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On the boundedness of invariant hyperbolic domains

Sur le caractère borné des domaines hyperboliques invariants

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Abstract. In this paper, we generalize a theorem of A. Kodama about boundedness of hyperbolic circular domains. We will prove that if K is a compact Lie group which acts linearly on $\mathbb{C}^n$ with $\mathscr{O}(\mathbb{C}^n)^K = \mathbb{C}$, and Ω is a K-invariant orbit convex domain in $\mathbb{C}^n$ which contains 0, then Ω is bounded if and only if Ω is Kobayashi hyperbolic.

Résumé. Dans cet article, nous généralisons un théorème de A. Kodama sur le caractère borné des domaines circulaires hyperboliques. Nous démontrons que si K est un groupe de Lie compact qui agit linéairement sur $\mathbb{C}^n$ et vérifie $\mathscr{O}(\mathbb{C}^n)^K = \mathbb{C}$, et si Ω est un domaine K-invariant orbitalement convexe de $\mathbb{C}^n$ qui contient 0, alors Ω est borné si et seulement s'il est hyperbolique au sens de Kobayashi.

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1. Introduction

In this note, we investigate the boundedness of invariant hyperbolic domains. In the following, by hyperbolicity we always mean Kobayashi hyperbolicity. A domain $\Omega \subset \mathbb{C}^n$ is said to be a starlike circular domain or balanced domain, if for any $z = (z_1, z_2, \ldots, z_n) \in \Omega$ and $t \in \mathbb{C}$ with $|t| \leq 1$, one has $tz := (tz_1, tz_2, \ldots, tz_n) \in \Omega$. A. Kodama [7] proved that:

Theorem 1. *For any starlike circular domain $\Omega \subset \mathbb{C}^n$, Ω is hyperbolic if and only if it is bounded.*

In the language of group actions, a circular domain is a special S^1-invariant domain. We want to study more general invariant domains with respect to compact Lie group actions.

Let G be a Lie group, act linearly on $\mathbb{C}^n$ by ρ, i.e. $\rho: G \to GL(\mathbb{C}^n)$ is a Lie group homomorphism. We may write gz or $g\cdot z$ for $\rho(g)z$ for $g \in G$ and $z = (z_1, \ldots, z_n)' \in \mathbb{C}^n$ as a column vector. A domain $\Omega \subset \mathbb{C}^n$ is called G-invariant if for any $z \in \Omega$ and $g \in G$, then $gz \in \Omega$.

For a compact Lie group K, denote by $K^{\mathbb{C}}$ the universal complexification of K. If K acts linearly on $\mathbb{C}^n$, then the K action induces a $K^{\mathbb{C}}$ linear action on $\mathbb{C}^n$. If $\mathfrak{k}$ is the Lie algebra of K, and $\exp: \mathfrak{k} \to K$ is the exponential map, then the first Cartan decomposition theorem says that, $K^{\mathbb{C}} = K \cdot \exp(i\mathfrak{k}) = \exp(i\mathfrak{k}) \cdot K$, and $K \cap \exp(i\mathfrak{k}) = \{e\}$, where e is the identity of G.

We need to recall the following definition (see [3] or [13]).

Definition 2. *Let K be a compact Lie group with Lie algebra $\mathfrak{k}$, and let $K^{\mathbb{C}}$ be the complexification of K. Let X be a $K^{\mathbb{C}}$-space. A K-invariant subset U of X is said to be orbit convex if for each $z \in U$ and $v \in i\mathfrak{k}$, such that* $\exp(v)\cdot z \in U$, *it follows that* $\exp(tv)\cdot z \in U$ *for all $t \in [0,1]$.*

Let $\mathscr{O}(\mathbb{C}^n)^K = \{f \in \mathscr{O}(\mathbb{C}^n) : f \circ \rho(k) = f, \forall\ k \in K\}$, i.e. $\mathscr{O}(\mathbb{C}^n)^K$ is the set of K-invariant entire functions.

