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New Properties of Multiplier Submodule Sheaves

Nouvelles Propriétés des Faisceaux de Sous-modules Multiplicateurs

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Abstract. In this note, we establish the strong openness and stability property of multiplier submodule sheaves associated to singular Nakano semi-positive metrics on holomorphic vector bundles, which generalizes the same properties for multiplier ideal sheaves associated to pseudo-effective line bundles.

Résumé. Dans cette note, nous établissons la conjecture forte d'ouverture et la stabilité des faisceaux de sous-modules multiplicateurs associés aux métriques semi-positives de Nakano singulières sur les fibrés vectoriels holomorphes, ce qui généralise les mêmes propriétés pour les faisceaux d'idéaux multiplicateurs associés aux fibrés en droites pseudo-effectifs.

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1. Introduction and main results

Let X be a complex manifold and (L,h) be a pseudo-effective holomorphic line bundle. Let $\mathscr{I}(h)$ be the sheaf of germs of holomorphic sections f such that $|f|_h^2$ is locally integrable, which is called the multiplier ideal sheaf. When $h=e^{-\varphi}$ locally, denote $\mathscr{I}(h)$ by $\mathscr{I}(\varphi)$.

S.T. Yau [19] once said that the notion of multiplier ideal sheaf plays a central role in modern higher-dimensional algebraic geometry. Studying properties of the multiplier ideal sheaf and its generalizations is important in several complex variables and complex algebraic geometry. Some basic properties of the multiplier ideal sheaf include coherence, integral closedness, Nadel vanishing theorem.

Some conjectures related to further useful properties of multiplier ideal sheaves, e.g., the openness conjecture and the strong openness conjecture were posed in [3, 4].

Conjecture 1 (Openness conjecture). *If $\mathscr{I}(h)=\mathscr{O}_X$, then*

$$\mathscr{I}_+(h)=\mathscr{I}(h);$$

Conjecture 2 (Strong openness conjecture).

$$\mathscr{I}_+(h)=\mathscr{I}(h)$$

where $\mathscr{I}_+(h):=\bigcup_j\mathscr{I}(h_j)=\mathscr{I}(h_{j_0})$, for any decreasing sequence of metrics h_j converging to h; in particular $\mathscr{I}_+(h):=\bigcup_{\varepsilon>0}\mathscr{I}(h^{1+\varepsilon})=\mathscr{I}(h^{1+\varepsilon_0})$; the fact that both equalities hold follows from the Noetherian property of coherent analytic sheaves.

There are many works related to these conjectures ([1, 7, 14]). The strong openness conjecture was proved by Guan–Zhou in [11] by using a different method from previous ones. In addition, Berndtsson in [1] established an effectiveness result in his proof of the openness conjecture and Guan–Zhou gave an effectiveness result of the strong openness conjecture in [10] (see also [8]). Here, it is meant by effectiveness that explicit conditions are found for h_{j_0} or ε_0, independently of the specific expressions of f and h.

Let $o\in D\subset\mathbb{C}^n$, $\varphi\in\mathrm{Psh}(D)$ and $f\subset\mathscr{O}(D)$, then the complex singularity exponent of φ with respect to f at o is $c_o^f(\varphi):=\sup\{c\ge 0\mid |f|^2e^{-c\varphi}$ is integrable near $o\}$. When $f\equiv 1$, denote $c_o^f(\varphi)$ by $c_o(\varphi)$. Then by the coherence of the multiplier ideal sheaves, the openness conjecture and the strong openness conjecture are equivalent to that

$$e^{-c_o(\varphi)\varphi}\quad\text{and}\quad |f|^2e^{-c_o^f(\varphi)\varphi}$$

are not integrable near o for $(f,o)\in\mathscr{I}(\varphi)_o$ respectively.

