INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

Paul-Emile Paradan

The Horn cone associated with symplectic eigenvalues

Volume 360 (2022), p. 1163-1168

https://doi.org/10.5802/crmath.383

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2022, Vol. 360, p. 1163-1168
https://doi.org/10.5802/crmath.383

Representation theory / *Théorie des représentations*

The Horn cone associated with symplectic eigenvalues

Paul-Emile Paradan[*, a]

[a] IMAG, Univ Montpellier, CNRS, France
E-mail: paul-emile.paradan@umontpellier.fr

Abstract. In this note, we show that the Horn cone associated with symplectic eigenvalues admits the same inequalities as the classical Horn cone, except that the equality corresponding to $\mathrm{Tr}(C) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$ is replaced by the inequality corresponding to $\mathrm{Tr}(C) \geq \mathrm{Tr}(A) + \mathrm{Tr}(B)$.

2020 Mathematics Subject Classification. 00X99.

Manuscript received 20 February 2022, revised 22 May 2022, accepted 24 May 2022.

1. Introduction

We consider $\mathbb{R}^{2n}$ equipped with its canonical symplectic structure $\Omega_n = \sum_{k=1}^n dx_k \wedge dx_{k+n}$. Recall that a family $(e_k)_{1\leq k\leq 2n}$ is a symplectic basis of $\mathbb{R}^{2n}$, if $\Omega_n(e_k, e_\ell) = 0$ if $|k-\ell| \neq n$ and $\Omega_n(e_k, e_{k+n}) = 1$, $\forall\, k$.

Williamson's theorem [18] says that any positive definite quadratic form $q : \mathbb{R}^{2n} \to \mathbb{R}$ can be written $q(v) = \sum_{k=1} \lambda_k (v_k^2 + v_{k+n}^2)$ where the (v_j) are the coordinates of the vector $v \in \mathbb{R}^{2n}$ relatively to a symplectic basis. The positive numbers λ_k, that one chooses so that

$$\lambda(q) := (\lambda_1 \geq \cdots \geq \lambda_n),$$

will be referred to as the *symplectic eigenvalues* of the quadratic form q. They correspond to the frequencies of the normal modes of oscillation for the linear Hamiltonian system generated by q.

The object of study of this note concerns the symplectic Horn cone, denoted $\mathrm{Horn}_{\mathrm{sp}}(n)$, that is defined as the set of triplets $(\lambda(q_1), \lambda(q_2), \lambda(q_1+q_2))$ where q_1, q_2 are positive definite quadratic forms on $\mathbb{R}^{2n}$.

Example 1. In dimension 2, the symplectic eigenvalue $\lambda(q)$ of a positive definite quadratic form $q(x_1, x_2) = ax_1^2 + bx_2^2 + cx_1x_2$ is equal to $\frac{1}{2}\sqrt{4ab - c^2}$. It is straightforward to show that $\mathrm{Horn}_{\mathrm{sp}}(1)$ is equal to the set of triplets (x, y, z) of positive numbers satisfying $x + y \leq z$.

[*] Corresponding author.

Our main Theorem states that $\mathrm{Horn}_{\mathrm{sp}}(n)$ is a convex polyhedral set. Before detailing it, let us recall some related results.

In [17], A. Weinstein showed that for non-increasing n-tuples of positive real numbers a and b, the set $\Delta_{\mathrm{sp}}(a,b) := \{\lambda(q_1+q_2)\,|\,\lambda(q_1)=a, \lambda(q_2)=b\}$ is closed, convex and locally polyhedral.

