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Harmonic analysis / *Analyse harmonique*

Integrability properties of quasi-regular representations of NA groups

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Abstract. Let $G = N \rtimes A$, where N is a graded Lie group and $A = \mathbb{R}^+$ acts on N via homogeneous dilations. The quasi-regular representation $\pi = \mathrm{ind}_A^G(1)$ of G can be realised to act on $L^2(N)$. It is shown that for a class of analysing vectors the associated wavelet transform defines an isometry from $L^2(N)$ into $L^2(G)$ and that the integral kernel of the corresponding orthogonal projector has polynomial off-diagonal decay. The obtained reproducing formula is instrumental for obtaining decomposition theorems for function spaces on nilpotent groups.

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1. Introduction

Let N be a connected, simply connected nilpotent Lie group and let $A = \mathbb{R}^+$ act on N via automorphic dilations. The semi-direct product $G = N \rtimes A$ acts unitarily on $L^2(N)$ via the quasi-regular representation $\pi = \mathrm{ind}_A^G(1)$ of G. For $g \in L^2(N)$, the associated wavelet transform $V_g : L^2(N) \to L^\infty(G)$ is defined as

$$V_g f(x,t) = \langle f, \pi(x,t) g\rangle, \quad (x,t) \in G.$$

A vector $g \in L^2(N)$ is said to be *admissible* if V_g is an isometry from $L^2(N)$ into $L^2(G)$.

Given an admissible vector $g \in L^2(N)$, the orthogonal projector P from $L^2(G)$ onto the closed subspace $V_g(L^2(N)) \subset L^2(G)$ is given by right convolution $P(F) = F * V_g g$. In particular, an element $F \in V_g(L^2(N))$, i.e., $F = V_g f$ for some $f \in L^2(N)$, satisfies the reproducing formula

$$V_g f = V_g f * V_g g. \tag{1}$$

The existence of admissible vectors for irreducible, square-integrable representations π is automatic by the orthogonality relations [10], but a non-trivial problem for reducible representations. For $N = \mathbb{R}^d$ and general dilation groups $A \leq \mathrm{GL}(d,\mathbb{R})$, the admissibility of quasi-regular representations is well-studied, see, e.g. [2, 20, 34] and the references therein. For non-commutative groups N, the admissibility problem is considered in, e.g. [7, 9, 19, 37].

This note is concerned with admissible vectors that are also integrable: A vector $g \in L^2(N)$ is said to be *integrable* if $\Delta_G^{-1/2} V_g g \in L^1(G)$, where $\Delta_G : G \to \mathbb{R}^+$ denotes the modular function on G. The significance of integrably admissible vectors is that $F := \Delta_G^{-1/2} V_g g$ forms a *projection* in $L^1(G)$ by (1), that is, $F = F * F = F^*$, with $F^* := \Delta_G^{-1} \overline{F(\cdot^{-1})}$.

The construction of projections in $L^1(G)$ arising from matrix coefficients is an ongoing research topic, and such projections provide (if they exist) a powerful tool for studying problems in non-commutative harmonic analysis. Among others, they play a vital role in the theory of atomic decompositions in Banach spaces [12, 27].

For the affine group $G = \mathbb{R} \rtimes \mathbb{R}^+$, the construction of projections in $L^1(G)$ goes back to [11]. The papers [8, 28, 32] consider groups $G = \mathbb{R}^d \rtimes A$ and provide criteria for the explicit construction of projections in $L^1(G)$ based on the dual action of A on $\mathbb{R}^d$; see also [21, 23]. The techniques of [28, 32] were used in [40] for the Heisenberg group $N = \mathbb{H}_1$ acted upon by automorphic dilations. For a stratified group N with canonical dilations, the existence of smooth admissible vectors was investigated in [25], although not linked to integrability.

The main concern of this note is the integrability of $\pi = \operatorname{ind}_A^{N \rtimes A}$ when N is a (possibly, non-stratified) graded Lie group. The main result obtained is the following:

Theorem 1. *Let $G = N \rtimes A$, where N is a graded Lie group and $A = \mathbb{R}^+$ acts on N via automorphic dilations. The quasi-regular representation $\pi = \operatorname{ind}_A^G(1)$ admits integrably admissible vectors, i.e., there exist vectors $g \in L^2(N)$ satisfying $\Delta_G^{-1/2} V_g g \in L^1(G)$ and*

$$\int_G |\langle f, \pi(x,t) g\rangle|^2 \, d\mu_G(x,t) = \|f\|_2^2, \qquad \text{for all } f \in L^2(N).$$

The integrably admissible vector g can be chosen to be Schwartz with all moments vanishing, in which case $V_g g \in L^1_w(G)$ for any polynomially bounded weight $w : G \to [1, \infty)$.