Demailly–Kollár proved the following semi-continuity theorem on plurisubharmonic functions in [4]: Let $\{\varphi_j\}\subset\mathrm{Psh}(D)$ converge to φ in $L^1_{\mathrm{loc}}(D)$, then $e^{-c\varphi_j}$ converges to $e^{-c\varphi}$ in L^1-norm near o for $c<c_o(\varphi)$. In [12], P.H. Hiep generalized it to the following result with an additional condition $\varphi_j\le\varphi$: Let $\{\varphi_j\}\subset\mathrm{Psh}(D)$ converge to φ in $L^1_{\mathrm{loc}}(D)$ with $\varphi_j\le\varphi$ and $f\subset\mathscr{O}(D)$ with $(f,o)\in\mathscr{I}(\varphi)_o$. Then $|f|^2e^{-c\varphi_j}$ converges to $|f|^2e^{-c\varphi}$ in L^1-norm near o for $c<c_o^f(\varphi)$.

In [9], Guan–Li–Zhou found and proved the following stability property of the multiplier ideal sheaves which is more general than the above theorems on semi-continuity: Let $\{\varphi_j\}\subset\mathrm{Psh}(D)$ converge to φ locally in measure with $\varphi_j\le 0$ and $\mathscr{I}(\varphi_j)_o\subset\mathscr{I}(\varphi)_o$, and $\{f_j\}\subset\mathscr{O}(D)$ compactly converge to $f\in\mathscr{O}(D)$ with $(f_j,o)\in\mathscr{I}(\varphi)_o$. Then $|f_j|^2e^{-c\varphi_j}$ converges to $|f|^2e^{-c\varphi}$ in L^1-norm near o for $c<c_o^f(\varphi)$.

Now, let us recall recent results of Deng–Ning–Wang–Zhou in [6].

Definition 3 ([6]). *We say that (E,h) satisfies the optimal L^2-estimate condition, if for any Stein coordinate open set U such that $E|_U$ is trivial, any Kähler form ω_U on U, any $\psi\in\mathrm{Spsh}(U)\cap C^\infty(U)$*

and any $\bar{\partial}$*-closed* $f \in L^2_{(n,1)}(U, \omega_U, E|_U, he^{-\psi})$*, there exists* $u \in L^2_{(n,0)}(U, \omega_U, E|_U, he^{-\psi})$ *such that* $\bar{\partial} u = f$ *and*

$$\int_U |u|^2_{\omega_U,h} e^{-\psi} dV_{\omega_U} \le \int_U \left\langle B^{-1}_{\omega_U,\psi} f, f \right\rangle_{\omega_U,h} e^{-\psi} dV_{\omega_U} \tag{1}$$

provided that the right hand side is finite, where $B_{\omega_U,\psi} = [\sqrt{-1}\partial\bar{\partial}\psi \otimes \mathrm{Id}_E, \Lambda_\omega]$.

In [6], Deng–Ning–Wang–Zhou prove that if h is C^2-smooth, then h satisfies the optimal L^2-estimate condition if and only if h is Nakano semi-positive in the usual sense. We observe that Deng–Ning–Wang–Zhou's results can actually answer the following problem.

Lempert's Problem In [15], Lempert asked whether a C^2-smooth hermitian metric, which locally is a limit of an increasing sequence of C^2-smooth Nakano semi-positive hermitian metrics, is also Nakano semi-positive.

Although it is hard to check that the Nakano semi-positivity is preserved under an increasing limit of metrics, it is easy to check that satisfying the optimal L^2-estimate condition is preserved under an increasing limit of metrics, that is, a C^2-smooth hermitian metric, which locally is a limit of an increasing sequence of C^2-smooth hermitian metrics satisfying the optimal L^2-estimate condition, also satisfies the optimal L^2-estimate condition. Hence by [6], the metric is Nakano semi-positive. Therefore, we obtain an affirmative solution to Lempert's problem.

In [2, 17], a singular hermitian metric on a holomorphic vector bundle is defined. It is natural to define a higher rank analogue of multiplier ideal sheaves.

Definition 4 ([2]). *The multiplier submodule sheaf* $\mathscr{E}(h)$ *of* $\mathscr{O}(E)$ *associated with a singular hermitian metric* h *on* E *is the submodule sheaf of the germs of* $s_x \in \mathscr{O}(E)_x$ *such that* $|s_x|^2_h$ *is integrable on some neighborhood of* x.