Recently, several authors have realized that some inequalities obtained long ago in the context of eigenvalues of Hermitian matrices still apply to symplectic eigenvalues:

- T. Hiroshima proved in [7] an analogue of Ky Fan inequalities:
$$\sum_{j=1}^{k} \lambda_j\left(q_1+q_2\right) \geq \sum_{j=1}^{k} \lambda_j\left(q_1\right) + \sum_{j=1}^{k} \lambda_j\left(q_2\right).$$
- In [8], T. Jain and H. Mishra obtained an analogue of Lidskii inequalities:
$$\sum_{j=1}^{k} \lambda_{i_j}\left(q_1+q_2\right) \geq \sum_{j=1}^{k} \lambda_{i_j}\left(q_1\right) + \sum_{j=1}^{k} \lambda_j\left(q_2\right)$$
for any subset $\{i_1 < i_2 < \cdots < i_k\}$.
- In [2], R. Bhatia and T. Jain obtained an analogue of the Weyl inequalities:
$$\lambda_{i+j-1}\left(q_1+q_2\right) \geq \lambda_i\left(q_1\right) + \lambda_j\left(q_2\right).$$

As the previous results suggest, we now explain the strong relationship between $\mathrm{Horn}_{\mathrm{sp}}(n)$ with the classical Horn cone. If A is a Hermitian $n\times n$ matrix, we denote by $\mathrm{s}(A) = (\mathrm{s}_1(A) \geq \cdots \geq \mathrm{s}_n(A))$ its spectrum. The Horn cone $\mathrm{Horn}(n)$ is defined as the set of triplets $(\mathrm{s}(A), \mathrm{s}(B), \mathrm{s}(A+B))$ where A, B are Hermitian $n \times n$ matrices.

Denote the set of cardinality r-subsets $I = \{i_1 < i_2 < \cdots < i_r\}$ of $[n] := \{1,\dots,n\}$ by $\mathscr{P}^n_r$. To each $I \in \mathscr{P}^n_r$ we associate:

- a weakly decreasing sequence of non-negative integers $\lambda(I) = (\lambda_1 \geq \cdots \geq \lambda_r)$ where $\lambda_a = n - r + a - i_a$ for $a \in [r]$.
- the irreducible representation $V_{\lambda(I)}$ of $GL_r(\mathbb{C})$ with highest weight $\lambda(I)$.

If $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ and $I \subset [n]$, we define $|x|_I = \sum_{i\in I} x_i$ and $|x| = \sum_{i=1}^n x_i$. Let us denote by $\mathbb{R}^n_+$ the set of weakly decreasing n-tuples of real numbers.

A. Klyachko [10] has shown that an element $(x,y,z) \in (\mathbb{R}^n_+)^3$ belongs to the cone $\mathrm{Horn}(n)$ if and only if it satisfies $|x|+|y| = |z|$ and
$$|x|_I + |y|_J \leq |z|_K \qquad ((\star)_{I,J,K})$$
for any $r < n$, for any $I, J, K \in \mathscr{P}^n_r$ such that the Littlewood-Richardson coefficient
$$c^K_{IJ} := \dim\left[V_{\lambda(I)} \otimes V_{\lambda(J)} \otimes V^*_{\lambda(K)}\right]^{GL_r(\mathbb{C})}$$
is non-zero. P. Belkale [1] showed that the inequalities $(\star)_{I,J,K}$ associated to the condition $c^K_{IJ} = 1$ are sufficient. Finally A. Knutson, T. Tao, and C. Woodward [11] have proved that this smaller list is actually minimal. We refer the reader to survey articles [3, 5] for details.

The main result of this note is the following Theorem. Let us denote by $\mathbb{R}^n_{++}$ the set of non-increasing n-tuples of positive real numbers.

Theorem 2. *An element $(x,y,z) \in (\mathbb{R}^n_{++})^3$ belongs to* $\mathrm{Horn}_{\mathrm{sp}}(n)$ *if and only if it satisfies*

(1) $|x| + |y| \leq |z|$,
(2) $(\star)_{I,J,K}$ *for all (I,J,K) of cardinality $r < n$ such that $c^K_{IJ} = 1$.*

