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Complex Analysis / *Analyse complexe*

L^2 estimates and existence theorems for $\overline{\partial}_b$ on Lipschitz boundaries of Q-pseudoconvex domains

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This research is dedicated to the memory of Professor Osama Abdelkader

Abstract. On a bounded q-pseudoconvex domain Ω in $\mathbb{C}^n$ with Lipschitz boundary $b\Omega$, we prove the L^2 existence theorems of the $\overline{\partial}_b$-operator on $b\Omega$. This yields the closed range property of $\overline{\partial}_b$ and its adjoint $\overline{\partial}_b^*$. As an application, we establish the L^2-existence theorems and regularity theorems for the $\overline{\partial}_b$-Neumann operator.

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Introduction and main results

Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary $b\Omega$. The Cauchy–Riemann operators $\overline{\partial}$ on $\mathbb{C}^n$ induce the tangential Cauchy–Riemann complex or $\overline{\partial}_b$ complex on $b\Omega$. On the boundaries of smooth bounded domains, there are several equivalent ways of defining the $\overline{\partial}_b$ complex. The $\overline{\partial}_b$ complex was first formulated by J. J. Kohn and H. Rossi in [26] for smooth boundaries to understand the holomorphic extension of CR-functions from the boundaries of complex manifolds. On a strictly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, the $\overline{\partial}_b$-complex has been studied in several articles (cf. [2, 5, 8, 17, 18, 27]). In the case of a weakly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, the L^2 and Sobolev estimates for $\overline{\partial}_b$ have been obtained by M.-C. Shaw in [33] for $1 \leqslant q < n-1$ and by H. P. Boas and M.-C. Shaw in [3] for $q = n-1$ (see also J. J. Kohn [25]). On the boundary of a weakly pseudoconvex domain, it was pointed out by J. P. Rosay in [32] that one can combine the results of J. J. Kohn and H. Rossi in [26] with those of J. J. Kohn in [24] to prove that the global solutions to the equation $\overline{\partial}_b u = f$ exists. Other results in this direction see Andreea C. Nicoara [31] and Phillip S. Harrington and Andrew Raich [14].

When the boundary is only Lipschitz, not every definition can be appropriately extended. On a Lipschitz boundary of a bounded domain in $\mathbb{C}^n$, the complex normal vector is defined almost everywhere on $b\Omega$. It was pointed out by D. Sullivan in [39] (see also N. Teleman in [40]) that on a real Lipschitz manifold, q-forms with L^2 coefficients and the de Rham complex are still well defined. Thus one can still define (p,q)-forms with $L^2(b\Omega)$ coefficients, denoted by $L^2_{p,q}(b\Omega)$. The $\overline{\partial}_b$ complex is then well defined as a closed densely defined operator from $L^2_{p,q-1}(b\Omega)$ to $L^2_{p,q}(b\Omega)$. In [13], Phillip S. Harrington has constructed a compact solution operator to the $\overline{\partial}_b$-operator on a pseudoconvex domain with Lipschitz boundary. On the same domain, the L^2 existence theorems of the $\overline{\partial}_b$-operator was established by Mei-Chi Shaw in [37]. The first purpose of the paper is to extend this result to Lipschitz boundaries of q-pseudoconvex domains. Our first main result is the following:

Theorem 1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded q-pseudoconvex domain with Lipschitz boundary $b\Omega$ and let $1 \leqslant q \leqslant n$. For every $\alpha \in L^2_{p,q}(b\Omega)$, where $0 \leqslant p \leqslant n$, $1 \leqslant q < n-1$, $n \geqslant 2$ such that*

$$\overline{\partial}_b \alpha = 0 \quad on \quad b\Omega,$$

there exists a $u \in L^2_{p,q-1}(b\Omega)$ satisfying $\overline{\partial}_b u = \alpha$ in the distribution sense in $b\Omega$. Moreover, there exists a constant C depending only on the diameter and the Lipschitz constant of Ω but is independent of α such that

$$\|u\|_{b\Omega} \leqslant C\|\alpha\|_{b\Omega}.$$

When $q = n-1$, for every $\alpha \in L^2_{p,n-1}(b\Omega)$ satisfies

$$\int_{b\Omega} \alpha \wedge \phi \, \mathrm{d}S = 0, \quad \textit{for any } \phi \in C^\infty_{n-p,0}(\overline{\Omega}) \cap \ker \overline{\partial}$$

the same conclusion holds.

The proof of the main theorem consists of three parts: first we prove the existence and the boundedness of the $\overline{\partial}$-Neumann operator N on Sobolev spaces $W^m(\Omega)$ for $-\frac{1}{2} \leqslant m \leqslant \frac{1}{2}$. This yields that the operators $\overline{\partial}N$ and $\overline{\partial}^* N$ and the Bergman projection P are bounded operators on $W^m(\Omega)$. Second, we study the solvability of the $\overline{\partial}$-problem in the Sobolev space $W^m(\Omega)$ with prescribed support in $\overline{\Omega}$, for $-\frac{1}{2} \leqslant m \leqslant \frac{1}{2}$. Third, by using the jump formula derived from the Bochner–Martinelli–Koppelman kernel, the main result follows.

The closed range property is related to existence and regularity results for $\overline{\partial}_b$. Independently, when $b\Omega$ is smooth and weakly pseudoconvex in $\mathbb{C}^n$, Mei-Chi Shaw in [33] and H. P. Boas and Mei-Chi Shaw in [3] proved that the range of $\overline{\partial}_b$ was closed on (p,q)-forms of degrees $1 \leqslant q < n-1$ and $q = n-1$, respectively. On a boundary of strongly pseudoconvex domain, the range of $\overline{\partial}_b$ is closed follows from J. J. Kohn and H. Rossi [26]. If Ω is Lipschitz pseudoconvex in $\mathbb{C}^n$ and if there exists a plurisubharmonic defining function in a neighborhood of $\overline{\Omega}$, the range of $\overline{\partial}_b$ is closed follows by Mei-Chi Shaw [37]. Other results in this direction see [31]. In [15], Phillip S. Harrington and Andrew Raich established sufficient conditions for the closed range of $\overline{\partial}$ (and $\overline{\partial}_b$) on not necessarily pseudoconvex domains (and their boundaries) in Stein manifolds. Also, Phillip S. Harrington and Andrew Raich established sufficient conditions for the closed range of $\overline{\partial}$ (and $\overline{\partial}_b$) on domains neither boundedness nor pseudoconvexity in $\mathbb{C}^n$ (see [16]).

As an application of Theorem 1, we prove that the ranges of $\overline{\partial}_b$ and its adjoint $\overline{\partial}_b^*$ are closed for Lipschitz boundaries of q-pseudoconvex domains.

Theorem 2. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded q-pseudoconvex domain with Lipschitz boundary $b\Omega$ and let $1 \leqslant q \leqslant n$. Then, one obtains*

(i) *$\overline{\partial}_b$ and $\overline{\partial}_b^*$ acting on $L^2_{p,q}(b\Omega)$ have closed range for every $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n-1$, $n \geqslant 2$.*
(ii) *The space of harmonic forms on the boundary $b\Omega$ vanishes, i.e.,*

$$\mathscr{H}_b^{p,q}(b\Omega) = \{0\}, \quad for \quad 0 \leqslant p \leqslant n \quad 1 \leqslant q < n-1.$$

When the unbounded operator is the $\overline{\partial}_b$ operator, the Hilbert space approach has been established by J.J. Kohn in [23] for strongly pseudoconvex domains and by L. Hörmander in [21] for pseudoconvex domain in a Stein manifold. When the boundary of a pseudoconvex domain is smooth, the Hodge decomposition on $b\Omega$ has been obtained by Mei-Chi Shaw in [36] for $1 \leqslant q < n-1$ and by H. P. Boas and Mei-Chi Shaw in [3] for $q = n-1$ (See also Mei-Chi Shaw in [37] for C^1 or Lipschitz boundaries).

In the end of the paper, we will prove that the $\overline{\partial}_b$-Laplacian, or Kohn Laplacian, $\Box_b = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b$ has closed range for (p,q)-forms when $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n-1$, $n \geqslant 2$. Thus there exists a bounded inverse operator for $\Box_b$, the $\overline{\partial}_b$-Neumann operator N_b, and we have the decomposition for $\overline{\partial}_b$ on $b\Omega$: $\alpha = \overline{\partial}_b\overline{\partial}_b^* N_b\alpha + \overline{\partial}_b^*\overline{\partial}_b N_b\alpha$ for any (p,q)-forms α with $L^2(b\Omega)$ coefficients.

Theorem 3. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded q-pseudoconvex domain with Lipschitz boundary $b\Omega$ and let $1 \leqslant q \leqslant n$. Then, for each $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n-1$, $n \geqslant 2$, there exists a bounded linear boundary operator $N_b : L^2_{p,q}(b\Omega) \longrightarrow L^2_{p,q}(b\Omega)$ such that*

(i) *$\mathscr{R}ang(N_b) \Subset \mathrm{Dom}\,\Box_b$ and $\Box_b N_b = N_b \Box_b = I$ on $\mathrm{Dom}\,\Box_b$.*
(ii) *For $\alpha \in L^2_{p,q}(b\Omega)$, we have $\alpha = \overline{\partial}_b\overline{\partial}_b^* N_b\alpha \oplus \overline{\partial}_b^*\overline{\partial}_b N_b\alpha$.*
(iii) *$\overline{\partial}_b N_b = N_b\overline{\partial}_b$ on $\mathrm{Dom}\,\overline{\partial}_b$, for $1 \leqslant q \leqslant n-1$.*
(iv) *$\overline{\partial}_b^* N_b = N_b\overline{\partial}_b^*$ on $\mathrm{Dom}\,\overline{\partial}_b^*$, for $2 \leqslant q \leqslant n$.*
(v) *If $\alpha \in L^2_{p,q}(b\Omega)$ and $\overline{\partial}_b\alpha = 0$, then $u = \overline{\partial}_b^* N_b\alpha$ is the unique solution to the equation $\overline{\partial}_b u = \alpha$ which is orthogonal to $\ker\overline{\partial}_b$.*

1. Notation and preliminaries

1.1. *Morrey–Kohn–Hörmander*

Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary $b\Omega$ and defining function ρ so that $|\partial\rho| = 1$ on $b\Omega$. Let $(z_1,\dots,z_n)$ be the complex coordinates for $\mathbb{C}^n$. Any (p,q)-form α on $\overline{\Omega}$ can be expressed as follows:

$$\alpha = {\sum_{I,J}}' \alpha_{I,J}\,\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J, \tag{1}$$

where $I = (i_1,\dots,i_p)$ and $J = (j_1,\dots,j_q)$ are multiindices and $\mathrm{d}z^I = \mathrm{d}z_{i_1} \wedge \cdots \wedge \mathrm{d}z_{i_p}$, $\mathrm{d}\bar{z}^J = \mathrm{d}\bar{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_q}$. The notation $\sum'$ means the summation over strictly increasing multiindices. Denote by $C^\infty(\mathbb{C}^n)$ the space of complex-valued C^∞ functions on $\mathbb{C}^n$ and $C^\infty_{p,q}(\mathbb{C}^n)$ the space of complex-valued differential forms of class C^∞ and of type (p,q) on $\mathbb{C}^n$, where $0 \leqslant p \leqslant n$, $0 \leqslant q \leqslant n$. Let

$$C^\infty_{p,q}(\overline{\Omega}) = \left\{ u\,|_{\overline{\Omega}} \;\middle|\; u \in C^\infty_{p,q}(\mathbb{C}^n) \right\}.$$

Denote $\mathscr{D}(\mathbb{C}^n)$, the space of C^∞-functions with compact support in $\mathbb{C}^n$. A form $u \in C^\infty_{p,q}(\mathbb{C}^n)$ is said to be has compact support in $\mathbb{C}^n$ if its coefficients belongs to $\mathscr{D}(\mathbb{C}^n)$. The subspace of $C^\infty_{p,q}(\mathbb{C}^n)$ which has compact support in $\mathbb{C}^n$ is denoted by $\mathscr{D}_{p,q}(\mathbb{C}^n)$. For $u, \alpha \in C^\infty_{p,q}(\mathbb{C}^n)$, the local inner product (u,α) is denoted by:

$$(u,\alpha) = {\sum_{I,J}}' u_{I,J}\,\overline{\alpha}_{I,J}$$

and (u,u) is defined by

$$(u,u) = |u|^2 = {\sum_{I,J}}' |u_{I,J}|^2.$$

The Cauchy–Riemann operator $\overline{\partial} : C^\infty_{p,q-1}(\Omega) \longrightarrow C^\infty_{p,q}(\Omega)$ is defined by

$$\overline{\partial}\alpha = {\sum_{I,J}}' \sum_{k=1}^{n} \frac{\partial\alpha_{I,J}}{\partial\bar{z}^k}\,\mathrm{d}\bar{z}^k \wedge \mathrm{d}z^I \wedge \mathrm{d}\bar{z}^J.$$

Recall that $L^2(\Omega)$ is the space of square-integrable functions on Ω with respect to the Lebesgue measure in $\mathbb{C}^n$ and $L^2_{p,q}(\Omega)$ is the space of (p,q)-forms with coefficients in $L^2(\Omega)$. If $u,\alpha \in L^2_{p,q}(\Omega)$, the L^2-inner product $\langle u,\alpha\rangle_\Omega$ and norm $\|u\|_\Omega$ are defined by

$$\langle u,\alpha\rangle_\Omega = \langle u,\alpha\rangle_{L^2_{p,q}(\Omega)} = \int_\Omega (u,\alpha)\,\mathrm{d}V = \int_\Omega u \wedge \star\overline{\alpha}$$

and

$$\|u\|^2_\Omega = \|u\|^2_{L^2_{p,q}(\Omega)} = \langle u,u\rangle_\Omega,$$

where $\mathrm{d}V$ is the volume element induced by the Hermitian metric and $\star : C^\infty_{p,q}(\mathbb{C}^n) \longrightarrow C^\infty_{n-q,n-p}(\mathbb{C}^n)$ is the Hodge star operator such that $\overline{\star u} = \star\overline{u}$ (that is $\star$ is a real operator) and $\star\star u = (-1)^{p+q}u$. For $u \in C^\infty_{p,q}(\Omega)$ and $\alpha \in \mathscr{D}_{p,q-1}(\Omega)$, the formal adjoint operator ϑ of $\overline{\partial} : C^\infty_{p,q-1}(\Omega) \longrightarrow C^\infty_{p,q}(\Omega)$, with respect to $\langle\cdot,\cdot\rangle_\Omega$, is defined by

$$\langle\overline{\partial}\alpha,u\rangle_\Omega = \langle\alpha,\vartheta u\rangle_\Omega.$$

