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Plethysm and a character embedding problem of Miller

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Abstract. We use a plethystic formula of Littlewood to answer a question of Miller on embeddings of symmetric group characters. We also reprove a result of Miller on character congruences.

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Given $d \geq 1$ and a partition $\lambda = (1^{m_1} 2^{m_2} 3^{m_3} \cdots)$ of a positive integer n, let $\boxplus^d(\lambda)$ be the partition of $d^2 \cdot n$ given by $\boxplus^d(\lambda) := (d^{dm_1} (2d)^{dm_2} (3d)^{dm_3} \cdots)$. The Young diagram of $\boxplus^d(\lambda)$ is obtained from that of λ by subdividing every box into a $d \times d$ grid, as suggested by the notation.

Let S_n be the symmetric group on n letters. For a partition $\lambda \vdash n$, let V^λ be the corresponding S_n-irreducible with character $\chi^\lambda : S_n \to \mathbb{C}$. For $d \geq 1$, define a new class function $\boxplus^d(\chi^\lambda)$ on S_n whose value on permutations of cycle type $\mu \vdash n$ is given by

$$\boxplus^d \left(\chi^\lambda\right)_\mu := \chi^{\boxplus^d(\lambda)}_{\boxplus^d(\mu)}. \tag{1}$$

Thus, the values of the class function $\boxplus^d(\chi^\lambda)$ on S_n are embedded inside the character table of the larger symmetric group $S_{d^2 \cdot n}$. A. Miller conjectured [4] that the class functions $\boxplus^d(\chi^\lambda)$ are genuine characters of (rather than merely class functions on) S_n. We prove that this is so in Theorem 1 using *plethysm* of symmetric functions.

In the arguments that follow, we use standard material on symmetric functions; for details see [3]. For $\mu \vdash n$, let $m_i(\mu)$ be the multiplicity of i as a part of μ and $z_\mu := 1^{m_1(\mu)} 2^{m_2(\mu)} \cdots m_1(\mu)! m_2(\mu)! \cdots$ be the size of the centralizer of a permutation $w \in S_n$ of cycle type μ.

Let $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ be the ring of symmetric functions in an infinite variable set $(x_1, x_2, \dots)$. Bases of Λ are indexed by partitions; we use the Schur basis $\{s_\lambda\}$ and power sum basis $\{p_\lambda\}$. The basis p_λ is *multiplicative*: if $\lambda = (\lambda_1, \lambda_2, \dots)$ then $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$. The transition matrix from the Schur to the power sum basis encodes the character table of S_n; for $\lambda \vdash n$ we have

$$s_\lambda = \sum_{\mu \vdash n} \frac{\chi^\lambda_\mu}{z_\mu} p_\mu.$$

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Let $\langle -,- \rangle$ be the *Hall inner product* on Λ with respect to which the Schur basis $\{s_\lambda\}$ is orthonormal. The power sums are orthogonal with respect to this inner product. We have $\langle p_\lambda, p_\mu \rangle = z_\lambda \cdot \delta_{\lambda,\mu}$ where δ is the Kronecker delta.

Write $R = \bigoplus_{n \geq 0} R_n$ where R_n is the space of class functions $\varphi : S_n \to \mathbb{C}$. The *characteristic map* $\mathrm{ch}_n : R_n \to \Lambda_n$ is given by $\mathrm{ch}_n(\varphi) = \frac{1}{n!} \sum_{w \in S_n} \varphi(w) \cdot p_{\mathrm{cyc}(w)}$ where $\mathrm{cyc}(w) \vdash n$ is the cycle type of $w \in S_n$. The map $\mathrm{ch} = \bigoplus_{n \geq 0} \mathrm{ch}_n$ is a linear isomorphism $R \to \Lambda$. The space R has an *induction product* given by $\varphi \circ \psi := \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (\varphi \otimes \psi)$ for all $\varphi \in R_n$ and $\psi \in R_m$. Under this product, the map $\mathrm{ch} : R \to \Lambda$ becomes a ring isomorphism. We record two properties of ch.

- We have $\mathrm{ch}(\chi^\lambda) = s_\lambda$, so that ch sends the irreducible character basis of R to the Schur basis of Λ.
- If $\varphi : S_n \to \mathbb{C}$ is any class function and $\mu \vdash n$, then

$$\left\langle \mathrm{ch}(\varphi), p_\mu \right\rangle = \text{ value of } \varphi \text{ on a permutation of cycle type } \mu. \tag{2}$$

Let $\psi^d : \Lambda \to \Lambda$ be the map $\psi^d : F(x_1, x_2, \ldots) \mapsto F(x_1^d, x_2^d, \ldots)$ which replaces each variable x_i with its d^{th} power x_i^d. The symmetric function $\psi^d(F)$ is the plethysm $p_d[F]$ of F into the power sum p_d. Let $\phi_d : \Lambda \to \Lambda$ be the adjoint of ψ^d characterized by $\langle \psi^d(F), G \rangle = \langle F, \phi_d(G) \rangle$ for all $F, G \in \Lambda$. In this note we apply the operators ψ^d and ϕ_d to character theory; see [6] for an application to the cyclic sieving phenomenon of enumerative combinatorics.