Admissible vectors that are Schwartz with all vanishing moments are known to exist already for stratified Lie groups [25, Corollary 1]. Theorem 1 provides a modest extension of this result to general graded Lie groups, and complements it with integrability properties of the associated matrix coefficients. More explicit (point-wise) localisation estimates for the matrix coefficients on homogeneous groups are also obtained; see Section 3 below for details.

The proof method for Theorem 1 resembles the construction of Littlewood–Paley functions and Calderón-type reproducing formulae. Most techniques can already be found in some antecedent form in [17] as pointed out throughout the text. Particular use is made of the (non-stratified) Taylor inequality and Hulanicki's theorem for Rockland operators. The use of a Rockland operator instead of a sub-Laplacian is essential for the proof method as the latter are no longer always homogeneous for non-stratified groups. The exploitation of homogeneity is the reason that the strategy fails for non-graded homogeneous groups (see Remark 8).

The motivation for Theorem 1 stems from the study of function spaces, and is twofold:

(i) The question whether there exist vectors yielding a reproducing kernel with suitable off-diagonal decay on homogeneous groups was posed in [27, Remark 6.6 (a)], where it was mentioned that this is a representation-theoretic problem rather than one of function spaces. The use of such vectors for function space theory, however, is due to the fact that the techniques [27] yield frames and atomic decompositions for Besov–Triebel–Lizorkin spaces. The same holds true for the recent sampling theorems in [38]. The admissible vectors provided by Theorem 1 satisfy the integrability conditions assumed in [27, 38] (see Section 3.3), and Theorem 1 solves the problem mentioned in [27, Remark 6.6 (a)] for graded Lie groups.

(ii) The differentiability properties of functions in terms of Banach spaces are well-studied on stratified Lie groups for several classes of spaces, including Lipschitz spaces [16, 33], Sobolev spaces [15, 39], Besov spaces [6, 22, 39] and Triebel–Lizorkin spaces [17, 30]. More recently, there has been an interest in such spaces on possibly non-stratified graded Lie groups, see, e.g. [1, 3, 5, 14]. This was a motivation to obtain Theorem 1 for graded groups, as it allows to apply the techniques [27,38] discussed in (i) to these new classes of spaces. Moreover, even for stratified groups, the integrability properties provided by Theorem 1 allow to apply the techniques [38] and bridge a gap between what has been established on the locality of the sampling expansions for stratified groups in [6, 22, 25, 27] and for the classical setting $N = \mathbb{R}^d$ in [18, 26]; see [26, 38] for more details on the discrepancy between [27] and [18, 26, 38].

The details on the applications of Theorem 1 to various functional spaces are beyond the scope of the present paper, and will be deferred to subsequent work.

Notation

The open and closed positive half-lines in $\mathbb{R}$ are denoted by $\mathbb{R}^+ = (0,\infty)$ and $\mathbb{R}_0^+ = [0,\infty)$, respectively. For functions $f_1, f_2 : X \to \mathbb{R}_0^+$, it is written $f_1 \lesssim f_2$ if there exists a constant $C > 0$ such that $f_1(x) \le C f_2(x)$ for all $x \in X$. The space of smooth functions on a Lie group G is denoted by $C^\infty(G)$ and the space of test functions by $C_c^\infty(G)$.

2. Preliminaries on homogeneous Lie groups

This section provides background on homogeneous groups. Standard references for the theory are the books [13, 17].

2.1. *Dilations*

Let $\mathfrak{n}$ be a real d-dimensional Lie algebra. A *family of dilations* on $\mathfrak{n}$ is a one-parameter family $\{D_t\}_{t>0}$ of automorphisms $D_t : \mathfrak{n} \to \mathfrak{n}$ of the form $D_t := \exp(A \ln t)$, where $A : \mathfrak{n} \to \mathfrak{n}$ is a diagonalisable linear map with positive eigenvalues $v_1, \dots, v_d$. If a Lie algebra $\mathfrak{n}$ is endowed with a family of dilations, then it is nilpotent.

A *homogeneous group* is a connected, simply connected nilpotent Lie group N whose Lie algebra $\mathfrak{n}$ admits a family of dilations. The number $Q := v_1 + \cdots + v_d$ is the *homogeneous dimension* of N. The exponential map $\exp_N : \mathfrak{n} \to N$ is a diffeomorphism, providing a global coordinate system on N. Dilations $\{D_t\}_{t>0}$ can be transported to a one-parameter group of automorphisms of N, which will be denoted by $\{\delta_t\}_{t>0}$. The associated action of $t \in \mathbb{R}^+$ on $x \in N$ will often simply be written as $tx = \delta_t(x)$.