Corollary 3. *Let $a, b \in \mathbb{R}^n_{++}$. An element $z \in \mathbb{R}^n_{++}$ belongs to $\Delta_{\mathrm{sp}}(a,b)$ if and only if it satisfies $|a|+|b| \leq |z|$ and $|a|_I + |b|_J \leq |z|_K$ for all (I,J,K) of cardinality $r < n$ such that $c^K_{IJ} = 1$.*

2. The causal cone of the symplectic Lie algebra

The $2n \times 2n$ matrix $J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ defines a complex structure on $\mathbb{R}^{2n}$ that is compatible with the symplectic structure Ω_n. The symplectic group $Sp(\mathbb{R}^{2n})$ is defined by the relation ${}^t g J_n g = J_n$. A matrix X belongs to the Lie algebra $\mathfrak{sp}(\mathbb{R}^{2n})$ of $Sp(\mathbb{R}^{2n})$ if and only the matrix $J_n X$ is symmetric. Moreover, $J_n X$ is positive if and only if $\Omega_n(Xv, v) \geq 0$, $\forall\ v \in \mathbb{R}^{2n}$.

We call an invariant convex cone C in $\mathfrak{sp}(\mathbb{R}^{2n})$ a causal cone if C is nontrivial, closed, and satisfies $C \cap -C = \{0\}$. A classical result [13, 14, 16] asserts that there are exactly two causal cones in $\mathfrak{sp}(\mathbb{R}^{2n})$: one, denoted by $\mathbf{C}(n)$, containing $-J_n$ and its opposite $-\mathbf{C}(n)$. The causal cone $\mathbf{C}(n)$ is determined by the following equivalent conditions : for $X \in \mathfrak{sp}(\mathbb{R}^{2n})$, we have

$$X \in \mathbf{C}(n) \iff J_n X \text{ is positive} \iff \operatorname{Tr}\left(X g J_n g^{-1}\right) \geq 0,\ \forall\ g \in Sp\left(\mathbb{R}^{2n}\right).$$

Now we explain how is parameterized the interior $\mathbf{C}(n)^0$ of $\mathbf{C}(n)$. From the definition above, we see first that $X \in \mathbf{C}(n)^0$ if and only if $J_n X$ is positive definite.

The Lie algebra of the maximal compact subgroup $K = Sp(2n, \mathbb{R}) \cap O(2n)$ is

$$\mathfrak{k} := \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix},\ {}^t A = -A,\ {}^t B = B \right\}.$$

If $\mu := (\mu_1, \cdots, \mu_n)$, we write $\Delta(\mu) = \operatorname{Diag}(\mu_1, \cdots, \mu_n)$ and $X(\mu) = \begin{pmatrix} 0 & \Delta(\mu) \\ -\Delta(\mu) & 0 \end{pmatrix}$. We work with the Cartan subalgebra $\mathfrak{t} := \{X(\mu), \mu \in \mathbb{R}^n\}$ of $\mathfrak{k}$ and the corresponding maximal torus $T \subset K$. The set of roots $\mathfrak{R}$ relatively to the action of T on $\mathfrak{sp}(\mathbb{R}^{2n}) \otimes \mathbb{C}$ are composed by the compact ones $\mathfrak{R}_c := \{\epsilon_i - \epsilon_j\}$ and the non compact ones $\mathfrak{R}_n = \{\pm(\epsilon_i + \epsilon_j)\}$. We work with the subsets of positive roots $\mathfrak{R}_c^+ := \{\epsilon_i - \epsilon_j, i < j\}$ and $\mathfrak{R}_n^+ := \{\epsilon_i + \epsilon_j\}$. The Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{t}$ is defined by the relations $\langle \alpha, \mu \rangle \geq 0$, $\forall\ \alpha \in \mathfrak{R}_c^+$, namely $\mu_1 \geq \cdots \geq \mu_n$. The subchamber $\mathscr{C}_n \subset \mathfrak{t}_+$ is defined by the conditions $\langle \beta, \mu \rangle > 0$, $\forall\ \beta \in \mathfrak{R}_n^+$. Thus $X(\mu) \in \mathscr{C}_n$ if and only if $\mu \in \mathbb{R}^n_{++}$.

