INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Oliver Thomas

Duality for K-analytic Group Cohomology of p-adic Lie Groups

Volume 360 (2022), p. 1213-1226

Published online: 8 December 2022

https://doi.org/10.5802/crmath.373

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2022, Vol. 360, p. 1213-1226
https://doi.org/10.5802/crmath.373

Number theory, Representation theory / *Théorie des nombres, Théorie des représentations*

Duality for K-analytic Group Cohomology of p-adic Lie Groups

Oliver Thomas [a]

[a] Department of Mathematics, University of Bahrain, P.O. Box 32038, Sukhair, Bahrain
E-mail: othomas@uob.edu.bh

Abstract. We prove a duality result for the analytic cohomology of Lie groups over non-archimedean fields acting on locally convex vector spaces by combining Tamme's non-archimedean van Est comparison morphism with Hazewinkel's duality result for Lie algebra cohomology.

Keywords. analytic cohomology, duality.

2020 Mathematics Subject Classification. 22E50, 46S10, 11S31.

Manuscript received 6 September 2021, revised 25 April 2022, accepted 29 April 2022.

Introduction

Let K be a non-archimedean complete field and G a K-analytic group, i.e., a group object in the category of K-analytic manifolds. Let furthermore V be a K-analytic representation of V and $C^\bullet(G,V)$ the complex of K-analytic inhomogeneous cochains of G with coefficients in V. Its cohomology is called the K-analytic cohomology of G with coefficients in V. If $K = \mathbb{Q}_p$ then Lazard showed that this is just continuous cohomology, cf. [10, V.(2.3.10)]. But if $K \neq \mathbb{Q}_p$, then this is no longer the case and the K-analytic cohomology differs from the $\mathbb{Q}_p$-analytic cohomology.

Assume for a moment that V is of finite dimension over K. Then for $d = \dim G$ we show the existence of a quasi-isomorphism

$$C^\bullet(G,V') \xrightarrow{\cong} C^\bullet(G,V)'[-d],$$

where $(-)' = \mathrm{Hom}_K(-,K)$. If V is not of finite dimension, then the functional analysis regrettably gets more complicated: We still get a morphism

$$C^\bullet(G,V'_b) \longrightarrow \mathrm{Hom}_K(C^\bullet(G,V),K)[-d]$$

where V'_b denotes the strong dual of V, but we need additional requirements for this morphism to be a quasi-isomorphism. For example, the Hahn–Banach theorem only holds for certain subclasses of non-archimedean fields, so taking the continuous dual is not always an exact functor. The precise statement of our main Theorem 53 includes more assumptions for these kinds of reasons, which we will explain in the first few sections.

Strategically, the proof of the duality result is charmingly straight-forward: Using [14], we compare analytic cohomology with Lie algebra cohomology. Hazewinkel (cf. [7]) showed a duality result for Lie algebra cohomology and plucking both results together then yields the result.

Technically things are more complicated. The van Est comparison between analytic cohomology and Lie algebra cohomology only yields an isomorphism of cohomology groups when the underlying group is sufficiently connected, something that of course isn't the case for p-adic groups. Showing that the duality of the Lie algebra cohomology correctly identifies the subspaces stemming from analytic cohomology is one issue, taking care of multiple topological subtleties not present in the archimedean world another.

Acknowledgements

This article is based upon parts of its author's PhD thesis (cf. [15]). We thank Otmar Venjakob for valuable discussions and insights and the referee for helpful suggestions.

1. Some Functional Analysis

We want to briefly recall some notions of non-archimedean functional analysis. We refer the reader to [2, 6, 12] for details. An excellent overview can also be found in [5]. In this section, we fix a complete non-archimedean field K with valuation ring $\mathcal{O}_K$.

1.1. *Foundations*

Definition 1. *K is called spherically complete if every decreasing sequence of closed balls has a non-empty intersection.*

Example 2. Every locally compact field is spherically complete. $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$, is not spherically complete.

Definition 3. *A lattice L in a K-vector space V is an $\mathcal{O}_K$-submodule of V, which satisfies*

$$V = \bigcup_{\lambda \in K} \lambda L.$$

Definition 4. *We call a topological K-vector space* locally convex *(or an LCVS), if it has a neighbourhood basis of lattices.*

Remark 5. Note that a subset M of a K-vector space is an $\mathcal{O}_K$-module if and only if for all $m, m' \in M$ and all λ, μ with $|\lambda|, |\mu| \leq 1$ also $\lambda m + \mu m' \in M$. This is the analogy to the usual notion of convexity. Requiring $\lambda m + (1-\lambda)m' \in M$ regrettably does not suffice.

Remark 6. Let V be a K-vector space. For every lattice L in V, there is an attached seminorm p_L defined by

$$p_L(v) = \inf_{\lambda \in K, a \in \lambda L} |\lambda|.$$

Conversely, for a seminorm $p\colon V \to \mathbb{R}$ and $\varepsilon > 0$ we can define a lattice

$$V_p(\varepsilon) = \{v \in V \mid p(v) < \varepsilon\}.$$

These constructions are inverse to one another in the following sense: For a family of seminorms $(p_i)_i$, the coarsest topology on V such that all p_i are continuous is the locally convex topology generated by the lattices $(V_{p_i}(\varepsilon))_{i,\varepsilon}$. Conversely, if V is locally convex, the topology on V is the coarsest topology, such that all $(p_L)_L$ are continuous, where L ranges over the open lattices in V. We refer to [12, Section I.4] for details.

