x, \gamma) < \epsilon \right\} \tag{1.2}$$

for the tube whose core is a smooth closed curve γ and whose normal section is a circle of radius ϵ.

Theorem 1.1 ([6], informal statement). *Take a collection of N disjoint tubes of the form (1.2), whose central curves can be knotted and linked. We assume that these tubes are thin in the sense that the thickness ϵ is small enough. Then one can slightly deform these tubes to obtain a collection of vortex tubes of a steady solution of the Euler equation in $\mathbb{R}^3$, which is smooth and falls off at infinity as $1/|x|$.*

A key feature of our approach is that we do not consider arbitrary solutions to the steady Euler equation, but a particular class known as (strong) *Beltrami fields*. These are the solutions of the equation

$$\operatorname{curl} u = \lambda u \tag{1.3}$$

in $\mathbb{R}^3$ for some nonzero constant λ. Notice that this immediately implies that u is an analytic, divergence-free vector field.

Of course, a first advantage of this equation over the general steady Euler equation (1.1) is that it is linear. A deeper reason to consider this class of solutions, however, is related to the so-called "structure theorem" of Arnold [1]. This theorem asserts that if u is a stationary solution which is not everywhere collinear with its vorticity (that is, if the Bernoulli function B is not constant), under mild technical assumptions the vortex lines are tangent to a family of invariant tori, cylinders and planes that defines a rigid structure closely related to that of an integrable Hamiltonian system with two degrees of freedom. For our purposes, the content of Arnold's theorem is that when u and ω are not collinear, there is not much freedom in choosing how the vortex lines and possible vortex tubes can sit in space, so it should be difficult to construct vortex tubes with complicated shapes. A straightforward consequence, e.g., is that the steady Euler equation (1.1) does not admit any

solutions for most choices of the function B. It can be proved that Beltrami fields with nonconstant proportionality factor, i.e., solutions of the equation

$$\operatorname{curl} u = f(x)\, u\,, \qquad \operatorname{div} u = 0\,,$$

are also very difficult to handle [7], so in order to keep things simple we are naturally led to consider Beltrami fields of the form (1.3) to construct steady solutions with nontrivial vortex tubes.

The paper is organized as follows. In Section 2 we will discuss the motivation to consider the problem of constructing thin steady vortex tubes and the history of the problem. The precise statement of Theorem 1.1 will be presented in Section 3 together with a number of comments and remarks. In Section 4 we will give some ideas on the proof of this result. Finally, Section 5 will be devoted to the comparison between the proof of the theorem for vortex tubes and that of our previous realization theorem for vortex lines [5].

2. Motivation and heuristics

The study of vortex tubes can be traced back to Helmholtz, who discovered the phenomenon of the transport of vorticity, and to Lord Kelvin, who developed an atomic theory in which atoms were understood as thin knotted vortex tubes in the ether, which was modeled as an ideal fluid using the Euler equation. Although this atomic theory was abandoned after some years, it was a major boon for the development of knot theory.

Specifically, a long-standing problem in this direction is Lord Kelvin's conjecture [12] that thin vortex tubes of arbitrarily complicated topology can arise in steady solutions to the Euler equation. This is consistent with a conjecture of Arnold, formulated right after his structure theorem [1], in which he claims that there should be Beltrami fields with very complicated structure, which is typically interpreted as the coexistence of invariant tori and chaotic regions.

Let us recall what is the heuristic argument, essentially due to Helmholtz [9], which resorts to the transport of vorticity to suggest the existence of knotted and linked vortex tubes. The basic idea is the following. Suppose that $u(x,t)$ is a time-dependent solution of the Euler equation. Then its vorticity satisfies the equation

$$\partial_t \omega = [\omega, u]\,,$$

with $[\cdot, \cdot]$ the commutator of vector fields. Therefore, the vorticity at time t can be expressed in terms of the vorticity $\omega_0(x)$ at time t_0 as

$$\omega(x,t) = (\phi_{t,t_0})_* \omega_0(x)\,, \tag{2.1}$$

where $(\phi_{t,t_0})_*$ denotes the push-forward of the non-autonomous flow of the velocity field between the times t_0 and t.