Thus ϑ can be expressed explicitly by

$$\vartheta u = (-1)^{p-1} \sum_{\substack{|I|=p\\|K|=q-1}}{}' \sum_{j=1}^n \frac{\partial u_{I,jK}}{\partial z^j}\,\mathrm{d}z^I \wedge \mathrm{d}\bar{z}^K. \tag{2}$$

The operator ϑ defined in (2) satisfies

$$\vartheta = -\star\partial\star. \tag{3}$$

Let $\overline{\partial} : \mathrm{Dom}\,\overline{\partial} \subset L^2_{p,q}(\Omega) \longrightarrow L^2_{p,q+1}(\Omega)$ be the maximal closed extensions of the original $\overline{\partial}$ and $\overline{\partial}^* : \mathrm{Dom}\,\overline{\partial}^* \subset L^2_{p,q}(\Omega) \longrightarrow L^2_{p,q-1}(\Omega)$ be the Hilbert space adjoint of $\overline{\partial}$. Let $\ker\overline{\partial} = \{\alpha \in \mathrm{Dom}\,\overline{\partial} : \overline{\partial}\alpha = 0\}$ and $\mathscr{R}\mathrm{ang}\,\overline{\partial} = \{\overline{\partial}\alpha : \alpha \in \mathrm{Dom}\,\overline{\partial}\}$, be the kernel and the range of $\overline{\partial}$, respectively. The complex Laplacian $\Box$ is defined by $\Box = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial} : L^2_{p,q}(\Omega) \longrightarrow L^2_{p,q}(\Omega)$ on $\mathrm{Dom}\,\Box = \{\alpha \in \mathrm{Dom}\,\overline{\partial} \cap \mathrm{Dom}\,\overline{\partial}^* : \overline{\partial}\alpha \in \mathrm{Dom}\,\overline{\partial}^* \text{ and } \overline{\partial}^*\alpha \in \mathrm{Dom}\,\overline{\partial}\}$. The space of harmonic forms $\mathscr{H}^{p,q}(\Omega)$ is defined by

$$\mathscr{H}^{p,q}(\Omega) = \{\alpha \in L^2_{p,q}(\Omega) \cap \mathrm{Dom}\,\overline{\partial} \cap \mathrm{Dom}\,\overline{\partial}^* : \overline{\partial}\alpha = \overline{\partial}^*\alpha = 0\}.$$

Let $\mathbb{H} : L^2_{p,q}(\Omega) \longrightarrow \ker\Box$ be the orthogonal projection from the space $L^2_{p,q}(\Omega)$ onto the space $\ker\Box$. The $\overline{\partial}$-Neumann operator

$$N : L^2_{p,q}(\Omega) \longrightarrow L^2_{p,q}(\Omega)$$

is defined as the inverse of the restriction of $\Box$ to $(\mathscr{H}^{p,q}(\Omega))^\perp$, i.e.,

$$N\alpha = \begin{cases} 0 & \text{if } \alpha \in \mathscr{H}^{p,q}(\Omega),\\ u & \text{if } \alpha = \Box u, \text{ and } u \perp \mathscr{H}^{p,q}(\Omega). \end{cases}$$

In other words, $N\alpha$ is the unique solution u to the equations $\mathbb{H}u = 0$, $\Box u = \alpha - \mathbb{H}\alpha$. The Bergman projection operator $P : L^2_{p,q}(\Omega) \longrightarrow \ker\overline{\partial}$ is the orthogonal projection of $L^2_{p,q}(\Omega)$ onto $\ker\overline{\partial}$. For any $0 \leqslant p \leqslant n$ and $1 \leqslant q \leqslant n$, P is represented in terms of N by the Kohn's formula

$$P = I - \overline{\partial}^*\overline{\partial}N. \tag{4}$$

Let $a = (a_1,\dots,a_n)$ be a multiindices, that is, $a_1,\dots,a_n$ are nonnegative integers. For $x \in \mathbb{R}^n$, one defines $x^a = x_1^{a_1}\dots x_n^{a_n}$ and D^a is the operator

$$D^a = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{a_1}\cdots\left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{a_n}.$$

Denote by $\mathscr{S}$ the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^n$, that is, $\mathscr{S}$ consists of all functions u which are smooth on $\mathbb{R}^n$ with $\sup_{x\in\mathbb{R}^n}|x^aD^bu(x)| < \infty$ for all multiindices a, b. The Fourier transform $\widehat{u}$ of a function $u \in \mathscr{S}$ is defined by

$$\widehat{u}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} u(x)\,e^{-i\,x.\xi}\,\mathrm{d}x,$$

where $x.\xi = \sum_{\nu=1}^{n} x_\nu \xi_\nu$ and $\mathrm{d}x = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ with $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$. If $u \in \mathscr{S}$, then $\widehat{u} \in \mathscr{S}$. The Sobolev space $W^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, is the completion of $\mathscr{S}$ under the Sobolev norm

$$\|u\|^2_{W^m(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|\xi|^2)^m |\widehat{u}|^2 \,\mathrm{d}\xi.$$

Denote by $W^m(\Omega)$, $m \geqslant 0$, the space of the restriction of all functions $u \in W^m(\mathbb{C}^n) = W^m(\mathbb{R}^{2n})$ to Ω and

$$\|u\|_{W^m(\Omega)} = \inf\left\{\|\alpha\|_{W^m(\mathbb{C}^n)},\ \alpha \in W^m(\mathbb{C}^n), \alpha|_\Omega = u\right\}$$

is the $W^m(\Omega)$-norm. Let $W_0^m(\Omega)$ be the completion of $\mathscr{D}(\Omega)$ under the $W^m(\Omega)$-norm. If Ω is a Lipschitz domain, $C^\infty(\overline{\Omega})$ is dense in $W^m(\Omega)$ with respect to the $W^m(\Omega)$-norm. If $0 \leqslant m \leqslant \frac{1}{2}$, $\mathscr{D}(\Omega)$ is dense in $W^m(\Omega)$ (cf. [11, Theorem 1.4.2.4]). Thus

$$W^m(\Omega) = W_0^m(\Omega), \text{ for } 0 \leqslant m \leqslant \frac{1}{2}.$$

For $m > 0$, one defines $W^{-m}(\Omega)$ to be the dual of $W_0^m(\Omega)$ and the norm of $W^{-m}(\Omega)$ is defined by

$$\|u\|_{W^{-m}(\Omega)} = \sup_{0 \neq \alpha \in W_0^m(\Omega)} \frac{|\langle u, \alpha\rangle_\Omega|}{\|\alpha\|_{W^m(\Omega)}}.$$

Denote by $W^m_{p,q}(\Omega)$, $m \in \mathbb{R}$, the Hilbert spaces of (p,q)-forms with $W^m(\Omega)$-coefficients and their norms are denoted by $\|u\|_{W^m(\Omega)}$. Noting that, for a bounded domain Ω, the generalized Schwartz inequality, for $u \in W^m(\Omega)$ and $\alpha \in W^{-m}(\Omega)$,

$$|\langle u, \alpha\rangle_\Omega| \leqslant \|u\|_{W^m(\Omega)} \|\alpha\|_{W^{-m}(\Omega)} \tag{5}$$

holds when $-\frac{1}{2} \leqslant m \leqslant \frac{1}{2}$.

Lemma 4 ([4]). *Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain with C^2 boundary and ρ be a C^2 defining function of Ω. Let $\varphi \in C^2(\overline{\Omega})$ with $\varphi \geqslant 0$. Then, for $\alpha \in C^\infty_{p,q}(\overline{\Omega}) \cap \operatorname{Dom} \overline{\partial}^*$ with $1 \leqslant q \leqslant n-1$, one obtains*

$$\begin{aligned}
\|\sqrt{\varphi}\,\overline{\partial}\alpha\|^2_\Omega + \|\sqrt{\varphi}\,\overline{\partial}^*\alpha\|^2_\Omega = {\sum_{I,J}}' \sum_{j,k=1}^{n} \int_{b\Omega} \varphi \frac{\partial^2 \rho}{\partial z^j \partial \bar{z}^k}\, \alpha_{I,jK}\, \overline{\alpha}_{I,kK}\, \mathrm{d}S \\
+ {\sum_{I,J}}' \sum_{k=1}^{n} \int_\Omega \varphi \left|\frac{\partial \alpha_{I,J}}{\partial \bar{z}^k}\right|^2 \mathrm{d}V + 2\operatorname{Re}\left({\sum_{I,K}}' \sum_{j=1}^{n} \frac{\partial \varphi}{\partial z^j}\, \alpha_{I,jK}\, \mathrm{d}\bar{z}_K, \overline{\partial}^*\alpha\right) \\
- {\sum_{I,K}}' \sum_{j,k=1}^{n} \int_\Omega \frac{\partial^2 \varphi}{\partial z^j \partial \bar{z}^k}\, \alpha_{I,jK}\, \overline{\alpha}_{I,kK}\, \mathrm{d}V.
\end{aligned} \tag{6}$$

The case of $\varphi \equiv 1$ is the classical Kohn–Morrey formula see [21, 23].

1.2. *The $\overline{\partial}_b$ complex on Lipschitz domains*

In this subsection, we introduce square-integrable (p,q)-forms on a Lipschitz boundary $b\Omega$ of a bounded domain Ω in $\mathbb{C}^n$ with distance function ρ. We equip $b\Omega$ with the induced metric from $\mathbb{C}^n$. A boundary $b\Omega$ of a bounded domain $\Omega \Subset \mathbb{C}^n$ is called Lipschitz if locally the boundary $b\Omega$ is the graph of a Lipschitz function. Let $\psi : \mathbb{R}^{2n-1} \longrightarrow \mathbb{R}$ be a function that satisfies the Lipschitz condition

$$|\psi(x) - \psi(x')| \leqslant M|x - x'|, \text{ for all } x, x' \in \mathbb{R}^{2n-1}. \tag{7}$$

The smallest $M > 0$ in which (7) holds is called the bound of the Lipschitz constant. A boundary $b\Omega$ of a bounded domain $\Omega \Subset \mathbb{C}^n$ is called Lipschitz if near every boundary point $p \in b\Omega$ there exists a neighborhood U of p such that, after a rotation,

$$\Omega \cap U = \{(x, x_{2n}) \in U \mid x_{2n} > \psi(x)\},$$

for some Lipschitz function ψ. By choosing finitely many balls $\{U_i\}$ covering $b\Omega$, the Lipschitz constant for a Lipschitz domain is the smallest M such that the Lipschitz constant is bounded by M in every ball U_i. A Lipschitz function is almost everywhere differentiable (see [7]).

Definition 5. *A bounded domain Ω with Lipschitz boundary $b\Omega$ in $\mathbb{C}^n$ is said to have a global Lipschitz defining function if there exists a Lipschitz function $\rho:\mathbb{C}^n \longrightarrow \mathbb{R}$ such that $\rho<0$ in Ω, $\rho>0$ outside $\overline{\Omega}$ and*

$$c_1<|\mathrm{d}\rho|<c_2 \ \text{ a.e. on } \ b\Omega, \tag{8}$$

where c_1, c_2 are positive constants.

We cover $b\Omega$ by finitely many boundary coordinate patches U_i where $i=1,\dots,k$. Let r_i be a local defining function on U_i which is locally a Lipschitz graph. Let $\chi_i\in C_0^\infty(U_i)$ be a partition of unity such that $\sum_i \chi_i=1$ in a neighborhood of $b\Omega$. We define $\rho=\sum_i \chi_i r_i$. Then ρ is a defining function for Ω.

Lemma 6 ([37]). *Let $\Omega\Subset\mathbb{C}^n$ be a bounded domain with Lipschitz boundary $b\Omega$. Then Ω has a global Lipschitz defining function ρ. Furthermore, the distance function to the boundary is comparable to $|\rho|$ for any global Lipschitz defining function ρ near the boundary.*

Let $C^\infty(b\Omega)$ be the space of the restriction of all smooth functions in $\mathbb{C}^n$ to $b\Omega$. For each m with $1\leqslant m\leqslant\infty$, one defines $\widetilde{L}^m_{p,q}(b\Omega)$ to be the space of (p,q)-forms in $\mathbb{C}^n$ such that each coefficient of α, when restricted to $b\Omega$, is in $L^m(b\Omega)$. Write α as in (1), then $\alpha\in\widetilde{L}^m_{p,q}(b\Omega)$ if and only if $\alpha_{I,J}|_{b\Omega}\in L^m(b\Omega)$ for each I,J. Let $\vee$ be the interior product which is the dual of the wedge product. Since the boundary is Lipschitz, the normal vector is defined almost everywhere and satisfies (8). If we fix $p\in b\Omega$, then for some neighborhood U of p we may locally choose an orthonormal coordinate patch $\{\mathrm{d}z_1,\dots,\mathrm{d}z_n\}$ defined almost everywhere in $U\cap\overline{\Omega}$ such that $\mathrm{d}\overline{z}_n=\overline{\partial}\rho$ (note that $|\overline{\partial}\rho|=\frac{1}{2}$ because we are using the metric where $|\mathrm{d}z^j|=1$, which is half the size induced by the usual Euclidean metric on $\mathbb{R}^n$). We define $L^m_{p,q}(b\Omega)\subset\widetilde{L}^m_{p,q}(b\Omega)$ as the space of all $\widetilde{L}^m_{p,q}$ such that $\mathrm{d}\overline{z}_n\vee\alpha=0$ almost everywhere on $b\Omega$.

Locally, if $\alpha\in\widetilde{L}^m_{p,q}(b\Omega\cap U)$, one can express

$$\alpha=\sum_{\substack{I,J\\ n\notin J}}{}'\alpha_{I,J}\mathrm{d}z^I\wedge\mathrm{d}\bar{z}^J+\sum_{\substack{I,J\\ n\in J}}{}'\alpha_{I,J}\mathrm{d}z^I\wedge\mathrm{d}\bar{z}^J,$$

where $\alpha_{I,J}$'s are $L^m(b\Omega\cap U)$ functions. Let τ denote the projection map

$$\tau:\widetilde{L}^m_{p,q}(b\Omega)\longrightarrow L^m_{p,q}(b\Omega)$$

defined by

$$\tau\alpha=\sum_{\substack{I,J\\ n\notin J}}{}'\alpha_{I,J}\mathrm{d}z^I\wedge\mathrm{d}\bar{z}^J. \tag{9}$$

Since changing basis will result in multiplication by $L^\infty(b\Omega\cap U)$ functions, the projection τ is well-defined since it is independent of the choice of $\{\mathrm{d}\bar{z}_1,\dots,\mathrm{d}\bar{z}_{n-1}\}$.