A *graded group* is a connected, simply connected nilpotent Lie group N whose Lie algebra $\mathfrak{n}$ admits an $\mathbb{N}$-gradation $\mathfrak{n} = \bigoplus_{j=1}^\infty \mathfrak{n}_j$, where $\mathfrak{n}_j$, $j = 1, 2, \dots$, are vector subspaces of $\mathfrak{n}$, almost all equal to $\{0\}$, and satisfying $[\mathfrak{n}_j, \mathfrak{n}_{j'}] \subset \mathfrak{n}_{j+j'}$ for $j, j' \in \mathbb{N}$. If, in addition, $\mathfrak{n}_1$ generates $\mathfrak{n}$, the group N is *stratified*. Canonical dilations $D_t : \mathfrak{n} \to \mathfrak{n}$, $t > 0$, can be defined through a gradation as $D_t(X) = t^j X$ for $X \in \mathfrak{n}_j$, $j \in \mathbb{N}$.

Henceforth, a homogeneous group N will be fixed with dilations $D_t := \exp(A \ln t)$. Haar measure will be denoted by μ_N. The eigenvalues $v_1, \dots, v_d$ of A will be listed in increasing order and it will be assumed (without loss of generality) that $v_1 \ge 1$. In addition, a basis $X_1, \dots, X_d$ of $\mathfrak{n}$ such that $AX_j = v_j X_j$ for $j = 1, \dots, d$ will be fixed throughout.

2.2. *Homogeneity*

A function $f : N \to \mathbb{C}$ is called ν-*homogeneous* ($\nu \in \mathbb{C}$) if $f \circ \delta_t = t^\nu f$ for $t > 0$. For all measurable functions $f_1, f_2 : N \to \mathbb{C}$,

$$\int_N f_1(x)(f_2 \circ \delta_t)(x)\, \mathrm{d}\mu_N(x) = t^{-Q} \int_N (f_1 \circ \delta_{1/t})(x) f_2(x)\, \mathrm{d}\mu_N(x)$$

provided the integral is convergent. The map $f \mapsto f \circ \delta_t$ is naturally extended to distributions.

A linear operator $T : C_c^\infty(N) \to (C_c^\infty(N))'$ is said to be homogeneous of degree $\nu \in \mathbb{C}$ if $T(f \circ \delta_t) = t^\nu (Tf) \circ \delta_t$ for all $f \in C_c^\infty(N)$ and $t > 0$.

A *homogeneous quasi-norm* on N is a continuous function $|\cdot| : N \to [0,\infty)$ that is symmetric, 1-homogeneous and definite. If $|\cdot|$ is a homogeneous quasi-norm on N, there is a constant $C > 0$ such that $|xy| \leq C(|x| + |y|)$ for all $x, y \in N$.

2.3. *Derivatives and polynomials*

A basis element $X_j \in \mathfrak{n}$ acts as a left-invariant vector field on $\mathfrak{n}$ by

$$X_j f(x) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} f(x \exp_N(sX_j))$$

for $f \in C^\infty(N)$ and $x \in N$. The first-order left-invariant differential operator X_j is homogeneous of degree v_j. For a multi-index $\alpha \in \mathbb{N}_0^d$, higher-order differential operators are defined by $X^\alpha := X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_d^{\alpha_d}$. The algebra of all left-invariant differential operators on N is denoted by $\mathscr{D}(N)$.

A function $P : N \to \mathbb{C}$ is a *polynomial* if $P \circ \exp_N$ is a polynomial on $\mathfrak{n}$. Denoting by $\xi_1, \ldots, \xi_d$ a dual basis of $X_1, \ldots, X_d$, the system $\eta_j = \xi_j \circ \exp_N^{-1}$, $j = 1, \ldots, d$, forms a global coordinate system on N. Each $\eta_j : N \to \mathbb{C}$ forms a polynomial on N, and any polynomial P on N can be written uniquely as

$$P = \sum_{\alpha \in \mathbb{N}_0^d} c_\alpha \eta^\alpha, \tag{2}$$

where all but finitely many $c_\alpha \in \mathbb{C}$ vanish and $\eta^\alpha := \eta_1^{\alpha_1} \eta_2^{\alpha_2} \cdots \eta_d^{\alpha_d}$ for a multi-index $\alpha \in \mathbb{N}_0^d$. The homogeneous degree of $\alpha \in \mathbb{N}_0^d$ is defined as $[\alpha] := v_1 \alpha + \cdots + v_d \alpha_d$ and the homogeneous degree of a polynomial P written as (2) is $d(P) := \max\{[\alpha] : \alpha \in \mathbb{N}_0^d \text{ with } c_\alpha \neq 0\}$.

For any $k \geq 0$, the set of polynomials P on N such that $d(P) \leq k$ is denoted by $\mathscr{P}_k$.