Definition 7. *A subset B of an LCVS V is called bounded if for any open lattice L in V there is a $\lambda \in K$ such that $B \subset \lambda L$.*

Proposition 8. *Every quasi-compact subset C of an LCVS V is bounded.*

Proof. Let L be an open lattice. By assumption, $V = \bigcup_{\lambda \in K} \lambda L$, so finitely many $\lambda_1 L, \ldots, \lambda_n L$ cover C. We can assume that none of the λ_i lie in $\mathscr{O}_K$. Then $C \subseteq \lambda_1 \cdots \lambda_n L$. □

Remark 9. If K is not locally compact, an LCVS over K does not have non-trivial compact $\mathscr{O}_K$-submodules.

Definition 10. *Let V be an LCVS. We call V bornological if a K-linear map $V \to W$ of LCVS is continuous if and only if it respects bounded subsets. V is called barrelled if every closed lattice is open.*

1.2. *Dual spaces*

Definition 11. *Let V, W be LCVS. We denote the set of continuous K-linear maps from V to W by $\mathscr{L}(V,W)$. For bounded subsets $B \subseteq V$ and open subsets $U \subseteq W$ we denote by $L(B,U) \subseteq \mathscr{L}(V,W)$ those continuous linear maps which map B into U. The families*

$$\{L(S,U) \mid S \subseteq V \text{ a single point, } U \subseteq W \text{ open}\}$$
$$\{L(C,U) \mid C \subseteq V \text{ compact, } U \subseteq W \text{ open}\}$$
$$\{L(B,U) \mid B \subseteq V \text{ bounded, } U \subseteq W \text{ open}\}$$

generate locally convex topologies on the space $\mathscr{L}(V,W)$ of continuous linear maps from V to W, which are called the weak, compact-open, *and* strong *topology respectively. The corresponding LCVS will be denoted by $\mathscr{L}_s(V,W)$, $\mathscr{L}_c(V,W)$, and $\mathscr{L}_b(V,W)$.*

Remark 12. The weak topology is coarser than the compact-open topology, which in turn is coarser than the strong topology.

Remark 13. Denote by $\mathbf{T}$ the category of Hausdorff topological spaces. (A variant of this remark also holds in the non-Hausdorff case.) For topological spaces X, Y we denote by $[X,Y]$ the set $\mathrm{Hom}_{\mathbf{T}}(X,Y)$ endowed with the compact-open topology. It is an easy exercise to check that for topological spaces X, Y, Z there is a well-defined map

$$\mathrm{Hom}_{\mathbf{T}}(X \times Y, Z) \longrightarrow \mathrm{Hom}_{\mathbf{T}}(X, [Y,Z])$$

sending f to

$$x \longmapsto \big(y \longmapsto f(x,y)\big).$$

However, the obvious candidate for an inverse

$$\mathrm{Hom}_{\mathbf{T}}(X,[Y,Z]) \longrightarrow \mathrm{Hom}_{\mathbf{Set}}(X \times Y, Z),$$

sending f to

$$(x,y) \longmapsto f(x)(y),$$

in general does not yield continuous maps! Formally speaking, not every topological space is exponentiable. In our setting, we would have a bijection if Y was locally compact, and locally compact spaces are the largest class for which this holds for all spaces X and Z. As LCVS are only locally compact if they are finite dimensional, we cannot use the adjointness properties of the compact-open topology. In fact, there is mostly no reason to look at the compact-open topology at all. Considering linear maps, the strong topology plays the same role, but better.

Proposition 14 (Hahn–Banach). *Let K be spherically complete, V a LCVS and W a linear subspace of V endowed with the subspace topology. Then every continuous linear map $W \to K$ extends to a continuous linear map $V \to K$.*

Proof. [12, Proposition 9.2, corollary 9.4]. □

There is also the following version of the Hahn–Banach theorem for LCVS of countable type.

Definition 15. *An LCVS V is said to be of countable type if for every continuous seminorm p on V its completion V_p at p has a dense subspace of countable algebraic dimension.*

Proposition 16. *Let V be an LCVS of countable type and W a sub-vector space endowed with the subspace topology. Then every continuous linear map $W \to K$ extends to a continuous linear map $V \to K$.*

Proof. [11, Corollary 4.2.6] □

Definition 17. *We say that Hahn–Banach holds for an LCVS V if K is spherically complete or V is of countable type.*

Remark 18. Spaces of countable type are stable under forming subspaces, linear images, projective limits, and countable inductive limits, cf. [11, Theorem 4.2.13].

Proposition 19 (Banach–Steinhaus). *Let V, W be LCVS. If V is barrelled, then every bounded subset $H \subseteq \mathscr{L}_s(V, W)$ is equicontinuous, i.e., for every open lattice $L' \subseteq W$ there exists an open lattice $L \subseteq V$ such that $f(L) \subseteq L'$ for every $f \in H$.*

Proof. [12, Proposition 6.15] □

Proposition 20. *Let G be a locally compact topological group and V a barrelled LCVS. Assume that G acts via linear maps on V. Then*

$$G \times V \longrightarrow V$$

is continuous if and only if it is separately continuous.

Proof. It is clear that a continuous group action is separately continuous.

Let $U \subseteq V$ be an open lattice and $g \in G, v \in V, gv \in U$. Let H be a compact neighbourhood of g with $Hv \subset U$, which exists by local compactness of G and separate continuity of the group action.

Consider the set $M = \{h\cdot - \mid h \in H\}$ of continuous linear maps $V \to V$. We want to show that it is bounded in the topology of pointwise convergence on $\mathrm{Hom}_{\mathrm{cts}}(V, V)$. For this matter, take $w \in V$, $S \subseteq V$ an open lattice, and denote by L those continuous linear maps $V \to V$ which map w into S. We need to show that there exists $\lambda \in K$ with $M \subseteq \lambda L$. As H is compact, $Hw \subseteq V$ is quasicompact, hence there exists $\lambda \in K$ such that $Hw \in \lambda^{-1}S$, i.e., $M \subseteq \lambda L$.

