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Positive families and Boolean chains of copies of ultrahomogeneous structures

Miloš S. Kurilić [a] **and Boriša Kuzeljević** [a]

[a] Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad

E-mails: milos@dmi.uns.ac.rs (M. S. Kurilić), borisha@dmi.uns.ac.rs (B. Kuzeljević)

Abstract. A family of infinite subsets of a countable set X is called *positive* iff it is closed under supersets and finite changes and contains a co-infinite set. We show that a countable ultrahomogeneous relational structure $\mathbb{X}$ has the strong amalgamation property iff the set $\mathbb{P}(\mathbb{X}) = \{A \subset X : \mathbb{A} \cong \mathbb{X}\}$ contains a positive family. In that case the family of large copies of $\mathbb{X}$ (i.e. copies having infinite intersection with each orbit) is the largest positive family in $\mathbb{P}(\mathbb{X})$, and for each $\mathbb{R}$-embeddable Boolean linear order $\mathbb{L}$ whose minimum is non-isolated there is a maximal chain isomorphic to $\mathbb{L} \setminus \{\min \mathbb{L}\}$ in $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$.

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1. Introduction

The purpose of this short note is twofold. One is to present some new results about positive families. The other one is to provide a natural context for the recent research from [11–13]. For a countably infinite set X, a family $\mathscr{P} \subset P(X)$ is called a *positive family on* X (see [10]) iff

(P1) $\mathscr{P} \subset [X]^\omega$,
(P2) $\mathscr{P} \ni A \subset B \subset X \Rightarrow B \in \mathscr{P}$,
(P3) $A \in \mathscr{P} \wedge |F| < \omega \Rightarrow A \setminus F \in \mathscr{P}$,
(P4) $\exists\, A \in \mathscr{P}\ |X \setminus A| = \omega$.

We regard a positive family $\mathscr{P}$ on X as a suborder of the partial order $\langle [X]^\omega, \subset \rangle$ (isomorphic to $\langle [\omega]^\omega, \subset \rangle$) and important examples of positive families are co-ideals: if $\mathscr{I} \subset P(\omega)$ is an ideal containing the ideal Fin of finite subsets of ω, then the set $\mathscr{I}^+ := P(\omega) \setminus \mathscr{I}$ of $\mathscr{I}$-positive sets is a positive family. Thus $[\omega]^\omega$ is the largest, while non-principal ultrafilters $\mathscr{U} \subset P(\omega)$ are the minimal positive families of this form. Also, $\mathscr{I}^+_{\mathrm{nwd}} = \{A \subset \mathbb{Q} : \operatorname{Int} \overline{A} \neq \emptyset\}$ and $\mathscr{I}^+_{\mathrm{lmz}} = \{A \subset \mathbb{Q} : \mu(\overline{A}) > 0\}$ are positive families on the set of rationals $\mathbb{Q}$, where $\overline{S}$, Int S and $\mu(S)$ denote the $\mathbb{R}$-closure, $\mathbb{R}$-interior and Lebesgue measure of a subset S of the real line $\mathbb{R}$ with the standard topology. Taking a non-maximal filter $\mathscr{F} \subset P(\omega)$ which extends the Fréchet filter we obtain a positive family which is not a co-ideal; another such example is the family $\mathrm{Dense}(\mathbb{Q})$ from Example 7; see also Theorem 5.

In our notation $\mathbb{P}(\mathbb{X}) = \{A \subset X : \mathbb{A} \cong \mathbb{X}\}$ denotes the set of all copies of a structure $\mathbb{X}$ contained in $\mathbb{X}$. The class of order types of maximal chains in the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ will be denoted by $\mathscr{M}_{\mathbb{X}}$. Let $\mathscr{C}_{\mathbb{R}}$ denote the class of order types of sets of the form $K \setminus \{\min K\}$, where $K \subset \mathbb{R}$ is a compact set such that $\min K$ is an accumulation point of K. Let $\mathscr{B}_{\mathbb{R}}$ be the subclass of order types from $\mathscr{C}_{\mathbb{R}}$ for which the corresponding compact set K is, in addition, nowhere dense. Main results from [12, 13] state that for a countable ultrahomogeneous partial order $\mathbb{P}$

$$\mathscr{M}_{\mathbb{P}} = \begin{cases} \mathscr{B}_{\mathbb{R}}, & \text{if } \mathbb{P} \text{ is a countable antichain,} \\ \mathscr{C}_{\mathbb{R}}, & \text{otherwise,} \end{cases}$$

while for a countable ultrahomogeneous graph $\mathbb{G}$ we have

$$\mathscr{M}_{\mathbb{G}} = \begin{cases} \mathscr{B}_{\mathbb{R}}, & \text{if } \mathbb{G} \text{ is a disjoint union of complete graphs,} \\ \mathscr{C}_{\mathbb{R}}, & \text{otherwise.} \end{cases}$$