If $M \in \mathfrak{sp}(\mathbb{R}^{2n})$, we denote by $\mathscr{O}_M := \{gMg^{-1}, g \in Sp(\mathbb{R}^{2n})\}$ the corresponding adjoint orbit.

Lemma 4.

(1) *$M \in \mathbf{C}(n)^0$ if and only if there exists $X \in \mathscr{C}_n$ such that $M \in \mathscr{O}_X$.*

(2) *Let $\mu \in \mathbb{R}^n_{++}$, and $M \in \mathscr{O}_{X(\mu)}$. The symplectic eigenvalues of the positive definite quadratic form $q(v) = {}^t v J_n M v = \Omega_n(Mv, v)$ are the numbers $\mu_1 \geq \cdots \geq \mu_n > 0$.*

Proof. The first point is a classical fact [14, 16]. If $M = gX(\mu)g^{-1}$ with $g \in Sp(\mathbb{R}^{2n})$, we see that

$$\Omega_n(Mv, v) = \Omega_n\left(X(\mu)g^{-1}v, g^{-1}v\right) = \sum_{k=1}^{n} \mu_k \left(v_k^2 + v_{k+n}^2\right)$$

where each v_j is the j^{th} coordinate of the vector $g^{-1}v$. □

Remark 5. In [15], we call the interior $\mathbf{C}(n)^0$ of $\mathbf{C}(n)$ the holomorphic cone, since any coadjoint orbit $\mathscr{O}_X \subset \mathbf{C}(n)^0$ admits a canonical structure of a Kähler manifold with a holomorphic action of K. These orbits are closely related to the holomorphic discrete series representations of the symplectic group $Sp(\mathbb{R}^{2n})$.

Thanks to the previous Lemma 4, we see that the symplectic Horn cone admits the alternative definition:

$$\operatorname{Horn}_{\mathrm{sp}}(n) = \left\{(x, y, z) \in \left(\mathbb{R}^n_{++}\right)^3 \,\middle|\, \mathscr{O}_{X(z)} \subset \mathscr{O}_{X(x)} + \mathscr{O}_{X(y)}\right\}.$$

In the next section, we explain the result of [15] concerning the determination of $\operatorname{Horn}_{\mathrm{sp}}(n)$.

3. Convexity results

The trace on $\mathfrak{gl}(\mathbb{R}^{2n})$ provides an identification between $\mathfrak{sp}(\mathbb{R}^{2n})$ and its dual $\mathfrak{sp}(\mathbb{R}^{2n})^*$: to $X \in \mathfrak{sp}(\mathbb{R}^{2n})$ we associate $\xi_X \in \mathfrak{sp}(\mathbb{R}^{2n})^*$ defined by $\langle \xi_X, Y\rangle = -\mathrm{Tr}(XY)$. Through this identification the causal cone $\mathbf{C}(n)$ becomes

$$\widetilde{\mathbf{C}}(n) := \left\{\xi \in \mathfrak{sp}\left(\mathbb{R}^{2n}\right)^* ; \langle \xi, \mathrm{Ad}(g)z\rangle \geq 0,\ \forall\, g \in Sp\left(\mathbb{R}^{2n}\right)\right\}$$

where $z = \frac{-1}{2}J_n$. The identification $\mathfrak{sp}(\mathbb{R}^{2n}) \simeq \mathfrak{sp}(\mathbb{R}^{2n})^*$ induces several identifications $\mathfrak{k} \simeq \mathfrak{k}^*$, $\mathfrak{t} \simeq \mathfrak{t}^*$ and $\mathfrak{t}_+ \simeq \mathfrak{t}_+^*$. In the latter cases the identifications are done through an invariant scalar product $(-,-)$ on $\mathfrak{k}^*$.The subchamber $\widetilde{\mathscr{C}}_n \subset \mathfrak{t}_+^*$ is defined by the conditions: $(\alpha, \xi) \geq 0$, $\forall\ \alpha \in \mathfrak{R}_c^+$, and $(\beta, \xi) > 0$, $\forall\ \beta \in \mathfrak{R}_n^+$.

