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Damir Yeliussizov

Dual Grothendieck polynomials via last-passage percolation

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Polynômes de Grothendieck duales par percolation de dernier passage

Damir Yeliussizov[a]

[a] Kazakh-British Technical University, 59 Tole bi st, Almaty 050000, Kazakhstan.
E-mail: yeldamir@gmail.com.

Abstract. The ring of symmetric functions has a basis of dual Grothendieck polynomials that are inhomogeneous K-theoretic deformations of Schur polynomials. We prove that dual Grothendieck polynomials determine column distributions for a directed last-passage percolation model.

Résumé. L'anneau de fonctions symétriques a une base de polynômes de Grothendieck duales qui sont des déformations K-théoriques non homogénes des polynômes de Schur. Nous prouvons que les polynômes de Grothendieck duales déterminent distributions des colonnes pour un modèle de percolation dirigée de dernier passage.

1. Introduction

In this note we show a surprising connection between (i) the dual Grothendieck polynomials that are deformations of Schur polynomials arising in K-theoretic Schubert calculus, and (ii) a directed last-passage percolation model (which can also be viewed as the corner growth model or a totally asymmetric simple exclusion process).

1.1. *Dual Grothendieck polynomials*

The ring of symmetric functions has an inhomogeneous basis $\{g_\lambda\}$ called the *dual Grothendieck polynomials*. The symmetric polynomials $g_\lambda(x_1, \dots, x_n)$ can be defined via the following combinatorial formula

$$g_\lambda(x_1, \dots, x_n) := \sum_{\pi : \mathrm{sh}(\pi) = \lambda} \prod_{i=1}^{n} x_i^{c_i(\pi)},$$

where the sum runs over *plane partitions* π of shape λ with largest entry at most n and $c_i(\pi)$ is the number of columns of π containing i. It is easy to see that $g_\lambda = s_\lambda +$ lower degree terms, where s_λ is the Schur polynomial. This basis was explicitly introduced and studied in [7] (and earlier implicitly in [3, 8]) in relation to the K-theory of Grassmannians. More properties of these functions can also be found in [13, 16].

1.2. *Directed last-passage percolation*

Let $W = (w_{ij})_{i,j \geq 1}$ be a random matrix with independent entries w_{ij} that have geometric distribution with parameters $q_j \in (0, 1)$, i.e.

$$\text{Prob}(w_{ij} = k) = (1 - q_j)\, q_j^k, \quad k \in \mathbb{N}.$$

A lattice path Π with vertices in $\mathbb{N}^2$ is called a *directed path* if it uses only steps of the form $(i, j) \to (i + 1, j), (i, j + 1)$. Define the *last-passage times $G(m, n)$* as follows:

$$G(m, n) := \max_{\Pi : (1,1) \to (m,n)} \sum_{(i,j) \in \Pi} w_{ij},$$

where the maximum is over directed paths Π from $(1, 1)$ to (m, n). The function G presents certain random growth. This probabilistic model, which can also be viewed as the corner growth model or a totally asymmetric simple exclusion process (TASEP), was studied intensively (especially in the i.i.d. case $q_j = q$), see [1, 4, 6, 10, 11] and references therein. Let us call the matrix $G = (G(m, n))_{m,n \geq 1}$ as the *percolation matrix*.

1.3. *Column distributions of the percolation matrix*

Our main result is the formula showing that joint distribution of elements along any column in the percolation matrix G is proportional to evaluations of dual Grothendieck polynomials. Let

$$\mathscr{P}_m := \{\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{N}^m : \lambda_1 \geq \cdots \geq \lambda_m \geq 0\}$$

be the set of integer partitions with at most m parts.

Theorem 1. *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathscr{P}_m$. The following formula holds*

$$\text{Prob}(G(m, n) = \lambda_1, \ldots, G(1, n) = \lambda_m) = g_\lambda(q_1, \ldots, q_n) \prod_{i=1}^n (1 - q_i)^m$$

On one hand, this formula can be viewed as a natural probabilistic interpretation of dual Grothendieck polynomials. On the other hand, it can also be used for computing distribution formulas in the percolation matrix. We prove Theorem 1 combinatorially, using certain bijection between plane partitions and integer matrices. We then give some applications. For example, we present new generating function identities for dual Grothendieck polynomials and determinantal formulas for distributions of the percolation matrix.