Let $\rho: S^1 \to GL(\mathbb{C}^n)$, $\rho(t) = \mathrm{diag}\{t, t, \ldots, t\}$, then S^1 acts linearly on $\mathbb{C}^n$ and $\mathscr{O}(\mathbb{C}^n)^{S^1} = \mathbb{C}$. A domain $\Omega \subset \mathbb{C}^n$ is said to be circular if it is S^1-invariant, i.e. for any $z = (z_1, z_2, \ldots, z_n) \in \Omega$ and $t \in S^1$, we have $tz := (tz_1, tz_2, \ldots, tz_n) \in \Omega$. It is easy to check that a domain Ω is a starlike circular domain if and only if Ω is an orbit convex S^1-invariant domain containing 0.

Studying problems in several complex variables from a point view of group actions is frequently very fruitful. In this note, we discuss the boundedness problem of hyperbolic invariant domains in a setting of group actions. For prerequisites about Lie groups and several complex variables, one is referred to [4, 8]. We consider a compact Lie group K acting linearly on $\mathbb{C}^n$ with $\mathscr{O}(\mathbb{C}^n)^K = \mathbb{C}$. Let Ω be a K-invariant domain containing 0. There are many interesting results about some problems in several complex variables related to group actions on Ω, see [2]–[3], [5], [10], [12]– [14] et al.

We will prove the following theorem which is a generalization of Kodama's theorem.

Theorem 3. *Let K be a compact Lie group which acts linearly on $\mathbb{C}^n$ with $\mathscr{O}(\mathbb{C}^n)^K = \mathbb{C}$. Let Ω be a K-invariant orbit convex domain in $\mathbb{C}^n$ containing 0. Then Ω is bounded if and only if Ω is hyperbolic.*

In the last section, we deal with the case for the Brody hyperbolic domains.

2. Some preliminary results

Let G be a Hermitian closed subgroup of $GL(n;\mathbb{C}^n)$ acting on $\mathbb{C}^n$ by multiplication.

Theorem 4 (The Hilbert–Mumford theorem (see [9])). *For $z \in \mathbb{C}^n$, the closure of the orbit Gz contains 0 if and only if there exist an algebraic group homomorphism $\varphi: \mathbb{C}^* \to G$ such that* $\lim_{\lambda\to 0}\varphi(\lambda)z = 0$.

Remark 5. For compact Lie group K which acts linearly on $\mathbb{C}^n$, we may identify K (and $K^{\mathbb{C}}$) with the image of K (and $K^{\mathbb{C}}$). After a linear transformation, we can assume that K (and $K^{\mathbb{C}}$) is a subgroup of $U(n)$ (and $GL(n;\mathbb{C}^n)$). Let $\mathfrak{k}$ be the Lie algebra of K, then $\mathfrak{k} \subset M(n,n,\mathbb{C})$, and we can give the usual norm on $i\mathfrak{k}$, i.e. $\|v\|^2 = Tr(v\overline{v}^t)$. For any $v \in i\mathfrak{k}$, we have $v = \overline{v}^t$. There is a $g \in U(n)$, such that $v = g\,\mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}g^{-1}$ with $\lambda_j \in \mathbb{R}$ for $j = 1, 2, \ldots, n$.

We need the following lemma (see [11, Theorem 12]), which is another form of the Hilbert–Mumford theorem, in order to suit our case. We give a proof in detail for completeness.

Lemma 6. *Let K be a compact Lie group, which acts linearly on $\mathbb{C}^n$. Let $\mathfrak{b}$ be the Cartan subalgebra of $\mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K. For $z \in \mathbb{C}^n$, if $0 \in \overline{K^{\mathbb{C}} z}$, then there is a $k \in K$ and $v \in i\mathfrak{b}$, such that* $\lim_{t\to+\infty} \exp(tv)kz = 0$.