Deng–Ning–Wang–Zhou's results also naturally lead to a definition of singular Nakano semi-positive metrics.

Definition 5 (see [13]). *We say that a singular hermitian metric* h *is singular Nakano semi-positive if*

(1) *h is singular Griffiths semi-positive;*
(2) *h satisfies the optimal L^2-estimate condition.*

When h is singular Nakano semi-positive one can show that $\mathscr{E}(h)$ is a coherent subsheaf of $\mathscr{O}(E)$ (see [13]) and $h(\det h)^\beta$ is also singular Nakano semi-positive. Hence one may define the "upper semicontinuous regularization" of the sheaf $\mathscr{E}(h)$ to be:

$$\mathscr{E}_+(h) = \bigcup_{\beta>0} \mathscr{E}\left(h(\det h)^\beta\right).$$

It is natural to ask the following question.

Question 6. Is $\mathscr{E}_+(h)$ equal to $\mathscr{E}(h)$ when h is singular Nakano semi-positive?

When E is a line bundle, then $\mathscr{E}(h) = \mathscr{I}(h)$, $\mathscr{E}_+(h) = \mathscr{I}_+(h)$, and the above question is just the strong openness conjecture. In this note, we announce affirmative answers to the above question and some related more general ones, and outline the main ideas of the proofs. For complete proofs of the theorems, the reader is referred to [16].

Since the question is local, we may assume $E = D \times \mathbb{C}^r$ where $D \ni o$ is Stein open in $\mathbb{C}^n$. Inspired by [10] and [8], we introduce some useful numbers as follows and establish an effectiveness result for the question.

Definition 7. *Let* $F \in H^0(D, E)$ *and* h *be singular Nakano semi-positive, we define the complex singularity exponent associated to* h *with respect to* F *at* o *to be*

$$c_o^F(h) := \sup\left\{ \beta \ge 0 \,\middle|\, (F, o) \in \mathscr{E}\left(h(\det h)^\beta\right)_o \right\},$$

and for any open subset $o \in U \subset D$,

$$C_{F,\beta}(U) := \inf\left\{ \int_U |\widetilde{F}|^2_{\frac{h}{\det h}} \;\middle|\; \widetilde{F} \in H^0(D,E) \quad and \quad \left(\widetilde{F} - F, o\right) \in \mathscr{E}\left(h(\det h)^{\beta}\right)_o \right\},$$

$$C_{F,\beta^+}(U) := \inf\left\{ \int_U |\widetilde{F}|^2_{\frac{h}{\det h}} \;\middle|\; \widetilde{F} \in H^0(D,E) \quad and \quad \left(\widetilde{F} - F, o\right) \in \bigcup_{\beta' > \beta} \mathscr{E}\left(h(\det h)^{\beta'}\right)_o \right\}.$$

One can prove that $C_{F,c_o^F(h)^+}(D) > 0$.

Theorem 8. *Assume $\varphi = -\log\det h < 0$ on D and $\varphi(o) = -\infty$. Then for any $F \in \mathscr{E}(h)$ satisfying $\int_D |F|^2_h < +\infty$, we have $F \in \mathscr{E}(he^{-\beta\varphi})_o$ if*

$$\theta(\beta) > \frac{\int_D |F|^2_h}{C_{F,c_o^F(h)^+}(D)}$$

where

$$\theta(\beta) := 1 + \int_0^{+\infty} \left(1 - \exp\left(-\int_t^{+\infty} \frac{ds}{se^{1+(1+\beta)s}}\right)\right) e^t dt.$$

In particular, $\mathscr{E}_+(h) = \mathscr{E}(h)$.

As an application of Theorem 8, we show the strong openness and stability property of multiplier submodule sheaves under two kinds of settings.