These results suggest that there might be a general theorem describing the classes $\mathscr{M}_{\mathbb{X}}$. The reason for focusing on ultrahomogeneous structures is that $\mathscr{M}_{\mathbb{X}} \subset \mathscr{C}_{\mathbb{R}}$ for an ultrahomogeneous $\mathbb{X}$ (see [13] for example). Still, there are pathological structures even in the class of ultrahomogeneous ones. For example, there are ultrahomogeneous structures without non-trivial copies (see [8, p. 399]). This kind of obstruction does not exist in the class of countable ultrahomogeneous relational structures whose age satisfies the strong amalgamation property (SAP). Recall the following equivalence (see [8, p. 399]): a countable ultrahomogeneous relational structure $\mathbb{X}$ satisfies SAP if and only if $X \setminus F \in \mathbb{P}(\mathbb{X})$, for each finite $F \subset X$.

Section 2 contains results about positive families. The central one is that for a countable ultrahomogeneous relational structure $\mathbb{X}$, there is a positive family $\mathscr{P}$ on X such that $\mathscr{P} \subset \mathbb{P}(\mathbb{X})$ if and only if the age of $\mathbb{X}$ satisfies SAP. From this result in Section 3 we deduce that the structures whose age satisfies SAP provide a natural context for investigating the phenomena we have described above.

Theorem 1. *If $\mathbb{X}$ is a countable ultrahomogeneous relational structure whose age satisfies SAP, then $\mathscr{B}_{\mathbb{R}} \subset \mathscr{M}_{\mathbb{X}} \subset \mathscr{C}_{\mathbb{R}}$.*

Since the class $\mathscr{B}_{\mathbb{R}}$ is quite rich, the previous result shows that many linear orders can be realized as maximal chains in $\mathbb{P}(\mathbb{X})$ in that case. For example, the reverse of every countable limit ordinal, or the order type of the Cantor set without 0. Note also that the countable complete graph $\mathbb{K}_\omega$ satisfies SAP, and that $\mathscr{M}_{\mathbb{K}_\omega} = \mathscr{B}_{\mathbb{R}}$. On the other hand, the Rado graph $\mathbb{G}_{\mathrm{Rado}}$ also satisfies SAP, but $\mathscr{M}_{\mathbb{G}_{\mathrm{Rado}}} = \mathscr{C}_{\mathbb{R}}$. This implies that it is not possible to narrow the interval of possibilities in Theorem 1. However, we do not know an answer to the following question.

Question 2. *Is there a countable ultrahomogeneous relational structure $\mathbb{X}$ whose age satisfies SAP, but such that $\mathscr{B}_{\mathbb{R}} \subsetneq \mathscr{M}_{\mathbb{X}} \subsetneq \mathscr{C}_{\mathbb{R}}$?*

We assume that the reader is familiar with Fraïssé theory. The theory itself was started in [5–7], while a detailed treatment is given in [8]. Besides the mentioned book, [13] is a good reference for all undefined notions. We will only comment on the notion of an orbit. Suppose that $\mathbb{X}$ is a relational structure and $F \subset X$ finite. We say that $x \sim_F y$ iff there is $g \in \mathrm{Aut}(\mathbb{X})$ such that $g \restriction F = \mathrm{id}_F$ and $g(x) = y$. Clearly, $\sim_F$ is an equivalence relation, and $\mathrm{orb}_F(x)$ denotes the class of an element x. The sets $\mathrm{orb}_F(x)$ are called the *orbits of* $\mathbb{X}$. We call a copy $A \in \mathbb{P}(\mathbb{X})$ *large* iff it has infinite intersection with each orbit of $\mathbb{X}$. For sets A and B, let $A \subset^* B$ denote the inclusion modulo finite, i.e. $A \subset^* B \Leftrightarrow |A \setminus B| < \omega$.