Theorem 1. *Let $d \geq 1$ and $\lambda \vdash n$. Consider the chain of subgroups $\Delta(S_n) \subseteq S_n^d \subseteq S_{dn}$ where $S_n^d = S_n \times \cdots \times S_n$ is the d-fold self-product of S_n and $\Delta(S_n)$ is the diagonal $\{(w, \ldots, w) : w \in S_n\}$ in S_n^d. Then $\boxplus^d(\chi^\lambda)$ is the character of the $\Delta(S_n) \cong S_n$ module*

$$\mathrm{Res}_{\Delta(S_n)}^{S_{dn}} \left(V^\lambda \circ \cdots \circ V^\lambda \right) \tag{3}$$

obtained by restricting the d-fold induction product $V^\lambda \circ \cdots \circ V^\lambda = \mathrm{Ind}_{S_n^d}^{S_{dn}} (V^\lambda \otimes \cdots \otimes V^\lambda)$ to $\Delta(S_n)$.

Proof. Let $\lambda, \mu \vdash n$ be two partitions and let $d \geq 1$. By (2) we have the class function value

$$\chi_{\boxplus^d(\mu)}^{\boxplus^d(\lambda)} = \left\langle s_{\boxplus^d(\lambda)}, p_{\boxplus^d(\mu)} \right\rangle = \left\langle s_{\boxplus^d(\lambda)}, \psi^d\left(p_\mu^d\right) \right\rangle = \left\langle \phi_d\left(s_{\boxplus^d(\lambda)}\right), p_\mu^d \right\rangle. \tag{4}$$

Littlewood [2, p. 340] proved (see also [1, Equation 13]) that for any partition $\nu \vdash dm$, with empty d-core, the image $\phi_d(s_\nu)$ is given by

$$\phi_d(s_\nu) = \epsilon_d(\nu) \cdot s_{\nu^{(1)}} \cdots s_{\nu^{(d)}} \tag{5}$$

where $\epsilon_d(\nu)$ is the d-sign of ν and $(\nu^{(1)}, \ldots, \nu^{(d)})$ is the d-quotient of ν. We refer the reader to [1,2] for definitions. In our context we have $\epsilon_d(\boxplus^d(\lambda)) = +1$ (since $\boxplus^d(\lambda)$ admits a d-ribbon tiling with only horizontal ribbons) and the d-quotient of $\boxplus^d(\lambda)$ is the constant d-tuple $(\lambda, \ldots, \lambda)$. Equation (5) reads

$$\phi_d\left(s_{\boxplus^d(\lambda)}\right) = s_\lambda^d. \tag{6}$$

Plugging (6) into (4) gives

$$\chi_{\boxplus^d(\mu)}^{\boxplus^d(\lambda)} = \left\langle \phi_d\left(s_{\boxplus^d(\lambda)}\right), p_\mu^d \right\rangle = \left\langle s_\lambda^d, p_\mu^d \right\rangle \tag{7}$$

which (thanks to (2)) agrees with the trace of $(w, \ldots, w) \in \Delta(S_n)$ on $V^\lambda \circ \cdots \circ V^\lambda$ for $w \in S_n$ of cycle type μ. □

If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition, let $d \cdot \lambda = (d\lambda_1, d\lambda_2, \ldots)$ be the partition obtained by multiplying every part of λ by d. The argument proving Theorem 1 applies to show that for $\lambda \vdash n$, the class function $\chi^{d \cdot \lambda} : S_n \to \mathbb{C}$ given by $(\chi^{d \cdot \lambda})_\mu := \chi_{d \cdot \mu}^{d \cdot \lambda}$ is a genuine character (although its module does not have such a nice description). It may be interesting to find other ways to discover characters of S_n embedded inside characters of larger symmetric groups.

In closing, we use plethysm to give a quick proof of a character congruence result of Miller [5, Thm. 1]. Miller gave an interesting combinatorial proof of the following theorem by introducing objects called "cascades".