Let $\Lambda_{p,q}(b\Omega)$ denote the restriction of $C^\infty_{p,q}(\mathbb{C}^n)$ to $b\Omega$. Define $\mathscr{B}_{p,q}(b\Omega)$ to be the subspace of $L^\infty_{p,q}(b\Omega)$ such that $\alpha\in\mathscr{B}_{p,q}(b\Omega)$ if and only if there exists $\widetilde{\alpha}\in\Lambda_{p,q}(b\Omega)$ such that $f=\tau\widetilde{\alpha}$. In other words, we have $\tau(\Lambda_{p,q}(\mathbb{C}^n))=\mathscr{B}_{p,q}(b\Omega)$. Obviously, $\mathscr{B}_{p,q}(b\Omega)\Subset L^\infty_{p,q}(b\Omega)\Subset L^2_{p,q}(b\Omega)$. Denote by $W^m_{p,q}(b\Omega)$, $0\leqslant m\leqslant 1$, the space of forms that are the completion of $\mathscr{B}_{p,q}(b\Omega)$-forms with $W^m(b\Omega)$-norms. This is well defined also for Lipschitz domains since on $b\Omega$, $W^1(b\Omega)$ is well defined and the boundary value of any function in $W^1(\Omega)$ to the boundary belongs to $W^{\frac{1}{2}}(b\Omega)$ (see [22]).

Lemma 7 ([30, Lemma 1.4]). *Let Ω be a bounded domain with Lipschitz boundary $b\Omega$ in $\mathbb{R}^n$. Then $C^\infty(b\Omega)$ is dense in $L^2(b\Omega)$ and $\Lambda_{p,q}(b\Omega)$ is dense in $\widetilde{L}^2_{p,q}(b\Omega)$ for every $0\leqslant p\leqslant n$, $0\leqslant q\leqslant n$. Also $\mathscr{B}_{p,q}(b\Omega)$ is a dense subset in $L^2_{p,q}(b\Omega)$ for every $0\leqslant p\leqslant n$, $0\leqslant q\leqslant n-1$.*

The Bochner–Martinelli–Koppelman transform on (p,q)-forms is defined as follows. Let

$$(\zeta - z) = (\zeta_1 - z_1, \dots, \zeta_n - z_n),$$
$$\mathrm{d}\zeta = (\mathrm{d}\zeta_1, \dots, \mathrm{d}\zeta_n).$$

Define

$$\langle \overline{\zeta} - \bar{z}, \mathrm{d}\zeta \rangle = \sum_{j=1}^{n} (\overline{\zeta}_j - \bar{z}_j)\, \mathrm{d}\zeta_j,$$
$$\langle \mathrm{d}\overline{\zeta} - \mathrm{d}\bar{z}, \mathrm{d}\zeta \rangle = \sum_{j=1}^{n} (\mathrm{d}\overline{\zeta}_j - \mathrm{d}\bar{z}_j)\, \mathrm{d}\zeta_j.$$

The Bochner–Martinelli–Koppelman kernel $K(\zeta, z)$ is defined by

$$\begin{aligned} K(\zeta, z) &= \frac{1}{(2\pi i)^n} \frac{\langle \overline{\zeta} - \bar{z}, \mathrm{d}\zeta \rangle}{|\zeta - z|^2} \wedge \left(\frac{\langle \mathrm{d}\overline{\zeta} - \mathrm{d}\bar{z}, \mathrm{d}\zeta \rangle}{|\zeta - z|^2} \right)^{n-1} \\ &= \sum_{q=0}^{n-1} K_q(\zeta, z), \end{aligned}$$

where $K_q(\zeta, z)$ denote the component of $K(\zeta, z)$ that is a (p,q)-form in z and an $(n-p, n-q-1)$-form in ζ.

When $n = 1$,

$$K(\zeta, z) = \frac{1}{2\pi i} \frac{\mathrm{d}\zeta}{\zeta - z}$$

is the Cauchy kernel. As in the Cauchy integral case, for any $f \in L^2_{p,q}(b\Omega)$ the Cauchy principal value integral $K_b f$ is defined as follows:

$$K_b f(z) = \lim_{\epsilon \longrightarrow 0^+} \int_{\substack{b\Omega \\ |\zeta - z| > \epsilon}} K_q(\zeta, z) \wedge f(\zeta),$$

whenever the limit exists. Denote by ν_z the outward unit normal to $b\Omega$ at z. Since $b\Omega$ is Lipschitz, ν_z exists almost everywhere on $b\Omega$. Then, for $z \in b\Omega$, one defines

$$K_b^- f(z) = \lim_{\epsilon \longrightarrow 0^+} \int_{b\Omega} K_q(\cdot, z - \epsilon v_z) \wedge f,$$
$$K_b^+ f(z) = \lim_{\epsilon \longrightarrow 0^+} \int_{b\Omega} K_q(\cdot, z + \epsilon v_z) \wedge f.$$

Proposition 8 ([37]). *Let Ω be a bounded domain in $\mathbb{C}^n$ with Lipschitz boundary. For $\alpha \in \mathscr{B}_{p,q}(b\Omega)$, $0 \leqslant q \leqslant n-1$, the following formula holds for almost every $z \in b\Omega$:*

$$\alpha(z) = \tau \lim_{\epsilon \longrightarrow 0^+} \left(\int_{b\Omega} K_q(\cdot, z - \epsilon v_z) \wedge \alpha - \int_{b\Omega} K_q(\cdot, z + \epsilon v_z) \wedge \alpha \right). \tag{10}$$

The $\overline{\partial}_b$-operator is defined distributionally as follows: for any $u \in L^2_{p,q}(b\Omega)$ and $\alpha \in L^2_{p,q+1}(b\Omega)$ we say that u is in $\mathrm{Dom}\,\overline{\partial}_b$ and $\overline{\partial}_b u = \alpha$ if and only if:

$$\int_{b\Omega} u \wedge \overline{\partial}\phi \,\mathrm{d}S = (-1)^{p+q} \int_{b\Omega} \alpha \wedge \phi \,\mathrm{d}S, \quad \text{for every } \ \phi \in C^\infty_{n-p,n-q-1}(\mathbb{C}^n).$$

Since $\overline{\partial}^2 = 0$, it follows that $\overline{\partial}_b^2 = 0$. Thus $\overline{\partial}_b$ is a complex and we have the following:

$$0 \longrightarrow L^2_{p,0}(b\Omega) \xrightarrow{\overline{\partial}_b} L^2_{p,1}(b\Omega) \xrightarrow{\overline{\partial}_b} L^2_{p,2}(b\Omega) \xrightarrow{\overline{\partial}_b} \dots \xrightarrow{\overline{\partial}_b} L^2_{p,n-1}(b\Omega) \longrightarrow 0.$$

Proposition 9 ([35]). *Let Ω be a bounded domain in $\mathbb{C}^n$ with Lipschitz boundary $b\Omega$. The $\overline{\partial}_b$ operator is a closed, densely defined, linear operator from $L^2_{p,q-1}(b\Omega)$ to $L^2_{p,q}(b\Omega)$, where $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n-1$.*

We need to define $\overline{\partial}_b^*$, the L^2 adjoint of $\overline{\partial}_b$. Again, we first define its domain:

Definition 10. $\mathrm{Dom}\,\overline{\partial}_b^*$ *is the subset of* $L^2_{p,q}(b\Omega)$ *composed of all forms* α *for which there exists a constant* $c>0$ *such that*

$$|\langle \alpha, \overline{\partial}_b u\rangle_{b\Omega}| \leqslant C\|u\|_{b\Omega},$$

for all $u \in \mathrm{Dom}\,\overline{\partial}_b$.

For all $\alpha \in \mathrm{Dom}\,\overline{\partial}_b^*$, let $\overline{\partial}_b^*\alpha$ be the unique form in $L^2_{p,q}(b\Omega)$ satisfying

$$\langle \overline{\partial}_b^*\alpha, u\rangle_{b\Omega} = \langle \alpha, \overline{\partial}_b u\rangle_{b\Omega},$$

for all $u \in \mathrm{Dom}\,\overline{\partial}_b$.

Definition 11. *Let* $\Box_b = \overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b : \mathrm{Dom}\,\Box_b \longrightarrow L^2_{p,q}(b\Omega)$ *the* $\overline{\partial}_b$*-Laplacian operator defined on* $\mathrm{Dom}\,\Box_b = \{\alpha \in L^2_{p,q}(b\Omega) : \alpha \in \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}_b^* : \overline{\partial}_b\alpha \in \mathrm{Dom}\,\overline{\partial}_b^*$ *and* $\overline{\partial}_b^*\alpha \in \mathrm{Dom}\,\overline{\partial}_b\}$.

Proposition 12 ([35, Proposition 1.3]). *The* $\overline{\partial}_b$*-Laplacian operator is a closed, densely defined self-adjoint operator.*

Let $\mathscr{H}_b^{p,q}(b\Omega)$ denote the space of harmonic forms on the boundary $b\Omega$, i.e.,

$$\mathscr{H}_b^{p,q}(b\Omega) = \{\alpha \in \mathrm{Dom}\,\Box_b : \overline{\partial}_b\alpha = \overline{\partial}_b^*\alpha = 0\}.$$

The space $\mathscr{H}_b^{p,q}(b\Omega)$ is a closed subspace of $\mathrm{Dom}\,\Box_b$ since $\Box_b$ is a closed operator. One defines the boundary operator or the $\overline{\partial}_b$-Neumann operator

$$N_b : L^2_{p,q}(b\Omega) \longrightarrow L^2_{p,q}(b\Omega),$$

as the inverse of the restriction of $\Box_b$ to $(\mathscr{H}_b^{p,q}(b\Omega))^{\perp}$, i.e.,

$$N_b\alpha = \begin{cases} 0 & \text{if } \alpha \in \mathscr{H}_b^{p,q}(b\Omega), \\ u & \text{if } \alpha = \Box_b u, \text{ and } u \perp \mathscr{H}_b^{p,q}(b\Omega). \end{cases}$$

In other words, $N_b\alpha$ is the unique solution u to the equations $\alpha = \Box_b u$ with $\alpha \perp \mathscr{H}_b^{p,q}(b\Omega)$ and we extend N_b by linearity.

1.3. *q-pseudoconvex domains*

In this subsection, we recall the following definition of q-subharmonic functions which has been introduced by H. Ahn and N. Q. Dieu in [1] (also see Lop-Hing Ho [20]). For a real valued C^2 function u defined on $U \subseteq \mathbb{C}^n$, Lop-Hing Ho [20] first defined q-subharmonicity of u on U and using this q-subharmonic function, he introduce the notion of weak q-convexity for domains with smooth boundaries. As Theorem 1.4 in [20], Ahn and Dieu [1] investigated a natural extension of these notions to the class of upper semicontinuous functions and q-pseudoconvex domains with non-smooth boundaries.

Definition 13 ([1]). *Let* u *be an upper semicontinuous function on* Ω*. Then we say that* u *is* q*-subharmonic on* Ω *if for every* q*-complex dimension space* H *and for every compact set* $D \Subset H\cap\Omega$*, the following holds: if* h *is a continuous harmonic function on* D *and* $h \geqslant u$ *on the boundary of* D*, then* $h \geqslant u$ *on* D*.*

Definition 14 ([1]). *The function* u *is called strictly* q*-subharmonic if for every* $U \Subset \Omega$ *there exists a constant* $C_U > 0$ *such that* $u - C_U|z|^2$ *is* q*-subharmonic.*

The following results gives some basic properties of q-subharmonic functions, follows the same lines as for plurisubharmonic functions, that will be used later (see [1, 20]).

Proposition 15. *Let Ω be a domain in $\mathbb{C}^n$ and let $1 \leqslant q \leqslant n$. Then we have*

(i) *If $\{u_\nu\}_{\nu=1}^{\infty}$ is a decreasing sequence of q-subharmonic functions then $u = \lim_{\nu\to+\infty} u_\nu$ is a q-subharmonic function.*

(ii) *Let $\chi(z) \in C_0^\infty(\mathbb{C}^n)$ be a function such that $\chi \geqslant 0$, $\int_{\mathbb{C}^n} \chi(z)\mathrm{d}V = 1$, $\chi(z)$ depends only on $|z|$ and vanishes when $|z| > 1$. Set $\chi_{\varepsilon_\nu}(z) = \varepsilon_\nu^{-2n}\chi(z/\varepsilon_\nu)$ for $\varepsilon_\nu \downarrow 0$. If u is a q-subharmonic function, then*

$$u_{\varepsilon_\nu}(z) = u * \chi_{\varepsilon_\nu}(z) = \int_\Omega u(\zeta)\,\chi_{\varepsilon_\nu}(z-\zeta)\,\mathrm{d}V(\zeta)$$

is smooth q-subharmonic on $\Omega_{\varepsilon_\nu} = \{z \in \Omega : d(z, b\Omega) > \varepsilon_\nu\}$. Moreover, $u_{\varepsilon_\nu} \downarrow u$ as $\varepsilon_\nu \downarrow 0$.

(iii) *If $u \in C^2(\Omega)$ such that $\frac{\partial^2 u}{\partial z^j \partial \bar{z}^k}(z) = 0$ for all $j \neq k$ and $z \in \Omega$. Then u is q-subharmonic if and only if $\sum_{j,k\in J} \frac{\partial^2 u}{\partial z^j \partial \bar{z}^k}(z) \geqslant 0$, for all $|J| = q$ and for all $z \in \Omega$.*

Proof. The proof of this proposition follows from properties of subharmonic functions. The proof of (ii) is exact as Proposition 1.2 in [1]. Similarly, it is easy to see that (i) and (iii) hold because these properties are true for subharmonic functions. □

Proposition 16. *Let Ω be a bounded domain in $\mathbb{C}^n$ and let q be an integer with $1 \leqslant q \leqslant n$. Let $u \in C^2(\Omega)$. Then the q-subharmonicity of u is equivalent to*

$$\sum_{|K|=q-1}' \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z^j \partial \bar{z}^k}\, \alpha_{jK}\,\overline{\alpha}_{kK} \geqslant 0,$$

for all $(0,q)$-forms $\alpha = \sum\limits_{|J|=q} \alpha_J \,\mathrm{d}\bar{z}^J$.

Proof. By Theorem 1.4 and Lemma 1.2 in [20], it is easy to see that this fact is true if $u \in C^2(\Omega)$. In the case u is arbitrary we note that the assertion is true for u_ε. Let $\varepsilon \searrow 0$ we obtain the assertion for u and the proof follows. □

The following examples of a q-subharmonic function which is not plurisubharmonic.