2. SAP, large copies and positive families

Theorem 3. *If $\mathbb{X}$ is a countable ultrahomogeneous structure $\mathbb{X}$ satisfying SAP, then a copy $A \in \mathbb{P}(\mathbb{X})$ is large iff it intersects each orbit of $\mathbb{X}$.*

Proof. Suppose that A is a copy of $\mathbb{X}$ intersecting all orbits of $\mathbb{X}$ and that the intersection $A \cap \mathrm{orb}_F(x) = F_1$ is finite, for some finite set $F \subset X$ and some $x \in X \setminus F$. Since $\mathbb{X}$ satisfies SAP we have $|\mathrm{orb}_F(x)| = \omega$ and, thus, we can assume that $x \notin F_1$. Now, $\mathrm{orb}_{F\cup F_1}(x) \subset \mathrm{orb}_F(x) \setminus F_1$ and, hence, $A \cap \mathrm{orb}_{F\cup F_1}(x) = \emptyset$, which is a contradiction. □

Note that the assumption that $\mathbb{X}$ has SAP can not be removed from the previous theorem, since (trivially) X intersects all orbits of $\mathbb{X}$.

Theorem 4. *For a countable ultrahomogeneous relational structure $\mathbb{X}$ the following conditions are equivalent:*

(a) *$\mathbb{X}$ satisfies the strong amalgamation property,*
(b) *$\mathbb{X}$ has a large copy,*
(c) *There is a positive family $\mathscr{P}$ on X such that $\mathscr{P} \subset \mathbb{P}(\mathbb{X})$,*
(d) *There is a co-infinite $A \in \mathbb{P}(\mathbb{X})$ such that $B \in \mathbb{P}(\mathbb{X})$, whenever $A \subset^* B \subset X$.*

Proof. (a) ⇔ (b)**.** Recall that $\mathbb{X}$ satisfies SAP iff all the orbits of $\mathbb{X}$ are infinite (cf. [2, Theorem 2.15, p. 37]). Then X is a large copy of $\mathbb{X}$. Conversely, if A is a large copy of $\mathbb{X}$, then A witnesses that all orbits of $\mathbb{X}$ are infinite; thus $\mathbb{X}$ satisfies SAP.

(a) ⇒ (c)**.** If $\mathbb{X}$ satisfies SAP, then the orbits of $\mathbb{X}$ are infinite and by Bernstein's Lemma (see [9, Lemma 2, p. 514], with ω instead of $\mathfrak{c}$) there are two disjoint sets $A_0, A_1 \subset X$ intersecting all orbits of $\mathbb{X}$, which implies that $A_0, A_1 \in \mathbb{P}(\mathbb{X})$ (see e.g. [14, Theorem 2.3]). By Theorem 3 A_0 and A_1 are large copies of $\mathbb{X}$ (alternatively, see [14, Theorem 3.2]). Now, $\mathscr{P} := \{A \in \mathbb{P}(\mathbb{X}) : A_0 \subset^* A\} \subset [X]^\omega$ and, since $A_1 \subset X \setminus A_0$, (P4) is true. If $\mathscr{P} \ni A \subset B \subset X$, then $A_0 \subset^* B$. In addition, for each orbit O of $\mathbb{X}$ we have $|A_0 \cap O| = \omega$ and, hence, $|B \cap O| = \omega$, which gives $B \in \mathbb{P}(\mathbb{X})$ (by [14, Theorem 2.3] again). Thus $B \in \mathscr{P}$ and (P2) is true. If $A \in \mathscr{P}$ and $F \subset X$ is a finite set, then, clearly, $A_0 \subset^* A \setminus F$ and, as above, $A \setminus F \in \mathbb{P}(\mathbb{X})$. Thus $A \setminus F \in \mathscr{P}$, (P3) is true and $\mathscr{P}$ is a positive family indeed.

(c) ⇒ (d)**.** If $\mathscr{P} \subset \mathbb{P}(\mathbb{X})$ is a positive family, then by (P4) there is a co-infinite set $A \in \mathscr{P}$ and, hence, $A \in \mathbb{P}(\mathbb{X})$. For $B \subset X$ such that $A \setminus B =: F$ is a finite set, by (P3) we have $\mathscr{P} \ni A \setminus F \subset B$ and, by (P2), $B \in \mathscr{P}$, thus $B \in \mathbb{P}(\mathbb{X})$.