From this expression for the vorticity it stems that the vortex lines at time t are diffeomorphic to those at time t_0. Therefore, one can attempt to construct the initial vorticity ω_0 with a vortex tube $\mathcal{T}$ of prescribed topology. This can be done as follows. Let γ be a smooth closed curve in $\mathbb{R}^3$ that will play the role of the core of the tube $\mathcal{T}$. It is known that there are two smooth functions f, g of compact support

in $\mathbb{R}^3$ such that γ is the transverse intersection

$$\gamma = f^{-1}(1) \cap g^{-1}(1) \, .$$

Using these functions, we can prescribe the initial vorticity as the divergence-free vector field

$$\omega_0 := \nabla f \times \nabla g \, .$$

Through the Biot–Savart operator, this initial vorticity corresponds to the initial velocity

$$u_0(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \omega_0(y)}{|x-y|^3} dy \, ,$$

which is smooth and falls off at infinity as

$$|D^j u_0(x)| < \frac{C_j}{|x|^2} \, .$$

Since the initial vorticity then satisfies the condition

$$\omega_0 \cdot \nabla F = 0$$

with

$$F(x) := (f(x) - 1)^2 + (g(x) - 1)^2 \, ,$$

it is not hard to check that, for small enough ϵ, the set

$$\mathcal{T} := \left\{ x : F(x) < \epsilon \right\}$$

is a vortex tube of the fluid at time 0. Therefore, *if* there is a smooth global solution to the Euler equation with initial datum u_0, the solution u has a vortex tube diffeomorphic to $\mathcal{T}$ at all times. In particular, *if* the fluid $u(x,t)$ evolves, for large times, into an equilibrium state, characterized by a steady solution to Euler $u_\infty(x)$, it is conceivable (although certainly not at all obvious) that this steady solution could have a vortex tube diffeomorphic to $\mathcal{T}$ too (notice, however, that the argument does not justify why one expects the tubes to be thin). Of course, these very nontrivial "ifs" prevent us from promoting this heuristic argument to a rigorous result.

In addition to this theoretical evidence, the firm belief among the physics community on the existence of steady thin knotted vortex tubes is based on experimental results, going back to Maxwell's recorded observations of what he called "water twists". Another recent related experiment is described in [10].

Furthermore, thin vortex tubes have long played a key role in the construction and numerical exploration of possible blow-up scenarios for the Euler equation. The reason is that, as long as the evolution of some initial datum under the Euler equation remains smooth for $t \in [0, T]$, Eq. (2.1) ensures that the set of vortex tubes at time $t_1 \in [0, T]$ is topologically equivalent to the initial set of vortex tubes. Therefore, if one can prove that there is a change of topology in the set of vortex tubes, e.g. by collapsing or merging tubes, it stems that there has been a singularity in the evolution. This idea has been explored using computer simulations, which in turn has led to rigorous results such as [3, 4]. A particularly influential scenario in this direction is due to Pelz [11], who discusses how an initial condition with a certain set of linked thin vortex tubes might lead to singularity formation in finite time. To our best knowledge, it remains an open problem to confirm or rule out this scenario.

3. Realization theorem for thin vortex tubes

The main result that we want to discuss in this note is the following realization theorem for thin vortex tubes, which is a precise version of the informal statement that we presented in the Introduction (Theorem 1.1):

Theorem 3.1. *Let $\gamma_1, \ldots, \gamma_N$ be N pairwise disjoint (possibly knotted and linked) closed curves in $\mathbb{R}^3$. For small enough ϵ, one can transform the collection of pairwise disjoint thin tubes*

$$\mathcal{T}_\epsilon(\gamma_1), \ldots, \mathcal{T}_\epsilon(\gamma_N)$$

by a diffeomorphism Φ of $\mathbb{R}^3$, arbitrarily close to the identity in any C^m norm, so that

$$\Phi[\mathcal{T}_\epsilon(\gamma_1)], \ldots, \Phi[\mathcal{T}_\epsilon(\gamma_N)]$$

are vortex tubes of a Beltrami field u, which satisfies the equation

$$\operatorname{curl} u = \lambda u$$

in $\mathbb{R}^3$ for some nonzero constant λ. Moreover, the field u decays at infinity as

$$|D^j u(x)| < \frac{C_j}{|x|} \quad \text{for all } j.$$

Before discussing the proof of this result, let us make a few comments:

- The reason why we are constructing thin tubes is not just that Lord Kelvin thought it should be possible; indeed, the fact that the vortex tubes are thin is a crucial ingredient in the proof of this result. What one can easily do, however, is to take tubes of different width $\mathcal{T}_{\epsilon_1}(\gamma_1), \ldots, \mathcal{T}_{\epsilon_N}(\gamma_N)$, as long as all the widths ϵ_i remain small.