2.4. *Schwartz space*

A function $f : N \to \mathbb{C}$ belongs to the Schwartz space $\mathscr{S}(N)$ if $f \circ \exp_N$ is a Schwartz function on $\mathfrak{n}$. A family of semi-norms on $\mathscr{S}(N)$ is given by

$$\|f\|_{\mathscr{S},K} = \sup_{|\alpha| \leq K, x \in N} (1 + |x|)^K |X^\alpha f(x)|, \quad K \in \mathbb{N}_0.$$

For simplicity, the parameter K is sometimes suppressed from the notation $\|\cdot\|_{\mathscr{S},K}$ and it is simply written $\|\cdot\|_{\mathscr{S}}$. The closed subspace of $\mathscr{S}(N)$ of functions with all moments vanishing is defined by

$$\mathscr{S}_0(N) = \left\{ f \in \mathscr{S}(N) : \int_N x^\alpha f(x) \mathrm{d}\mu_N(x) = 0, \quad \forall\, \alpha \in \mathbb{N}_0^d \right\}.$$

For arbitrary $f \in \mathscr{S}(N)$, it will be written $\check{f}(x) := \overline{f(x^{-1})}$ and $f_t(x) := t^{-Q} f(t^{-1}x)$ for $t > 0$.

The dual space $\mathscr{S}'(N)$ of $\mathscr{S}(N)$ is the space of tempered distributions on N. If $f \in \mathscr{S}'(N)$ and $\varphi \in \mathscr{S}(N)$, the conjugate-linear evaluation is denoted by $\langle f, \varphi \rangle$. If well-defined, the evaluation is also written as $\langle f, \varphi \rangle = \int_N f(x) \overline{\varphi(x)}\, \mathrm{d}\mu_N(x)$ and extends the L^2-inner product. Convolution is defined by $f * \varphi(x) := \langle f, \check{\varphi}(x^{-1} \cdot) \rangle$ and $\varphi * f(x) := \langle f, \check{\varphi}(\cdot x^{-1}) \rangle$ for $x \in N$.

3. Matrix coefficients of quasi-regular representations

This section is devoted to point-wise estimates and integability properties of the matrix coefficients of a quasi-regular representation.

3.1. *Quasi-regular representation*

Let N be a homogeneous Lie group and let $A = \mathbb{R}^+$ be the multiplicative group. Then A acts on N via automorphic dilations $A \ni t \mapsto \delta_t \in \mathrm{Aut}(N)$. The semi-direct product $G = N \rtimes A$ is defined via the operations

$$(x,t)(y,u) = (x\delta_t(y), tu), \quad (x,t)^{-1} = (\delta_{t^{-1}}(x^{-1}), t^{-1}).$$

Identity element in G is $e_G = (e_N, 1)$. The group G is an exponential Lie group, that is, the exponential map $\exp_G : \mathfrak{g} \to G$ is a diffeomorphism, see, e.g. [19, Proposition 5.27].

The quasi-regular representation $\pi = \mathrm{ind}_A^G(1)$ of G acts unitarily on $L^2(N)$ by

$$\pi(x,t)f = t^{-Q/2} f(t^{-1}(x^{-1}\cdot)), \quad (x,t) \in N \times A,$$

for $f \in L^2(N)$. Note that $\pi(x,t) = L_x D_t$, where $L_x f = f(x^{-1}\cdot)$ and $D_t f = t^{-Q/2} f(t^{-1}(\cdot))$.

A detailed account on the representation theory of quasi-regular representations of exponential groups can be found in [7, 35, 37], but these results will not be used in this paper.

3.2. *Point-wise estimates*

For $f_1, f_2 \in L^2(N)$, denote the associated matrix coefficient by

$$V_{f_2} f_1(x,t) = \langle f_1, \pi(x,t) f_2 \rangle, \quad (x,t) \in N \rtimes A.$$

The following result provides point-wise estimates for a class of matrix coefficients.

Proposition 2. *Let $f_1, f_2 \in \mathscr{S}_0(N)$ and $K, M \in \mathbb{N}$ be arbitrary.*

(i) *For all $(x,t) \in N \rtimes A$ with $t \leq 1$,*

$$|V_{f_2} f_1(x,t)| \lesssim t^{Q/2+M}(1+|x|)^{-K} \|f_1\|_{\mathscr{S}} \|f_2\|_{\mathscr{S}}. \tag{3}$$

(ii) *For all $(x,t) \in N \rtimes A$ with $t \geq 1$,*

$$|V_{f_2} f_1(x,t)| \lesssim t^{-(Q/2+M)}(1+|x|)^{-K} \|f_1\|_{\mathscr{S}} \|f_2\|_{\mathscr{S}}. \tag{4}$$

The implicit constants in (3) *and* (4) *are group constants that depend further only on M, K.*

Proof. Throughout the proof, a Schwartz semi-norm $\|\cdot\|_{\mathscr{S},N}$ is simply denoted by $\|\cdot\|_N$.