(d) ⇒ (a)**.** Suppose that $A \subset X$ is a copy given by (d). Then for each finite set $F \subset X$ we have $A \subset^* X \setminus F$. Thus, by (d), $X \setminus F \in \mathbb{P}(\mathbb{X})$. Now [4, Theorem 2] implies that the structure $\mathbb{X}$ satisfies SAP. □

Now we turn to maximal positive families.

Theorem 5. *Let $\mathbb{X}$ be a countable ultrahomogeneous relational structure satisfying SAP. If $\mathscr{P}_{\max} := \big\{A \in \mathbb{P}(\mathbb{X}) : \forall B \subset X \ (A \subset^* B \Rightarrow B \in \mathbb{P}(\mathbb{X}))\big\}$, then*

(a) *$\mathscr{P}_{\max}$ is the largest positive family on X contained in $\mathbb{P}(\mathbb{X})$;*
(b) *$\mathscr{P}_{\max} = \{A \in \mathbb{P}(\mathbb{X}) : \forall B \subset X \ (A \subset B \Rightarrow B \in \mathbb{P}(\mathbb{X}))\}$;*
(c) *$\mathscr{P}_{\max} = \{A \subset X : A$ intersects all the orbits of $\mathbb{X}\}$;*
(d) *$\mathscr{P}_{\max} = \{A \subset X : A$ is a large copy of $\mathbb{X}\}$.*

Proof. (a). $\mathscr{P}_{\max}$ satisfies condition (P1), because $\mathscr{P}_{\max} \subset \mathbb{P}(\mathbb{X}) \subset [X]^{\omega}$.

(P2) Assuming that $\mathscr{P}_{\max} \ni A \subset C \subset X$ we show that $C \in \mathscr{P}_{\max}$. Let $C \subset^* B \subset X$. Then $A \subset^* B$ as well. Since $A \in \mathscr{P}_{\max}$, both $C \in \mathbb{P}(\mathbb{X})$ and $B \in \mathbb{P}(\mathbb{X})$ hold. Thus $C \in \mathscr{P}_{\max}$ indeed.

(P3) Let $A \in \mathscr{P}_{\max}$ and $F \in [X]^{<\omega}$. Let $A \setminus F \subset^* B \subset X$. Since $A \in \mathbb{P}(\mathbb{X})$, by [4, Theorem 2], $A \setminus F \in \mathbb{P}(\mathbb{X})$. Note that $A \subset^* A \setminus F$ implies $A \subset^* B$. Now from $A \in \mathscr{P}_{\max}$ follows $B \in \mathbb{P}(\mathbb{X})$. Thus $A \setminus F \in \mathscr{P}_{\max}$.

(P4) By Theorem 4, there is a co-infinite set $A \in \mathscr{P}_{\max}$.

Now we show that $\mathscr{P}_{\max}$ is the largest positive family. Let $\mathscr{P} \subset \mathbb{P}(\mathbb{X})$ be a positive family on X. We prove $\mathscr{P} \subset \mathscr{P}_{\max}$, so let $A \in \mathscr{P}$ and $A \subset^* B \subset X$. Then $F := A \setminus B$ is a finite set. Since $\mathscr{P}$ satisfies (P3), we have $A \cap B = A \setminus F \in \mathscr{P}$. By (P2) we have $B \in \mathscr{P}$. This implies $B \in \mathbb{P}(\mathbb{X})$ because $\mathscr{P} \subset \mathbb{P}(\mathbb{X})$. So $A \in \mathscr{P}_{\max}$.

(b). Clearly, $\mathscr{P} := \{A \in \mathbb{P}(\mathbb{X}) : \forall B \subset X \ (A \subset B \Rightarrow B \in \mathbb{P}(\mathbb{X}))\} \supset \mathscr{P}_{\max}$. To prove the reverse inclusion, take any $A \in \mathscr{P}$ and $B \subset X$ such that $A \subset^* B$. Then $F = A \setminus B \in [X]^{<\omega}$ and $A \subset B \cup F$. Definition of $\mathscr{P}$ implies $B \cup F \in \mathbb{P}(\mathbb{X})$. Since F is finite, Theorem 2 in [4] implies that $B \in \mathbb{P}(\mathbb{X})$ is as required.