- Since any Beltrami field satisfies the equation

  $$\Delta u + \lambda^2 u = 0,$$

  it is clear that our solutions cannot have finite energy (that is, $u \notin L^2(\mathbb{R}^3)$). In fact, it is an open question whether the Euler equation admits any (nonzero) steady solutions with finite energy. However, the solutions we construct have an optimal decay rate at infinity among all Beltrami fields and are, of course, analytic by the standard regularity theorems. In this direction, we notice that a Liouville theorem for Beltrami fields with a possibly nonconstant proportionality factor has been recently established by Nadirashvili [13].

- In the physics literature, it is always assumed that there are *stable* thin vortex tubes. Therefore, it is extremely convenient that the vortex tubes we construct are indeed stable in two different ways. On the one hand, they are *Lyapunov-stable*, meaning that the vortex lines passing through a suitably small neighborhood of each torus do not leave another neighborhood of the torus, which we can take as small as we wish. On the other hand, our vortex tubes are *structurally stable* in the sense that the invariant tori are preserved, up to a small diffeomorphism, under suitably small (divergence-free) perturbations of the field u.

- Although for simplicity we have not included it in the statement, the method of proof of Theorem 3.1 gives much information about the behavior of vortex lines in the vortex tubes. In particular, one can prove that:

  (i) In the interior of $\Phi[\mathcal{T}_\epsilon(\gamma_i)]$ there are uncountably many nested tori invariant under the Beltrami field u. On each of these invariant tori, the field u is ergodic.

  (ii) The set of invariant tori has positive Lebesgue measure in a small neighborhood of the boundary $\partial\Phi[\mathcal{T}_\epsilon(\gamma_i)]$.

  (iii) In the region bounded by any pair of these invariant tori there are infinitely many closed vortex lines, not necessarily of the same knot type as the curve γ_i.

  (iv) The image of the core curve γ_i under the diffeomorphism Φ is a closed vortex line of u.

  An immediate consequence of this is that we recover our previous result on the existence of knotted vortex lines in steady solutions to the Euler equation [5]. It is worth emphasizing, however, that the proofs of these results are totally different, as we will discuss in Section 5, which is reflected in the fact that the behavior of the vortex lines in a neighborhood of the knots in the aforementioned reference and in a neighborhood of the central curve $\Phi(\gamma_i)$ is very different too.

- We do not claim that all the vortex tubes of u are close, in a certain sense, to the tubes $\mathcal{T}_\epsilon(\gamma_i)$: in principle, the Beltrami field u can have many vortex tubes outside this region.

- The parameter λ in the theorem cannot be chosen freely: it must be of order $\mathcal{O}(\epsilon^3)$ (and nonzero). It is worth emphasizing that the case of $\lambda = 0$ is quite exceptional and corresponds to the case of potential flows, for which the velocity field is the gradient of a harmonic function. It can be checked that a potential flow cannot have any closed integral curves.

- Notice that
$$w(x,t) := \mathrm{e}^{-\lambda^2 t} u(x)$$
is a solution to the Navier-Stokes equation in $\mathbb{R}^3$ with viscosity 1, i.e.,
$$\partial_t w + (w \cdot \nabla)w = \Delta w - \nabla P, \qquad \operatorname{div} w = 0.$$
Therefore, the theorem automatically implies the existence of stationary knotted vortex tubes for Navier-Stokes (which appear in time-dependent solutions, however).

- Beltrami fields, and the question of which kind of structures can arise from their integral curves, do not only appear in fluid mechanics. As a matter of fact, these fields also play an important role in electrodynamics, where they are known as force-free magnetic fields [2].