Proof. As the lemma is obviously true for $z = 0$, in the following, we assume $z \neq 0$. We may assume that K is a closed subgroup of $U(n)$. As $\mathfrak{b}$ is the Cartan subalgebra of $\mathfrak{k}$, there is a $g \in U(n)$, such that for any $v \in i\mathfrak{b}$,

$$v = g\Lambda(v)g^{-1}$$

where $\Lambda(v) = \mathrm{diag}\{\lambda_1(v), \lambda_2(v), \dots, \lambda_n(v)\}$, $\lambda_j(v) \in \mathbb{R}$ for $j = 1, 2, \dots, n$. By a theorem of Lie algebra, $\mathfrak{k} = \cup_{k\in K} k\mathfrak{b}k^{-1}$, and since $K^{\mathbb{C}} = \exp(i\mathfrak{k}) \cdot K$ and $0 \in \overline{K^{\mathbb{C}} z}$, there is a sequence $\{a_l\}$, $\{v_l\}$ and $\{k_l\}$ with $a_l > 0$, $v_l \in i\mathfrak{b}$, $\|v_l\| = 1$ and $k_l \in K$ for $l = 1, 2, \dots$, such that

$$\lim_{l\to\infty} \exp(a_l v_l) k_l z = 0.$$

We may write $v_l = g\Lambda_l g^{-1}$, where $\Lambda_l = \mathrm{diag}\{\lambda_{l1}, \lambda_{l2}, \dots, \lambda_{ln}\}$ with $\lambda_{lj} \in \mathbb{R}$ for $j = 1, 2, \dots, n$ and $l = 1, 2, \dots$. Since $\|v_l\| = 1$ and $U(n)$ and K are compact, after a subsequence, we may assume that $\lim_{l\to\infty} \Lambda_l = \Lambda_0$ and $\lim_{l\to\infty} k_l = k$. So $k \in K$ and $v_0 := g\Lambda_0 g^{-1} \in i\mathfrak{k}$. Set $w_l = (w_{l1}, w_{l2}, \dots, w_{ln}) := g^{-1}k_l z$, then $\lim_{l\to\infty} w_l = g^{-1}kz$. Put $w_0 = (w_{01}, w_{02}, \dots, w_{0n}) := g^{-1}kz$ and $\Lambda_0 = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

If $w_{0j} \neq 0$ for some $j \in \{1, 2, \dots, n\}$,

$$\|\exp(a_l v_l) k_l z\| = \|\exp(a_l \Lambda_l) w_l\| \geq e^{a_l \lambda_{lj}} |w_{lj}|.$$

As $\lim_{l\to\infty} \exp(a_l v_l) k_l z = 0$ and $a_l > 0$, then we have $\lambda_j \leq 0$; furthermore, if $\lambda_j = 0$, then $\lambda_{lj} < 0$ for $l > N_j$. So, take $v = v_0 + v_N \in i\mathfrak{b}$ for some N large enough,

$$v = g\,\mathrm{diag}\{\lambda_1 + \lambda_{N1}, \lambda_2 + \lambda_{N2}, \dots, \lambda_n + \lambda_{Nn}\} g^{-1},$$

satisfying $\lambda_j + \lambda_{Nj} < 0$, whenever $w_{0j} \neq 0$ for $j \in \{1, 2, \dots, n\}$. Therefore,

$$\lim_{t\to+\infty} \exp(tv)kz = 0. \qquad \square$$

Lemma 7. *Let K be a compact Lie group, which acts linearly on $\mathbb{C}^n$. Let $\mathfrak{b}$ be the Cartan subalgebra of $\mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K. If $\mathscr{O}(\mathbb{C}^n)^K = \mathbb{C}$, there are $v_1, v_2, \dots, v_s \in i\mathfrak{b}$, such that for any $z \in \mathbb{C}^n$, $z \neq 0$, there is an $l_z \in \{1, 2, \dots, s\}$ and $k_z \in K$, satisfying* $\lim_{t\to+\infty} \exp(tv_{l_z}) k_z z = 0$.

Proof. We may assume that K is a closed subgroup of $U(n)$. As $\mathfrak{b}$ is the Cartan subalgebra of $\mathfrak{k}$, there is a $g \in U(n)$, such that for any $v \in i\mathfrak{b}$,

$$v = g\Lambda(v)g^{-1}$$

where $\Lambda(v) = \mathrm{diag}\{\lambda_1(v), \lambda_2(v), \dots, \lambda_n(v)\}$, $\lambda_j(v) \in \mathbb{R}$ for $j = 1, 2, \dots, n$.