Through $\mathfrak{sp}(\mathbb{R}^{2n}) \simeq \mathfrak{sp}(\mathbb{R}^{2n})^*$, the symplectic Horn cone becomes

$$\mathrm{Horn}_{\mathrm{hol}}\left(Sp\left(\mathbb{R}^{2n}\right)\right) := \left\{(\xi_1, \xi_2, \xi_3) \in \left(\widetilde{\mathscr{C}}_n\right)^3 \,\middle|\, \mathscr{O}_{\xi_3} \subset \mathscr{O}_{\xi_1} + \mathscr{O}_{\xi_2}\right\}.$$

Here we have kept the notations of [15].

We have a Cartan decomposition $\mathfrak{sp}(\mathbb{R}^{2n}) = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\mathfrak{p} := \left\{\begin{pmatrix} A & B \\ B & -A\end{pmatrix},\ {}^tA = A,\ {}^tB = B\right\}.$$

We denote by $\mathfrak{p}^+$ the vector space $\mathfrak{p}$ equipped with the complex structure $\mathrm{ad}(z)$ and the compatible symplectic structure $\Omega_{\mathfrak{p}^+}(Y, Y') := -\mathrm{Tr}(J_n[Y, Y'])$: here $\Omega_{\mathfrak{p}^+}(Y, [z, Y]) > 0$ for any $Y \neq 0$.

The action of the maximal compact subgroup $K \subset Sp(\mathbb{R}^{2n})$ on $(\mathfrak{p}^+, \Omega_{\mathfrak{p}^+})$ is Hamiltonian with moment map

$$\Phi_{\mathfrak{p}^+} : \mathfrak{p}^+ \to \mathfrak{k}^*$$

defined by $\langle \Phi_{\mathfrak{p}^+}(Y), X\rangle = \frac{1}{2}\Omega_{\mathfrak{p}^+}([X, Y], Y)$. If $Y = \left(\begin{smallmatrix} A & B \\ B & -A\end{smallmatrix}\right)$, we see that $\langle \Phi_{\mathfrak{p}^+}(Y), J_n\rangle = \mathrm{Tr}(A^2 + B^2) = \frac{1}{2}\|Y\|^2$. Hence the moment map $\Phi_{\mathfrak{p}^+}$ is a proper map.

We consider the following action of the group K^3 on the manifold $K \times K$:

$$(k_1, k_2, k_3) \cdot (g, h) = \left(k_1 g k_3^{-1}, k_2 h k_3^{-1}\right).$$

The action of K^3 on the cotangent bundle $N := T^*(K \times K)$ is Hamiltonian with moment map $\Phi_N : N \to \mathfrak{k}^* \times \mathfrak{k}^* \times \mathfrak{k}^*$ defined by the relations[1]

$$\Phi_N\left(g_1, \eta_1; g_2, \eta_2\right) = \left(-g_1\eta_1, -g_2\eta_2, \eta_1 + \eta_2\right).$$

Finally we consider the Hamiltonian K^3-manifold $N \times \mathfrak{p}^+$, where $\mathfrak{p}^+$ is equipped with the symplectic structure $\Omega_{\mathfrak{p}^+}$. The action is defined by the relations: $(k_1, k_2, k_3) \cdot (g, h, X) = (k_1 g k_3^{-1}, k_2 h k_3^{-1}, k_3 X)$. Let us denote by $\Phi : N \times \mathfrak{p}^+ \to \mathfrak{k}^* \times \mathfrak{k}^* \times \mathfrak{k}^*$ the moment map relative to the K^3-action:

$$\Phi\left(g_1, \eta_1; g_2, \eta_2, Y\right) = \left(-g_1\eta_1, -g_2\eta_2, \eta_1 + \eta_2 + \Phi_{\mathfrak{p}^+}(Y)\right). \tag{1}$$

Since Φ is proper map, the Convexity Theorem [9, 12] tell us that

$$\Delta\left(N \times \mathfrak{p}^+\right) := \mathrm{Image}(\Phi) \bigcap \mathfrak{t}_+^* \times \mathfrak{t}_+^* \times \mathfrak{t}_+^*$$

is a closed, convex, and locally polyhedral set.