4. Some ideas on the proof

Ultimately, Theorem 3.1 is a result on the existence of invariant tori (with a very concrete geometry) in steady solutions to the Euler equation in $\mathbb{R}^3$, which combines ideas from partial differential equations and dynamical systems. For concreteness, to explain the gist of the proof we will concentrate on constructing a solution for which we are prescribing just one vortex tube $\mathcal{T}_\epsilon \equiv \mathcal{T}_\epsilon(\gamma)$. There is no loss of generality in assuming that the curve γ is analytic.

The heart of the matter is that we do not have any standard tools to construct solutions of the Eq. (1.3) in $\mathbb{R}^3$ that contain the prescribed invariant torus $\mathcal{T}_\epsilon$. What might be expected to be easier, of course, is to construct a solution v of (1.3) only in the interior of the tube with the condition that u be tangent to $\partial\mathcal{T}_\epsilon$, since this is some kind of boundary value problem. The connection between the solution v in the tube and the global problem considered in the theorem is that we will show that, given v, one can always construct a solution u in the whole space $\mathbb{R}^3$ which is close to v in $C^k(\mathcal{T}_\epsilon)$. Hence, in order to ensure that the solution u has a vortex tube close to $\mathcal{T}_\epsilon$ we need to prove a result on the preservation of the invariant torus $\partial\mathcal{T}_\epsilon$ under suitably small perturbations of v. This is a subtle problem, since the verification of the (KAM-type) nondegeneracy conditions for the preservation of these tori hinges on a fine analysis of the solutions to the boundary problem. Furthermore, that we can actually satisfy the nondegeneracy conditions is not something we infer from a choice of boundary data, but something we extract from the equation using in a crucial way the geometry of the tube $\mathcal{T}_\epsilon$ (and, in particular, the smallness of ϵ).

In the rest of this section we will sketch how one can implement this strategy to prove Theorem 3.1; the proof can be found in [6]. This will be done in three deeply interrelated steps, which correspond to the construction of the solution v in the tube (which we will call "local" solution), to the preservation of the invariant torus $\partial\mathcal{T}_\epsilon$ and to the approximation of v by a global solution u.

Step 1. As we have said, we will obtain the local Beltrami field v as the unique solution to certain boundary value problem for the Beltrami equation. To specify this problem, let us fix a (nonzero) harmonic field h in $\mathcal{T}_\epsilon$, which satisfies

$$\operatorname{div} h = 0 \quad \text{and} \quad \operatorname{curl} h = 0$$

in the tube and is tangent to the boundary. By Hodge theory, it is standard that there is a unique harmonic field in $\mathcal{T}_\epsilon$ up to a multiplicative constant. For concreteness, let us assume that $\|h\|_{L^2(\mathcal{T}_\epsilon)} = 1$.

The boundary problem we will then consider is

$$\operatorname{curl} v = \lambda v \tag{4.1}$$

in $\mathcal{T}_\epsilon$, supplemented with the boundary condition $v \cdot \nu = 0$ and a condition on the harmonic part of v such as

$$\int_{\mathcal{T}_\epsilon} v \cdot h \, dx = 1 \,.$$

Notice that in this boundary problem we are specifying the normal component of v on the boundary (which we set to zero, to ensure that $\partial\mathcal{T}_\epsilon$ is an invariant torus) but *not* the tangential component. This will be important later on.

Through a duality argument, it is not hard to prove that for any λ outside some discrete set, and in particular whenever $|\lambda|$ is smaller than some ϵ-independent constant, there is a unique solution to this problem. An easy consequence of the proof is that the field v becomes close to h for small λ, in the sense that

$$\|v - h\|_{H^k(\mathcal{T}_\epsilon)} \leqslant C_{k,\epsilon}|\lambda| \,. \tag{4.2}$$

The problem now is that, when one tries to verify the conditions for the preservation of the invariant torus $\partial \mathcal{T}_\epsilon$ under small perturbations of v, one realizes that the above existence result is far from enough: the robustness of the invariant torus depends on KAM arguments, which require very fine information on the behavior of v in a neighborhood of $\partial \mathcal{T}_\epsilon$. Intuitively, the idea is that the fact that thin vortex tubes can be realized in steady solutions to the Euler equation depends strongly on the geometry of the tubes, while the above existence result is valid for a wide class of domains, such as thick, irregular tubes with large bumps.