Example 17. One of the most typical examples of q-subharmonic function which is not plurisubharmonic is

$$u(z) = -\sum_{\nu=1}^{q-1} |z_\nu|^2 + (q-1)\sum_{\nu=q}^{n} |z_\nu|^2.$$

Indeed, for $q = 2$, we have

$$\left(\frac{\partial^2 u}{\partial z^j \partial \bar{z}^k}\right) = \begin{bmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Then, following Proposition 15 (iii), u is 2-subharmonic. More precisely

$$\sum_{l=1}^{n} \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z^j \partial \bar{z}^k}\, \alpha_{jl}\,\overline{\alpha}_{kl} = \sum_{l=1}^{n} \left(-|\alpha_{1l}|^2 + \sum_{j=2}^{n} |\alpha_{jl}|^2 \right) = \sum_{l=2}^{n} \sum_{j=2}^{n} |\alpha_{jl}|^2 \geqslant 0,$$

for all $(0,2)$-forms $\alpha = \sum_{|J|=2}' \alpha_J \,\mathrm{d}\bar{z}^J$ (because $\alpha_{ll} = 0$, $l = 1, 2, \ldots, n$). Thus, u is 2-subharmonic.

Example 18. Let $d > 1$ and $1 < q \leqslant d$. Consider the function

$$u(z) = |z|^2 - q|z_1|^2 = \sum_{j=1}^{d} |z_j|^2 - q|z_1|^2, \quad z \in \mathbb{C}^d.$$

It is easy to see that $\sum_{j=1}^{q} \frac{\partial^2 u}{\partial z^j \partial \bar{z}^j}(z) = 0$ and by Proposition 15 it follows that u is q-subharmonic. However, u is not plurisubharmonic. Indeed, let $L = \{(z_1, 0, \ldots, 0)\} \subset \mathbb{C}^d$ be a complex line. Then $u|_L = (1-q)|z_1|^2$ is not subharmonic, and the desired conclusion follows.

Example 19 ([12]). Let q be an integer with $1 \leqslant q \leqslant n$ and let λ_k, $k = 1,2,\dots,q$, are complex numbers such that $\sum_{k=1}^{q} |\lambda_k|^2 > 0$ and $\lambda_k \neq 0$, for $k = q+1,\dots,n$. Then the function

$$u(z) = 2\left\{\mathrm{Re}\left(\sum_{k=1}^{q} \lambda_k z_k\right)\right\}^2 + \sum_{k=q+1}^{n} |\lambda_k z_k|^2$$

is 1-subharmonic and strictly q-subharmonic on $\mathbb{C}^n$. Moreover, if $q > 1$ then u is not strictly $(q-1)$-subharmonic on any open set of $\mathbb{C}^n$.

According to Lop-Hing Ho (see [20, Definition 2.1]), a smoothly bounded domain Ω is called weakly q-convex if Ω has a defining function ρ such that for every $z \in b\Omega$ one obtains

$$\sum_{|K|}{}' \sum_{j,k} \frac{\partial^2 \rho}{\partial z^j \partial \bar{z}^k} \alpha_{jK} \overline{\alpha}_{kK} \geqslant 0,$$

for every $(0,q)$-forms $\alpha = \sum_{|J|=q} \alpha_J \, \mathrm{d}\bar{z}^J$ such that

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z^j} \alpha_{jK} = 0 \text{ for all } |K| = q-1.$$

Definition 20. *A (Lipschitz) domain $\Omega \Subset \mathbb{C}^n$ is said to be q-pseudoconvex if there exists q-subharmonic exhaustion (Lipschitz) function on Ω. Moreover, a C^2 smooth bounded domain Ω is called strictly q-convex if it admits a C^2 smooth defining function which is strictly q-subharmonic on a neighbourhood of $\overline{\Omega}$.*

Remark 21.

(i) Note that Ω is pseudoconvex if and only if it is 1-pseudoconvex, since 1-subharmonic function is just plurisubharmonic.
(ii) By Theorem 2.4 in Lop-Hing Ho [20], every weakly q-convex domain (see [20, Definition 2.1]) is q-pseudoconvex.
(iii) If $\Omega \Subset \mathbb{C}^n$ is a q-pseudoconvex domain, $1 \leqslant q \leqslant n$, and if $b\Omega$ is of class C^2, then Ω is weakly q-convex (see Lop-Hing Ho [20]).
(iv) Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain satisfy the $Z(q)$ condition. Thus Ω is strictly q-pseudoconvex.
(v) Every n-dimensional connected non compact complex manifold has a strongly subharmonic exhaustion function with respect to any hermitian metric ω. Thus, every open set in $\mathbb{C}^n$ is n-pseudoconvex (see Greene and Wu [10]).

2. Existence of the $\overline{\partial}$-Neumann operator

This section deals with the existence of the $\overline{\partial}$-Neumann operator N on q-pseudoconvex domains in $\mathbb{C}^n$.

Lemma 22 ([1, 12]). *Let $\Omega \Subset \mathbb{C}^n$ be a q-pseudoconvex domain, $1 \leqslant q \leqslant n$. Then Ω has a C^∞-smooth strictly q-subharmonic exhaustion function. More precisely, there are strictly q-pseudoconvex domains, Ω_ν's, $\nu = 1,2,\dots,$ with smooth boundary satisfying*

$$\Omega = \cup_{\nu=1}^{\infty} \Omega_\nu, \ \ \Omega_\nu \Subset \Omega_{\nu+1} \Subset \Omega \textit{ for all } \nu.$$

Proof. Let u be q-subharmonic exhaustion function for Ω. By induction one can choose a sequence $\{a_\nu\}_{\nu\geqslant 1} \uparrow \infty$ such that the open sets $U_\nu := \{u < a_\nu\}$ satisfy $U_\nu \Subset U_{\nu+1} \Subset \Omega$. Next, for each ν we choose $\varepsilon_\nu > 0$ so small that

$$d(U_{\nu-1}, bU_\nu) > \varepsilon_\nu, \ \ d(U_\nu, bU_{\nu+1}) > \varepsilon_\nu, \ \ U_{\nu+1} \Subset \Omega_{\varepsilon_\nu},$$

where $\Omega_{\varepsilon_\nu} = \{z : d(z, b\Omega) > \varepsilon_\nu\}$. Put

$$u_\nu(z) = \int_\Omega u(\zeta)\,\chi_{\varepsilon_\nu}(z-\zeta)\,\mathrm{d}V(\zeta),\ \ \forall\, z \in \Omega_{\varepsilon_\nu}.$$

Since $u < u_\nu$ on Ω_{ε_ν} we deduce that

$$\Omega_\nu = \{z \in \Omega_{\varepsilon_\nu} : u_\nu(z) > a_\nu\} \subset U_\nu \Subset U_{\nu+1} \Subset \Omega_{\varepsilon_\nu} \subset \Omega.$$

We claim that $U_{\nu-1} \subset \Omega_\nu$. Indeed, let $z \in U_{\nu-1}$. Then we have $\mathbb{B}(z,\varepsilon_\nu) \subset U_\nu$ and, hence,

$$u_\nu(z) = \int_{\mathbb{B}(z,\varepsilon_\nu)} u(\zeta)\,\chi_{\varepsilon_\nu}(z-\zeta)\,\mathrm{d}V(\zeta) < a_\nu.$$

This proves the claim, and therefore $\Omega = \cup_{\nu=1}^\infty \Omega_\nu$, Finally, it is easy to see that Ω_ν is a q-pseudoconvex domain with the smooth strictly q-subharmonic exhaustion function $\varphi_\nu(z) := \frac{1}{a_\nu - u_\nu(z)} + |z|^2$. □

Following Lemmas 6 and 22, as Lemma 2.1 in [29], we prove the following lemma:

Lemma 23. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded q-pseudoconvex domain with Lipschitz boundary $b\Omega$. There exists an exhaustion $\{\Omega_\nu\}$ of Ω such that*

(i) *there exists a Lipschitz function $\rho : \mathbb{C}^n \longrightarrow \mathbb{R}$ such that $\rho < 0$ in Ω, $\rho > 0$ outside $\overline{\Omega}$ and satisfies* (8).

(ii) *$\{\Omega_\nu\}$ is an increasing sequence of relatively compact subsets of Ω and $\Omega = \cup_\nu \Omega_\nu$.*

(iii) *Each Ω_ν, $\nu = 1,2,\ldots,$ is strictly q-pseudoconvex domains, i.e., each Ω_ν has a C^∞ strictly q-subharmonic defining function ρ_ν on a neighbourhood of $\overline{\Omega}$, such that*

$$\sum_{I,|K|}{}' \sum_{j,k} \frac{\partial^2 \rho_\nu}{\partial z^j \partial \bar{z}^k}\, \alpha_{I,jK}\, \overline{\alpha}_{I,kK} \geqslant c_0 |\alpha|^2$$

$\alpha \in C^\infty_{p,q}(\overline{\Omega}_\nu) \cap \mathrm{Dom}\,\overline{\partial}^*_\nu$ with $q \geqslant 1$ and c_0 is independent of ν.

(iv) *There exist positive constants c_1, c_2 such that $c_1 \leqslant |\nabla \eta_\nu| \leqslant c_2$ on $b\Omega_\nu$, where c_1, c_2 are independent of ν.*

Proof. Using Lemma 6, there exists a global Lipschitz defining function ρ for Ω satisfying (8) and ρ is obtained as above by a partition of unity of defining functions r_i which is a Lipschitz graph. Choose a function $\chi(z) \in C_0^\infty(\mathbb{C}^n)$ as in Proposition 15. Let δ_ν be a sequence of small numbers with $\delta_\nu \searrow 0$. For each δ_ν, one defines

$$\Omega_{\delta_\nu} = \{z \in \Omega \,|\, \rho(z) < -\delta_\nu\}.$$

Then Ω_{δ_ν} is a sequence of relatively compact open subsets of Ω with union equal to Ω. For each δ_ν, one defines, for $0 < \epsilon < \delta_\nu$ and $z \in \Omega_{\delta_\nu}$,

$$\rho_\epsilon(z) = \rho * \chi_\epsilon(z) = \int \rho(z - \epsilon\zeta)\,\chi(\zeta)\,\mathrm{d}V(\zeta).$$

Then, $\rho_\epsilon \in C^\infty(\Omega_{\delta_\nu})$ and $\rho_\epsilon \searrow \rho$ on Ω_{δ_ν}. Since ρ is q-subharmonic, it follows that, from Proposition 15, that ρ_ϵ is q-subharmonic.

Each ρ_{ϵ_ν} is well defined if $0 < \epsilon_\nu < \delta_{\nu+1}$ for $z \in \Omega_{\delta_{\nu+1}}$. Let $c_2 = \sup_\Omega |\nabla\rho|$, then for ϵ_ν sufficiently small, one obtains

$$\rho(z) < \rho_{\epsilon_\nu}(z) < \rho(z) + c_2\epsilon_\nu \ \text{ on } \ \Omega_{\delta_{\nu+1}}.$$

For each ν we choose

$$\epsilon_\nu = \frac{1}{2c_2}(\delta_{\nu-1} - \delta_\nu) \text{ and } t_\nu \in (\delta_{\nu+1} - \delta_\nu).$$

Define

$$\Omega_\nu = \{z \in \mathbb{C}^n \,|\, \rho_{\epsilon_\nu} < -t_\nu\}.$$

Since $\rho(z) < \rho_{\epsilon_\nu}(z) < -t_\nu < -\delta_{\nu+1}$, we have that $\Omega_{\delta_{\nu+1}} \supset \Omega_\nu$. Also, if $z \in \Omega_{\delta_{\nu-1}}$, then $\rho_{\epsilon_\nu}(z) < \rho(z) + c_2\epsilon_\nu < -\delta_\nu < -t_\nu$. Thus we have $\Omega_{\delta_{\nu+1}} \supset \Omega_\nu \supset \Omega_{\delta_{\nu-1}}$ and (ii) is satisfied.

Each Ω_ν is defined by $\eta_\nu = \rho_{\epsilon_\nu} + t_\nu$ which is strictly q-subharmonic in Ω_ν and (iii) is satisfied. That the subdomain Ω_ν has smooth boundary will follow from (iv).

To prove (iv), it is easy to see that $|\nabla\eta_\nu| \leqslant c_2$ in $b\Omega_\nu$. To show that $|\nabla\eta_\nu|$ is uniformly bounded from below, we note $b\Omega$ satisfies the uniform interior cone property. Then there exists a conic neighborhood Γ with vertex $0 \in \mathbb{C}^n$ such that for any unit vector $\xi \in \Gamma + \{p\}$, $-\langle\nabla\rho,\xi\rangle_p > c_0$ a.e. in $U \cap b\Omega$, where c_0 is a positive constant independent of p if U is sufficiently small. There exist a finite covering $\{V_\mu\}_{1\leqslant\mu\leqslant K}$ of $b\Omega$, a finite set of unit vectors $\{\xi_\mu\}_{1\leqslant\mu\leqslant K}$ and $c_1 > 0$ such that the inner product $\langle\nabla\rho,\xi_\mu\rangle \geqslant c_1 > 0$ a.e. for $z \in V_\mu$, $1 \leqslant \mu \leqslant K$. Since this is preserved by convolution, (iv) is proved. Thus the proof follows. □

Theorem 24. *Let $\Omega \Subset \mathbb{C}^n$ be a q-pseudoconvex domain, $1 \leqslant q \leqslant n$. Then, for any $1 \leqslant q \leqslant n$, there exists a bounded linear operator $N : L^2_{p,q}(\Omega) \longrightarrow L^2_{p,q}(\Omega)$ which satisfies the following properties:*

(i) *$\mathscr{R}ang N \Subset \operatorname{Dom}\Box$, $N\Box = I$ on $\operatorname{Dom}\Box$.*
(ii) *For any $\alpha \in L^2_{p,q}(\Omega)$, we have $\alpha = \bar\partial\bar\partial^* N\alpha \oplus \bar\partial^*\bar\partial N\alpha$.*
(iii) *$\bar\partial N = N\bar\partial$ on $\operatorname{Dom}\bar\partial$, $1 \leqslant q \leqslant n-1$, $n \geqslant 2$.*
(iv) *$\bar\partial^* N = N\bar\partial^*$ on $\operatorname{Dom}\bar\partial^*$, $2 \leqslant q \leqslant n$.*
(v) *If δ is the diameter of Ω, we have the following estimates:*

$$\begin{aligned}
\|N\alpha\|_\Omega &\leqslant \frac{e\delta^2}{q}\|\alpha\|_\Omega,\\
\|\bar\partial N\alpha\|_\Omega &\leqslant \sqrt{\frac{e\delta^2}{q}}\|\alpha\|_\Omega,\\
\|\bar\partial^* N\alpha\|_\Omega &\leqslant \sqrt{\frac{e\delta^2}{q}}\|\alpha\|_\Omega.
\end{aligned} \tag{11}$$