As $\mathscr{O}(\mathbb{C}^n)^K = \mathbb{C}$, by [10, Fact 2.1], we can get $0 \in \overline{K^{\mathbb{C}} z}$ for any $z \in \mathbb{C}^n$. By Lemma 6, for any $z \in \mathbb{C}^n$, $z \neq 0$, there is a $k_z \in K$ and $v \in i\mathfrak{b}$, such that $\lim_{t\to+\infty} \exp(tv) k_z z = 0$. As $\exp(tv) = g\,\exp(t\Lambda(v))g^{-1}$, therefore,

$$\lim_{t\to+\infty} g\,\exp(t\Lambda(v))g^{-1} k_z z = \lim_{t\to+\infty} \exp(tv) k_z z = 0.$$

Let

$$w_z = (w_1, w_2, \dots, w_n) := g^{-1} k_z z.$$

Then set $a_z := (a_1, a_2, \dots, a_n)$ with

$$a_j := \begin{cases} 0 & \text{if } w_j = 0 \\ 1 & \text{if } w_j \neq 0 \end{cases}$$

and $A = \{a_z : z \in \mathbb{C}^n, z \neq 0\}$. As $\sharp A < 2^n$, and for each $a \in A$, we may choose

$$v_a \in i\mathfrak{b}, v_a = g\,\mathrm{diag}\{\lambda_1(v_a), \lambda_2(v_a), \dots, \lambda_n(v_a)\} g^{-1}$$

such that $\lambda_j(v_a) < 0$ if $a_j = 1$ for $j \in \{1, 2, \dots, n\}$. So we get

$$\lim_{t\to+\infty} \exp(tv_{a_z})k_z z = 0.$$

Then taking the index of A as $\{1, 2, \dots, s\}$, we prove the lemma. □

3. Hyperbolicity and boundedness

The aim of this subsection is to prove Theorem 3. For convenience, we restate it here.

Theorem 8 (= Theorem 3). *Let K be a compact Lie group, which acts linearly on $\mathbb{C}^n$ with $\mathscr{O}(\mathbb{C}^n)^K = \mathbb{C}$. Let Ω be a K-invariant orbit convex domain in $\mathbb{C}^n$ containing 0. Then Ω is bounded if and only if Ω is hyperbolic.*

Proof. It is well known (see [6]) that if Ω is bounded, then Ω is hyperbolic.

In the following, we prove the 'if' part. We may assume the unit ball $B \subset \Omega$ and K is a closed subgroup of $U(n)$. Suppose that Ω is unbounded and hyperbolic, and we will get a contradiction.

Let $z_l \in \Omega$ for $l = 1, 2, \dots$ such that $\|z_l\| > 1$ and $\lim_{l\to\infty} \|z_l\| = \infty$. By Lemma 7, we can find a subsequence $\{x_l\}_{l=1}^{\infty}$ of $\{z_l\}_{l=1}^{\infty}$ such that there is a $v \in \{v_j : j = 1, 2, \dots, s\}$, satisfying for each x_l, there is a $k_l \in K$, such that $\lim_{t\to+\infty} \exp(tv)k_l x_l = 0$ for any $l = 1, 2, \dots$.

Choose $b_l > 0$ satisfying $\|\exp(b_l v)k_l x_l\| = 1/3$ for each l. As $\lim_{l\to\infty} \|x_l\| = \infty$, we have $\lim_{l\to\infty} b_l = \infty$. Since Ω is orbit convex, so

$$f_l(\lambda) = \exp((1-\lambda)b_l v)k_l x_l : \Delta \to \Omega$$

is holomorphic, where $\Delta = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. Notice $\|f_l(0)\| = \frac{1}{3}$ and $\|f_l(1)\| > 1$, we can choose $t_l \in (0,1)$, such that $\|f_l(t_l)\| = \frac{1}{2}$. As in the proof of Lemma 7, we write $v = g\,\mathrm{diag}\{\lambda_1(v), \lambda_2(v), \dots, \lambda_n(v)\}g^{-1}$ with $g \in U(n)$. Let

$$w_l = (w_{l1}, w_{l2}, \dots, w_{ln}) := g^{-1}k_l x_l,$$

then $\lambda_j(v) < 0$, if $w_{lj} \neq 0$ for some $l \in \mathbb{N}$, $j \in \{1, 2, \dots, n\}$. Set