Proposition 19 now shows the existence of an open lattice L' such that $HL' \subseteq U$, or in other words, $H \times L' \subseteq \mathrm{mult}^{-1}(U)$. □

Definition 21. *The dual space of an LCVS V is the vector space of continuous K-linear functions $V \to K$ and will be denoted by V'.*

We denote by $V'_s = \mathscr{L}_s(V, K)$ the dual space equipped with the weak topology, which is the topology of pointwise convergence.

$V'_c = \mathscr{L}_c(V, K)$ will denote the dual space equipped with the compact-open topology, which is also the topology of uniform convergence on compact subsets.

The strong dual will be denoted by $V'_b = \mathscr{L}_b(V, K)$ and is defined as the topology of uniform convergence on bounded subsets of V.

Remark 22. Note that for both the weak and strong duals, the dual of a direct sum of LCVS is the product of its duals. However, only for the strong dual is the dual of a product of LCVS the sum of its duals (cf. [12, Propositions 9.10, 9.11]).

Note that by [12, Lemma 6.4], V'_c can be defined as the coarsest topology on V' such that for every quasi-compact $K \subseteq V$ the map

$$\begin{aligned} V' &\longrightarrow \mathbb{R} \\ v' &\longmapsto \sup_{v\in C} |v'(v)|_K \end{aligned}$$

is continuous.

1.3. *Strictness*

Definition 23. *A linear map* $V \to W$ *of LCVS is called* strict *if the induced map*

$$V/\ker f \longrightarrow \operatorname{im} f$$

with the quotient topology on $V/\ker f$ *and the subspace topology on* $\operatorname{im} f$ *is an isomorphism.*

Remark 24. Open linear maps are clearly strict, but strictness is remarkably bad behaved in general: Neither the sum nor the composition of strict maps needs to be strict again.

Definition 25. *A sequence of LCVS*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is called exact if it is exact as a sequence of vector spaces and if the involved maps are all strict.

Proposition 26. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence of LCVS. If Hahn–Banach holds for B*, then the induced sequence of abelian groups*

$$0 \longrightarrow C' \longrightarrow B' \longrightarrow A' \longrightarrow 0$$

is also exact.

Proof. Let

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0$$

be exact. It is clear that

$$0 \longrightarrow C' \xrightarrow{\pi^*} B'$$

is exact. Let $f\colon B \to K$ be in the kernel of ι^*, i.e., $\iota A \subseteq \ker f$. This induces a map $B/\iota A \to K$, which by strictness is a map $C \to K$.

It remains to show surjectivity of ι^*, i.e., the existence of a map $\tilde{f}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \iota A & \xrightarrow{\cong} & A & \xrightarrow{f} & K \\ \cap\!\!| & & & \nearrow_{\tilde f} & \\ B & & & & \end{array}$$

But this extension exists by Proposition 14 or Proposition 16. □

Definition 27. *A Fréchet space is an LCVS which is isomorphic to the countable projective limit of Banach spaces.*

Remark 28. A space is Fréchet if and only if it is a complete LCVS whose topology is induced by a translation-invariant metric if and only if it is a Hausdorff topological K-vector space whose topology is induced by a countable family of semi-norms for which every Cauchy sequence converges.

Proposition 29 (Open-mapping theorem). *Let $f: V \to W$ be a continuous surjective linear map from a Fréchet space to a barrelled Hausdorff LCVS. Then f is open.*

Proof. [12, Proposition 8.6] □

Definition 30. *An LCVS is called an LF-space if it is the direct limit of a countable family of Fréchet spaces, the colimit being formed in the category of locally convex vector spaces.*

Remark 31. LF-spaces are Hausdorff.

Proposition 32 (Open-mapping theorem for LF-spaces). *Every continuous surjective linear map between LF-spaces is open.*

Proof. [12, Proposition 8.8] □

Proposition 33. *If a continuous linear map $f: V \to W$ between LF spaces has finite-dimensional cokernel, it is strict.*

Proof. Note that this does not follow immediately from Proposition 32, as we do not know that $\operatorname{im} f$ is again LF.

Take finitely many independent vectors whose projection to the cokernel form a basis of the cokernel. Their span in W will be called X. As X is finite dimensional, it is especially also LF and hence so is $V \oplus X$. The map

$$V \oplus X \xrightarrow{f \oplus \mathrm{id}} W$$

is then surjective, linear and continuous; thus it is open by Proposition 29. □

2. Analytic Actions of Lie Groups

We continue with a fixed non-archimedean field K.

Definition 34. *Let V a LCVS. A continuous (linear) inclusion of a separated Banach space into V is called a BH-space for V (cf. [9, Definition 1.2.1]).*

Definition 35. *Let M be a finite-dimensional analytic manifold over K. A map $f: M \to V$ into a LCVS V is called locally analytic if for every $x \in M$ the following holds: There exists a open neighbourhood $x \in U$ and a BH-space $W \hookrightarrow V$ containing the image of U such that the induced map*

$$f|_U : U \longrightarrow W$$

in local charts is given by a convergent power series with coefficients in W (cf. [9, Definition 2.1.7]).