Proof. We first prove the theorem for Ω with C^2 boundary. Let ρ be the defining function of Ω. Following Definitions 14 and 20, there exists $\alpha \in C^\infty_{p,q}(\bar\Omega) \cap \operatorname{Dom}\bar\partial^*$ with $q \geqslant 1$, such that

$$\sum_{I,K}{}' \sum_{j,k=1}^n \int_{b\Omega} \frac{\partial^2\rho}{\partial z^j \partial\bar z^k}\, \alpha_{I,jK}\,\bar\alpha_{I,kK}\,\mathrm{d}S \geqslant 0.$$

If we replace φ by $1 - e^\psi$, where ψ is an arbitrary twice continuously differentiable non-positive function, and after applying the Cauchy–Schwarz inequality (5), for $m = 0$, to the term in (6) involving first derivatives of φ, we find

$$\|\sqrt{\varphi}\,\bar\partial\alpha\|^2_\Omega + \|\sqrt{\varphi}\,\bar\partial^*\alpha\|^2_\Omega \geqslant \sum_{I,K}{}' \sum_{j,k=1}^n \int_\Omega e^\psi \frac{\partial^2\psi}{\partial z^j\partial\bar z^k}\,\alpha_{I,jK}\,\bar\alpha_{I,kK}\,\mathrm{d}V - \|e^{\psi/2}\bar\partial^*\alpha\|_\Omega.$$

Since $\varphi + e^\psi = 1$ and $\varphi \leqslant 1$, it follows that

$$\|\bar\partial\alpha\|^2_\Omega + \|\bar\partial^*\alpha\|^2_\Omega \geqslant \sum_{I,K}{}' \sum_{j,k=1}^n \int_\Omega e^\psi \frac{\partial^2\psi}{\partial z^j\partial\bar z^k}\,\alpha_{I,jK}\,\bar\alpha_{I,kK}\,\mathrm{d}V,$$

for all $\alpha \in C^\infty_{p,q}(\bar\Omega) \cap \operatorname{Dom}\bar\partial^*$ and for $q \geqslant 1$. If $z_0 \in \Omega$, and $\psi(z) = -1 + |z - z_0|^2/\delta^2$, where $\delta = \sup_{z,z'\in\Omega}|z - z'|$ is the diameter of the bounded domain Ω, then the preceding inequality implies

$$\|\alpha\|^2_\Omega \leqslant \left(\frac{e\delta^2}{q}\right)\left(\|\bar\partial\alpha\|^2_\Omega + \|\bar\partial^*\alpha\|^2_\Omega\right).$$

This estimate was derived when α is continuous and differential on $\bar\Omega$. Thus, by density it holds for all square-integrable forms $\alpha \in \operatorname{Dom}\bar\partial \cap \operatorname{Dom}\bar\partial^*$. Thus, for $q \geqslant 1$, one obtains

$$\|\alpha\|_\Omega \leqslant \left(\frac{e\delta^2}{q}\right)\|\Box\alpha\|_\Omega. \tag{12}$$

For a general q-pseudoconvex domain, from Lemma 22, one can exhaust Ω by a sequence of strictly q-pseudoconvex domains with C^∞ boundary $b\Omega$. We write $\Omega=\cup_\nu\Omega_\nu$, where each Ω_ν is a bounded strictly q-pseudoconvex domains with C^∞ boundary and $\Omega_\nu\Subset\Omega_{\nu+1}\Subset\Omega$ for each ν. Let δ_ν be the diameter of Ω_ν and let $\Box_\nu$ be the complex Laplacian on each Ω_ν. Thus (12) holds on each Ω_ν. That is, there exists a $\alpha_\nu\in C^\infty_{p,q}(\overline{\Omega}_\nu)\cap\operatorname{Dom}\overline{\partial}^*_\nu$ with $q\geqslant 1$, such that

$$\|\alpha_\nu\|_{\Omega_\nu}\leqslant\left(\frac{e\delta_\nu^2}{q}\right)\|\Box_\nu\alpha_\nu\|_{\Omega_\nu}\leqslant\left(\frac{e\delta^2}{q}\right)\|\Box\alpha\|_\Omega.$$

We can choose a subsequence of α_ν, still denoted by α_ν, such that $\alpha_\nu\longrightarrow\alpha$ weakly in $L^2_{p,q}(\Omega)$. Furthermore, α satisfies the estimate

$$\|\alpha\|_\Omega\leqslant\liminf\left(\frac{e\delta_\nu^2}{q}\right)\|\Box_\nu\alpha_\nu\|_{\Omega_\nu}\leqslant\left(\frac{e\delta^2}{q}\right)\|\Box\alpha\|_\Omega. \tag{13}$$

Since $\Box$ is a linear closed densely defined operator, then, from Theorem 1.1.1 in [21], $\mathscr{R}\text{ang}(\Box)$ is closed. Thus, from (1.1.1) in [21] and the fact that $\Box$ is self adjoint, one obtains

$$L^2_{p,q}(\Omega)=\overline{\partial}\,\overline{\partial}^*\operatorname{Dom}\Box\oplus\overline{\partial}^*\overline{\partial}\operatorname{Dom}\Box.$$

Since $\Box:\operatorname{Dom}\Box\longrightarrow\mathscr{R}\text{ang}(\Box)=L^2_{p,q}(\Omega)$ is one to one on $\operatorname{Dom}\Box$ from (13), there exists a unique bounded inverse operator $N:\mathscr{R}\text{ang}(\Box)\longrightarrow\operatorname{Dom}\Box$ such that $N\Box\alpha=\alpha$ on $\operatorname{Dom}\Box$. Also, from the definition of N, one obtains $\Box N=I$ on $L^2_{p,q}(\Omega)$. Thus (i) and (ii) are satisfied. To show that $\overline{\partial}N=N\overline{\partial}$ on $\operatorname{Dom}\overline{\partial}$, by using (ii), we have $\overline{\partial}\alpha=\overline{\partial}\,\overline{\partial}^*\overline{\partial}N\alpha$, for $\alpha\in\operatorname{Dom}\overline{\partial}$. Thus

$$N\overline{\partial}\alpha=N\overline{\partial}\,\overline{\partial}^*\,\overline{\partial}N\alpha=N(\overline{\partial}^*\,\overline{\partial}+\overline{\partial}\,\overline{\partial}^*)\,\overline{\partial}N\alpha=\overline{\partial}N\alpha.$$

And, by similar argument, $\overline{\partial}^*N=N\overline{\partial}^*$ on $\operatorname{Dom}\overline{\partial}^*$ for $2\leqslant q\leqslant n$. By using (iii) and the condition on α, $\overline{\partial}\alpha=0$, we have $\overline{\partial}N\alpha=N\overline{\partial}\alpha=0$. Then, by using (ii), one obtains $\alpha=\overline{\partial}\,\overline{\partial}^*N\alpha$. Thus the form $u=\overline{\partial}^*N\alpha$ satisfies the equation $\overline{\partial}u=\alpha$. Since $\mathscr{R}\text{ang}\,N\subset\operatorname{Dom}\Box$, then by applying (13) to $N\alpha$ instead of α, (11) follows. Thus the proof follows. □

3. Sobolev estimates for the $\overline{\partial}$-Neumann problem

Let Ω be a domain with C^1-boundary $b\Omega$, and let ρ be a C^1-defining function of Ω. Assume that $\mathscr{E}_{p,q}(b\Omega)$ is the space of the restriction to $b\Omega$ of all (p,q)-forms with $C^1(\overline{\Omega})$-coefficients which are pointwise orthogonal to the ideal generated by $\overline{\partial}\rho$ and $\digamma_{p,q}(b\Omega)$ is the space of the restriction to $b\Omega$ of all (p,q)-forms that are multiples of $\overline{\partial}\rho$. Denote by $\urcorner_1$, the projection $\urcorner_1:C^1_{p,q}(\overline{\Omega})\longrightarrow\mathscr{E}_{p,q}(b\Omega)$ and $\urcorner_2$, the projection $\urcorner_2:C^1_{p,q}(\overline{\Omega})\longrightarrow\digamma_{p,q}(b\Omega)$. In particular, $\urcorner_1\oplus\urcorner_2=\urcorner$, where $\urcorner$ is simply the restriction map from $C^1_{p,q}(\overline{\Omega})$ to the boundary. If Ω has only Lipschitz boundary $b\Omega$, the operators $\urcorner_1$ and $\urcorner_2$ are also defined almost everywhere on $b\Omega$.

Lemma 25 ([30]). *Let Ω be a domain with C^1-boundary $b\Omega$, and let ρ be a C^1-defining function of Ω. For any $f\in L^2_{p,q}(\Omega)$, the restriction maps $\urcorner(\overline{\partial}N)$, $\urcorner P$ and $\urcorner(\overline{\partial}^*N)$ can be extended as bounded operators from $L^2_{p,q}(\Omega)$ to $W^{-\frac12}_{p,q+1}(b\Omega)$, $W^{-\frac12}_{p,q}(b\Omega)$ and $W^{-\frac12}_{p,q-1}(b\Omega)$, respectively.*

Denote by $\mathscr{R}_1$ and $\mathscr{R}_2$, the restriction maps of $\overline{\partial}^*N$ and $\overline{\partial}N$ to $b\Omega$, respectively. For any $\alpha\in L^2_{p,q}(\Omega)$, $\mathscr{R}_1f=\urcorner(\overline{\partial}^*N\alpha)$ and $\mathscr{R}_2f=\tau(\overline{\partial}N\alpha)$. From Lemma 25, one obtains

$$\mathscr{R}_1:L^2_{p,q}(\Omega)\longrightarrow W^{-\frac12}_{p,q-1}(b\Omega)$$

and

$$\mathscr{R}_2:L^2_{p,q}(\Omega)\longrightarrow W^{-\frac12}_{p,q+1}(b\Omega).$$

Let $T_1 : W^{\frac{1}{2}}_{p,q-1}(b\Omega) \longrightarrow L^2_{p,q}(\Omega)$ be the dual of $\mathscr{R}_1$ and be defined as follows: For a fixed $\alpha \in W^{\frac{1}{2}}_{p,q-1}(b\Omega)$ and for any $u \in L^2_{p,q}(\Omega)$, we have, using Lemma 25, that

$$\left|\int_{b\Omega} (\overline{\partial}^* Nu, \alpha)\,\mathrm{d}S\right| \leqslant C\|\mathscr{R}_1 u\|_{W^{-\frac{1}{2}}(b\Omega)} \|\alpha\|_{W^{\frac{1}{2}}(b\Omega)} \leqslant C\|u\|_\Omega,$$

where C depends on α. Thus there exists an element $g = T_1\alpha \in L^2_{p,q}(\Omega)$ such that

$$\int_{b\Omega} (\overline{\partial}^* Nu, \alpha)\mathrm{d}S = \langle u, T_1\alpha\rangle_\Omega, \text{ for any } u \in L^2_{p,q}(\Omega).$$

Let $T_2 : W^{\frac{1}{2}}_{p,q+1}(b\Omega) \longrightarrow L^2_{p,q}(\Omega)$ be the dual of $\mathscr{R}_2$, such that for any $\alpha \in W^{\frac{1}{2}}_{p,q+1}(b\Omega)$

$$\int_{b\Omega} (\overline{\partial} Nu, \alpha)\mathrm{d}S = \langle u, T_2\alpha\rangle_\Omega, \text{ for any } u \in L^2_{p,q}(\Omega).$$

Lemma 26 ([30, Lemma 4.3]). *Let Ω be a domain with C^∞-boundary such that Ω has a C^∞-plurisubharmonic defining function ρ. For any $\alpha \in W^{\frac{1}{2}}_{p,q-1}(b\Omega)$, $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n$, we have the following estimate:*

$$\int_\Omega (-\rho)|T_1\alpha|^2\mathrm{d}V \leqslant C\int_{b\Omega} |\alpha|^2\mathrm{d}S. \tag{14}$$

Also for any $\alpha \in W^{\frac{1}{2}}_{p,q+1}(b\Omega)$, $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n$, one obtains

$$\int_\Omega (-\rho)|T_2\alpha|^2\mathrm{d}V \leqslant C\int_{b\Omega} |\alpha|^2\mathrm{d}S, \tag{15}$$

where C is a constant depending only on the Lipschitz constant of Ω and the diameter of Ω.

Theorem 27. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded q-pseudoconvex domain with a Lipschitz boundary $b\Omega$ and let $1 \leqslant q \leqslant n$. Then, for $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n$ and $-\frac{1}{2} \leqslant m \leqslant \frac{1}{2}$, the operators N, $\overline{\partial}N$, $\overline{\partial}^* N$ and the Bergman projection P are bounded on $W^m_{p,q}(\Omega)$ and satisfies the following estimates: there exists $C > 0$ such that for any $\alpha \in W^m_{p,q}(\Omega)$,*

$$\begin{gathered} \|N\alpha\|_{W^m_{p,q}(\Omega)} \leqslant C\|\alpha\|_{W^m_{p,q}(\Omega)}, \\ \|\overline{\partial}N\alpha\|_{W^m_{p,q+1}(\Omega)} + \|\overline{\partial}^* N\alpha\|_{W^m_{p,q-1}(\Omega)} \leqslant C\|\alpha\|_{W^m_{p,q}(\Omega)}, \\ \|P\alpha\|_{W^m_{p,q}(\Omega)} \leqslant C\|\alpha\|_{W^m_{p,q}(\Omega)}, \end{gathered} \tag{16}$$

where C depends only on the Lipschitz constant and the diameter of Ω, but is independent of α.