(c). Let $\mathscr{P}_1 := \{A \subset X : A$ intersects all the orbits of $\mathbb{X}\}$. We check if $\mathscr{P}_1$ is a positive family on X. By Theorem 2.3 in [14], $\mathscr{P}_1 \subset \mathbb{P}(\mathbb{X}) \subset [X]^{\omega}$, so (P1) holds.

(P2) If $\mathscr{P}_1 \ni A \subset B \subset X$, then B intersects all the orbits of $\mathbb{X}$. So $B \in \mathscr{P}_1$.

(P3) Let $A \in \mathscr{P}_1$, $F \in [X]^{<\omega}$, and let O be an orbit of $\mathbb{X}$. Since $\mathbb{X}$ satisfies SAP, Theorem 3 implies $|A \cap O| = \omega$. So $(A \setminus F) \cap O \neq \emptyset$, and $A \setminus F \in \mathscr{P}_1$.

(P4) follows from [14, Theorem 3.2].

By the maximality of $\mathscr{P}_{\max}$, as proved in (a), we have $\mathscr{P}_1 \subset \mathscr{P}_{\max}$. So we still have to prove $\mathscr{P}_{\max} \subset \mathscr{P}_1$. Take any $A \in \mathscr{P}_{\max}$, any $F \in [X]^{<\omega}$, and any $x \in X \setminus F$. We will find $y \in A \cap \mathrm{orb}_F(x)$, which proves that $A \in \mathscr{P}_1$. Definition of $\mathscr{P}_{\max}$ implies that $A_1 := A \cup F \cup \{x\} \in \mathbb{P}(\mathbb{X})$. Since $\mathbb{X}$ satisfies SAP, by Theorem 2.15 in [2, p. 37] applied to the structure $\mathbb{A}_1$ we know that the orbit of x over F in $\mathbb{A}_1$ is infinite. Hence there is $y \in A_1 \setminus (F \cup \{x\})$, and $g \in \mathrm{Aut}(\mathbb{A}_1)$ such that $g \restriction F = \mathrm{id}_F$ and $g(x) = y$. Let $\varphi := g \restriction (F \cup \{x\})$. Since $\mathbb{X}$ is ultrahomogeneous, there is $f \in \mathrm{Aut}(\mathbb{X})$ such that $\varphi \subset f$. Hence, $f \restriction F = \mathrm{id}_F$ and $f(x) = y$. Thus $y \in \mathrm{orb}_F(x)$. Since $y \in A_1 \setminus (F \cup \{x\})$ we have $y \in A \cap \mathrm{orb}_F(x)$ as required.

(d). It follows from (c) and Theorem 3. □

Example 6. Following the terminology of Fraïssé, a relational structure $\mathbb{X}$ is called *constant* iff $\mathrm{Aut}(\mathbb{X}) = \mathrm{Sym}(X)$. Since each isomorphism between finite substructures of $\mathbb{X}$ can be extended to a bijection, $\mathbb{X}$ is ultrahomogeneous. In addition, for a finite $F \subset X$ and $x \in X \setminus F$ we have $\mathrm{orb}_F(x) = X \setminus F$. So each countable constant relational structure $\mathbb{X}$ is ultrahomogeneous and satisfies SAP. Moreover, since each injection from X to X is an embedding, $\mathbb{X}$ has the following extreme property: $\mathscr{P}_{\max} = \mathbb{P}(\mathbb{X}) = [X]^{|X|}$. It is easy to see that $\mathbb{X}$ is constant iff each of its relations is definable by a (quantifier-free) first order formula whose unique non-logical symbol is the equality. For example, there are four countable binary constant structures: $\langle \omega, \emptyset \rangle$, $\langle \omega, \omega^2 \rangle$, $\langle \omega, \Delta_\omega \rangle$ and $\langle \omega, \omega^2 \setminus \Delta_\omega \rangle$ and the last one is defined by the formula $\neg v_0 = v_1$. As another example, the formula $\varphi := v_0 = v_1 \vee v_1 = v_2 \vee \neg v_2 = v_3$ defines a quaternary constant relation.