Theorem 9. *Let h be a singular hermitian metric which is locally bounded below by a continuous hermitian metric and $\{h_j\}$ a sequence of singular Nakano semi-positive metrics on E. If $h_j \geq h$ and $-\log\det h_j$ converges to $-\log\det h$ locally in measure, then $\sum_j \mathscr{E}(h_j) = \mathscr{E}(h)$. In particular, $\mathscr{E}(h)$ is coherent.*

Moreover, let $\{F_j\} \subset H^0(D,E)$ satisfy that $(F_j, o) \in \mathscr{E}(h)_o$ and F_j compactly converge to $F \in H^0(D,E)$. Then for any given $p \in (0, 1 + c_o^F(h))$, $|F_j|^2_{h_j}$ converges to $|F|^2_h$ in L^p in some neighborhood of o.

In [15], Lempert proves the above theorem for a decreasing sequence $\{h_j\}$ of singular hermitian metrics whose Nakano curvatures dominate 0, where h_j is called a singular hermitian metric whose Nakano curvature dominates 0 if there exists a sequence $\{h_{j_k}\}$ of C^2-smooth Nakano semi-positive hermitian metrics which increasingly converges to h_j. It is easy to check that a singular hermitian metric whose Nakano curvature dominates 0 is singular Nakano semi-positive in the sense of Definition 5.

For such an h in the Theorem 9, we have that $\bigcup_{\beta>0} \mathscr{E}(h \det h^{\beta}) = \mathscr{E}(h)$ although h may not be singular Nakano semi-positive. In fact, by the strong Noetherian property, $\sum_j \mathscr{E}(h_j)_o = \sum_{j \leq j_o} \mathscr{E}(h_j)_o$. Applying Theorem 8 to these finite singular Nakano semi-positive metrics $\{h_j\}_{j \leq j_o}$, we obtain that there exists some $\beta > 0$ such that $\mathscr{E}(h_j)_o = \mathscr{E}(h_j \det h_j^{\beta})_o$ for any $j \leq j_o$. Then

$$\mathscr{E}(h)_o = \sum_j \mathscr{E}\left(h_j\right)_o = \sum_{j \leq j_o} \mathscr{E}\left(h_j\right)_o = \sum_{j \leq j_o} \mathscr{E}\left(h_j \det h_j^{\beta}\right)_o \subset \mathscr{E}\left(h \det h^{\beta}\right)_o.$$

Theorem 10. *Let h be a singular Nakano semi-positive metric and $\{h_j\}$ a sequence of singular Griffiths semi-positive metrics on E. If $h_j \geq h$ and $-\log\det h_j$ converges to $-\log\det h$ locally in measure, then $\bigcup_j \mathscr{E}(h_j) = \mathscr{E}(h)$.*

Moreover, let $\{F_j\} \subset H^0(D,E)$ satisfy that $(F_j, o) \in \mathscr{E}(h)_o$ and F_j compactly converge to $F \in H^0(D,E)$. Then for any given $p \in (0, 1 + c_o^F(h))$, $|F_j|^2_{h_j}$ converges to $|F|^2_h$ in L^p in some neighborhood of o.

2. Ideas of the proofs

In this section, we present the main ideas in the proofs of the theorems. Complete proofs can be found in [16].

Noticing that $\frac{h}{\det h}$ is singular Griffiths semi-negative, we decompose the singular Griffiths semi-positive metric h as

$$h = \frac{h}{\det h} e^{-\varphi} \quad \text{and hence} \quad |F|_h^2 = |F|^2_{\frac{h}{\det h}} e^{-\varphi}$$

where the weight $\varphi = -\log\det h$ is plurisubharmonic. Firstly, we get a convergence lemma with respect to the singular hermitian metric $\frac{h}{\det h}$.

Lemma 11. *Let h be a singular hermitian metric on a trivial holomorphic vector bundle E over D and $F_j \in H^0(D,E)$. Assume that h is bounded below by a continuous hermitian metric and $\varphi = -\log\det h$ is plurisubharmonic. If for any relatively compact subset U of D there exists C_U independent of j such that*

$$\int_U |F_j|^2_{\frac{h}{\det h}} \leq C_U < +\infty \;\forall\, j,$$

then there exists a subsequence of $\{F_j\}$ compactly converging to $F_0 \in H^0(D,E)$. Moreover, if $C_U \leq C$ for any U, then $\int_D |F_0|^2_{\frac{h}{\det h}} \leq C$.

