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Moduli space of rank one logarithmic connections over a compact Riemann surface

Anoop Singh [a]

[a] Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhusi, Prayagraj 211 019, India.

E-mail: anoopsingh@hri.res.in.

Abstract. Let $\mathscr{M}_X$ denote the moduli space of rank one logarithmic connections singular over a finite subset S of a compact Riemann surface X with fixed residues. We study the rational functions into $\mathscr{M}_X$. We prove that there is a natural compactification of $\mathscr{M}_X$ and the Picard group of $\mathscr{M}_X$ is isomorphic to the Picard group of $\mathrm{Pic}^d(X)$.

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1. Introduction

The moduli space of logarithmic connection has been constructed in [4]. Let X be a compact Riemann surface. In [1], several properties, like Picard group, algebraic functions and compactification have been studied for the moduli space of logarithmic connections in a holomorphic vector bundle over X with singularity exactly over one point. Let S be a finite subset of X and $\mathscr{M}_X$ denote the moduli space of rank one logarithmic connection singular over S with fixed residues. In [6], it has been proved that there is a natural symplectic structure on $\mathscr{M}_X$ and there are no nonconstant algebraic functions on $\mathscr{M}_X$. In the present article, we reconsider the moduli space $\mathscr{M}_X$ and try to study the biholomorphic class of the $\mathscr{M}_X$, see Section 3. We prove that any rational map from a normal variety to $\mathscr{M}_X$ is defined on the whole of the normal variety, see Section 4. Also, we show, by explicit manipulation that there is a natural compactification of $\mathscr{M}_X$ and compute the Picard group of $\mathscr{M}_X$, see Section 5.

2. Preliminaries

Let X be a compact connected Riemann surface and $S = \{x_1, \dots, x_m\}$ be a finite subset consisting of distinct points of X. The set S will be fixed throughout this article. We denote by $S = x_1 + \cdots + x_m$ the reduced effective divisor on X associated to the finite set S. Let $\Omega^1_X(\log S)$ denote the sheaf of logarithmic differential 1-forms along S, see [5]. Notion of logarithmic connection was introduced by P. Deligne in [2]. Let L be a holomorphic line bundle over X. We will denote the fibre of L over any point $x \in X$ by $L(x)$. A logarithmic connection on L singular over S is a $\mathbf{C}$-linear map

$$D : L \to \Omega^1_X(\log S) \otimes L \tag{1}$$

which satisfies the Leibniz identity

$$D(fs) = fD(s) + \mathrm{d}f \otimes s, \tag{2}$$

where, f is a local section of $\mathscr{O}_X$ and s is a local section of L.

Let $\mathrm{Res}(D, x_\beta)$ denote the *residue* of the logarithmic connection D at point $x_\beta \in S$ (see [2] for the details). If D and D' are two logarithmic connections on L singular over S with

$$\mathrm{Res}(D, x_\beta) = \mathrm{Res}(D', x_\beta), \tag{3}$$

then $D' = D + \theta$, where $\theta \in \mathrm{H}^0(X, \Omega^1_X)$. Thus, the space of all logarithmic connections on a given holomorphic line bundle L, singular over S, and satisfying (3) is an affine space for $\mathrm{H}^0(X, \Omega^1_X)$.

3. Biholomorphic class of the moduli space

In this section, we will assume that $\mathrm{genus}(X) = g \geq 2$. For each point $x_j \in S$, fix a complex number $r_j \in \mathbf{C}$, for $j = 1, \dots, m$. Let d be a fix integer which denotes the degree of a line bundle over X. By a pair (L, D) over X, we mean that L is a line bundle over X of degree d and D is a logarithmic connection in L singular over S with residues to be given complex numbers $r_j \in \mathbf{C}$ for each $x_j \in S$.

Let $\mathscr{M}_X$ denote the moduli space of pairs (L, D) over X (see [4]).

Let $X_0 = X \setminus S$ and fix a point $x_0 \in X_0$. Now, consider the fundamental group

$$\pi_1(X_0, x_0) = \left\{ a_1, b_1, \dots, a_g, b_g, \gamma_1, \dots, \gamma_m \,\middle|\, \prod_{i=1}^{g} [a_i, b_i] \gamma_1 \dots \gamma_m = 1 \right\}$$

of X_0 at x_0, where a_i, b_i for $i = 1, \dots, g$ are generators of fundamental group of genus g compact Riemann surface X and γ_j are the homotopy class of loops at x_0 around x_j's. Let $\mathscr{B}_g = \{\rho : \pi_1(X_0, x_0) \to \mathbf{C}^* \mid \rho(\gamma_j) = \exp(2\pi\sqrt{-1} r_j)\}$. Now, the representation space $\mathrm{Hom}(\pi_1(X_0, x_0), \mathbf{C}^*)$ is complex algebraic variety in a natural way and $\mathscr{B}_g$ is a closed subvariety of representation space. Moreover, $\mathscr{B}_g$ is smooth, because the representations are irreducible. The space $\mathscr{B}_g$ does not depend on the point x_0 of X_0.