The map $\mu \mapsto X(\mu)$ defines an isomorphism of $\mathbb{R}^n$ with $\mathfrak{t} \simeq \mathfrak{t}^*$ that induces an identification of $\mathbb{R}^n_{++}$ with $\mathscr{C}_n \simeq \widetilde{\mathscr{C}}_n$. Recall that on $\mathfrak{t}^* \simeq \mathbb{R}^n$, we have a natural involution that sends $\mu = (\mu_1, \ldots, \mu_n)$ to $\mu^* := (-\mu_n, \ldots, -\mu_1)$. The following result is proved in [15, Theorem B].

Theorem 6. *An element* $(x, y, z) \in (\mathbb{R}^n_{++})^3$ *belongs to* $\mathrm{Horn}_{\mathrm{hol}}(Sp(\mathbb{R}^{2n}))$ *if and only if*

$$\left(x, y, z^*\right) \in \Delta\left(N \times \mathfrak{p}^+\right).$$

[1] We use the identification $T^*K \simeq K \times \mathfrak{k}^*$ given by left translations.

Recall that a Hermitian matrix M majorizes another Hermitian matrix M if $M - M'$ is positive semidefinite (its eigenvalues are all nonnegative). In this case, we write $M \geq M'$.

Proposition 7. *Let $(x, y, z) \in (\mathbb{R}^n_+)^3$. Then $(x, y, z^*) \in \Delta(N \times \mathfrak{p}^+)$ if and only if there exist Hermitian matrices A, B, C such that $\mathrm{s}(A) = x$, $\mathrm{s}(B) = y$, $\mathrm{s}(C) = z$ and $C \geq A + B$.*

Proof. The map $\left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right) \mapsto A - iB$ defines an isomorphism between K and the unitary group $U(n)$. Let us denote by $S^2(\mathbb{C}^n)$ the vector space of complex $n \times n$ symmetric matrices that is equipped with the following action of $U(n)$: $k \cdot M = kM^t k$. The map $\left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right) \mapsto A - iB$ defines an isomorphism between the K-module $\mathfrak{p}^+$ and the $U(n)$-module $S^2(\mathbb{C}^n)$. Through this identifications the moment map $\Phi_{\mathfrak{p}^+} : \mathfrak{p}^+ \to \mathfrak{k}^*$ becomes the map $\Phi_{S^2} : S^2(\mathbb{C}^n) \to \mathfrak{u}(n)$ defined by the relations

$$\Phi_{S^2}(M) = -2iM\overline{M}.$$

So we know that the moment polytope Δ relative to the Hamiltonian action of $U(n)^3$ on $T^*U(n) \times T^*U(n) \times S^2(\mathbb{C}^n)$ is equal to $\Delta(N \times \mathfrak{p}^+)$. A small computation shows that $(x, y, z^*) \in \Delta$ if and only if there exist Hermitian matrices A, B, C and $M \in S^2(\mathbb{C}^n)$ such that

$$\mathrm{s}(A) = x, \quad \mathrm{s}(B) = y, \quad \mathrm{s}(C) = z \quad \text{and} \quad A + B + 2M\overline{M} = C.$$

The existence of $M \in S^2(\mathbb{C}^n)$ satisfying the condition $A + B + 2M\overline{M} = C$ is equivalent to $C \geq A + B$. The proof is then completed. □

S. Friedland [4] considered the following question: *which eigenvalues* $(\mathrm{s}(A), \mathrm{s}(B), \mathrm{s}(C))$ *can occur if* $C \geq A + B$. His solution was in terms of linear inequalities, which includes Klyachko's inequalities, a trace inequality and some additional inequalities. Later, W. Fulton [6] proved the additional inequalities are unnecessary. Let us summarizes their result in the following theorem.