$$h_l(t) := \|f_l(t)\|^2 = \Sigma_{j=1}^n e^{2(1-t)b_l\lambda_j(v)}|w_{lj}|^2, \text{ for } \quad t \in [0,1],$$

then $h_l(0) = 1/9$, $h_l(t_l) = 1/4$. We have $h_l''(t) \geq 0$, and

$$\begin{aligned} h_l'(0) &= 2b_l\Sigma_{j=1}^n(-\lambda_j(v))e^{2b_l\lambda_j(v)}|w_{lj}|^2 \\ &\geq 2b_l\lambda\Sigma_{j=1}^n e^{2b_l\lambda_j(v)}|w_{lj}|^2 \\ &= 2b_l\lambda h_l(0) \\ &\to \infty, as\ l \to \infty, \end{aligned} \tag{1}$$

where $\lambda = \min\{-\lambda_j(v) : \lambda_j(v) < 0 \text{ for } j \in \{1, 2, \dots, n\}\} > 0$. Since

$$h_l(t_l) \geq h_l(0) + h_l'(0)t_l,$$

we have

$$\begin{aligned} t_l &\leq \frac{h_l(t_l) - h_l(0)}{h_l'(0)} \\ &= \frac{5}{36h_l'(0)} \\ &\to 0 \text{ as } l \to \infty \text{ by (1)}. \end{aligned} \tag{2}$$

By the compactness of $\{z \in \mathbb{C}^n : \|z\| = 1/3\}$ and $\{z \in \mathbb{C}^n : \|z\| = 1/2\}$, we have a subsequence $\{f_{l_p}(0)\}$ of $\{f_l(0)\}$ and $\{f_{l_p}(t_{l_p})\}$ of $\{f_l(t_l)\}$ with $\lim_{p\to\infty} f_{l_p}(0) = y_0 \in \Omega$ and $\lim_{p\to\infty} f_{l_p}(t_{l_p}) = y_1 \in \Omega$. By the distance decreasing property of holomorphic mappings with respect to the Kobayashi distances,

$$\mathbf{k}_\Omega(f_{l_p}(0), f_{l_p}(t_{l_p})) \leq \mathbf{k}_\Delta(0, t_{l_p}). \tag{3}$$

Taking $p \to \infty$, by (2) and (3), we have $\mathbf{k}_\Omega(y_0, y_1) = 0$. Since $y_0 \neq y_1$, we get a contradiction. Hence, Ω must be bounded. □

Corollary 9. *Let K be a compact Lie group, which acts linearly on $\mathbb{C}^n$ with $\mathscr{O}(\mathbb{C}^n)^K = \mathbb{C}$. Let Ω be a K-invariant pseudoconvex domain in $\mathbb{C}^n$ containing 0. Then Ω is bounded if and only if Ω is hyperbolic.*

Proof. It is proved in [10, Theorem 2.4] that Ω is orbit convex, so Corollary is followed by the above theorem. □

4. Some results about Brody hyperbolicity

In this section, we discuss Brody hyperbolicity. A domain without nonconstant holomorphic mappings from $\mathbb{C}$ into it is said to be Brody hyperbolic. It is well known that hyperbolic domains are Brody hyperbolic, the reverse is true for compact complex manifolds but is false in generally. We focus on invariant domains.

Theorem 10. *Let K be a compact Lie group acting linearly on $\mathbb{C}^n$ with $\mathscr{O}(\mathbb{C}^n)^K = \mathbb{C}$. Let Ω be an unbounded K-invariant taut domain in $\mathbb{C}^n$ containing 0. Then there is a nonconstant holomorphic mapping $f : \mathbb{C} \to \Omega$.*

Proof. We may assume that the closed unit ball $\overline{B}$ is contained in Ω and that K is a closed subgroup of $U(n)$.