Let $K, M \in \mathbb{N}$ and let $P = P_{x,M} \in \mathscr{P}_M$ denote the Taylor polynomial of $f \in \mathscr{S}(N)$ at $x \in N$ of homogeneous degree M. By Taylor's inequality [13, Theorem 3.1.51], there exist constants $c, C > 0$ such that for all $x, y \in N$,

$$|f(xy) - P(y)| \leq C \sum_{\substack{|\alpha| \leq M'+1 \\ [\alpha] > M}} |y|^{[\alpha]} \sup_{|z| \leq c^{M'+1}|y|} |(X^\alpha f)(xz)|,$$

where $M' := \max\{|\alpha| : \alpha \in \mathbb{N}_0^d \text{ with } [\alpha] \leq M\}$. For $|\alpha| \leq M'+1$ and $x, y \in N$,

$$\begin{aligned}
\sup_{|z| \leq c^{M'+1}|y|} |(X^\alpha f)(xz)| &\leq \|f\|_{K+M'+1} \sup_{|z| \leq c^{M'+1}|y|} (1+|xz|)^{-K} \\
&\lesssim \|f\|_{K+M'+1} \sup_{|z| \leq c^{M'+1}|y|} (1+|x|)^{-K}(1+|z|)^K \\
&\lesssim \|f\|_{K+M'+1} (1+|x|)^{-K}(1+|y|)^K,
\end{aligned}$$

where the second line follows from the Peetre-type inequality [17, Lemma 1.10]. Thus,

$$|f(xy) - P(y)| \lesssim \|f\|_{K+M'+1}(1+|x|)^{-K} \sum_{\substack{|\alpha| \leq M'+1 \\ [\alpha] > M}} |y|^{[\alpha]}(1+|y|)^K \tag{5}$$

for all $x, y \in N$.

(i) Let $(x,t) \in N \rtimes A$ with $t \leq 1$. Then, using that $f_2 \in \mathscr{S}_0(N)$,

$$|V_{f_2} f_1(x,t)| = \left| \int_N f_1(xy) D_t \check{f}_2(y^{-1}) \, \mathrm{d}\mu_N(y) \right| \leq \int_N |f_1(xy) - P(y)| \left| D_t \check{f}_2(y^{-1}) \right| \mathrm{d}\mu_N(y).$$

Applying (5) thus gives

$$\begin{aligned}
|V_{f_2} f_1(x,t)| &\lesssim \|f_1\|_{K+M'+1} (1+|x|)^{-K} t^{-Q/2} \sum_{\substack{|\alpha| \leq M'+1 \\ [\alpha] > M}} \int_N |y|^{[\alpha]} |\check{f}_2(t^{-1} y^{-1})| (1+|y|)^K \, \mathrm{d}\mu_N(y) \\
&= \|f_1\|_{K+M'+1} (1+|x|)^{-K} t^{Q/2} \sum_{\substack{|\alpha| \leq M'+1 \\ [\alpha] > M}} \int_N |ty|^{[\alpha]} |\check{f}_2(y^{-1})| (1+|ty|)^K \, \mathrm{d}\mu_N(y) \\
&\lesssim \|f_1\|_{K+M'+1} (1+|x|)^{-K} t^{Q/2+M} \int_N |f_2(y)| (1+|y|)^{K+Q(M'+1)} \, \mathrm{d}\mu_N(y), \qquad (6)
\end{aligned}$$

where the last inequality uses $[\alpha] \leq Q|\alpha| \leq Q(M'+1)$. The integral in (6) can be estimated by

$$\begin{aligned}
\int_N |f_2(y)| (1+|y|)^{K+Q(M'+1)} \, \mathrm{d}\mu_N(y) &\leq \|f_2\|_{K+Q(M'+1)+Q+1} \int_N (1+|y|)^{-Q-1} \, \mathrm{d}\mu_N(y) \\
&\lesssim \|f_2\|_{K+Q(M'+1)+Q+1}, \qquad (7)
\end{aligned}$$

where convergence of the integral follows by using polar coordinates [17, Proposition 1.15]; see also [17, Corollary 1.17]. A combination of (7) and (6) yields the desired claim (3).