Proof. To prove (16) for $m = \frac{1}{2}$, we approximate Ω as Lemma 23 by a sequence of subdomains $\Omega_\nu = \{\rho < -\epsilon_\nu\}$ such that each Ω_ν is strictly q-pseudoconvex domains with C^∞ smooth boundary, i.e, each Ω_ν has a C^∞ strictly q-subharmonic defining function ρ_ν such that (ii) and (iii) in Lemma 23. Thus, we can apply Lemma 26 on each Ω_ν. We use T_1^ν, T_2^ν, $\mathscr{R}_1^\nu$, $\mathscr{R}_2^\nu$, and N^ν, to denote the corresponding operators on each Ω_ν. By applying (14) and (15) to T_1^ν, T_2^ν, for any $\alpha \in W^{\frac{1}{2}}_{p,q-1}(b\Omega_\nu)$, where $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n$, we have

$$\int_{\Omega_\nu} (-\rho_\nu)|T_1^\nu\alpha|^2\mathrm{d}V_\nu \leqslant C\int_{b\Omega_\nu} |\alpha|^2\mathrm{d}S_\nu, \tag{17}$$

where C can be chosen independent of ν. Also for any $\alpha \in W^{\frac{1}{2}}_{p,q+1}(b\Omega_\nu)$, where $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n$, one obtains

$$\int_{\Omega_\nu} (-\rho_\nu)|T_2^\nu\alpha|^2\mathrm{d}V_\nu \leqslant C\int_{b\Omega_\nu} |\alpha|^2\mathrm{d}S_\nu. \tag{18}$$

Using Lemma 26, T_1^ν is bounded from $L^2_{p,q-1}(b\Omega_\nu)$ to $W^{-\frac{1}{2}}_{p,q-1}(\Omega_\nu)$ and T_2^ν is bounded from $L^2_{p,q+1}(b\Omega_\nu)$ to $W^{-\frac{1}{2}}_{p,q}(\Omega_\nu)$. Also, the bounds depend only on the Lipschitz constant and the diameter of the domain. Thus, from (17) and (18) we have from duality, for any $\alpha \in W^{\frac{1}{2}}_{p,q}(\Omega_\nu)$,

$$\|\mathscr{R}_1^\nu \alpha\|_{L^2_{p,q-1}(b\Omega_\nu)} \leqslant C\|\alpha\|_{W^{\frac{1}{2}}_{p,q}(\Omega_\nu)}, \tag{19}$$

and

$$\|\mathscr{R}_1^\nu \alpha\|_{L^2_{p,q+1}(b\Omega_\nu)} \leqslant C\|\alpha\|_{W^{\frac{1}{2}}_{p,q}(\Omega_\nu)}, \tag{20}$$

where C is a constant independent of ν. For any $\alpha \in W^{\frac{1}{2}}_{p,q}(\Omega_\nu)$ and by using the trace theorem for elliptic equations (cf. [22, 38]), from (19) and (20), one obtains

$$\|\bar\partial^* N^\nu \alpha\|_{W^{\frac{1}{2}}_{p,q-1}(\Omega_\nu)} \leqslant C\|\alpha\|_{W^{\frac{1}{2}}_{p,q}(\Omega_\nu)}, \tag{21}$$

and

$$\|\bar\partial N^\nu \alpha\|_{W^{\frac{1}{2}}_{p,q+1}(\Omega_\nu)} \leqslant C\|\alpha\|_{W^{\frac{1}{2}}_{p,q}(\Omega_\nu)}, \tag{22}$$

where C is a constant independent of ν. Passing to the limit, one obtains from (21) and (22) that

$$\|\bar\partial^* N \alpha\|_{W^{\frac{1}{2}}_{p,q-1}(\Omega)} \leqslant C\|\alpha\|_{W^{\frac{1}{2}}_{p,q}(\Omega)}, \tag{23}$$

and

$$\|\bar\partial N \alpha\|_{W^{\frac{1}{2}}_{p,q+1}(\Omega)} \leqslant C\|\alpha\|_{W^{\frac{1}{2}}_{p,q}(\Omega)}, \tag{24}$$

Using Theorem 24 (iii) and (iv), one can write

$$N = (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)N^2 = (\bar\partial N)(\bar\partial^* N) + (\bar\partial^* N)(\bar\partial N).$$

It follows from (23) and (24) that

$$\|N\alpha\|_{W^{\frac{1}{2}}_{p,q}(\Omega)} \leqslant C\|\alpha\|_{W^{\frac{1}{2}}_{p,q}(\Omega)}. \tag{25}$$

By virtue of Kohn's formula (4), we have

$$\|P\alpha\|_{W^{\frac{1}{2}}_{p,q}(\Omega)} \leqslant C\|f\|_{W^{\frac{1}{2}}_{p,q}(\Omega)}. \tag{26}$$

This proves the continuity of P in $W^{\frac{1}{2}}_{p,q}(\Omega)$. Thus (16) follows for $m = \frac{1}{2}$. Using duality, the estimates (23) to (26) also hold for $m = -\frac{1}{2}$. The other cases follow from interpolation. □

4. The L^2 $\bar\partial$-Cauchy problem

Using the duality relations pertaining to the $\bar\partial$-Neumann problem, one can solve the Cauchy problem for $\bar\partial$ on q-pseudoconvex domains. This method was first used in [26] for smooth forms on strongly pseudo-convex domains. As Theorem 9.1.2 in [6] (cf. [34, Proposition 2.7]), the following result is proved:

Theorem 28. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded q-pseudoconvex domain with a Lipschitz boundary $b\Omega$ and let $1 \leqslant q \leqslant n$. Then, for every $\alpha \in L^2_{p,q}(\mathbb{C}^n)$ which is supported in $\overline{\Omega}$ such that*

$$\bar\partial\alpha = 0, \quad \textit{for } 1 \leqslant q \leqslant n-1, \tag{27}$$

one can find $u \in L^2_{p,q-1}(\mathbb{C}^n)$ such that $\bar\partial u = \alpha$ in the distribution sense in $\mathbb{C}^n$ with u supported in $\overline{\Omega}$ and

$$\|u\|_{\mathbb{C}^n} \leqslant C\|\alpha\|_{\mathbb{C}^n},$$

where C depends only on the Lipschitz constant and the diameter of Ω, but is independent of α.

Proof. Let $\alpha \in L^2_{p,q}(\mathbb{C}^n)$ which is supported in $\overline{\Omega}$, then $\alpha \in L^2_{p,q}(\Omega)$. From Theorem 24, the $\overline{\partial}$-Neumann operator $N_{n-p,n-q}$ exists for $n-q \geqslant 1$. Since $N_{n-p,n-q} = \Box^{-1}_{n-p,n-q}$ on $\mathscr{R}\mathrm{ang}\Box_{n-p,n-q}$ and $\mathscr{R}\mathrm{ang}\, N_{n-p,n-q} \subset \mathrm{Dom}\Box_{n-p,n-q}$, then $N_{n-p,n-q} \star \overline{\alpha} \in \mathrm{Dom}\Box_{n-p,n-q} \subset L^2_{n-p,n-q}(\Omega)$, for $q \leqslant n-1$. Thus, one can define $u \in L^2_{p,q-1}(\Omega)$ by

$$u = -\star\, \overline{\overline{\partial} N_{n-p,n-q} \star \overline{\alpha}}. \tag{28}$$

Extending u to $\mathbb{C}^n$ by defining $u = 0$ in $\mathbb{C}^n \setminus \overline{\Omega}$. We want to prove that the extended form u satisfies the equation $\overline{\partial} u = \alpha$ in the distribution sense in $\mathbb{C}^n$. To do that we need first clear that

$$\overline{\partial}^* (\star \overline{\alpha}) = 0 \text{ on } \Omega.$$

For $\eta \in \mathrm{Dom}\,\overline{\partial} \subset L^2_{n-p,n-q-1}(\Omega)$, one obtains

$$\langle \overline{\partial}\eta, \star\overline{\alpha}\rangle_\Omega = \int_\Omega \overline{\partial}\eta \wedge \star\star\alpha = (-1)^{(p+q)(p+q-1)} \int_\Omega \alpha \wedge \overline{\partial}\eta = (-1)^{p+q} \langle \alpha, \star\partial\overline{\eta}\rangle_\Omega.$$

Since $\vartheta = \overline{\partial}^*$ on $\mathscr{D}_{p,q}(\Omega)$, when ϑ acts in the distribution sense and $\mathscr{D}_{p,q}(\Omega)$ is dense in $\mathrm{Dom}\,\overline{\partial} \cap \mathrm{Dom}\,\overline{\partial}^*$ in the graph norm (cf. [21]), then from (3) one obtains

$$\langle \overline{\partial}\eta, \star\overline{\alpha}\rangle_\Omega = \langle \alpha, \overline{\partial}^* \star \overline{\eta}\rangle_{\mathbb{C}^n} = \langle \overline{\partial}\alpha, \star\overline{\eta}\rangle_{\mathbb{C}^n} = 0$$

because α is supported in $\overline{\Omega}$. It follows that $\overline{\partial}^* (\star\overline{\alpha}) = 0$ on Ω. Using Theorem 24 (iv), one obtains

$$\overline{\partial}^* N_{n-p,n-q}(\star\overline{\alpha}) = N_{n-p,n-q-1}\overline{\partial}^* (\star\overline{\alpha}) = 0.$$

Thus, from (3) and (28), one obtains

$$\begin{aligned}
\overline{\partial} u &= -\overline{\partial \star \overline{\partial} N_{n-p,n-q} \star \overline{\alpha}} \\
&= (-1)^{p+q+1} \overline{\star\star\partial\star\overline{\partial} N_{n-p,n-q} \star \overline{\alpha}} \\
&= (-1)^{p+q} \overline{\star \overline{\partial}^* \overline{\partial} N_{n-p,n-q} \star \overline{\alpha}} \\
&= (-1)^{p+q} \overline{\star (\overline{\partial}^*\overline{\partial} + \overline{\partial}\,\overline{\partial}^*) N_{n-p,n-q} \star \overline{\alpha}} \\
&= (-1)^{p+q} \overline{\star\star\overline{\alpha}} \\
&= \alpha
\end{aligned} \tag{29}$$

in the distribution sense in Ω. Since $u = 0$ in $\mathbb{C}^n \setminus \Omega$, then for $g \in \mathrm{Dom}\,\overline{\partial}^* \subset L^2_{p,q}(\mathbb{C}^n)$, and from (3) and (29), one obtains

$$\begin{aligned}
\langle u, \overline{\partial}^* g\rangle_{\mathbb{C}^n} &= \langle u, \overline{\partial}^* g\rangle_\Omega \\
&= \langle \star\overline{\overline{\partial}^* g}, \star\overline{u}\rangle_\Omega \\
&= -\langle \overline{\star\star\partial\star g}, \star\overline{u}\rangle_\Omega \\
&= (-1)^{p+q} \langle \star\overline{g}, \overline{\partial}^* \star \overline{u}\rangle_\Omega \\
&= \langle \overline{\partial} u, g\rangle_\Omega \\
&= \langle \alpha, g\rangle_\Omega \\
&= \langle \alpha, g\rangle_{\mathbb{C}^n}.
\end{aligned}$$

Thus $\overline{\partial} u = \alpha$ in the distribution sense in $\mathbb{C}^n$. □

As Proposition 9.1.3 in [6], the following result is proved:

Proposition 29. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded q-pseudoconvex domain with a Lipschitz boundary $b\Omega$ and let $1 \leqslant q \leqslant n$. Then, for $-\frac{1}{2} \leqslant m < \frac{1}{2}$ and for any $\alpha \in L^2_{p,n}(\mathbb{C}^n)$ such that α is supported in $\overline{\Omega}$ and*

$$\int_\Omega \alpha \wedge g = 0 \quad \textit{for any} \quad g \in L^2_{n-p,0}(\Omega) \cap \ker\overline{\partial}, \tag{30}$$

one can find $u \in L^2_{p,n-1}(\mathbb{C}^n)$ such that $\overline{\partial} u = \alpha$ in the distribution sense in $\mathbb{C}^n$ with u is supported in $\overline{\Omega}$ and

$$\|u\|_{\mathbb{C}^n} \leqslant C\|\alpha\|_{\mathbb{C}^n},$$

where C depends only on the Lipschitz constant and the diameter of Ω, but is independent of α.

Proof. To prove this result we need to prove that condition (30) is equivalent to $\overline{\partial}\alpha = 0$. Fist, we see that (30) implies (27). To see that, if we take $g = \overline{\partial} \star \beta$ for some $\beta \in C^\infty_{p,n+1}(\mathbb{C}^n)$ in (30). It is clear that $g \in \ker \overline{\partial}$. By (30) and the fact $\overline{\partial} \star \beta \in \ker \overline{\partial}$, we see that

$$\langle \alpha, \overline{\partial}^* \beta \rangle_{\mathbb{C}^n} = \int_\Omega \alpha \wedge \star(\overline{\partial}^* \beta) = (-1)^{p+n+1} \int_\Omega \alpha \wedge \overline{\partial}(\star\beta) = 0$$

for any $\beta \in C^\infty_{p,n+1}(\mathbb{C}^n)$, where we used the equality $\star(\star\alpha) = (-1)^{p+n}\alpha$ for $u = \overline{\partial} \star \beta \in C^\infty_{p,n+1}(\mathbb{C}^n)$. This implies that $\overline{\partial}\alpha = 0$ in the distribution sense in $\mathbb{C}^n$.