An important simplification is suggested by the estimate (4.2): if we take small nonzero values of λ, it should be enough to understand the behavior of the harmonic field h, since the local solution v is going to look basically like this field (more refined estimates are needed to fully exploit this fact, but this is the basic idea.)

Therefore, our next goal is to estimate various analytic properties of the harmonic field h. To simplify this task, we will introduce coordinates adapted to the tube $\mathcal{T}_\epsilon$, which essentially correspond to an arc-length parametrization of the curve γ and to rectangular coordinates in a transverse section of the tube defined using a Frenet frame. Thus we consider an angular coordinate α, taking values in $\mathbb{S}^1_\ell := \mathbb{R}/\ell\mathbb{Z}$ (with ℓ the length of the curve γ), and rectangular coordinates $y = (y_1, y_2)$ taking values in the unit 2-disk $\mathbb{D}^2$.

To extract information about h, we start with a good guess of what h should look like: one can check that there is some function of the form $1 + \mathcal{O}(\epsilon)$ such that the vector field

$$h_0 := [1 + \mathcal{O}(\epsilon)] \left(\partial_\alpha + \tau\, \partial_\theta \right)$$

is "almost harmonic", in the sense that it is curl-free, tangent to the boundary and satisfies

$$\rho := -\operatorname{div} h_0 = \mathcal{O}(\epsilon) \,.$$

The actual form of h_0 and ρ is important, but we will not write these details to keep the exposition simple.

From the above considerations we infer that the harmonic field is given by

$$h = h_0 + \nabla\psi \,,$$

where ψ solves the boundary value problem

$$\Delta\psi = \rho \quad \text{in } \mathcal{T}_\epsilon, \qquad \partial_\nu\psi|_{\partial\mathcal{T}_\epsilon} = 0, \qquad \int_{\mathcal{T}_\epsilon} \psi\, dx = 0 \,. \tag{4.3}$$

In the derivation of the result on preservation of the invariant torus we will need to use that ψ is of the following form:

- $\psi = \mathcal{O}(\epsilon^2)$

- $D_y\psi = (\text{certain explicit function}) + \mathcal{O}(\epsilon^4)$

- $\partial_\theta \psi = (\text{certain explicit function}) + \mathcal{O}(\epsilon^5)$

The explicit expressions for the derivatives of ψ are important too, but we will omit them so as not to obscure the main points of the proof.

What is clear is that a useful tool to analyze h is the boundary value problem for the Laplacian on scalar functions with zero mean and zero Neumann boundary conditions in the thin tube $\mathcal{T}_\epsilon$, cf. Eq. (4.3). When written in the natural coordinates (α, y), we obtain a boundary value problem in the domain $\mathbb{S}^1_\ell \times \mathbb{D}^2$, the coefficients of the Laplacian in these coordinates depending on the geometry of the tube strongly through its thickness ϵ and the curvature and torsion of the core curve γ_i.

Since all the functions that appear in the proof are smooth, the key parameter one needs to control is the width ϵ. Hence our main tool in the analysis of ψ will be estimates for the L^2 norm of ψ and and its derivatives that are *optimal* with respect to the parameter ϵ. The reason for this is that estimates of the form

$$\|\psi\|_{H^{k+2}(\mathcal{T}_\epsilon)} \leqslant C_{\epsilon,k} \|\rho\|_{H^k(\mathcal{T}_\epsilon)}$$

are of little use to us because for the preservation of the torus we will need to be very careful in dealing with powers of the small parameter ϵ. In particular, in order to be able to compute the desired dynamical quantities for the local Beltrami field later on, it is crucial to distinguish between estimates for derivatives of ψ with respect to the "slow" variable α and the "fast" variable y, and even to trade some of the gain of derivatives associated with the elliptic equation (4.3) (in some cases) for an improvement of the dependence on ϵ of the constants. Estimates optimal with respect to ϵ are also derived for the problem (4.1) to help us exploit the connection between Beltrami fields with small λ and harmonic fields.

Step 2. To analyze the robustness of the invariant torus $\partial \mathcal{T}_\epsilon$ of the local solution v, the natural tool is KAM theory. At first, it may not be immediate to see why we can apply KAM-type arguments, as v is a divergence-free vector field in a three-dimensional domain and KAM theory is usually discussed in the context of integrable Hamiltonian systems in even-dimensional spaces.