2. Proof of the main theorem

2.1. *Plane partitions and $\mathbb{N}$-matrices*

An $\mathbb{N}$-*matrix* is a matrix of nonnegative integers with only finitely many nonzero elements. A *plane partition* is an $\mathbb{N}$-matrix $\pi = (\pi_{ij})_{i,j \geq 1}$ such that

$$\pi_{ij} \geq \pi_{i+1\, j}, \quad \pi_{ij} \geq \pi_{i\, j+1}, \quad i, j \geq 1.$$

The *shape* of π is defined as $\text{sh}(\pi) := \{(i, j) : \pi_{ij} > 0\}$.

Given a plane partition π, define the *descent level sets*

$$D_{i\ell} := \{j : \pi_{ij} = \ell > \pi_{i+1\,j}\},$$

i.e. $D_{i\ell}$ is the set of column indices of the entry ℓ in ith row of π that are strictly larger than the entry below. Let $d_{i\ell} := |D_{i\ell}|$ and $D := (d_{i\ell})_{i,\ell \geq 1}$.

Define the map $\Phi : \{\text{plane partitions}\} \to \{\mathbb{N}\text{-matrices}\}$ by setting

$$\Phi(\pi) = D.$$

For example,

$$\Phi : \begin{array}{|c|c|c|} \hline 4 & 4 & 2 \\ \hline 4 & 2 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \longmapsto \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

In fact, Φ is a bijection; we can uniquely reconstruct π given the matrix D. We refer to [14, 15] for more on this bijection. Denote by $\mathrm{PP}(m, n)$ the set of plane partitions with at most m rows and largest entry at most n. In particular, if $\pi \in \mathrm{PP}(m, n)$, then $\mathrm{sh}(\pi) \in \mathscr{P}_m$ and $D = \Phi(\pi)$ has at most m rows and n columns that are nonzero.

Lemma 2. *Let $W = (w_{i\ell})_{i,\ell=1}^{m,n}$ be an $m \times n$ matrix, where $w_{i\ell}$ are independent geometrically distributed random variables with parameters q_ℓ. Let $\pi \in \mathrm{PP}(m, n)$. Then*

$$\mathrm{Prob}(W = \Phi(\pi)) = \prod_{\ell=1}^{n} (1 - q_\ell)^m q_\ell^{c_\ell(\pi)}$$

where $c_\ell(\pi)$ is the number of columns of π containing ℓ.

Proof. Let $(d_{i\ell}) = \Phi(\pi)$, i.e. $d_{i\ell} = |\{j : \pi_{ij} = \ell > \pi_{i+1\,j}\}|$ for $i \in [1, m], \ell \in [1, n]$. Since the entries of W are independent we obtain that

$$\mathrm{Prob}(W = \Phi(\pi)) = \prod_{i=1}^{m} \prod_{\ell=1}^{n} \mathrm{Prob}(w_{i\ell} = d_{i\ell}) = \prod_{i=1}^{m} \prod_{\ell=1}^{n} (1 - q_\ell)\, q_\ell^{d_{i\ell}} = \prod_{\ell=1}^{n} (1 - q_\ell)^m q_\ell^{c_\ell(\pi)}$$

as $\sum_i d_{i\ell} = c_\ell(\pi)$, as needed. $\qquad\square$

Lemma 3. *Let $\pi \in \mathrm{PP}(m, n)$ and $\Phi(\pi) = D = (d_{i\ell})$. Let $\lambda = \mathrm{sh}(\pi)$ be the shape of π. We have for all $k \in [1, m]$*

$$\lambda_k = \max_{\Pi:(k,1)\to(m,n)} \sum_{(i,\ell)\in\Pi} d_{i\ell}, \tag{1}$$

where the maximum is over directed paths Π from $(k, 1)$ to (m, n).