Simulating the approach of Guan–Zhou in [10], essentially we only need to prove the following lemma.

Lemma 12. *Let $o \in D \subset \mathbb{C}^n$ be a Stein open set, E be a trivial holomorphic vector bundle over D with singular Nakano semi-positive metric h. Assume $\varphi = -\log\det h < 0$ and $\varphi(o) = -\infty$. Then for given $t_0 > 0$, $B \in (0,1]$, $\beta \geq 0$, and $F_{t_0} \in H^0(D,E)$ with*

$$\int_{\{\varphi < -t_0\}} |F_{t_0}|^2_{\frac{h}{\det h}} < +\infty,$$

there exists

$$\widetilde{F} = \widetilde{F}_{t_0,B,\beta} \in H^0(D,E) \quad \text{such that} \quad \left(\widetilde{F} - F_{t_0}, o\right) \in \mathscr{E}\left(h(\det h)^\beta\right)_o$$

and

$$\int_D \left|\widetilde{F} - \left(1 - b_{t_0,B}(\varphi)\right) F_{t_0}\right|^2_{h(\det h)^\beta} e^{-\chi_{t_0,B}(\varphi)} \leq \frac{t_0 + B}{B} e^{1+(1+\beta)(t_0+B)} \int_{\{-t_0 - B \leq \varphi < -t_0\}} |F_{t_0}|^2_{\frac{h}{\det h}}$$

where $\chi_{t_0,B}(t) = \frac{1}{t_0+B}\int_0^t b_{t_0,B}(s)ds$, $b_{t_0,B}(t) = \int_{-\infty}^t \frac{1}{B}\mathbb{1}_{\{-t_0-B<s<-t_0\}}ds$.

It is worth mentioning that the constant $\frac{t_0+B}{B}e^{1+(1+\beta)(t_0+B)}$ is not as good as the constant $\frac{e^{(1+\beta)(t_0+B)}-1}{B}$ given by Guan–Zhou [8], who used a more precise calculation on the Chern curvature. However, in [17] Raufi points out that the Chern curvature of a singular Nakano semi-positive metric does NOT exist as a current in general. Fortunately, our constant also works in the proof of Theorem 8 since $\theta(\beta) = 1 + \int_0^{+\infty}(1 - e^{-g_\beta(t)})e^t dt$ goes to infinity as $\beta \to 0^+$, where $g_\beta(t) = \int_t^{+\infty} \frac{ds}{se^{1+(1+\beta)s}}$.

In the proof of the Lemma 12, we do not know whether there exists a sequence of smooth Nakano semi-positive metrics increasingly converging to h. Via a standard approximation procedure, we need to show that

$$\int_{\{-t_0-B\leq\varphi_j<-t_0\}\cap D_k} |F_{t_0}|^2_{h\det h^\beta}$$

is finite for j large enough where the smooth strictly plurisubharmonic function φ_j decreasingly converges to φ and $\{D_k\}$ is a relatively compact Stein open exhaustion of D.

This is owing to the following facts:

- $|F_{t_0}|^2_{\frac{h}{\det h}}$ is locally bounded;

- $A = \operatorname{Supp}\mathscr{O}_X/\mathscr{I}((1+\beta)\varphi) \subset \{\varphi = -\infty\}$ is an analytic subset and $e^{-(1+\beta)\varphi}$ is locally integrable outside A;
- $\{\varphi_j \ge -t_0 - B\}$ decreasingly converges to $\{\varphi \ge -t_0 - B\}$;
- $\bigcap_j (A \cap \overline{D}_k \cap \{\varphi_j \ge -t_0 - B\}) = \emptyset$.