Theorem 8 ([4, 6]). *A triple $x, y, z \in \mathbb{R}^n_+$ occurs as the eigenvalues of n by n Hermitian matrices A, B, C with $C \geq A + B$ if and only it satisfies $|x| + |y| \leq |z|$ and $(\star)_{I,J,K}$ for all (I, J, K) of cardinality $r < n$ such that $c^K_{IJ} = 1$.*

The combination of Theorems 6 and 8 with Proposition 7 completes the proof of Theorem 2.

References

[1] P. Belkale, "Local systems on $\mathbb{P} - S$ for S a finite set", *Compos. Math.* **129** (2001), no. 1, p. 67-86.

[2] R. Bhatia, T. Jain, "Variational principles for symplectic eigenvalues", *Can. Math. Bull.* **64** (2021), no. 3, p. 553-559.

[3] M. Brion, "Restrictions de representations et projections d'orbites coadjointes (d'après Belkale, Kumar et Ressayre)", 2011, Séminaire Bourbaki, http://www.bourbaki.ens.fr/TEXTES/1043.pdf.

[4] S. Friedland, "Finite and infinite dimensional generalizations of Klyachko's theorem", *Linear Algebra Appl.* **319** (2000), no. 1-3, p. 3-22.

[5] W. Fulton, "Eigenvalues, invariant factors, highest weights, and Schubert calculus", *Bull. Am. Math. Soc.* **37** (2000), no. 3, p. 209-249.

[6] ———, "Eigenvalues of majorized Hermitian matrices and Littlewood-Richardson coefficients", *Linear Algebra Appl.* **319** (2000), no. 1-3, p. 23-36.

[7] T. Hiroshima, "Additivity and multiplicativity properties of some Gaussian channels for Gaussian inputs", *Phys. Rev. A* **73** (2006), no. 1, article no. 012330 (9 pages).

[8] T. Jain, H. K. Mishra, "Derivatives of symplectic eigenvalues and a Lidskii type theorem", *Can. J. Math.* **74** (2020), no. 2, p. 457-485.

[9] F. Kirwan, "Convexity properties of the moment mapping III", *Invent. Math.* **77** (1984), p. 547-552.

[10] A. A. Klyachko, "Stable bundles, representation theory and Hermitian operators", *Sel. Math., New Ser.* **4** (1998), no. 3, p. 419-445.

[11] A. Knutson, T. Tao, C. Woodward, "The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determine facets of the Littlewood-Richardson cone", *J. Am. Math. Soc.* **17** (2004), no. 1, p. 19-48.

[12] E. Lerman, E. Meinrenken, S. Tolman, C. Woodward, "Non-Abelian convexity by symplectic cuts", *Topology* **37** (1998), no. 2, p. 245-259.

[13] S. M. Paneitz, "Invariant convex cones and causality in semisimple Lie algebras and groups", *J. Funct. Anal* **43** (1981), p. 313-359.

[14] ———, "Determination of invariant convex cones in simple Lie algebras", *Ark. Mat.* **21** (1983), p. 217-228.

[15] P.-E. Paradan, "Horn problem for quasi-hermitian Lie groups", *J. Inst. Math. Jussieu* (2022), p. 1-27.

[16] È. B. Vinberg, "Invariant convex cones and orderings in Lie groups", *Funct. Anal. Appl.* **14** (1980), p. 1-10.

[17] A. Weinstein, "Poisson geometry of discrete series orbits and momentum convexity for noncompact group actions", *Lett. Math. Phys.* **56** (2001), no. 1, p. 17-30.

[18] J. Williamson, "On the algebraic problem concerning the normal forms of linear dynamical systems", *Am. J. Math.* **58** (1936), p. 141-163.