Definition 36. *A group object in the category of (finite-dimensional K-analytic) K-manifolds is called a Lie group over K.*

Definition 37. *Let G be a Lie group over K and V a separated LCVS. A continuous action $G \times V \to V$ by continuous linear maps is called analytic if every orbit map $g \mapsto gv$ is locally analytic. It is called equi-analytic if it is analytic and the contragradient action on the dual space $G \times V' \to V'$ is analytic with respect to the strong topology on V'.*

Proposition 38. *An analytic action $G \times V \to V$ is equi-analytic if V is of finite dimension.*

Proof. It suffices to show that every orbit map

$$g \longmapsto v'(g^{-1} \cdot -)$$

is locally analytic. As linear combinations of locally analytic maps are clearly locally analytic, this follows if for all $v' \in V'$ and $v \in V$ the map

$$g \longmapsto v'(g^{-1} v)$$

is analytic. But as $g \mapsto g^{-1}$ and $g \mapsto gv$ are locally analytic by assumption, this follows immediately from the linearity of v'. □

Lemma 39. *Let φ be a continuous endomorphism of V. Then it induces a continuous map $\varphi' \colon V'_b \to V'_b$.*

Proof. We need to show that if $B \subseteq V$ is bounded and $U \subseteq K$ is open, then also $(\varphi')^{-1}(L(B,U)) = L(\varphi(B), U)$ is open. But a continuous map clearly maps bounded sets to bounded sets. □

3. Duality for Lie Algebras

For the general theory of Lie algebras and Lie groups we refer to [3, 13]. In this section, we fix a complete non-archimedean field K of characteristic zero and a Lie group G over K. We also consider its attached Lie algebra $\mathfrak{g}$ with Lie bracket $[-,-]$. The adjoint action of G on $\mathfrak{g}$ by differentiating conjugation maps will be denoted by $\mathrm{Ad}(-)$, the adjoint action of $\mathfrak{g}$ on itself given by $x \mapsto [x,-]$ will be denoted by $\mathrm{ad}(-)$.

Definition 40. *For a $\mathfrak{g}$-module M we define the Chevalley–Eilenberg complex*

$$\mathfrak{C}^\bullet(\mathfrak{g}, V) = \mathrm{Hom}\left(\overset{\bullet}{\bigwedge} \mathfrak{g}, V\right)$$

concentrated in non-negative degrees by considering the differential

$$\mathrm{d}\colon \mathfrak{C}^n(\mathfrak{g}, V) \longrightarrow \mathfrak{C}^{n+1}(\mathfrak{g}, V)$$

given by

$$\begin{aligned} \mathrm{d}f(x_1 \wedge \cdots \wedge x_{n+1}) = \sum_i (-1)^{i+1} x_i f(x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_{n+1}) \\ + \sum_{i<j} (-1)^{i+j} f([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_{n+1}). \end{aligned}$$

As usual, $\widehat{x_i}$ means omitting x_i etc.

Definition 41. *Let V be a $\mathfrak{g}$-module. Define V^{tw} as the vector space V with a $\mathfrak{g}$-action given by*

$$x \cdot^{\mathrm{tw}} v = xv - \mathrm{Tr}(\mathrm{ad}(x))\, v,$$

where Tr *is the trace map.*

Proposition 42. *For $V \neq 0$, $V^{\mathrm{tw}} = V$ if and only if $H^{\dim \mathfrak{g}}(\mathfrak{g}, K) \neq 0$. If $\mathfrak{g}$ is abelian or nilpotent, $V^{\mathrm{tw}} = V$.*

Proof. [7, Corollary 2] □

In applications, this is very often the case.

Proposition 43. *If G is compact, then $V^{\mathrm{tw}} = V$.*

Proof. As for a compact group G the left and right Haar measures coincide, [3, Section III.3.16] implies that

$$\det \operatorname{Ad} g = 1$$

for all $g \in G$. By [3, Section III.4.5], we see that for all x in a neighbourhood of zero of $\mathfrak{g}$

$$\operatorname{Ad}(\phi(x)) = \exp(\operatorname{ad} x),$$

where ϕ is a local exponential map from this neighbourhood into G. Here, exp is the usual exponential map of K extended to matrices. Applying the determinant, we see that

$$1 = \det \exp(\operatorname{ad} x) = \exp(\operatorname{Tr} \operatorname{ad} x)$$

so $\operatorname{Tr} \operatorname{ad} x = 0$ in a neighbourhood of the identity. Choosing a basis of $\mathfrak{g}$ in this neighbourhood, we see that indeed

$$\operatorname{Tr}(\operatorname{ad} x) = 0$$

for all $x \in G$ and hence $V^{\mathrm{tw}} = V$. □

Remark 44. The argument of Proposition 43 shows that if $\operatorname{Tr} \operatorname{ad} x = 0$ for all $x \in \mathfrak{g}$, then $\det \operatorname{Ad}(g) = 1$ for all g in a neighbourhood of the identity. If G is a connected Lie group over $\mathbb{R}$ or $\mathbb{C}$, then $\det \operatorname{Ad} g = 1$ for all $g \in G$. [3, Section III.3.16] then implies that the left and right Haar measures of G coincide.

Definition 45. *For any natural number n we denote by $\lceil n \rceil$ the ordered set $\lceil n \rceil = \{1, \dots, n\}$. For an injective morphism of ordered sets $\phi \colon \lceil k \rceil \to \lceil d \rceil$ there exists a unique morphism of ordered sets*

$$\phi^* \colon \lceil d-k \rceil \longrightarrow \lceil d \rceil$$

such that $\lceil d \rceil = \operatorname{im} \phi \cup \operatorname{im} \phi^*$. We then define

$$\operatorname{sgn} \phi = (-1)^{\sum_{i=1}^{d-k} \phi^*(i) - i}.$$

Remark 46. It is easy to see that for an injective morphism of ordered sets $\phi \colon \lceil k \rceil \to \lceil d \rceil$ the following holds:

$$\operatorname{sgn}(\phi) \cdot \operatorname{sgn}(\phi^*) = (-1)^{k(d-k)},$$

cf. [16, Lemma 10.17].