In the proof of the "strong openness" part of Theorem 9, $\varphi_j = -\log\det h_j \le \varphi$ converges to φ locally in measure, hence we may assume that φ_j converges to φ almost everywhere. In a similar manner as in the proof of Lemma 12 we also can obtain

$$\int_D \left|\widetilde{F} - \left(1 - b_{t,B}\left(\varphi_j\right)\right) F\right|^2_{h_j} e^{-\chi_{t,B}(\varphi_j)} \le \frac{t+B}{B} e^{1+t+B} \int_{D\cap\{-t-B<\varphi_j\le -t\}} |F|^2_{\frac{h_j}{\det h_j}},$$

We observe that $h_j \ge h$ implies $\frac{h_j}{\det h_j} \le \frac{h}{\det h}$ and $\varphi_j \le \varphi$ implies $\lim_j \mathbb{1}_{\{\varphi_j \le -t\}} = \mathbb{1}_{\{\varphi \le -t\}}$ almost everywhere. By a contradiction argument, we obtain that $\sum_j \mathscr{E}(h_j)_o \supseteq \mathscr{E}(h)_o$ for any $o \in X$.

In order to prove the "strong openness" part of Theorem 10, the following lemmas turn out to be useful.

Lemma 13. *Let E be a trivial holomorphic vector bundle over $D \ni o$ with singular Nakano semi-positive metric h. Then the following are equivalent:*

(1) $|F|^2_{h(\det h)^\beta}$ *is integrable near o for some* $\beta > 0$;

(2) $(|F|^2_h)^p$ *is integrable near o for some* $p > 1$.

Lemma 14. *Let φ and φ_j be plurisubharmonic functions. If $\varphi_j \le \varphi$ converges to φ locally in measure, then $e^{\varphi-\varphi_j}$ converges to* 1 *in L^p_{loc} for any* $0 < p < +\infty$.

The first lemma is based on Skoda's integrability theorem and Hölder's inequality. The second lemma follows from a combination of Skoda's integrability theorem, Demailly's approximation theorem and the Guan–Li–Zhou stability theorem for the multiplier ideal sheaves.

For the sake of proving the "stability parts" of Theorem 9 and Theorem 10, we need to consider the following setting. Let E be a trivial holomorphic vector bundle over an open set $o \in \Omega \subset \mathbb{C}^n$, h be a singular hermitian metric on E satisfying:

(1) h is bounded below by a continuous hermitian metric;
(2) $\varphi := -\log\det h$ equals to some plurisubharmonic function almost everywhere;
(3) $\mathscr{E}(h)$ is coherent;
(4) $\bigcup_{\beta>0} \mathscr{E}(he^{-\beta\varphi}) = \mathscr{E}(h)$.

Let $\{e_k\}_{k=1}^{k_0}$ be a minimal generating set of $\mathscr{E}(h)_o$. Then by (4), there exists $\varepsilon_0 > 0$ and a neighborhood U of o such that $\int_U |e_k|^2_h e^{-\varepsilon_0\varphi} < \infty$ for $1 \le k \le k_0$. Taking a Stein open set $o \in D \Subset U$ such that $D = \{\bar{z} \mid z \in D\}$ and $\varphi < 0$ on D, We consider the inner space

$$\mathscr{H}(D,o) = \left\{ f \in H^0(D,E) \,\middle|\, \int_D |f|^2_{\frac{h}{\det h}} < +\infty \quad \text{and} \quad (f,o) \in \mathscr{E}(h)_o \right\}.$$

It follows from (1), (2) and Lemma 11 that $\mathscr{H}(D,o)$ is a Hilbert space. Then $\{e_k\}_{k=1}^{k_0} \subset \mathscr{H}(D,o)$, and after the Schmidt orthogonalization, we may assume that $\{e_k\}_{k=1}^{k_0}$ is orthonormal in $\mathscr{H}(D,o)$, and then extend $\{e_k\}_{k=1}^{k_0}$ to an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of $\mathscr{H}(D,o)$.