The key here is to consider the Poincaré (or first return) map of v. To define this map, we take a normal section of the tube $\mathcal{T}_\epsilon$, say $\{\alpha = 0\}$. Given a point x_0 in this section, the Poincaré map Π associates to x_0 the point where the vortex line $x(t)$ with initial condition $x(0) = x_0$ cuts the section $\{\alpha = 0\}$ for the first positive time. The analysis in Step 1 gives that the harmonic field h is of the form

$$h = \partial_\alpha + \tau(\alpha)\,(y_1\,\partial_2 - y_2\,\partial_1) + \mathcal{O}(\epsilon)\,, \tag{4.4}$$

so with a little work one can prove that the Poincaré map is well defined for small enough ϵ and λ. Identifying this section with the disk $\mathbb{D}^2$ via the coordinates y, this defines the Poincaré map as a diffeomorphism

$$\Pi : \mathbb{D}^2 \to \mathbb{D}^2\,.$$

Since the vector field v is divergence-free, one can prove that the Poincaré map preserves some measure on the disk.

Notice that the invariant torus $\partial \mathcal{T}_\epsilon$ manifests itself as an invariant circle (namely, $\partial \mathbb{D}^2$) of the Poincaré map, which is measure-preserving. To establish the robustness of the invariant torus $\partial \mathcal{T}_\epsilon$, we will resort to an existing KAM theorem [8] to prove that

the invariant circle of Π is preserved under small measure-preserving perturbations of Π. After taking care of several technicalities that will be disregarded here, thanks to this theorem we can conclude that the invariant torus $\partial \mathcal{T}_\epsilon$ is robust provided two conditions are met: that the rotation number of Π on the invariant circle is Diophantine and that Π satisfies a nondegeneracy condition (namely, that its normal torsion is nonzero).

We will not attempt to define these objects here and explain why these conditions appear in the KAM theorem. What we want to emphasize is that computing the rotation number ω_Π and normal torsion $\mathcal{N}_\Pi$ of the Poincaré map amounts to obtaining quantitative information about the trajectories of v. This is a hard, messy, lengthy calculation that we carry out by combining an iterative approach to control the integral curves of the associated dynamical system (i.e., the vortex lines) with small parameter ϵ and the PDE estimates, optimal with respect to ϵ, that we obtained for v. The final formulas are

$$\omega_\Pi = \int_0^\ell \tau(\alpha)\, d\alpha + \mathcal{O}(\epsilon^2)\,,$$

$$\mathcal{N}_\Pi = -\frac{5\pi\epsilon^2}{8} \int_0^\ell \kappa(\alpha)^2\, \tau(\alpha)\, d\alpha + \mathcal{O}(\epsilon^3)\,, \tag{4.5}$$

where κ and τ respectively denote the curvature and torsion of the curve γ. These expressions allow us to prove that for a "generic" curve γ the rotation number is Diophantine and the normal torsion is nonzero, so the hypotheses of the KAM theorem are satisfied. Hence the invariant torus $\partial \mathcal{T}_\epsilon$ of the local Beltrami field v is robust: if u is a divergence-free vector field in a neighborhood of the tubes that is close enough to v in a suitable sense (e.g., in a high enough C^k norm), then u also has an invariant tube diffeomorphic to $\mathcal{T}_\epsilon$, and moreover the corresponding diffeomorphism can be taken close to the identity.

It is worth mentioning that the formula (4.5) provides some intuition about the question of why one needs to be so careful with the dependence on ϵ of the various estimates: the normal torsion, which must be nonzero, is of order $\mathcal{O}(\epsilon^2)$. Another way of understanding this is by looking at the expression (4.4) for the harmonic field, which implies that our local solution v is an ϵ-small perturbation of the most degenerate kind of vector field from the point of view of KAM theory: a field with constant rotation number.

Step 3.

To complete the proof of the theorem, we show that there is a solution u of Eq. (1.3) in $\mathbb{R}^3$ close to the local solution:

$$\|u - v\|_{C^k(\mathcal{T}_\epsilon)} < \delta\,.$$

Besides, the field u falls off at infinity as

$$|D^j u(x)| < \frac{C_j}{|x|}\,.$$

As we have mentioned, this decay is optimal in the class of Beltrami fields and ensures that u belongs to the Lorentz space $L^{3,\infty}(\mathbb{R}^3)$.