Theorem 2. (Miller) *Let $d \geq 1$. For any partitions $\lambda \vdash n$ and $\mu \vdash dn$, we have*

$$\chi^{\boxplus^d(\lambda)}_{d\cdot\mu} \equiv 0 \mod d!. \tag{8}$$

Furthermore, suppose $\lambda, \nu \vdash n$ with $d \nmid n$. Then

$$\chi^{\boxplus^d(\lambda)}_{d^2\cdot\nu} = 0. \tag{9}$$

Proof. Arguing as in the proof of Theorem 1, we have

$$\chi^{\boxplus^d(\lambda)}_{d\cdot\mu} = \langle s_{\boxplus^d(\lambda)}, p_{d\cdot\mu}\rangle = \left\langle s_{\boxplus^d(\lambda)}, \psi^d(p_\mu)\right\rangle = \langle \phi_d\left(s_{\boxplus^d(\lambda)}\right), p_\mu\rangle = \left\langle s^d_\lambda, p_\mu\right\rangle \tag{10}$$

where the last equality used Equation (6). We have $s_\lambda = \sum_{\rho\vdash n} \frac{\chi^\lambda_\rho}{z_\rho} p_\rho$ so that

$$\chi^{\boxplus^d(\lambda)}_{d\cdot\mu} = \left\langle s^d_\lambda, p_\mu\right\rangle = \left\langle \left(\sum_{\rho\vdash n} \frac{\chi^\lambda_\rho}{z_\rho} p_\rho\right)^d, p_\mu\right\rangle. \tag{11}$$

We expand far right of (11) using the orthogonality of the p's to obtain

$$\left\langle \left(\sum_{\rho\vdash n} \frac{\chi^\lambda_\rho}{z_\rho} p_\rho\right)^d, p_\mu\right\rangle = \sum_{(\mu_{(1)},\ldots,\mu_{(d)})} \frac{z_\mu}{z_{\mu_{(1)}}\cdots z_{\mu_{(d)}}} \times \chi^\lambda_{\mu_{(1)}}\cdots\chi^\lambda_{\mu_{(d)}} \tag{12}$$

where the sum is over all d-tuples $(\mu_{(1)},\ldots,\mu_{(d)})$ of partitions of n whose multiset of parts equals μ. In particular, (12) is zero unless every part of μ is $\leq n$; we assume this going forward. We want to show that (12) is divisible by $d!$. To show this, we examine what happens when some of the entries in a tuple $(\mu_{(1)},\ldots,\mu_{(d)})$ coincide.

Fix a d-tuple $(\mu_{(1)},\ldots,\mu_{(d)})$ of partitions of n whose multiset of parts is μ. The ratio of z's in the corresponding term on the RHS of (12) is a product of multinomial coefficients

$$\frac{z_\mu}{z_{\mu_{(1)}}\cdots z_{\mu_{(d)}}} = \binom{m_1(\mu)}{m_1\left(\mu_{(1)}\right),\ldots,m_1\left(\mu_{(d)}\right)}\cdots\binom{m_n(\mu)}{m_n\left(\mu_{(1)}\right),\ldots,m_n\left(\mu_{(d)}\right)}. \tag{13}$$

Let $\sigma = (\sigma_1,\ldots,\sigma_r) \vdash d$ be the partition of d obtained by writing the entry multiplicities in the d-tuple $(\mu_{(1)},\ldots,\mu_{(d)})$ in weakly decreasing order. For example, if $n = 3$, $d = 5$, and our d-tuple of partitions of n is $(\mu_{(1)},\ldots,\mu_{(5)}) = ((2,1),(3),(1,1,1),(3),(2,1))$, then $\sigma = (2,2,1)$. Each multinomial coefficient in (13) for which $m_i(\mu) > 0$ is divisible by $\sigma_1!\cdots\sigma_r!$. Since each part of μ is $\leq n$, at least one $m_i(\mu) > 0$ and the whole product (13) of multinomial coefficients is divisible by $\sigma_1!\cdots\sigma_r!$. Thus, the sum of the terms in (12) indexed by rearrangements of $(\mu_{(1)},\ldots,\mu_{(d)})$ is divisible by $\binom{d}{\sigma_1,\ldots,\sigma_r}\cdot\sigma_1!\cdots\sigma_r! = d!$, so that (12) itself is divisible by $d!$. This proves the first part of the theorem.

For the second part of the theorem, let $\lambda, \nu \vdash n$ where $d \nmid n$. Arguing as above, we have

$$\chi^{\boxplus^d(\lambda)}_{d^2\cdot\nu} = \left\langle \left(\sum_{\rho\vdash n} \frac{\chi^\lambda_\rho}{z_\rho} p_\rho\right)^d, p_{d\cdot\nu}\right\rangle. \tag{14}$$

Since $d \nmid n$, each partition $\rho \vdash n$ appearing in the first argument of the inner product in (14) has at least one part not divisible by d. Since the p's are an orthogonal basis of Λ, we see that (14) = 0, proving the second part of the theorem. □

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