Using the same argument in the case of line bundles(see [5] or [9]), one can prove that for any $V \Subset D$ there exists $k_V \ge k_0$ and $C_V > 0$ such that

$$\sum_{k=1}^{\infty} |e_k|^2_{\frac{h}{\det h}} \le C_V \sum_{k=1}^{k_V} |e_k|^2_{\frac{h}{\det h}} \quad \text{on } V.$$

Moreover, by taking V small enough, we can request that $k_V = k_0$, which is sharp. By the way, the sharp result comes from the minimality of the generating set $\{e_k\}_{k=1}^{k_0}$.

Then the "stability parts" of Theorem 9 and Theorem 10 can be reduced to the following theorem.

Theorem 15. *Let h satisfy* (1)-(4) *and h_j be singular Griffiths semi-positive on E with $h_j \geq h$ and $F_j \in H^0(\Omega, E)$ with $(F_j, o) \in \mathscr{E}(h)_o$. Assume that $\varphi_j = -\log\det h_j$ converges to $\varphi = -\log\det h$ locally in measure and F_j compactly converges to $F \in H^0(\Omega, E)$. Then for any given $p \in (0, 1 + c_o^F(h))$, $|F_j|^2_{h_j}$ converges to $|F|^2_h$ in L^p in some neighborhood of o.*

Proof. For any given $p \in (0, 1+c_o^F(h))$, we can find $p < 1+\varepsilon_0 < 1+c_o^F(h)$, U, D, $\{e_k\}$ and V as above. Since $|F_j|^2_{h_j}$ converges to $|F|^2_h$ locally in measure, by [18] we only need to show that $\{|F_j|^2_{h_j}\}_j$ is bounded in $L^{\widetilde{p}}(W)$ for some $p < \widetilde{p} < 1 + \varepsilon_0$ and small neighborhood $W \subset V$ of o.

Note that $|F_j|^2_{h_j} \leq |F_j|^2_h e^{\varphi - \varphi_j}$. By Hölder's inequality, one has

$$\int_W \left(|F_j|^2_h e^{\varphi-\varphi_j}\right)^{\widetilde{p}} \leq \left(\int_W \left(|F_j|^2_h\right)^{p_0\widetilde{p}}\right)^{1/p_0} \left(\int_W e^{q_0 p\varphi - q_0 p\varphi_j}\right)^{1/q_0},$$

where $p_0 = (1+\varepsilon_0)/\widetilde{p} > 1$ and $1/p_0 + 1/q_0 = 1$.

It follows from Lemma 14 (2) that we only need to show that $\{|F_j|^2_h\}_j$ is bounded in $L^{1+\varepsilon_0}(W)$.

Firstly

$$\left|F_j\right|^2_{\frac{h}{\det h}} \leq M$$

holds on D by (1) and the fact that F_j compactly converges to F on Ω. Note that $F_j|_D \in \mathscr{H}(D, o)$, we write $F_j = \sum_k a_j^k e_k$ on D where $a_j^k \in \mathbb{C}$ and

$$\sum_{k=1}^{\infty} \left|a_j^k\right|^2 = \int_D \left|F_j\right|^2_{\frac{h}{\det h}} \leq \mathrm{Vol}(D) M.$$

Then we obtain

$$\begin{aligned}\int_W \left(|F_j|^2_h\right)^{1+\varepsilon_0} &\leq M^{\varepsilon_0} \int_W |F_j|^2_h e^{-\varepsilon_0\varphi} \leq M^{\varepsilon_0} \int_W \sum_{k=1}^{\infty} \left|a_j^k\right|^2 \sum_{k=1}^{\infty} |e_k|^2_{\frac{h}{\det h}} e^{-(1+\varepsilon_0)\varphi} \\ &\leq C_V M^{\varepsilon_0} \sum_{k=1}^{\infty} \left|a_j^k\right|^2 \int_W \sum_{k=1}^{k_0} |e_k|^2_{\frac{h}{\det h}} e^{-(1+\varepsilon_0)\varphi} \leq C_V M^{1+\varepsilon_0} \mathrm{Vol}(D) \int_W \sum_{k=1}^{k_0} |e_k|^2_h e^{-\varepsilon_0\varphi}.\end{aligned}$$

□

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