Proof. Take an arbitrary directed path Π from $(k, 1)$ to (m, n). Then the descent level sets $D_{i\ell}$ for $(i, \ell) \in \Pi$ are pairwisely disjoint. Using this property and since $i \geq k$ for all $(i, \ell) \in \Pi$, we obtain

$$\sum_{(i,\ell)\in\Pi} d_{i\ell} = \sum_{(i,\ell)} |\{j : \pi_{ij} = \ell > \pi_{i+1\,j}\}| \leq \lambda_k. \tag{2}$$

On the other hand, suppose the k-th row of π has entries $(\ell_1 \geq \cdots \geq \ell_s > 0)$ where $s = \lambda_k$. Assume the entries $\ell_1, \ldots, \ell_s$ end in rows $i_1 \geq \cdots \geq i_s$ of π. Then there is a directed path Π from $(k, 1)$ to (m, n) containing all points $(i_s, \ell_s), \ldots, (i_1, \ell_1)$. The weight of any such path is at least $\sum_j d_{i_j\ell_j} \geq s = \lambda_k$. Combining this with the inequality (2) we obtain (1). $\qquad\square$

Proof of Theorem 1. As we are interested only in joint distribution of the last-passage times $(G(m, n), G(m-1, n), \ldots, G(1, n))$ we can restrict the source random matrix W to the first m rows and n columns. Consider w_{ij} as geometric with parameter q_{n-j+1}. By rotation symmetry it is obvious that the corresponding last-passage times produced from the matrix

$W' = (w_{m-i+1, n-j+1})_{i,j=1}^{m,n}$ have the same distributions. Now, by definition of g_λ and using Lemma 2 we obtain that

$$\prod_{\ell=1}^{n} (1-q_\ell)^m g_\lambda(q_1, \ldots, q_n) = \sum_{\pi \in PP(n,m), \, sh(\pi) = \lambda} \prod_{\ell=1}^{n} (1-q_\ell)^m q_\ell^{c_\ell(\pi)}$$

$$= \sum_{\pi \in PP(n,m), \, sh(\pi) = \lambda} \text{Prob}(W' = \Phi(\pi))$$

By Lemma 3 if $W' = \Phi(\pi)$ and $sh(\pi) = \lambda$, we have $G(m,n) = \lambda_1, \ldots, G(1,n) = \lambda_m$. Therefore,

$$\sum_{\pi \in PP(n,m), \, sh(\pi) = \lambda} \text{Prob}(W' = \Phi(\pi)) = \text{Prob}(G(m,n) = \lambda_1, \ldots, G(1,n) = \lambda_m)$$

and hence the result is proved. $\square$

Next we discuss some applications.

3. Last-passage time distributions

3.1. *Parameter symmetry*

Since the polynomials g_λ are symmetric we obtain

Corollary 4. *The distribution*

$$\text{Prob}(G(m,n) = \lambda_1, \ldots, G(1,n) = \lambda_m)$$

is invariant under permutations of the parameters $(q_1, \ldots, q_n)$.

3.2. *Formulas for last-passage distributions*

First, using branching formulas for g_λ, we easily obtain the following distribution formula as well.

Corollary 5. *We have*

$$\text{Prob}(G(m,n) \le \lambda_1, \ldots, G(1,n) \le \lambda_m) = \prod_{i=1}^{n} (1-q_i)^m g_\lambda(1, q_1, \ldots, q_n).$$

Proof. Theorem 1 combined with the branching relation

$$\sum_{\mu \subset \lambda} g_\mu(q_1, \ldots, q_n) = g_\lambda(1, q_1, \ldots, q_n)$$

easily imply the given formula. $\square$

Using Jacobi–Trudi-type determinantal identities for g_λ (see [13]) we get the next formulas.