Proposition 47. *Let M be a finite dimensional vector space with basis $e_1, \dots, e_d$. For an injective morphism of ordered sets $\phi \colon \lceil k \rceil \to \lceil d \rceil$ define*

$$e_\phi = e_{\phi(1)} \wedge \cdots \wedge e_{\phi(k)} \in \bigwedge^k M.$$

For the chosen basis also define the K-linear isomorphism

$$\star \colon \bigwedge^k M \longrightarrow \bigwedge^{d-k} M$$

given by

$$\star e_\phi = \operatorname{sgn}(\phi^*) e_{\phi^*}.$$

Then for any invertible endomorphism A of M the following holds:

$$\det A \cdot ((A^{-1})^t \circ \star) = \star \circ A.$$

Proof. This is a straight-forward piece of linear algebra, but we could not find a reference for $k \neq 1$.

Let $\phi, \psi \colon \lceil k \rceil \to \lceil d \rceil$ be injective maps of ordered sets. For a matrix A denote by $A_{\phi,\psi}$ the matrix with entries $(a_{\phi(i),\psi(j)})_{i,j \in \lceil k \rceil}$. Now a straight forward calculation (or [4, Proposition 9 in III.8.5]) shows that for fixed ψ, we have

$$A e_\psi = \sum_\phi (\det A_{\phi,\psi}) e_\phi, \tag{♠}$$

where ϕ ranges over the injective maps of ordered sets $\lceil k \rceil \to \lceil d \rceil$. We hence also get

$$(A^{-1})^t \star e_\psi = \operatorname{sgn}(\psi^*) \sum_{\phi^*} \det((A^{-1})^t_{\phi^*,\psi^*}) e_{\phi^*},$$

where ϕ^* ranges over the injective maps of ordered sets $\lceil d-k \rceil \to \lceil d \rceil$. Applying $\star$ to (♠), we are reduced to showing

$$\operatorname{sgn}(\psi^*) \cdot \det(A) \cdot \det(A^{-1})^t_{\phi^*,\psi^*} = \operatorname{sgn}(\phi^*) \cdot \det A_{\phi,\psi}.$$

For $k = 1$, this is precisely Cramer's rule.

Generally, for a matrix B, the submatrix $B_{\phi,\psi}$ can be considered as a linear map from the span of $e_{\phi(1)}, \dots, e_{\phi(k)}$ to the span of $e_{\psi(1)}, \dots, e_{\psi(k)}$. Denote this linear map by $B^{\mathrm{res}}_{\phi,\psi}$. Define $B^{\mathrm{ext}}_{\phi,\psi}$ via

$$B^{\mathrm{ext}}_{\phi,\psi} e_{\phi(i)} = B^{\mathrm{res}}_{\phi,\psi} e_{\phi(i)},$$
$$B^{\mathrm{ext}}_{\phi,\psi} e_{\phi^*(i)} = e_{\psi^*(i)}.$$

It is clear that $\det B^{\mathrm{ext}}_{\phi,\psi} = \varepsilon \det B_{\phi,\psi}$, with $\varepsilon = (-1)^{\sum_i \phi^*(i)+\psi^*(i)}$ so

$$\begin{aligned}\varepsilon \det(B_{\phi,\psi}) \cdot e_1 \wedge \cdots \wedge e_n &= B^{\mathrm{ext}}_{\phi,\psi} e_1 \wedge \cdots \wedge B^{\mathrm{ext}}_{\phi,\psi} e_n \\ &= \operatorname{sgn}(\phi^*) \cdot B^{\mathrm{res}}_{\phi,\psi} e_\phi \wedge e_{\psi^*}\end{aligned}$$

By anticommutativity of the exterior algebra, we see that indeed

$$B^{\mathrm{res}}_{\phi,\psi} e_\phi \wedge e_{\psi^*} = B e_\phi \wedge e_{\psi^*},$$

so

$$\varepsilon \cdot \det(B_{\phi,\psi}) \cdot e_1 \wedge \cdots \wedge e_n = \operatorname{sgn}(\phi^*) \cdot B e_\phi \wedge e_{\psi^*}. \tag{♣}$$

If B is invertible, we can apply B^{-1} and get

$$\begin{aligned}\varepsilon \det(B_{\phi,\psi}) \cdot \det(B^{-1}) \cdot e_1 \wedge \cdots \wedge e_n &= \operatorname{sgn}(\phi^*) \cdot e_\phi \wedge B^{-1} e_{\psi^*} \\ &= \operatorname{sgn}(\phi^*)(-1)^{k(d-k)} B^{-1} e_{\psi^*} \wedge e_\phi.\end{aligned}$$

Applying (♣) to $(B^{-1})_{\psi^*,\phi^*}$, we find that

$$B^{-1} e_{\psi^*} \wedge e_\phi = \varepsilon' \cdot \operatorname{sgn}(\psi) \cdot \det((B^{-1})_{\psi^*,\phi^*}) \cdot e_1 \wedge \cdots \wedge e_n$$

with $\varepsilon' = (-1)^{\sum_i \phi(i)+\psi(i)}$. As clearly $\varepsilon\varepsilon' = 1$, we get

$$\det(B_{\phi,\psi}) \cdot \det(B^{-1}) = \operatorname{sgn}(\phi^*) \cdot (-1)^{k(d-k)} \cdot \operatorname{sgn}(\psi) \cdot \det((B^{-1})_{\psi^*,\phi^*}),$$

and using Remark 46, this becomes

$$\operatorname{sgn}(\phi^*) \cdot \det(B_{\phi,\psi}) \cdot \det(B^{-1}) = \operatorname{sgn}(\psi^*) \cdot \det((B^{-1})_{\psi^*,\phi^*}),$$

which is exactly what we needed to show. □

Theorem 48. *Let G be compact and V be a LCVS with a continuous action by $\mathfrak{g}$. If $\mathfrak{g}$ is of dimension d, then there is a G-equivariant and in V functorial isomorphism of complexes*

$$\mathfrak{C}^\bullet(\mathfrak{g}, V') \cong \mathfrak{C}^\bullet(\mathfrak{g}, V)'[-d].$$

Proof. In [7], Hazewinkel shows that as abstract vector spaces

$$\mathfrak{C}^\bullet(\mathfrak{g}, (V^{\mathrm{tw}})^*) \cong \mathfrak{C}^\bullet(\mathfrak{g}, V)^*[-d],$$

where $(-)^* = \operatorname{Hom}_K(-, K)$. While it is easy to check that the isomorphism respects continuous maps, it is not immediate at all that it is G-equivariant. The proof itself is a brutal calculation.