(ii) Note that $|V_{f_2} f_1(x,t)| = |V_{f_1} f_2((x,t)^{-1})|$ for $(x,t) \in N \rtimes A$. Hence, if $t \geq 1$, then it follows by part (i) with $M_0 := M + K$ that

$$\begin{aligned}
|V_{f_2} f_1(x,t)| &\lesssim t^{-(Q/2+M_0)} (1 + t^{-1}|x|)^{-K} \|f_1\|_{K+M_0'+1} \|f_2\|_{K+Q(M_0'+1)+Q+1} \\
&\leq t^{-Q/2-M} t^{-K} t^K (1+|x|)^{-K} \|f_1\|_{K+M_0'+1} \|f_2\|_{K+Q(M_0'+1)+Q+1},
\end{aligned}$$

showing (4). This completes the proof. □

The estimates provided by Proposition 2 recover the well-known polynomial localisation for wavelet transforms when $N = \mathbb{R}$, see, e.g. [29, Section 11-12]. A similar use of the Taylor inequality for (compactly supported) atoms can be found in [17, Theorem 2.9].

3.3. *Analysing vectors*

Left Haar measure on G is given by $\mu_G(x,t) = t^{-(Q+1)} \mathrm{d}\mu_N(x) \mathrm{d}t$ and the modular function is given by $\Delta_G(x,t) = t^{-Q}$. The measure μ_G is used to define the Lebesgue space $L^p(G) = L^p(G, \mu_G)$ for $p \in [1,\infty]$, and $\|\cdot\|_p$ will denote the p-norm.

A measurable function $w : G \to [1,\infty)$ is said to be a *weight* if it is submultiplicative, i.e., $w((x,t)(y,u)) \lesssim w(x,t) w(y,u)$ for $(x,t),(y,u) \in G$. A weight w is called *polynomially bounded* if

$$w(x,t) \lesssim (1+|x|)^k (t^m + t^{-m'}), \quad (x,t) \in G, \qquad (8)$$

for some $k, m, m' \geq 0$. Given such a weight w, the weighted Lebesgue space $L^1_w(G)$ consists of all $F \in L^1(G)$ satisfying $\|F\|_{L^1_w} := \|Fw\|_1 < \infty$.

In [12,27,38], the space of *w-analysing vectors* of π, defined by

$$\mathscr{A}_w := \left\{ g \in L^2(N) : V_g g \in L^1_w(G) \right\},$$

plays a prominent role.

The following result provides a simple criterion for analysing vectors:

Lemma 3. *Suppose $g \in \mathscr{S}_0(N)$. Then $g \in \mathscr{A}_w$ for any polynomially bounded weight function $w : G \to [1,\infty)$. In particular, the representation $\pi = \mathrm{ind}_A^G(1)$ is integrable.*

Proof. Let $k, m, m' \geq 0$ be such that $w(x,t) \lesssim (1+|x|)^k(t^m + t^{-m'})$ for all $(x,t) \in G$. Then, choosing $K, M, M' \in \mathbb{N}$ sufficiently large, it follows by Proposition 2 that

$$\begin{aligned}\|V_g g\|_{L^1_w} &\lesssim \int_0^\infty \int_N V_g g(x,t)(1+|x|)^k(t^m + t^{-m'})\,\mathrm{d}\mu_N(x)\frac{\mathrm{d}t}{t^{Q+1}} \\ &\lesssim \int_0^1 t^{Q/2+M'-m'} t^{-(Q+1)}\mathrm{d}t + \int_1^\infty t^{-(Q/2+M)+m} t^{-(Q+1)}\mathrm{d}t < \infty.\end{aligned}$$

This shows that $g \in \mathscr{A}_w$, and thus π is w-integrable. □

4. Admissible vectors

A vector $g \in L^2(N)$ is said to be *admissible* for the quasi-regular representation $(\pi, L^2(N))$ if the map

$$V_g : L^2(N) \to L^\infty(G), \quad f \mapsto \langle f, \pi(\cdot)g\rangle$$

is an isometry into $L^2(G)$.

4.1. *Reproducing formulae*

The following observation relates admissibility to a Calderón-type reproducing formula.

Lemma 4. *Let $g \in \mathscr{S}(N)$ with $\int_N g(x)\,\mathrm{d}\mu_N(x) = 0$. Then g is admissible if, and only if,*

$$f = \int_0^\infty f * \check{g}_t * g_t \,\frac{\mathrm{d}t}{t} \equiv \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho f * \check{g}_t * g_t\,\frac{\mathrm{d}t}{t}, \quad f \in \mathscr{S}(N), \tag{9}$$

with convergence in $\mathscr{S}'(N)$.