Example 7. For the rational line, $\langle \mathbb{Q}, < \rangle$, the orbits are open intervals. Thus

$$\mathscr{P}_{\max} = \mathrm{Dense}(\mathbb{Q}) := \{A \subset \mathbb{Q} : \forall p, q \in \mathbb{Q} \ (p < q \Rightarrow A \cap (p,q)_{\mathbb{Q}} \neq \emptyset)\}.$$

This means that the fact that the rational line can be split into countably many disjoint dense sets is a special case of Theorem 3.2 in [14], while the fact that there is a continuum-sized almost disjoint family of dense subsets of the rational line is a special case of Theorem 4.1 in [14].

3. Boolean maximal chains of copies

Here we prove Theorem 1 and present some applications. Let $\mathbb{X}$ be a countable ultrahomogeneous relational structure satisfying SAP. As already mentioned $\mathcal{M}_{\mathbb{X}} \subset \mathcal{C}_{\mathbb{R}}$ is known (for example, take a look at [13, Theorem 2.2]). The remaining part of the statement follows from the next proposition.

Theorem 8. *If $\mathbb{X}$ is a countable ultrahomogeneous relational structure satisfying SAP, then $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}_{\mathbb{X}}$.*

Proof. Suppose that $\mathbb{L}$ is such that $\operatorname{otp}(\mathbb{L}) \in \mathcal{B}_{\mathbb{R}}$. Let $\mathbb{L}' = \mathbb{L} \cup \{-\infty\}$ where $\{-\infty\}$ is the minimum of $\mathbb{L}'$. By Theorem 3 in [11], $\mathbb{L}'$ is isomorphic to an $\mathbb{R}$-embeddable complete linear order whose minimum is non-isolated. Since $\mathbb{X}$ satisfies SAP, by Theorem 5(d) $\mathcal{P} = \{A \subset X : A \text{ is a large copy of } X\}$ is a positive family contained in $\mathbb{P}(\mathbb{X})$. Theorem 3.2 in [14] guaranties that $\bigcap \mathcal{P} = \emptyset$. Hence, Theorem 3.6(a) in [12] implies that there is a maximal chain $\mathcal{L}$ in $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ isomorphic to $\mathbb{L}$. Thus $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}_{\mathbb{X}}$. □

Example 9. Countable ultrahomogeneous digraphs have been classified by Cherlin [3]. Referring to the list given in [1] and [15], we mention some structures satisfying SAP, i.e. structures to which Theorem 1 can be applied.

- All countable ultrahomogeneous partial orders except the posets $\langle C_n, \prec_n \rangle$, for $2 \le n < \omega$, where $C_n = \mathbb{Q} \times n$ and $\langle q_1, k_1 \rangle \prec_n \langle q_2, k_2 \rangle \Leftrightarrow q_1 <_{\mathbb{Q}} q_2$ (thus, C_n is a $\mathbb{Q}$-chain of antichains of size n).
- All countable ultrahomogeneous tournaments: the rational line $\mathbb{Q}$; the random tournament $\mathbb{T}^\infty$; and the local order $\langle S(2), \to \rangle$, where $S(2)$ is a countable dense subset of the unit circle, such that no two of its points are antipodal, and $x \to y$ iff the counterclockwise angle between x and y is less than π.
- All Henson's digraphs with forbidden sets of tournaments;
- The digraphs Γ_n, for $n > 1$, where Γ_n is the Fraïssé limit of the amalgamation class of all finite digraphs not embedding the empty digraph of size n.
- Two "sporadic" primitive digraphs $S(3)$ and $\mathcal{P}(3)$. The digraph $S(3)$ is defined as the local order $S(2)$, but with angle $2\pi/3$. The digraph $\mathcal{P}(3)$ has a more complicated definition; it is precisely defined in [3, p. 76].
- The imprimitive digraphs $n * I_\infty$, for $2 \le n \le \omega$. The digraph $n * I_\infty$ is obtained from a countable complete n-partite graph by randomly orienting its edges.
- The digraph which is a semigeneric variant of $\omega * I_\infty$ with a parity constraint, i.e. it is a countable ultrahomogeneous digraph in which non-relatedness is an equivalence relation and for any two pairs A_1, A_2 taken from distinct equivalence classes, the number of edges from A_1 to A_2 is even.

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