We have a map

$$\Phi : \mathscr{M}_X \to \mathscr{B}_g \tag{4}$$

sending any pair $(L, D) \in \mathscr{M}_X$ to its monodromy representation $\rho : \pi_1(X_0, x_0) \to \mathbf{C}^*$ such that $\rho(\gamma_j) = \exp(2\pi\sqrt{-1} r_j)$. Note that both the spaces $\mathscr{M}_X$ and $\mathscr{B}_g$ have underlying structure of complex manifold.

Therefore, we have

Proposition 1. *The map $\Phi : \mathscr{M}_X \to \mathscr{B}_g$ defined in* (4) *is a biholomorphism. Further, if $(Y, T = \{y_1, \dots, y_m\})$ is another compact connected Riemann surface of genus $g \geq 2$, with finite subset T consisting of m elements and $\mathscr{M}_Y$ is the correspoding moduli space of logarithmic connections singular over T with same set of residues, then the two complex manifolds $\mathscr{M}_X$ and $\mathscr{M}_Y$ are biholomorphic.*

4. Rational functions into the moduli space

Let $\mathcal{M}_X$ be the moduli space described in Section 3. Let

$$p : \mathcal{M}_X \to \mathrm{Pic}^d(X) \tag{5}$$

be the projection defined by sending any pair (L, D) to L, where $\mathrm{Pic}^d(X)$ is a Picard variety of line bundles of degree d. Fix a pair $(L_0, D_0) \in \mathcal{M}_X$. Let L_0^* be the dual line bundle of L_0 and D_0^* denote the logarithmic connection in L_0^*, induced from D_0. Then D_0^* is sigular over S with $\mathrm{Res}(D_0^*, x_j) = -r_j$ for all $j = 1, \dots, m$.

Now, for any $(L, D) \in \mathcal{M}_X$, $L_0^* \otimes L$ is a holomorphic line bundle with holomorphic connection $\nabla = D_0^* \otimes \mathbb{1}_L + \mathbb{1}_{L_0^*} \otimes D$. Note that $\mathrm{Res}(\nabla, x_j) = 0$, for all $j = 1, \dots, m$.

Let $\mathcal{M}_X^h$ denote the moduli space of pairs (L, D), where L is a holomorphic line bundle of degree 0, and D is a holomorphic connection in L. Then we have a map

$$\Psi_{(L_0, D_0)} : \mathcal{M}_X \to \mathcal{M}_X^h \tag{6}$$

sending (L, D) to $(L_0^* \otimes L, D_0^* \otimes \mathbb{1}_L + \mathbb{1}_{L_0^*} \otimes D)$, which is an isomorphism. Therefore, the structure theory for both the moduli spaces are same. Note that the moduli space $\mathcal{M}_X$ is an algebraic group. We have an extension of group schemes

$$0 \to \mathrm{H}^0(X, \Omega_X^1) \to \mathcal{M}_X \to J(X) \to 0 \tag{7}$$

From the above extension (7) of $J(X)$ by $\mathrm{H}^0(X, \Omega_X^1)$, it is clear that there are no rational curves on $\mathcal{M}_X$. Now, we have following Theorem, which can be proved either using Hartogs' Theorem or [3, Theorem 9.4, p. 103].

Theorem 2. *Let $f : Z \dashrightarrow \mathcal{M}_X$ be a rational map from a normal variety Z to the moduli space $\mathcal{M}_X$. Suppose that f is defined in a complement of a subvariety of Z of codimension ≥ 2. Then f is defined on whole of Z.*

Proof. From Proposition 1, $\mathcal{M}_X$ is biholomorphic to the affine variety $\mathcal{B}_g$. So, we get $f : Z \dashrightarrow \mathcal{B}_g$. By hypothesis, f is defined in a complement of a subvariety of Z of codimension ≥ 2, so the conclusion follows from Hartogs' Theorem. □

Remark 3. Alternatively, given a rational map $f : Z \dashrightarrow \mathcal{M}_X$, defines a rational map $Z \dashrightarrow \mathcal{M}_X \to J(X)$ by composition. Now, using [3, Theorem 9.4, p. 103], which says that any rational map from a normal variety to an abelian variety is defined on whole of the domain, and hence we are done.