Corollary 6. *The following formulas hold* [1]

$$\text{Prob}(G(m,n) = \lambda_1, \ldots, G(1,n) = \lambda_m) = \prod_{i=1}^{n} (1-q_i)^m \det\left[e_{\lambda_i'-i+j}(1^{\lambda_i'-1}, q_1, \ldots, q_n) \right]_{i,j=1}^{\lambda_1},$$

$$= \prod_{i=1}^{n} (1-q_i)^m \det\left[h_{\lambda_i-i+j}(1^{i-1}, q_1, \ldots, q_n) \right]_{i,j=1}^{m}.$$

Corollary 7 (Single point distributions via Schur polynomials). *We have*

$$\text{Prob}(G(m,n) \le a) = \prod_{i=1}^{n} (1-q_i)^m s_{(a^m)}(1^m, q_1, \ldots, q_n)$$

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[1]Here e_n is the elementary symmetric polynomials, h_n is the complete homogeneous symmetric polynomials, $1^m = (1, \ldots, 1)$ repeated m times, and λ' is the conjugate partition of λ.

Proof. From Corollary 5 and the first determinantal formula in Corollary 6 which coincides with the Jacobi–Trudi determinant for $s_{(a^m)}(1^m, q_1, \ldots, q_n)$ we have

$$\text{Prob}(G(m,n) \le a) = \prod_{i=1}^{n}(1-q_i)^m g_{(a^m)}(1, q_1, \ldots, q_n) = \prod_{i=1}^{n}(1-q_i)^m s_{(a^m)}(1^m, q_1, \ldots, q_n)$$

as neeeded. ∎

Remark 8. From this formula, via the Jacobi–Trudi identity one can also obtain Toeplitz as well as Fredholm determinantal expressions using the Borodin–Okounkov formula [2].

Remark 9. These distribution formulas were presented in the special i.i.d. case $q_j = q$ in [15].

Remark 10. We should also note that there are alternative methods for computing the discussed distributions. There is a well-known connection between last-passage percolation and Schur polynomials, see [4, 6]. Such formulas for the distribution of $G(m,n)$ were first obtained in [4] and used for their asymptotic analysis, see also [6]. In addition, it can be observed that the column distributions $(G(m,n), \ldots, G(1,n))$ correspond to distributions of first rows of partitions in appropriately specialized *Schur processes* [9], see also [6]. Here one relies on the Robinson–Schensted–Knuth (RSK) correspondence. The symmetry from Corollary 4 can be seen that way, as well as similar formulas for last-passage time distributions can be derived.

4. Generating series identities for g_λ

By Theorem 1 we can define the probability distribution $\text{P}_{m,n}$ on the set of integer partitions $\mathscr{P}_m$ by setting

$$\text{P}_{m,n}(\lambda) := \prod_{i=1}^{n}(1-q_i)^m g_\lambda(q_1, \ldots, q_n), \quad \lambda \in \mathscr{P}_m.$$

In particular, since $\sum_{\lambda \in \mathscr{P}_m} \text{P}_{m,n}(\lambda) = 1$ we immediately obtain the following identity for dual Grothendieck polynomials (it can also be found in [14, 15]).

Corollary 11. *The following identity holds*

$$\sum_{\lambda \in \mathscr{P}_m} g_\lambda(q_1, \ldots, q_n) = \prod_{i=1}^{n}(1-q_i)^{-m}.$$

Next, observe that we have the following *marginal distributions* for the parts λ_k:

$$\text{P}_{m,n}(\lambda_k \le a) = \text{Prob}(G(m-k+1,n) \le a)$$

which give a *shift invariance* property

$$\text{P}_{m,n}(\lambda_k \le a) = \text{P}_{m-k+1,n}(\lambda_1 \le a)$$

In particular, the last part λ_m has distribution as $\sum_{i=1}^{n} W_i$ for independent W_i geometrically distributed with parameter q_i.

We now present a new more general identity for dual Grothendieck polynomials.