To see G-equivariance, choose a basis $(e_i)_i$ of $\mathfrak{g}$ and define the star operator as in Proposition 47. Hazewinkel's isomorphism stems from the following pairing:

$$\begin{aligned} \langle -,- \rangle \colon \operatorname{Hom}_K\Big(\bigwedge^k \mathfrak{g}, V'\Big) \times \operatorname{Hom}_K\Big(\bigwedge^{d-k} \mathfrak{g}, V\Big) &\longrightarrow K \\ (a,b) &\longmapsto \langle a,b\rangle = \sum_\phi a(e_\phi)(b(\star e_\phi)) \end{aligned}$$

We need to show that

$$\langle ga, b\rangle = \langle a, g^{-1}b\rangle$$

for all $g \in G$. Write A for $\operatorname{Ad}(g)$. Then

$$\begin{aligned} \langle gx, y\rangle &= \sum_\phi (g.x)(e_\phi)(y(\star e_\phi)) \\ &= \sum_\phi (g(x(A^{-1}e_\phi)))(y(\star e_\phi)) \\ &= \sum_\phi x(A^{-1}e_\phi)(g^{-1}y(\star e_\phi)) \end{aligned}$$

and

$$\langle x, g^{-1}y\rangle = \sum_\phi x(e_\phi)(g^{-1}y(A\star e_\phi)).$$

In both cases, ϕ runs over the injective increasing maps $\lceil k\rceil \to \lceil d\rceil$.

By considering the finite dimensional subspace of V generated by all $y(\star e_\phi)$ and their images under g^{-1}, we can consider a finite dimensional vector space instead, i.e.,

$$\langle gx, y\rangle = \sum_i (A^{-1}e_i')^t X^t G^{-1} Y \star e_i'$$

and

$$\langle x, g^{-1}y\rangle = \sum_i e_i'^t X^t G^{-1} Y A \star e_i'$$

for appropriate matrices $X, G^{-1}, Y, \star$ and $(e_i')_i$ the canonical basis of $K^{\binom{d}{k}}$. (The matrix G^{-1} will not be invertible in general, even though the notation does suggest this.) As $A\star = \star(A^{-1})^t$ by Propositions 43 and 47, equality follows, as the trace is invariant under cyclic permutations. □

Remark 49. The isomorphism constructed in Theorem 48 depends on the choice of basis of the Lie algebra $\mathfrak{g}$. Let $(e_i)_i$ and $(f_i)_i$ be two bases of $\mathfrak{g}$ and $T\colon e_i \mapsto f_i$ the corresponding automorphism of $\mathfrak{g}$. Denote by $\star_e$ and $\star_f$ the respective isomorphisms from Proposition 47. Then for the respective pairings

$$\langle a,b\rangle_e = \sum_\phi a(e_\phi)(b(\star_e e_\phi))$$

and

$$\langle a,b\rangle_f = \sum_\phi a(f_\phi)(b(\star_f f_\phi))$$

inducing the isomorphism in the proof of Theorem 48 it holds that

$$\langle -,-\rangle_f = \det(T)\langle -,-\rangle_e.$$

4. Tamme's Comparison Results

We will summarise the results from [14] which we need as follows:

Theorem 50. *Let K be a complete non-archimedean field of characteristic zero. Let G be a Lie group over K and V a barrelled LCVS with an analytic action of G. Then there is a functorial morphism*

$$C^\bullet(G,V) \longrightarrow \mathfrak{C}^\bullet(\mathfrak{g},V)$$

from the locally analytic cochains of G with coefficients in V to the Chevalley–Eilenberg complex of the Lie algebra $\mathfrak{g}$ of G with coefficients in V.

For an open subgroup $U \leq G$, we denote its Lie algebra by $\mathfrak{g}(U)$. Above morphism induces for all n isomorphisms

$$\varinjlim_{U \leq_o G} H^n(U,V) \cong \varinjlim_{U} H^n(\mathfrak{g}(U),V) = H^n(\mathfrak{g},V).$$

The adjoint action of G on $\mathfrak{g}$ together with the action of G on V induce an action of G on the Chevalley–Eilenberg complex and on the Lie algebra cohomology groups. If G is compact, then above morphism of complexes induces an isomorphism

$$H^n(G,V) \cong H^n(\mathfrak{g},V)^G$$

for all n.

Proof. [14, Sections 3-5] □

5. The Duality Theorem

Lemma 51. *Let G be a finite group acting linearly on an L-vector space V. If the order of G is invertible in L, the composition of the canonical inclusion and projection*

$$V^G \longrightarrow V \longrightarrow V_G$$

is an isomorphism.

Proof. By Maschke's theorem, $L[G]$ is a semisimple ring. Therefore there exists an $L[G]$-submodule W of V with $V = V^G \oplus W$. Without loss of generality we can assume that W is irreducible. Denote by I the augmentation ideal in $L[G]$. Then

$$V_G = V^G/IV^G \oplus W/IW = V^G \oplus W/IW$$

and as W is irreducible, IW is either 0 or W. If $IW = 0$, then $W \subseteq V^G$ and hence $W = 0$ by assumption, so $W/IW = 0$ in any case. □

Fix now a complete non-archimedean field K of characteristic zero and a Lie group G over K, which acts equi-analytically on an LCVS V.