Proof. Under the assumptions on g, it follows by [17, Theorem 1.65] that

$$H_{\varepsilon,\rho}(z) := \int_\varepsilon^\rho \check{g}_t * g_t(z)\,\frac{\mathrm{d}t}{t}, \quad z \in N,$$

converges in $\mathscr{S}'(N)$ to a distribution $H := \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} H_{\varepsilon,\rho}$ which is smooth on $N \setminus \{e_N\}$ and homogeneous of degree $-Q$. Let $f \in \mathscr{S}(N)$. Then

$$\begin{aligned}\|V_g f\|_2^2 &= \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho \int_N |f * D_t \check{g}(x)|^2\,\mathrm{d}\mu_G(x,t) \\ &= \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho \int_N\int_N\int_N f(y)\check{g}_t(y^{-1}x)\overline{\check{g}_t(z^{-1}x)f(z)}\,\mathrm{d}\mu_N(z)\mathrm{d}\mu_N(y)\mathrm{d}\mu_N(x)\frac{\mathrm{d}t}{t} \\ &= \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho \int_N\int_N f(y)\check{g}_t * g_t(y^{-1}z)\overline{f(z)}\,\mathrm{d}\mu_N(y)\mathrm{d}\mu_N(z)\frac{\mathrm{d}t}{t} \\ &= \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_N f * H_{\varepsilon,\rho}(z)\overline{f(z)}\,\mathrm{d}\mu_N(z) \\ &= \int_N f * H(z)\overline{f(z)}\,\mathrm{d}\mu_N(z),\end{aligned}$$

where the last equality used that $f * H_{\varepsilon,\rho} \to f * H$ in $\mathscr{S}'(N)$ as $\varepsilon \to 0$ and $\rho \to \infty$.

The map $f \mapsto f * H$ is bounded on $L^2(N)$ by [17, Theorem 6.19]. Hence $V_g : \mathscr{S}(N) \to L^2(G)$ is well-defined, and it follows that

$$\int_G |\langle f, \pi(x,t)g\rangle|^2\,\mathrm{d}\mu_G(x,t) = \langle f * H, f\rangle, \quad f \in L^2(N). \tag{10}$$

Thus g is admissible if, and only if, $\langle f * H, f\rangle = \langle f, f\rangle$ for all $f \in L^2(N)$. Polarisation yields that this is equivalent to (9), which completes the proof. □

The calculations in the proof of Lemma 4 are classical, see, e.g. [17, Theorem 7.7].

4.2. *Rockland operators*

This section provides background on spectral multipliers for Rockland operators, see, e.g. [13, Chapter 4] for a detailed account. The stated results will be used in Section 4.3 below for the construction of admissible vectors.

Let $\mathscr{L} \in \mathscr{D}(N)$ be positive and formally self-adjoint. Then $\mathscr{L}$ is essentially self-adjoint on $L^2(N)$, and $\mathscr{L}$ will also denote its self-adjoint extension. Let $E_{\mathscr{L}}$ be the spectral measure of $\mathscr{L}$. For $m \in L^\infty(\mathbb{R}_0^+)$, the operator

$$m(\mathscr{L}) := \int_{\mathbb{R}_0^+} m(\lambda)\, \mathrm{d}E_{\mathscr{L}}(\lambda)$$

is a left-invariant bounded linear operator on $L^2(N)$. By the Schwartz kernel theorem, the action of $m(\mathscr{L})$ on $\mathscr{S}(N)$ is given by

$$m(\mathscr{L})f = f * K_{m(\mathscr{L})}, \quad f \in \mathscr{S}(N),$$

where $K_{m(\mathscr{L})} \in \mathscr{S}'(N)$ is the associated convolution kernel.

A *Rockland operator* is a homogeneous differential operator $\mathscr{L} \in \mathscr{D}(N)$ of positive degree that is hypoelliptic, i.e. for every distribution $f \in (C_c^\infty(N))'$ and every open set $U \subseteq N$, the condition $(\mathscr{L} f)|_U \in C^\infty(U)$ implies that $f|_U \in C^\infty(U)$. Positive Rockland operators are well-known to exist on any graded Lie group.

The following theorem is the key result used to construct admissible Schwartz functions.

Theorem 5 (Hulanicki [31]). *Let N be a graded Lie group. Let $\mathscr{L} \in \mathscr{D}(N)$ be a positive Rockland operator and let $|\cdot| : N \to [0,\infty)$ be a fixed homogeneous quasi-norm on N.*

For any $M_1 \in \mathbb{N}$, $M_2 \geq 0$, there exist $C = C(M_1, M_2) > 0$ and $k = k(M_1, M_2), k' = k'(M_1, M_2) \in \mathbb{N}_0$ such that, for any $m \in C^k(\mathbb{R}_0^+)$, the kernel $K_{m(\mathscr{L})}$ of $m(\mathscr{L})$ satisfies

$$\sum_{[\alpha] \leq M_1} \int_G |X^\alpha K_{m(\mathscr{L})}(x)|(1+|x|)^{M_2}\, \mathrm{d}\mu_N(x) \leq C \sup_{\substack{\lambda>0 \\ \ell=0,\dots,k \\ \ell'=0,\dots,k'}} (1+\lambda)^{\ell'} |\partial_\lambda^\ell m(\lambda)|.$$

Corollary 6. *Let $\mathscr{L} \in \mathscr{D}(N)$ be a positive Rockland operator.*

(i) *If $m \in \mathscr{S}(\mathbb{R}_0^+)$, then $K_{m(\mathscr{L})} \in \mathscr{S}(N)$.*
(ii) *If $m \in \mathscr{S}(\mathbb{R}_0^+)$ vanishes near the origin, then $K_{m(\mathscr{L})} \in \mathscr{S}_0(N)$.*

4.3. *Existence of admissible vectors*

The following result yields a class of Schwartz vectors that are admissible.