As Ω is taut, then Ω is pseudoconvex, therefore, it is orbit convex (see [10, Theorem 2.4]). Since Ω is unbounded, by the proof of Theorem 8, we may find $z_l \in \Omega$ for $l = 1, 2, \ldots$ satisfying the following two conditions:

(1) $\|z_l\| > 0$ and $\lim_{l\to\infty} \|z_l\| = \infty$,
(2) there is a $v \in \mathfrak{k}$, such that for each l, $\lim_{t\to+\infty} \exp(tv)z_l = 0$.

We may write $v = g\,\mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}g^{-1}$, where $g \in U(n)$ and $\lambda_j \in \mathbb{R}$ for $j = 1, 2, \ldots, n$. Let $w_l = (w_{l1}, w_{l2}, \ldots, w_{ln}) := g^{-1}z_l$. By condition (2), we can get that if $w_{lj} \neq 0$ for some $j \in \{1, 2, \ldots, n\}$ and some $l \in \mathbb{N}$, then $\lambda_j < 0$. Since

$$\begin{aligned} \exp(tv)z_l &= g\ \exp(t \cdot \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\})g^{-1}z_l \\ &= g(e^{t\lambda_1}w_{l1}, e^{t\lambda_2}w_{l2}, \ldots, e^{t\lambda_n}w_{ln}), \end{aligned}$$

the equation $\|\exp(tv)z_l\| = 1$ for $t > 0$, has only one solution which is denoted by a_l. As $\lim_{l\to\infty}\|z_l\| = \infty$, we get $\lim_{l\to\infty} a_l = +\infty$. Since Ω is orbital convex,

$$f_l(\xi) = \exp(\xi v)\exp(a_l v)z_l : H_l \to \Omega$$

is holomorphic on H_l, where $H_l = \{\xi = x + iy \in \mathbb{C} : x > -a_l\}$. As $\{\exp(a_l v)z_l\} \subset \overline{B}$, we may find its subsequence $\{\exp(a_{l_p}v)z_{l_p}\}$, such that

$$\lim_{p\to\infty} \exp(a_{l_p}v)z_{l_p} = z_0 \in \Omega.$$

So

$$\exp(\xi v)z_0 = \lim_{p\to\infty} f_{l_p}(\xi).$$

Let $f(\xi) = \exp(\xi v)z_0$. By the tautness of Ω, and $\bigcup_p H_{l_p} = \mathbb{C}$, we get a holomorphic mapping $f : \mathbb{C} \to \Omega$. As

$$\|z_0\| = \lim_{p\to\infty}\|\exp(a_{l_p}v)z_{l_p}\| = 1,$$

write $g^{-1}z_0 = (\zeta_1, \zeta_2, \ldots, \zeta_n)$, then there is a j_0, such that $\zeta_{j_0} \neq 0$. Since

$$\begin{aligned}(\zeta_1, \zeta_2, \ldots, \zeta_n) &= g^{-1} \lim_{p\to\infty} \exp(a_{l_p} v) z_{l_p} \\ &= \lim_{p\to\infty} (e^{a_{l_p}\lambda_1} w_{l_p 1}, e^{a_{l_p}\lambda_2} w_{l_p 2}, \ldots, e^{a_{l_p}\lambda_n} w_{l_p n})\end{aligned}$$

and $\zeta_{j_0} \neq 0$, we get that $w_{l_p j_0} \neq 0$ for $p >> 1$, hence $\lambda_{j_0} < 0$. Note that

$$f(\xi) = \exp(\xi v) z_0 = g(e^{\xi\lambda_1}\zeta_1, e^{\xi\lambda_2}\zeta_2, \ldots, e^{\xi\lambda_n}\zeta_n),$$

we can get that f is nonconstant. □

K. Azukawa [1] gives an example showing that, there is a pseudoconvex circular domain in $\mathbb{C}^2$ containing 0, which is Brody hyperbolic but not hyperbolic. Hence, his domain must be unbounded. However, one has the following reformulation of Theorem 10:

Corollary 11 (= Theorem 10). *Let K be a compact Lie group, which acts linearly on $\mathbb{C}^n$ with $\mathcal{O}(\mathbb{C}^n)^K = \mathbb{C}$. Let Ω be a K-invariant taut domain in $\mathbb{C}^n$ containing 0. Then Ω is Brody hyperbolic if and only if it is bounded.*

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