Theorem 12. *Let $k \in [1, m]$ and $a \in \mathbb{N}$. The following identity holds*

$$\sum_{\lambda \in \mathscr{P}_m, \lambda_k \le a} g_\lambda(q_1, \ldots, q_n) = \prod_{i=1}^{n}(1-q_i)^{1-k} s_{(a^{m-k+1})}(1^{m-k+1}, q_1, \ldots, q_n).$$

Proof. Recall that we have the marginal distributions

$$\text{P}_{m,n}(\lambda_k \le a) = \text{Prob}(G(m-k+1,n) \le a)$$

Using Corollary 7, we have

$$\text{Prob}(G(m-k+1,n) \le a) = \prod_{i=1}^{n} (1-q_i)^{m-k+1} s_{(a^{m-k+1})}(1^{m-k+1}, q_1, \ldots, q_n).$$

On the other hand, by definition of the distribution $\text{P}_{m,n}$ above, we get

$$\text{P}_{m,n}(\lambda_k \le a) = \prod_{i=1}^{n} (1-q_i)^m \sum_{\lambda \in \mathscr{P}_m, \lambda_k \le a} g_\lambda(q_1, \ldots, q_n)$$

Combining the last two identities we obtain the needed.   □

Corollary 13. *For $k = m$ we obtain the following identity*

$$\sum_{\lambda \in \mathscr{P}_m, \lambda_m = a} g_\lambda(q_1, \ldots, q_n) = \prod_{i=1}^{n} (1-q_i)^{1-m} h_a(q_1, \ldots, q_n)$$

Proof. For $k = m$ we have

$$\sum_{\lambda \in \mathscr{P}_m, \lambda_m \le a} g_\lambda(q_1, \ldots, q_n) = \prod_{i=1}^{n} (1-q_i)^{1-m} h_a(1, q_1, \ldots, q_n).$$

Now the following recurrence relation for the polynomials h,

$$h_a(1, q_1, \ldots, q_n) - h_{a-1}(1, q_1, \ldots, q_n) = h_a(q_1, \ldots, q_n)$$

then gives the needed identity.   □

Remark 14. There is one more connection of dual Grothendieck polynomials with the corner growth model via *positive specializations* of $\{g_\lambda\}$, presented in [17]. The distribution $\text{P}_{m,n}$ can also be extended for any positive specialization as we discuss it for a special example in the next section.

5. Plancherel limit and longest increasing subsequences

Consider the specialization $q_i = \gamma/n$ for all $i \in [1, n]$, $\gamma > 0$ and let $n \to \infty$. We obtain

$$\lim_{n \to \infty} \text{P}_{m,n}(\lambda) = e^{-m\gamma} \lim_{n \to \infty} g_\lambda(\underbrace{\gamma/n, \ldots, \gamma/n}_{n \text{ times}}) = e^{-m\gamma} \sum_n \text{P}_{\text{gpl},m,n}(\lambda) \frac{(m\gamma)^n}{n!},$$

where $\text{P}_{\text{gpl},m,n}(\lambda)$ is a probability distribution on the set of partitions $\lambda \subset (n^m)$, defined below.

To define it, we need a generalization of standard Young tableaux. A plane partition π is called a *strict tableau* (ST) if for some n, each entry $i \in [n] := \{1, \ldots, n\}$ appears in exactly one column of π. We then say that $[n]$ is a *content* of π. Let $\text{ST}(\lambda, n)$ be the set of ST of shape λ with content $[n]$ and $f_\lambda(n) = |\text{ST}(\lambda, n)|$.

Lemma 15 (see [14]). *We have*

$$\text{P}_{\text{gpl},m,n}(\lambda) = \frac{f_\lambda(n)}{m^n}$$

is a well-defined probability measure on the set of integer partitions $\lambda \subset (n^m)$.

Let $W_{n,m}$ be the set of words of length n in the alphabet $[m]$. For a word $w = w_1 \cdots w_n \in W_{m,n}$, a *weakly increasing subsequence* is a sequence of the form

$$w_{i_1} \le \cdots \le w_{i_k}, \quad 1 \le i_1 < \cdots < i_k \le n,$$

where k is its length. Let $L_i(w)$ be the length of the longest weakly increasing subsequence of w using the letters $\{m-i+1, \ldots, m\}$. In particular, $L_1(w)$ is just the number of m's in w and $L_m(w)$ is the length of the longest weakly increasing subsequence of w.

Consider the uniform probability measure on $W_{n,m}$. Then we have the following analogue of Theorem 1 in this Plancherel limit regime.