Proposition 7. *Let N be a graded Lie group and let $\mathscr{L} \in \mathscr{D}(N)$ be a positive Rockland operator of degree ν. Let $K_{m(\mathscr{L})}$ be the convolution kernel of a multiplier $m \in \mathscr{S}(\mathbb{R}_0^+)$ satisfying*

$$\int_0^\infty |m(t)|^2 \frac{\mathrm{d}t}{t} = \nu. \tag{11}$$

Then $g := K_{m(\mathscr{L})} \in \mathscr{S}(N)$ is an admissible vector for $\pi = \mathrm{ind}_A^{N \rtimes A}(1)$.

Proof. Let $m \in \mathscr{S}(\mathbb{R}_0^+)$ be as in the statement, so that

$$\int_0^\infty |m(\lambda t^\nu)|^2 \frac{\mathrm{d}t}{t} = \frac{1}{\nu} \int_0^\infty |m(t)|^2 \frac{\mathrm{d}t}{t} = 1, \quad \text{for all } \lambda > 0. \tag{12}$$

By Corollary 6, $g := K_{m(\mathscr{L})} \in \mathscr{S}(N)$, and it suffices to show the reproducing formula (9). Define $H_{\varepsilon,\rho} := \int_\varepsilon^\rho \check{g}_t * g_t \, t^{-1} dt$ for $0 < \varepsilon < \rho < \infty$. Let $f_1, f_2 \in \mathscr{S}(N)$. Then

$$\langle f_1 * H_{\varepsilon,\rho}, f_2 \rangle = \int_\varepsilon^\rho \langle f_1 * \check{g}_t * g_t, f_2 \rangle \frac{\mathrm{d}t}{t} = \int_\varepsilon^\rho \langle f_1 * (\check{g} * g)_t, f_2 \rangle \frac{\mathrm{d}t}{t}. \tag{13}$$

The spectral theorem implies that $\check{g} * g = K_{\overline{m}(\mathscr{L})} * K_{m(\mathscr{L})} = K_{|m|^2(\mathscr{L})}$. In addition, the homogeneity of $\mathscr{L}$ yields that $(\check{g} * g)_t = K_{|m|^2(t^\nu \mathscr{L})}$ for all $t > 0$, see, e.g. [13, Corollary 4.1.16]. Combining this with (13) gives

$$\begin{aligned}\langle f_1 * H_{\varepsilon,\rho}, f_2 \rangle &= \int_\varepsilon^\rho \left\langle |m|^2(t^\nu \mathscr{L}) f_1, f_2 \right\rangle \frac{\mathrm{d}t}{t} = \int_\varepsilon^\rho \int_0^\infty |m(t^\nu \lambda)|^2 \,\mathrm{d}\langle E_{\mathscr{L}}(\lambda) f_1, f_2 \rangle \frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \int_\varepsilon^\rho |m(t^\nu \lambda)|^2 \frac{\mathrm{d}t}{t} \mathrm{d}\langle E_{\mathscr{L}}(\lambda) f_1, f_2 \rangle.\end{aligned}$$

Hence, by the identity (12),

$$\lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \langle f_1 * H_{\varepsilon,\rho}, f_2 \rangle = \int_0^\infty \int_0^\infty |m(t^\nu \lambda)|^2 \frac{\mathrm{d}t}{t} \mathrm{d}\langle E_{\mathscr{L}}(\lambda) f_1, f_2 \rangle = \langle f_1, f_2 \rangle.$$

An application of Lemma 4 therefore yields that g is admissible. □

Spectral multipliers for sub-Laplacians on stratified groups were used for constructing admissible vectors in [25]. See also [24] for similar discrete Littlewood–Paley decompositions.

Remark 8. The use of a *homogeneous* operator is essential in the proof of Proposition 7 to guarantee that the spectral dilates $m(t\,\cdot)$, $t > 0$, of a multiplier $m \in \mathscr{S}(\mathbb{R}_0^+)$ yield a convolution kernel $K_{m(t\mathscr{L})}$ that is compatible with automorphic dilations $\{\delta_t\}_{t>0}$. For non-homogeneous operators, other techniques seem required, see, e.g. [4, 36].

4.4. *Proof of Theorem 1*

Theorem 1 follows from combining Lemma 3, Corollary 6 and Proposition 7.

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