Lemma 52. *Let R be a K-algebra. Assume that V carries the structure of an R-module and that the operation of G on V is R-linear. If $H^i(\mathfrak{g},V)$ is finitely generated over R, then there is an open subgroup of G which acts trivially on $H^i(\mathfrak{g},V)$.*

Proof. By Theorem 50,

$$\varinjlim_{U \leq_o G, \mathrm{res}} H^i(U,V) = H^i(\mathfrak{g},V),$$

which is R-linear by our assumptions. Taking preimages of the finitely many generators in $H^i(\mathfrak{g},V)$, we see that there is an open subgroup $U \leq G$ such that $H^i(U,V) \to H^i(\mathfrak{g},V)$ is surjective. This U then operates trivially on $H^i(\mathfrak{g},V)$. □

Theorem 53. *If G is compact and V, V_b' barrelled, we get a functorial (in V) morphism of complexes*

$$C^\bullet(G, V_b') \longrightarrow \mathrm{Hom}_K(C^\bullet(G,V),K)[-d].$$

If the following are satisfied:

- *an open subgroup of G operates trivially on the Lie algebra cohomology,*
- *the differentials in the Chevalley–Eilenberg complex are strict, and*
- *Hahn–Banach holds for V*

then this morphism induces isomorphisms

$$H^i(G, V_b') \cong H^{d-i}(G,V)',$$

if we endow the locally analytical cohomology groups with the subspace topology of the Lie algebra cohomology stemming from the Chevalley–Eilenberg complex.

Proof. By Theorem 50 we have morphisms

$$C^\bullet(G, V_b') \longrightarrow \mathfrak{C}^\bullet(\mathfrak{g}, V_b')$$

and

$$\mathrm{Hom}_K(\mathfrak{C}^\bullet(\mathfrak{g}, V), K) \longrightarrow \mathrm{Hom}_K(C^\bullet(G,V),K).$$

As G is compact, we can employ Theorem 48 without having to twist the Lie algebra action (cf. Proposition 43). We therefore get a G-equivariant isomorphism

$$\mathfrak{C}^\bullet(\mathfrak{g}, V_b') \cong \mathrm{Hom}_{K,\mathrm{cts}}(\mathfrak{C}^\bullet(\mathfrak{g}, V), K)[-d].$$

Composition with the inclusion

$$\mathrm{Hom}_{K,\mathrm{cts}}(\mathfrak{C}^\bullet(\mathfrak{g}, V), K) \subseteq \mathrm{Hom}_K(\mathfrak{C}^\bullet(\mathfrak{g}, V), K)$$

then yields the comparison morphism, which is clearly functorial in V. If the differentials in the complex $\mathfrak{C}(\mathfrak{g}, V)$ are strict and Hahn–Banach holds for V, we get a G-equivariant isomorphism on the level of cohomology:

$$H^i(\mathfrak{g}, V_b') \longrightarrow \mathrm{Hom}_{K,\mathrm{cts}}(H^{d-i}(\mathfrak{g}, V), K).$$

Especially we get the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & (H^{d-i}(\mathfrak{g},V)^G)' & \overset{\cong}{\dashrightarrow} & \\
 & & & & \uparrow\!\uparrow & & \\
H^i(G,V_b') & \longrightarrow & H^i(\mathfrak{g},V_b') & \xrightarrow{\cong} & H^{d-i}(\mathfrak{g},V)' & \longrightarrow & H^{d-i}(G,V)' \\
 & \overset{\cong}{\dashrightarrow} & \hookuparrow & & \uparrow & & \\
 & & H^i(\mathfrak{g},V_b')^G & \xrightarrow{\cong} & (H^{d-i}(\mathfrak{g},V)_G)' & &
\end{array}$$

The dashed isomorphisms are again instances of Theorem 50. If an open subgroup of G acts trivially on the Lie algebra cohomology, then the composition

$$(H^{d-i}(\mathfrak{g}, V)_G)' \longrightarrow H^{d-i}(\mathfrak{g}, V) \longrightarrow (H^{d-i}(\mathfrak{g}, V)^G)'$$

is an isomorphism by Lemma 51 and the claim follows. □

Remark 54. As before the morphism depends on the choice of a basis of $\mathfrak{g}$, cf. Remark 49.

Corollary 55. *Let G be a compact Lie group of dimension d acting analytically on a finite dimensional K-vector space V. Then we have a functorial quasi-isomorphism*

$$C^\bullet(G, V^*) \xrightarrow{\cong} C^\bullet(G,V)^*[-d].$$

Proof. By Proposition 38, we are in the setting of Theorem 53. If V is finite-dimensional, we see that $\mathfrak{C}(\mathfrak{g}, V')$ is a complex of finite dimensional vector spaces and the analytic cohomology groups are therefore finite-dimensional as well by Theorem 50. For all cohomology groups involved, their abstract duals hence coincide with their continuous duals and the result follows. □

Remark 56. Functoriality in Theorem 53 means the following: Let V, W be LCVS with equi-analytic actions of G on them. Assume that V, W, V'_b, W'_b are all barrelled. Given a G-equivariant continuous linear map $\varphi\colon V \to W$, we get a commutative diagram:

$$\begin{array}{ccc} C^\bullet(G, W'_b) & \longrightarrow & \mathrm{Hom}_K(C^\bullet(G,W),K)[-d] \\ \big\downarrow{\scriptstyle C(G,\varphi')} & & \big\downarrow{\scriptstyle \mathrm{Hom}_K(C(G,\varphi),K)} \\ C^\bullet(G, V'_b) & \longrightarrow & \mathrm{Hom}_K(C^\bullet(G,V),K)[-d] \end{array}$$

That the maps involved are well-defined follows from Lemma 39.

Remark 57. Of course a quasi-isomorphism would be a nicer result in the setting of Theorem 53.