Corollary 4. *Every rational map $f : \mathbf{CP}^n \dashrightarrow \mathcal{M}_X$ from projective space to $\mathcal{M}_X$ is constant.*

Proof. As $\mathbf{CP}^n$ is a normal variety, from Theorem 2 f is defined on whole of $\mathbf{CP}^n$. Moreover, $\mathbf{CP}^n$ is $\mathbf{CP}^1$ connected and $\mathcal{M}_X$ does not contain any rational curve, which is equivalent to the fact that any rational map from $\mathbf{CP}^1$ to $\mathcal{M}_X$ is constant. Thus f is constant. □

5. Compactification and the Picard group of moduli space

The following proposition can be proved in more general context of any extension of an abelian variety by an additive group scheme using similar technique. Here, we restrict ourselves to the moduli space $\mathcal{M}_X$.

Proposition 5. *There exists an algebraic vector bundle $\pi : E \to \mathrm{Pic}^d(X)$ such that $\mathcal{M}_X$ is embedded in $\mathbf{P}(E)$ with $\mathbf{P}(E) \setminus \mathcal{M}_X$ as the hyperplane at infinity.*

Proof. Let $p : \mathscr{M}_X \to \mathrm{Pic}^d(X)$ be the map as defined in (5). Then for any $L \in \mathrm{Pic}^d(X)$, the fiber $p^{-1}(L)$ is an affine space modelled on $\mathrm{H}^0(X, \Omega^1_X)$. In fact, $p : \mathscr{M}_X \to \mathrm{Pic}^d(X)$ is a $\Omega^1_{\mathrm{Pic}^d(X)}$-torsor. Since the dual of an affine space is a vector space, so the dual $p^{-1}(L)^\vee = \{\varphi : p^{-1}(L) \to \mathbf{C} \mid \varphi \text{ is an affine linear map}\}$ is a vector space over $\mathbf{C}$. Define a sheaf E on $\mathrm{Pic}^d(X)$ as follows;

For every Zariski open subset U of $\mathrm{Pic}^d(X)$, sections of E over U is the following set

$$E(U) = \left\{ f : p^{-1}(U) \to \mathbf{C} \text{ is a regular function} : f|_{p^{-1}(L)} \in p^{-1}(L)^\vee \right\}.$$

Clearly, E is an $\mathscr{O}_{\mathrm{Pic}^d(X)}$-module. Moreover E is a locally free sheaf over $\mathrm{Pic}^d(X)$, and hence we get an algebraic vector bundle $\pi : E \to \mathrm{Pic}^d(X)$

Let $(L, D) \in \mathscr{M}_X$, and define a map $\Phi_{(L,D)} : p^{-1}(L)^\vee \to \mathbf{C}$, by $\Phi_{(L,D)}(\varphi) = \varphi[(L,D)]$, which is nothing but the evaluation map. Now, the kernel $\mathrm{Ker}(\Phi_{(L,D)})$ defines a hyperplane in $p^{-1}(L)^\vee$ denoted by $H_{(L,D)}$. Let $\mathbf{P}(E)$ be the projective bundle defined by hyperplanes in the fiber of π. Then, we have natural projection

$$\widetilde{\pi} : \mathbf{P}(E) \to \mathrm{Pic}^d(X) \tag{8}$$

induced from π. Define a map

$$\iota : \mathscr{M}_X \to \mathbf{P}(E) \tag{9}$$

by sending (L, D) to the hyperplane $H_{(L,D)}$, which is clearly an open embedding. Set $Y = \mathbf{P}(E) \setminus \mathscr{M}_X$. Then $\widetilde{\pi}^{-1}(L) \cap Y$ is a linear hyperplane in $\widetilde{\pi}^{-1}(L)$ for every $L \in \mathrm{Pic}^d(X)$, and hence Y is a hyperplane at infinity. This completes the proof. □