If Ω has Lipschitz boundary, $C^\infty_{n-p,1}(\overline{\Omega})$ is dense in dom $\overline{\partial}$ in the graph norm. This follows essentially from Friedrichs lemma (see [21, Proposition 1.2.3]). From the definition of $\overline{\partial}^*$, one obtains that $\star\alpha \in \mathrm{Dom}\,\overline{\partial}^*$ and $\overline{\partial}^*(\star\alpha) = 0$. For any $g \in L^2_{n-p,n-q}(\Omega) \cap \ker \overline{\partial}$, using Theorem 28 (since $1 \leqslant q < n$), there exist $u \in L^2_{n-p,n-q-1}(\Omega)$ such that $\overline{\partial} u = g$ in Ω. This implies that

$$\overline{\int_\Omega g \wedge \alpha} = \langle \star\alpha, g\rangle = \langle \star\alpha, \overline{\partial} u\rangle = \langle \overline{\partial}^* \star \alpha, u\rangle = 0,$$

for any $g \in L^2_{n-p,n-q}(\Omega) \cap \ker \overline{\partial}$. Thus α satisfies (30). Thus the proof follows. □

As Proposition 3.4 in [37], we prove the following result:

Lemma 30. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded q-pseudoconvex domain with Lipschitz boundary $b\Omega$ and let $1 \leqslant q \leqslant n$. The set of $\overline{\partial}$-closed forms in $L^2_{p,q}(\mathbb{C}^n)$ with support in $\overline{\Omega}$ is dense in the set of $\overline{\partial}$-closed forms in $W^m_{p,q}(\mathbb{C}^n)$ with support in $\overline{\Omega}$.*

Proof. By this we mean that if $f \in W^m_{p,q}(\mathbb{C}^n)$ with f is supported in $\overline{\Omega}$ and $\overline{\partial} f = 0$ on Ω, one can construct a sequence $f_\nu \in L^2_{p,q}(\mathbb{C}^n)$ with support in $\overline{\Omega}$ such that $f_\nu \longrightarrow f$ in $W^m_{p,q}(\mathbb{C}^n)$ and $\overline{\partial} f_\nu \longrightarrow \overline{\partial} f$ in $W^m_{p,q+1}(\mathbb{C}^n)$. This is possible by the well-known method of Friedrichs (see [9] or [21] or [6, Appendix D]) as follows: first assume that the domain Ω is star-shaped and $0 \in \Omega$. Since Ω is Lipschitz, locally it is star-shaped. This can be done from the usual regularization by convolution. We approximate f first by dilation componentwise. Let $\Omega^\epsilon = \{(1+\epsilon)z \mid z \in \Omega\}$ and

$$f^\epsilon = f\left(\frac{z}{1+\epsilon}\right),$$

where the dilation is performed for each component of f. Then $\Omega \Subset \Omega^\epsilon$ and $f^\epsilon \in W^m_{p,q}(\Omega^\epsilon)$. Also $\overline{\partial} f^\epsilon \longrightarrow \overline{\partial} f$ in $W^m_{p,q+1}(\mathbb{C}^n)$. Choose a function $\chi(z) \in C^\infty_0(\mathbb{C}^n)$ as in Proposition 15. Extend $f \in W^m_{p,q}(\mathbb{C}^n) \cap \ker \overline{\partial}$ to be 0 outside Ω. By regularizing f^ϵ componentwise as before, one can find a family of $f_{(\epsilon)} \in W^m_{p,q+1}(\mathbb{C}^n)$ defined by

$$f_{(\epsilon)}(z) = f\left(\frac{z}{1+\epsilon}\right) * \chi_{\delta_\epsilon} = \int_{\mathbb{C}^n} f(w)\, \chi_{\delta_\epsilon}\left(\frac{z-w}{1+\epsilon}\right) \mathrm{d}V_w,$$

where $\delta_\epsilon \searrow 0$ as $\epsilon \searrow 0$ and δ_ϵ is chosen sufficiently small. The convolution is performed on each component of f. In the first integral defining f_j, we can differentiate under the integral sign to show that f_j is $C^\infty(\mathbb{C}^n)$. From Young's inequality for convolution, we have

$$\|f_{(\epsilon)}\| \leqslant \|f\|.$$

Since $f_{(\epsilon)} \longrightarrow f$ uniformly when $f \in C^\infty_0(\mathbb{C}^n)$, a dense subset of $W^m(\mathbb{C}^n)$, we have that

$$f_{(\epsilon)} \longrightarrow f \quad \text{in } W^m(\mathbb{C}^n) \text{ for every } f \in W^m(\mathbb{C}^n).$$

Obviously, this implies that $f_{(\epsilon)} \longrightarrow f$ in $W^m_{p,q+1}(\mathbb{C}^n)$.

Let δ_ν be a sequence of small numbers with $\delta_\nu \searrow 0$. For each δ_ν, we define

$$\Omega_{\delta_\nu} = \{z \in \Omega \mid \rho(z) < -\delta_\nu\}.$$

Then Ω_{δ_ν} is a sequence of relatively compact open subsets of Ω with union equal to Ω. When the boundary is Lipschitz, one can use a partition of unity $\{\zeta_\nu\}_{\nu=1}^N$, with each ζ_ν supported in an open set U_ν such that $U_\nu \cap \Omega$ is star-shaped. We then regularize $\zeta_\nu f$ in U_ν as before. Thus, there exists a sequence $\alpha_\nu \in C^\infty_{p,q}(\Omega)$ with compact support in Ω such that $\alpha_\nu \longrightarrow f$ and $\overline{\partial}\alpha_\nu \longrightarrow 0$ in $W^m(\mathbb{C}^n)$. Applying Theorem 28 to each $\overline{\partial}\alpha_\nu$, we obtain $u_\nu \in L^2_{p,q}(\mathbb{C}^n)$ which is supported in $\overline{\Omega}$ and $\overline{\partial}u_\nu = \overline{\partial}\alpha_\nu$ in $\mathbb{C}^n$. Also using Theorem 28, we have

$$\|u_\nu\|_{W^m(\Omega_\nu)} \leqslant C\|\overline{\partial}\alpha_\nu\|_{W^m(\Omega_\nu)}.$$

Letting $h_\nu = \alpha_\nu - u_\nu$, we have that $h_\nu \in L^2_{p,q}(\mathbb{C}^n)$, which is supported in $\overline{\Omega}$, $\overline{\partial}h_\nu = 0$ in $\mathbb{C}^n$ and $h_\nu \longrightarrow f$ in the $W^m(\mathbb{C}^n)$ norm. Thus the proof follows. □

Theorem 31. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded q-pseudoconvex domain with Lipschitz boundary $b\Omega$ and let $1 \leqslant q \leqslant n$. Then, for $-\frac{1}{2} \leqslant m < \frac{1}{2}$ and for every $\alpha \in W^m_{p,q}(\mathbb{C}^n)$ which is supported in $\overline{\Omega}$ such that $\overline{\partial}\alpha = 0$, for $1 \leqslant q \leqslant n-1$, one can find $u \in W^m_{p,q-1}(\mathbb{C}^n)$ such that $\overline{\partial}u = \alpha$ in the distribution sense in $\mathbb{C}^n$ with u supported in $\overline{\Omega}$ and*

$$\|u\|_{W^m_{p,q-1}(\mathbb{C}^n)} \leqslant C\|\alpha\|_{W^m_{p,q}(\mathbb{C}^n)}, \tag{31}$$

where C depends only on the Lipschitz constant and the diameter of Ω, but is independent of α.

Proof. For $m = 0$, Theorem 31 follows from Theorem 28. Since Ω has Lipschitz boundary, there is a bounded extension operator from $W^m(\Omega) \longrightarrow W^m(\mathbb{C}^n)$ for $0 \leqslant m < \frac{1}{2}$ (see e.g. [11]). Let $\widetilde{\alpha} \in W^m_{p,q}(\mathbb{C}^n)$ be the extension of α so that $\widetilde{\alpha}|_\Omega = \alpha$ with

$$\|\widetilde{\alpha}\|_{W^m(\mathbb{C}^n)} \leqslant C\|\alpha\|_{W^m(\Omega)}.$$

This is obviously true for $m = 0$. Notice that this is not true for $m = \frac{1}{2}$ (see [28]).

We show that there is a bounded extension operator from $W^{-m}(\Omega) \longrightarrow W^{-m}(\mathbb{C}^n)$ for $0 < m \leqslant \frac{1}{2}$ by using the fact that $\mathscr{D}(\Omega)$ is dense in $W^m(\Omega)$. Thus, from (5), for any $g \in W^{-m}(\Omega)$ and for $0 < m \leqslant \frac{1}{2}$, we have

$$\|\widetilde{\alpha}\|_{W^{-m}(\mathbb{C}^n)} = \sup_{g \in W^0(\Omega)} \frac{|\langle\widetilde{\alpha}, g\rangle_{\mathbb{C}^n}|}{\|g\|_{W^m(\mathbb{C}^n)}} \leqslant \sup_{g \in W^0(\Omega)} \frac{|\langle\alpha, g\rangle_\Omega|}{\|g\|_{W^m(\Omega)}} = \|\alpha\|_{W^{-m}(\Omega)}.$$

Then for any $\alpha \in W^{-m}(\Omega)$, $-\frac{1}{2} \leqslant m < \frac{1}{2}$, $\widetilde{\alpha}$ can be identified as a distribution in $W^m(\mathbb{C}^n)$ by setting $\alpha = 0$ outside Ω.

Assume that $\alpha \in W^m_{p,q}(\mathbb{C}^n)$ be any $\overline{\partial}$-closed form with support in $\overline{\Omega}$ and $0 \leqslant m < \frac{1}{2}$. Let u be defined by (28). Using Theorem 27, one obtains

$$\|u\|_{W^m(\Omega)} \leqslant C\| \star \alpha\|_{W^m(\Omega)} = C\|\alpha\|_{W^m(\Omega)},$$

where C depends only on the Lipschitz constant and the diameter of Ω, but is independent of α. Setting $u = 0$ outside Ω. It follows that u is in $W^m(\mathbb{C}^n)$ and u satisfies (31) for any $\alpha \in W^m_{p,q}(\mathbb{C}^n)$ with $0 \leqslant m < \frac{1}{2}$.

To show that $\overline{\partial}u = \alpha$ in $\mathbb{C}^n$, one use an approximation argument of Lemma 30. Let $\alpha_\nu \longrightarrow \alpha$ in $W^m_{p,q}(\mathbb{C}^n)$ such that the support of α_ν is in $\overline{\Omega}$. Set

$$u_\nu = \begin{cases} -\star\, \overline{\overline{\partial} N_{n-p,n-q} \star \overline{\alpha}_\nu} & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \mathbb{C}^n \backslash \Omega. \end{cases} \tag{32}$$

Then, for each j, by applying Theorem 27 to obtain a $(p,q-1)$-form u_ν such that $\overline{\partial}u_\nu=\alpha_\nu$ in $\mathbb{C}^n$. From Lemma 30, u_ν converges to u in $W^m_{p,q-1}(\Omega)$. Extending u to be zero outside $\overline{\Omega}$, u satisfies (31) and $\overline{\partial}u=f$ in the distribution sense in $\mathbb{C}^n$. This proves the theorem for $0\leqslant m<\frac{1}{2}$.

Now, we prove the theorem for any $\overline{\partial}$-closed $\alpha\in W^m_{p,q}(\mathbb{C}^n)$ with support in $\overline{\Omega}$, $-\frac{1}{2}\leqslant m<0$, by approximate α by $\overline{\partial}$-closed forms α_ν in $L^2_{p,q}(\mathbb{C}^n)$ such that α_ν has support in $\overline{\Omega}$ and $\alpha_\nu\longrightarrow\alpha$ in $W^m_{p,q}(\Omega)$. Set u and u_ν as in (28) and (32). For each j, applying Theorem 28 to obtain a $(p,q-1)$-form u_ν such that $\overline{\partial}u_\nu=\alpha_\nu$ in $\mathbb{C}^n$. Using Theorem 27, one obtains

$$\|u\|_{W^m(\Omega)}\leqslant C\|\star\alpha\|_{W^m(\Omega)}=C\|\alpha\|_{W^m(\Omega)},$$

where C depends only on the Lipschitz constant and the diameter of Ω, but is independent of α. From Lemma 30, u_ν converges to u in $W^m_{p,q-1}(\Omega)$. Extending u to be zero outside $\overline{\Omega}$. Then u satisfies (31) and $\overline{\partial}u=\alpha$ in the distribution sense in $\mathbb{C}^n$. Thus the proof follows. □

As Proposition 3.5 in [37], we prove the following result:

Proposition 32. *Let $\Omega\Subset\mathbb{C}^n$ be a bounded q-pseudoconvex domain with Lipschitz boundary $b\Omega$ and let $1\leqslant q\leqslant n$. Then, for any $\alpha\in W^m_{p,n}(\mathbb{C}^n)$, $-\frac{1}{2}\leqslant m<\frac{1}{2}$, such that α is supported in $\overline{\Omega}$ and*

$$\int_\Omega \alpha\wedge g=0 \quad \textit{for any}\quad g\in C^\infty_{n-p,0}(\overline{\Omega})\cap\ker\overline{\partial} \tag{33}$$

one can find $u\in W^m_{p,n-1}(\mathbb{C}^n)$ such that $\overline{\partial}u=\alpha$ in the distribution sense in $\mathbb{C}^n$ with u supported in $\overline{\Omega}$ and

$$\|u\|_{W^m(\mathbb{C}^n)}\leqslant C\|\alpha\|_{W^m(\mathbb{C}^n)}, \tag{34}$$

where C depends only on the Lipschitz constant and the diameter of Ω, but is independent of α.

Proof. Let $\alpha\in W^m_{p,n}(\mathbb{C}^n)$ such that α is supported in $\overline{\Omega}$ and α satisfies (33). Define u as in (28) as follows

$$u=-\star\,\overline{\overline{\partial}N_{n-p,n-q}\star\overline{\alpha}}. \tag{35}$$

Now, we extend u to $\mathbb{C}^n$ by defining $u=0$ in $\mathbb{C}^n\setminus\overline{\Omega}$. From Theorem 28, $u\in W^m_{p,n-1}(\mathbb{C}^n)$ and satisfies (34). From Lemma 30, we have $C^\infty_{n-p,0}(\overline{\Omega})\cap\ker\overline{\partial}$ is dense in $W^m_{n-p,0}(\mathbb{C}^n)\cap\ker(\overline{\partial})$. Thus, Condition (33) implies that $P_{n-p,0}\star\overline{\alpha}=0$.

To show that $\overline{\partial}u=\alpha$ in $\mathbb{C}^n$, we use an approximation argument as before. Let $\alpha_\nu\longrightarrow\alpha$ in $W^m_{p,n}(\mathbb{C}^n)$ such that the support of α_ν is in $\overline{\Omega}$. Let

$$h_\nu=\alpha_\nu-\star P_{n-p,0}\star\overline{\alpha}_\nu.$$

Using Lemma 30, h_ν converges to α in $W^m_{n-p,0}(\Omega)$. Each h_ν is in $L^2_{p,n}(\Omega)$ and satisfies

$$\int_\Omega h_\nu\wedge g=0\quad\text{for any}\quad g\in L^2_{n-p,0}(\Omega)\cap\ker\overline{\partial}.$$

Define

$$u_\nu=\begin{cases}-\star\overline{\overline{\partial}N_{n-p,0}\star\overline{h}_\nu} & \text{if } z\in\Omega,\\ 0 & \text{if } z\in\mathbb{C}^n\setminus\Omega.\end{cases}$$

Then, for each j, by applying Proposition 29 to obtain a $(p,n-1)$-form u_ν such that $\overline{\partial}u_\nu=h_\nu$ in $\mathbb{C}^n$ and

$$\|u_\nu\|_{W^m(\mathbb{C}^n)}\leqslant C\|h_\nu\|_{W^m(\mathbb{C}^n)}.$$

From Lemma 30, u_ν converges to u in $W^m_{p,n-1}(\Omega)$. Extending u to be zero outside $\overline{\Omega}$, u satisfies (34) and $\overline{\partial}u=\alpha$ in the distribution sense in $\mathbb{C}^n$. Thus the proof follows. □

5. Proof of the main theorems

Following the construction in [37, Lemma 4.1]. Let B be a large ball in $\mathbb{C}^n$ such that $\overline{\Omega} \Subset B$. Let $\Omega^+ = B \setminus \overline{\Omega}$ and $\Omega^- = \Omega$. In [19], a Martinelli–Bochner–Koppelman type kernel is constructed, and in [36] it is shown that the transformation induced by this kernel satisfies a jump formula (10). As a result, there exists an integral kernel $K_q(\zeta, z)$ of type (p,q) in z and $(n-p, n-q-1)$ in ζ satisfying a Martinelli–Bochner–Koppelman formula

$$\int_{b\Omega} K_q(\zeta,z) \wedge \alpha(\zeta) = \begin{cases} \alpha^+(z) & \text{if } z \in \Omega^+, \\ \alpha^-(z) & \text{if } z \in \Omega^-, \end{cases}$$

where $\alpha^+(z) = K^+\alpha(z)$ if $z \in \Omega^+$ and $\alpha^-(z) = K^-\alpha(z)$ if $z \in \Omega$ (see [19, Section 2.3]). Let $\alpha \in L^2_{p,q}(b\Omega)$, $0 < q \leqslant n-1$, and $\overline{\partial}_b \alpha = 0$ in $b\Omega$. If, we extend $\alpha^+(z)$ and $\alpha^-(z)$ to $b\Omega$ by considering non-tangential limits, we have the jump formula

$$\tau(\alpha^+(z) - \alpha^-(z)) = \alpha(z) \ \text{ on } \ b\Omega, \tag{36}$$

where τ is defined in (9). Since α^+ and α^- have non-tangential boundary values in L^2, then from Lemma 4.1 in [37], $\alpha^+ \in W^{\frac{1}{2}}_{p,q}(\Omega^+)$ and $\alpha^- \in W^{\frac{1}{2}}_{p,q}(\Omega)$ such that $\overline{\partial}\alpha^+ = 0$ in Ω^+ and $\overline{\partial}\alpha^- = 0$ in Ω. Furthermore, we have

$$\|\alpha^+\|_{W^{\frac{1}{2}}(\Omega^+)} \leqslant C\|\alpha\|_{b\Omega}, \tag{37}$$

and

$$\|\alpha^-\|_{W^{\frac{1}{2}}(\Omega)} \leqslant C\|\alpha\|_{b\Omega}, \tag{38}$$

where the constant C depends only on the Lipschitz constant of Ω but is independent of α.