The obvious strategy would be topologise $C^\bullet(G,V)$ in such a manner that the differentials are strict and that the cohomology groups are topologically identical to the topology on the Lie algebra cohomology. The same argument as above would then (under the additional hypotheses on the Chevalley–Eilenberg complex and the Lie algebra cohomology) yield a quasi-isomorphism

$$C^\bullet(G, V') \longrightarrow \mathrm{Hom}_{K,\mathrm{cts}}(C^\bullet(G,V),K).$$

There is a natural topology on the space of locally analytic maps from a locally analytic manifold to a LCVS, cf. [9, Satz 2.1.10]. It is however not obvious to us that the comparison morphism in Theorem 50 induces a topological isomorphism. Strictness is then another hard question.

The examples we had in mind when we started to work on this duality result were (φ,Γ)-modules. (Indeed from [1, Theorem D] we get a lot of naturally occurring equi-analytic representations over different ground fields in infinite-dimensional LCVS.) From the study of these examples we know that showing strictness is often very hard (cf. the discussion in the introduction of [8]), so we consider the requirement that the differentials in the Chevalley–Eilenberg complex to be strict an actual obstacle.

A good strategy to show the strictness of differentials in a complex is usually to exploit Proposition 33 if the coefficients are LF-spaces. For this however you need previous knowledge of the cohomology groups being finite dimensional.

Remark 58. If $K = \mathbb{Q}_p$ and V is a finite-dimensional $\mathbb{Q}_p$-vector space, then by [10, V.(2.3.10)] analytic cohomology is just continuous cohomology. Theorem 53 is then a possible way to phrase Poincaré duality, which however does *not* coincide with Poincaré duality due to Lazard (cf. [10, V.(2.5.8)]). Poincaré duality there is an *integral* phenomenon and the dual is given by $\mathrm{Hom}_{\mathbb{Z}_p,\mathrm{cts}}(V, \mathbb{Q}_p/\mathbb{Z}_p)$.

Example 59. Let V be any barrelled LCVS and G a compact abelian Lie group over K of dimension d. The trivial action of G on V is of course equi-analytic. The Lie algebra $\mathfrak{g}$ of G then operates by zero on V. The differentials in the Chevalley–Eilenberg complex $\mathfrak{C}^\bullet(\mathfrak{g}, V)$ are all zero. Theorem 50 then yields that $H^i(G,V) \cong H^i(\mathfrak{g},V) = \mathrm{Hom}_K(\bigwedge^i \mathfrak{g}, V)$ and the isomorphism $H^i(G, V'_b) \cong H^{d-i}(G,V)'$ of Theorem 53 stems from the (choice of basis dependent) pairing

$$\wedge\colon \left(\bigwedge^i \mathfrak{g}\right)^* \times \left(\bigwedge^{d-i} \mathfrak{g}\right)^* \longrightarrow K.$$

References

[1] L. Berger, “Multivariable (φ, Γ)-modules and locally analytic vectors”, *Duke Math. J.* **165** (2016), no. 18, p. 3567-3595.

[2] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIII. Variétés différentielles et analytiques. Fascicule de résultats (Paragraphes 1 à 7)*, Actualités Scientifiques et Industrielles, vol. 1333, Hermann, 1967, 97 pages.

[3] ———, *Elements of Mathematics. Lie groups and Lie algebras. Chapters 1–3*, Springer, 1989, Translated from the French, Reprint of the 1975 edition, xviii+450 pages.

[4] ———, *Elements of Mathematics.Algebra I. Chapters 1–3*, Springer, 1998, Translated from the French, Reprint of the 1989 English translation, xxiv+709 pages.

[5] R. Crew, “Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve”, *Ann. Sci. Éc. Norm. Supér.* **31** (1998), no. 6, p. 717-763.

[6] M. Emerton, “Locally analytic vectors in representations of locally p-adic analytic groups”, *Mem. Am. Math. Soc.* **248** (2017), no. 1175, p. iv+158.

[7] M. Hazevinkel′, “A duality theorem for cohomology of Lie algebras”, *Mat. Sb., N. Ser.* **83 (125)** (1970), p. 639-644.

[8] B. Kupferer, O. Venjakob, “Herr-complexes in the Lubin–Tate setting”, *Mathematika* **68** (2022), no. 1, p. 74-147.

[9] C. T. Féaux de Lacroix, *Einige Resultate über die topologischen Darstellungen p-adischer Liegruppen auf unendlich dimensionalen Vektorräumen über einem p-adischen Körper*, Schriftenreihe des Mathematischen Instituts der Universität Münster. 3. Serie, vol. 23, Univ. Münster, Mathematisches Institut, 1999, x+111 pages.

[10] M. Lazard, “Groupes analytiques p-adiques”, *Publ. Math., Inst. Hautes Étud. Sci.* (1965), no. 26, p. 389-603.

[11] C. Perez-Garcia, W. H. Schikhof, *Locally convex spaces over non-Archimedean valued fields*, Cambridge Studies in Advanced Mathematics, vol. 119, Cambridge University Press, 2010, xiv+472 pages.

[12] P. Schneider, *Nonarchimedean functional analysis*, Springer Monographs in Mathematics, Springer, 2002, vi+156 pages.

[13] J.-P. Serre, *Lie algebras and Lie groups*, second ed., Lecture Notes in Mathematics, vol. 1500, Springer, 1992, 1964 lectures given at Harvard University, viii+168 pages.

[14] G. Tamme, “On an analytic version of Lazard’s isomorphism”, *Algebra Number Theory* **9** (2015), no. 4, p. 937-956.

[15] O. Thomas, “On Analytic and Iwasawa Cohomology”, PhD Thesis, Ruprecht Karl University of Heidelberg, 2019.

[16] ———, “Cohomology of Topologised Monoids”, *Math. Z.* **301** (2022), p. 2793-2840.