Theorem 16. *We have*

$$\text{Prob}(L_m = \lambda_1, \ldots, L_1 = \lambda_m) = \text{P}_{\text{gpl},m,n}(\lambda).$$

A combinatorial version of this result (an analogue of Green's theorem for RSK), which can be turned into this statement, is proved in [14]. The distribution of L_m was studied in [12] from which we obtain that for fixed m the limiting distribution of the first row λ_1 satisfies

$$\lim_{n \to \infty} \text{P}_{\text{gpl},m,n}\left(\frac{\lambda_1 - n/m}{\sqrt{2n/m}} \leq t\right) = \text{P}_{\text{GUE}_m^0}(\lambda_{\max} \leq t),$$

where the r.h.s. is the distribution of the largest eigenvalue in $m \times m$ traceless Gaussian unitary ensemble (GUE). In addition, note that λ_m has binomial distribution with parameters n and $1/m$ and hence after proper scaling it converges to normal distribution. Now, what is limiting joint distribution of the properly scaled shape λ when m is fixed? This would compare to the results in [5] on limiting distribution of the shape for a random word under the RSK correspondence, which converges to the spectrum of traceless GUE.

Acknowledgements

References

[1] Y. Baryshnikov, "GUEs and queues", *Probab. Theory Relat. Fields* **119** (2001), no. 2, p. 256-274.

[2] A. Borodin, A. Okounkov, "A Fredholm determinant formula for Toeplitz determinants", *Integral Equations Oper. Theory* **37** (2000), no. 4, p. 386-396.

[3] A. S. Buch, "A Littlewood–Richardson rule for the K-theory of Grassmannians", *Acta Math.* **189** (2002), no. 1, p. 37-78.

[4] K. Johansson, "Shape fluctuations and random matrices", *Commun. Math. Phys.* **209** (2000), no. 2, p. 437-476.

[5] ———, "Discrete orthogonal polynomial ensembles and the Plancherel measure", *Ann. Math.* **153** (2001), no. 1, p. 259-296.

[6] ———, "Random growth and random matrices", in *3rd European congress of mathematics (ECM)*, Progress in Mathematics, vol. 201, Birkhäuser, 2001, p. 445-456.

[7] T. Lam, P. Pylyavskyy, "Combinatorial Hopf algebras and K-homology of Grassmannians", *Int. Math. Res. Not.* **2007** (2007), no. 24, article ID rnm125 (48 pages).

[8] C. Lenart, "Combinatorial aspects of the K-theory of Grassmannians", *Ann. Comb.* **4** (2000), no. 1, p. 67-82.

[9] A. Okounkov, N. Reshetikhin, "Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram", *J. Am. Math. Soc.* **16** (2003), no. 3, p. 581-603.

[10] D. Romik, *The surprising mathematics of longest increasing subsequences*, Institute of Mathematical Statistics Textbooks, vol. 4, Cambridge University Press, 2015.

[11] T. Seppäläinen, "Lecture notes on the corner growth model", unpublished notes, available at https://www.researchgate.net/publication/228814673_Lecture_Notes_on_the_Corner_Growth_Model, 2009.

[12] C. A. Tracy, H. Widom, "On the distributions of the lengths of the longest monotone subsequences in random words", *Probab. Theory Relat. Fields* **119** (2001), no. 3, p. 350-380.

[13] D. Yeliussizov, "Duality and deformations of stable Grothendieck polynomials", *J. Algebr. Comb.* **45** (2017), no. 1, p. 295-344.

[14] ———, "Enumeration of plane partitions by descents", https://arxiv.org/abs/1911.03259, 2019.

[15] ———, "Random plane partitions and corner distributions", https://arxiv.org/abs/1910.13378, 2019.

[16] ———, "Symmetric Grothendieck polynomials, skew Cauchy identities, and dual filtered Young graphs", *J. Comb. Theory, Ser. A* **161** (2019), p. 453-485.

[17] ———, "Positive specializations of symmetric Grothendieck polynomials", *Adv. Math.* **363** (2020), article ID 107000 (35 pages).