Since Ω^+ is a bounded Lipschitz domain, there exists a continuous linear operator E from $W^m(\Omega^+)$ into $W^m(\mathbb{C}^n)$, for any $m \geqslant 0$, such that for any $g \in W^m(\Omega^+)$,

$$Eg|_{\Omega^+} = g$$

(see Stein [38, Chapter 6] or Grisvard [11]).

Extend α^+ from Ω^+ to $E\alpha^+$ componentwise on B such that

$$\|E\alpha^+\|_{W^{\frac{1}{2}}(B)} \leqslant C\|\alpha^+\|_{W^{\frac{1}{2}}(\Omega^+)}.$$

Since $\overline{\partial}\alpha^+ = 0$ on Ω^+, we have $\overline{\partial}E\alpha^+ = 0$ on Ω^+. Set $f = \overline{\partial}E\alpha^+$ in Ω and zero outside. Then $\overline{\partial} f = 0$ and f is supported in $\overline{\Omega}$. Furthermore, we have

$$\|f\|_{W^{-\frac{1}{2}}(\Omega)} \leqslant C\|E\alpha^+\|_{W^{\frac{1}{2}}(B)} \leqslant C\|\alpha^+\|_{W^{\frac{1}{2}}(\Omega^+)}. \tag{39}$$

When $1 \leqslant q \leqslant n-2$, one defines

$$h = \begin{cases} -\star\overline{\overline{\partial}N_{n-p,n-q}\star\overline{f}} & \text{if } \ z \in \Omega, \\ 0 & \text{if } \ z \in \mathbb{C}^n\setminus\Omega. \end{cases} \tag{40}$$

It follows, from Theorem 31, that $\overline{\partial}h = f$ and h is supported in $\overline{\Omega}$.

Also from (16) and (39), one obtains

$$\|h\|_{W^{-\frac{1}{2}}(\Omega)} \leqslant C\|f\|_{W^{-\frac{1}{2}}(\Omega)} \leqslant C\|\alpha^+\|_{W^{\frac{1}{2}}(\Omega^+)}. \tag{41}$$

Set

$$\widetilde{\alpha}^+ = E\alpha^+ - h \text{ in } B.$$

Then $\widetilde{\alpha}^+ = \alpha^+ - h = \alpha^+$ in Ω^+. Since $f = \overline{\partial}E\alpha^+$ in Ω, we have $\overline{\partial}\widetilde{\alpha}^+ = 0$ in B. Then from (41) and Theorem 27, one obtains

$$\begin{aligned} \|\widetilde{\alpha}^+\|_{W^{-\frac{1}{2}}(B)} &\leqslant C\left(\|E\alpha^+\|_{W^{-\frac{1}{2}}(B)} + \|h\|_{W^{-\frac{1}{2}}(\Omega)}\right) \\ &\leqslant C\|\alpha^+\|_{W^{\frac{1}{2}}(\Omega^+)}, \end{aligned}$$

for some constant C independent of α. Thus, for $1 \leqslant q \leqslant n-2$, one obtains

$$\|\widetilde{\alpha}^+\|_{W^{-\frac{1}{2}}(B)} \leqslant C\|\alpha^+\|_{W^{\frac{1}{2}}(\Omega^+)}. \tag{42}$$

When $q = n-1$, notice that $\overline{\partial}\alpha^- = 0$ in Ω. By using the jump formula (10), for every $\phi \in C^\infty_{n-p,0}(\overline{\Omega}) \cap \ker\overline{\partial}$, one obtains

$$\begin{aligned}\int_\Omega f \wedge \phi &= \int_\Omega \overline{\partial}(E\alpha^+ \wedge \phi)\\ &= \int_{b\Omega} K_b^+ \alpha \wedge \phi\\ &= \int_{b\Omega} K_b^- \alpha \wedge \phi + \int_{b\Omega} \alpha \wedge \phi\\ &= \int_\Omega \overline{\partial}\alpha^- \wedge \phi + \int_{b\Omega} \alpha \wedge \phi\\ &= \int_{b\Omega} \alpha \wedge \phi\\ &= 0.\end{aligned}$$

This implies that $f = \overline{\partial}E\alpha^+$ satisfies Condition (33). Defining h as in (40) and using Proposition 32, one obtains that $\overline{\partial}h = f$ in $\mathbb{C}^n$ and h is supported in $\overline{\Omega}$. Repeating the arguments as before, thus estimate (42) holds also for $q = n-1$. Thus, for $1 \leqslant q \leqslant n-1$, one obtains

$$\|\widetilde{\alpha}^+\|_{W^{-\frac{1}{2}}(B)} \leqslant C\|\alpha^+\|_{W^{\frac{1}{2}}(\Omega^+)}.$$

Then, from (37) and for $1 \leqslant q \leqslant n-1$, one obtains

$$\|\widetilde{\alpha}^+\|_{W^{-\frac{1}{2}}(B)} \leqslant C\|\alpha\|_{b\Omega}. \tag{43}$$

Since α satisfies (36) and $\overline{\partial}\alpha^+ = 0$ in Ω^+ and $\overline{\partial}\alpha^- = 0$ in Ω, then $\overline{\partial}_b\alpha = 0$ in $b\Omega$. Let

$$u = u^+ - u^- \ \text{ on } \ b\Omega.$$

First we solve $\overline{\partial}u^+ = \alpha^+$ on Ω^+. To do that, we use the canonical solution operator to define

$$u^+ = \overline{\partial}^* N^B \widetilde{\alpha}^+.$$

Then $\overline{\partial}u^+ = \widetilde{\alpha}^+ = 0$ in B. Using the interior regularity for N^B, u^+ gains a derivative on compact subsets of B, so u^+ has components in $W^{\frac{1}{2}}(b\Omega)$ on a neighborhood of $\overline{\Omega}$. From (43), we have for any $\xi \in \mathscr{D}(B)$ such that $\xi = 1$ in a neighborhood of $\overline{\Omega}$,

$$\|\xi u^+\|_{W^{\frac{1}{2}}(B)} \leqslant C\|\alpha^+\|_{W^{-\frac{1}{2}}(B)} \leqslant C\|\alpha\|_{b\Omega}, \tag{44}$$

for some constant C independent of α. Restricting u^+ to $b\Omega$, using (44) and the trace theorem again, one obtains

$$\|u^+\|_{b\Omega} \leqslant C\|\xi\alpha^+\|_{W^{\frac{1}{2}}(B)} \leqslant C\|\alpha\|_{b\Omega}, \tag{45}$$

for some constant C independent of α.

To solve $\overline{\partial}u^- = \alpha^-$ on Ω, one defines

$$u^- = \overline{\partial}^* N^\Omega \alpha^-.$$

Then $\overline{\partial}u^- = \alpha^-$. It follows from (38) and Theorem 27 that

$$\|u^-\|_{W^{\frac{1}{2}}(\Omega)} \leqslant C\|\alpha^-\|_{W^{\frac{1}{2}}(\Omega)} \leqslant C\|\alpha\|_{b\Omega}$$

for some constant C independent of α.

Since u^- satisfies a system of elliptic equations, it can be treated like harmonic functions. Thus u^- has boundary value in $L^2(b\Omega)$. Restricting u^- to $b\Omega$, using the trace theorem for smooth domains (see [22]), we have

$$\|u^-\|_{b\Omega} \leqslant C\|u^-\|_{W^{\frac{1}{2}}(\Omega)} \leqslant C\|\alpha\|_{b\Omega}. \tag{46}$$

Then $\overline{\partial}_b u = v$ on $b\Omega$. Also from (45) and (46), we have

$$\|u\|_{b\Omega} \leqslant C\|\alpha\|_{b\Omega}$$

for some constant C independent of α. Thus the proof of Theorem 1 follows.

Now, we prove Theorems 2 and 3. The proof of (i) in Theorem 2 follows directly from the main theorem and Theorem 1.1.1 in [21]. To prove (ii) in Theorem 2, we need to claim that, for all $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n-2$,

$$\mathscr{H}_b^{p,q}(b\Omega) = \ker\overline{\partial}_b \cap \ker\overline{\partial}_b^* = \{0\}. \tag{47}$$

To prove (47), let $\alpha \in \mathscr{H}_b^{p,q}(b\Omega) = \ker\overline{\partial}_b \cap \ker\overline{\partial}_b^*$. Then $\alpha = \overline{\partial}_b u$ for some $u \in L^2_{p,q-1}(b\Omega)$ by the main theorem. Since $\alpha \in \ker\overline{\partial}_b^*$, then

$$\langle\alpha,\alpha\rangle_{b\Omega} = \langle\overline{\partial}_b u,\alpha\rangle_{b\Omega} = \langle u,\overline{\partial}_b^*\alpha\rangle_{b\Omega} = 0.$$

We have $\mathscr{H}_b^{p,q}(b\Omega) = \{0\}$.

From Theorem 2, the range of $\overline{\partial}_b$, denoted by $\mathscr{R}\mathrm{ang}\,\overline{\partial}_b$, is closed in every degree. Then, one obtains $\ker\overline{\partial}_b = \mathscr{R}\mathrm{ang}\,\overline{\partial}_b^*$ and the following orthogonal decomposition:

$$L^2_{p,q}(b\Omega) = \ker\overline{\partial}_b \oplus \mathscr{R}\mathrm{ang}\,\overline{\partial}_b^* = \mathscr{R}\mathrm{ang}\,\overline{\partial}_b^* \oplus \mathscr{R}\mathrm{ang}\,\overline{\partial}_b^*.$$

Repeating the arguments of Theorem 8.4.10 in [6], one can prove that for every $\alpha \in \mathrm{Dom}\,\overline{\partial}_b \cap \mathrm{Dom}\,\overline{\partial}_b^*$,

$$\begin{aligned}\|\alpha\|^2_{b\Omega} &\leqslant C\left(\|\overline{\partial}_b\alpha\|^2_{b\Omega} + \|\overline{\partial}_b^*\alpha\|^2_{b\Omega}\right)\\ &= C\langle\Box_b\alpha,\alpha\rangle_{b\Omega}\\ &\leqslant C\|\Box_b\alpha\|\|\alpha\|_{b\Omega},\end{aligned}$$

i.e.,

$$\|\alpha\|_{b\Omega} \leqslant C\|\Box_b\alpha\|_{b\Omega}. \tag{48}$$

Since $\Box_b$ is a linear closed densely defined operator, then, from Theorem 1.1.1 in [21], $\mathscr{R}\mathrm{ang}\,\Box_b$ is closed. Thus, from (1.1.1) in [21] and the fact that $\Box_b$ is self adjoint, we have the Hodge decomposition

$$L^2_{p,q}(b\Omega) = \mathscr{R}\mathrm{ang}\,\Box_b \oplus \mathscr{H}_b^{p,q}(b\Omega) = \overline{\partial}_b\overline{\partial}_b^*\,\mathrm{Dom}\,\Box_b \oplus \overline{\partial}_b^*\overline{\partial}_b\,\mathrm{Dom}\,\Box_b.$$

Since $\Box_b$ is one to one on $\mathrm{Dom}\,\Box_b$ from (48), then there exists a unique bounded inverse operator

$$N_b : \mathscr{R}\mathrm{ang}\,\Box_b \longrightarrow \mathrm{Dom}\,\Box_b \cap (\mathscr{H}_b^{p,q}(b\Omega))^\perp$$

such that $N_b\Box_b\alpha = \alpha$ on $\mathrm{Dom}\,\Box_b$. We can write $N_b\Box_b = I$ on $\mathrm{Dom}\,\Box_b \cap (\mathscr{H}_b^{p,q}(b\Omega))^\perp$. From the definition of N_b, extend N_b to $L^2_{p,q}(b\Omega)$ to obtain $\Box_b N_b = I$ on $L^2_{p,q}(b\Omega)$. Thus N_b satisfies (i) and (ii).

To show that $\overline{\partial}_b^* N_b = N_b\overline{\partial}_b^*$ on $\mathrm{Dom}\,\overline{\partial}_b^*$. By using (ii), we have $\overline{\partial}_b^*\alpha = \overline{\partial}_b^*\overline{\partial}_b\overline{\partial}_b^* N_b\alpha$, for $\alpha \in \mathrm{Dom}\,\overline{\partial}_b^*$. Thus

$$N_b\overline{\partial}_b^*\alpha = N_b\overline{\partial}_b^*\overline{\partial}_b\overline{\partial}_b^* N_b\alpha = N_b(\overline{\partial}_b^*\overline{\partial}_b + \overline{\partial}_b\overline{\partial}_b^*)\overline{\partial}_b^* N_b\alpha = \overline{\partial}_b^* N_b\alpha.$$

A similar arguments shows that $\overline{\partial}_b N_b = N_b\overline{\partial}_b$ on $\mathrm{Dom}\,\overline{\partial}_b$. Thus the proof of Theorem 3 follows.

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