The way we should think about this result is as some kind of Runge theorem for Eq. (1.3). Let us recall that the classical Runge theorem asserts that, if we have an

analytic function in an open subset S of the complex plane, we can approximate it by an entire function provided that the complement of the closure of S is connected. Incidentally, let us point out that the analog of this connectedness condition for the set where the local solution is defined is what forces us to define the local solution via a boundary value problem instead of through a (much simpler) Cauchy–Kowalewski theorem.

The Runge theorem has been extended to elliptic equations by Lax and Malgrange. Notice, however, that Eq. (1.3) is not elliptic and that we additionally need to obtain some decay at infinity, which one does not have in the results of Runge, Lax or Malgrange. Basically, the proof of our Runge-type theorem consists of two steps. In the first step we use functional-analytic methods to approximate the field v by an auxiliary vector field w that satisfies the elliptic equation $\Delta w = -\lambda^2 w$ in a large ball of $\mathbb{R}^3$ that contains all the tubes. In the second step, we define the approximating global Beltrami field u in terms of a truncation of a suitable series representation of the field w and a simple algebraic trick.

Putting all three steps together, this gives the outline of the proof of Theorem 3.1.

5. Realization of vortex lines

To conclude, we shall recall our previous realization theorem for (possibly knotted and linked) vortex lines [5] and discuss the differences and similarities of the proofs. The statement is as follows:

Theorem 5.1. *Let $\gamma_1, \ldots, \gamma_N$ be N pairwise disjoint (possibly knotted and linked) closed curves in $\mathbb{R}^3$. Then for any nonzero real constant λ one can transform these curves by a C^∞ diffeomorphism Φ of $\mathbb{R}^3$, arbitrarily close to the identity in any C^k norm, so that $\Phi(\gamma_1), \ldots, \Phi(\gamma_N)$ is a set of closed vortex lines of a Beltrami field u, which satisfies $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$ and falls off as $|D^j u(x)| < C_j/|x|$.*

The proof of both this result and of the realization theorem for vortex tubes (Theorem 3.1) is based on the construction of a local Beltrami field with a collection of robust invariant sets (either invariant tori, in the case of vortex tubes, or periodic trajectories, for vortex lines), which is then approximated by a global Beltrami field. However, the implementation of this basic principle is totally different.

To begin with, the robustness of periodic trajectories in Theorem 5.1 relies on the notion of hyperbolicity, which is essentially a nondegeneracy condition to apply the implicit function theorem. On the contrary, in the case of vortex tubes, the robustness of the invariant tori requires a considerably more sophisticated KAM argument.

Most importantly, we could say that the nondegeneracy condition of periodic trajectories in Theorem 5.1 is something we prescribe, while for vortex tubes this nondegeneracy cannot be granted a priori: we have to extract it from the equation, and is only true for a certain large (indeed, "generic") set of core curves γ_i. The reason for this discrepancy is that, in order to construct a local Beltrami field with prescribed nondegenerate periodic trajectories γ_i, it is enough to prove a suitable

analog of the Cauchy–Kowalewski theorem for the curl operator. On the contrary, in the case of vortex tubes the construction of the local solution with a robust set of prescribed invariant tori requires the analysis of the boundary value problem and the KAM argument described in Steps 1 and 2.

It should be remarked that the decay of the field u stated in Theorem 5.1 cannot be found in the original paper [5]. The reason for this is that in [5] we used an approximation theorem for Beltrami fields weaker than the one we have developed for the Step 3 of the proof of Theorem 3.1, so that we could not provide any control whatsoever on the growth at infinity of the global Beltrami field. With the new approximation theorem, one readily obtains the statement of Theorem 5.1. The proof of these approximation theorems is quite different in that the old theorem was valid for the equation of Beltrami fields in any open Riemannian three-dimensional manifold, while the new one exploits the symmetries of Euclidean space to ensure that the global Beltrami field falls off at infinity in an optimal way.

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Instituto de Ciencias Matemáticas
Consejo Superior de Investigaciones Científicas
28049 Madrid, Spain
aenciso@icmat.es, dperalta@icmat.es