Thus, from the Proposition 5, $\mathbf{P}(E)$ is the natural compactification of the moduli space $\mathscr{M}_X$. Further, p defined in (5), induces a homomorphism of Picards groups

$$p^* : \mathrm{Pic}(\mathrm{Pic}^d(X)) \to \mathrm{Pic}(\mathscr{M}_X) \tag{10}$$

that sends a line bundle ξ over $\mathrm{Pic}^d(X)$ to a line bundle $p^*\xi$ over $\mathscr{M}_X$. We have

Theorem 6. *The homomorphism $p^* : \mathrm{Pic}(\mathrm{Pic}^d(X)) \to \mathrm{Pic}(\mathscr{M}_X)$ defined in* (10) *is an isomorphism.*

Proof. First we show that p^* in (10) is injective. Let $\Xi \to \mathrm{Pic}^d(X)$ be a line bundle such that $p^*\Xi$ is a trivial line bundle over $\mathscr{M}_X$. Giving a trivialization of $p^*\Xi$ is equivalent to giving a nowhere vanishing section of $p^*\Xi$ over $\mathscr{M}_X$. Fix $\zeta \in \mathrm{H}^0(\mathscr{M}_X, p^*\Xi)$ a nowhere vanishing section. Take any point $z \in \mathrm{Pic}^d(X)$. Then,

$$\zeta|_{p^{-1}(z)} : p^{-1}(z) \to \Xi(z)$$

is a nowhere vanishing map. Notice that $p^{-1}(z) \cong \mathbf{C}^g$ and $\Xi(z) \cong \mathbf{C}$. Now, any nowhere vanishing algebraic function on an affine space $\mathbf{C}^g$ is a constant function, that is, $\zeta|_{p^{-1}(z)}$ is a constant function and hence corresponds to a non-zero vector $\alpha_z \in \Xi(z)$. Since ζ is constant on each fiber of p, the trivialization ζ of $p^*\Xi$ descends to a trivialization of the line bundle Ξ over $\mathrm{Pic}^d(X)$, and hence giving a nowhere vanishing section of Ξ over $\mathrm{Pic}^d(X)$. Thus, Ξ is a trivial line bundle over $\mathrm{Pic}^d(X)$. It remains to show that p^* is surjective.

Let $\Theta \to \mathscr{M}_X$ be an algebraic line bundle. Since $\mathscr{M}_X \hookrightarrow \mathbf{P}(E)$ follows from (9), in the proof of above Proposition 5, we can extend Θ to a line bundle Θ' over $\mathbf{P}(E)$. Further, from the morphism $\widetilde{\pi} : \mathbf{P}(E) \to \mathrm{Pic}^d(X)$ in (8) in the above Proposition 5 we have

$$\mathrm{Pic}(\mathbf{P}(E)) \cong \widetilde{\pi}^* \mathrm{Pic}(\mathrm{Pic}^d(X)) \oplus \mathbf{Z}\mathscr{O}_{\mathbf{P}(E)}(1). \tag{11}$$

Therefore,

$$\Theta' = \widetilde{\pi}^* L \otimes \mathscr{O}_{\mathbf{P}(E)}(l) \tag{12}$$

where L is a line bundle over $\mathrm{Pic}^d(X)$ and $l \in \mathbf{Z}$. Since $Y = \mathbf{P}(E) \setminus \mathscr{M}_X$ is the hyperplane at infinity, again from (11) the line bundle $\mathscr{O}_{\mathbf{P}(E)}(Y)$ associated to the divisor Y can be expressed as

$$\mathscr{O}_{\mathbf{P}(E)}(Y) = \widetilde{\pi}^* L_1 \otimes \mathscr{O}_{\mathbf{P}(E)}(1) \tag{13}$$

for some line bundle L_1 over $\mathrm{Pic}^d(X)$. Now, from (12) and (13), we get

$$\Theta' = \widetilde{\pi}^*(L \otimes (L_1^\vee)^{\otimes l}) \otimes \mathcal{O}_{\mathbf{P}(E)}(lY). \tag{14}$$

Since, the restriction of the line bundle $\mathcal{O}_{\mathbf{P}(E)}(Y)$ to the compliment $\mathbf{P}(E) \setminus Y = \mathcal{M}_X$ is the trivial line bundle and restriction of $\widetilde{\pi}$ to $\mathcal{M}_X$ is the map p defined in (5), therefore, we have

$$\Theta = p^*(L \otimes (L_1^\vee)^{\otimes l}). \tag{15}$$

This completes the proof. □

Corollary 7. *The homomorphism $p^* : J(\mathrm{Pic}^d(X)) \to J(\mathcal{M}_X)$ is an isomorphism of Jacobians.*

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