INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Tatsien Li and Bopeng Rao

Uniqueness theorem for partially observed elliptic systems and application to asymptotic synchronization

Volume 358, issue 3 (2020), p. 285-295

Published online: 10 July 2020

https://doi.org/10.5802/crmath.31

Les Comptes Rendus. Mathématique *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2020, 358, nº 3, p. 285-295
https://doi.org/10.5802/crmath.31

Optimal Control / *Contrôle Optimal*

Uniqueness theorem for partially observed elliptic systems and application to asymptotic synchronization

Théorème d'unicité de systèmes elliptiques partiellement observés et application à la synchronisation asymptotique

Tatsien Li [a] **and Bopeng Rao** [a, b, c]

[a] Shanghai Key Laboratory for Contemporary Applied Mathematics; Nonlinear Mathematical Modeling and Methods Laboratory, School of Mathematical Sciences, Fudan University, Shanghai 200433, China

[b] Institut de Recherche Mathématique Avancée, Université de Strasbourg, 67084 Strasbourg, France

[c] School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China.

E-mails: dqli@fudan.edu.cn, bopeng.rao@math.unistra.fr.

Abstract. We show that under Kalman's rank condition, the observability of a scalar equation implies the uniqueness of solution to a system of elliptic operators. Using this result, we establish the asymptotic synchronization by groups for second order evolution systems.

Résumé. Nous montrons que sous la condition du rang de Kalman, l'observabilité d'une équation scalaire implique l'unicité de la solution d'un système d'opérateurs elliptiques. En utilisant ce résultat, nous établissons la synchronisation asymptotique par groupes de systèmes d'évolution du second ordre.

Funding. This work was partially supported by National Natural Science Foundation of China under Grant 11831011.

Manuscript received 6th January 2020, accepted 2nd March 2020.

1. Uniqueness Theorem

Let $\mathcal{H}$ and $\mathcal{V}$ be two Hilbert spaces such that $\mathcal{V} \subset \mathcal{H}$ with dense and compact imbedding. Let L be the duality mapping from $\mathcal{V}$ into its dual space $\mathcal{V}'$ such that

$$\langle L\phi, \psi\rangle_{\mathcal{V}',\mathcal{V}} = (\phi, \psi)_{\mathcal{V}}, \quad \forall\, \phi, \psi \in \mathcal{V}. \tag{1}$$

By Riesz–Fréchet's representation theorem, L is an isomorphism from $\mathcal{V}$ onto the dual space $\mathcal{V}'$. Let γ be a linear continuous operator from $\mathcal{V}$ into $\mathcal{V}'$, such that

$$\langle \gamma\phi, \psi\rangle_{\mathcal{V}',\mathcal{V}} = \langle \gamma\psi, \phi\rangle_{\mathcal{V}',\mathcal{V}}, \quad \forall\, \phi, \psi \in \mathcal{V} \tag{2}$$

and

$$\langle \gamma\phi, \phi\rangle_{\mathcal{V}',\mathcal{V}} \geqslant 0; \qquad \langle \gamma\phi, \phi\rangle_{\mathcal{V}',\mathcal{V}} = 0 \quad \text{if and only if} \quad \gamma\phi = 0. \tag{3}$$

Finally, denoting by I_N the identity of $\mathbb{R}^N$, we define the following operators $\mathscr{L}$ and $\mathscr{G}$ of diagonal form:

$$\mathscr{L} = LI_N, \quad \mathscr{G} = \gamma I_N. \tag{4}$$

The first objective of the present paper is to find a simple and efficient characterization for the uniqueness of solution to the over-determined system with variable $\Phi = (\phi^{(1)}, \dots, \phi^{(N)})^T$:

$$\mathscr{L}\Phi + A\Phi = \beta^2\Phi \tag{5}$$

associated with the condition of observation:

$$D^T\mathscr{G}\Phi = 0, \tag{6}$$

where $\beta \in \mathbb{R}$, D is a matrix of order $N \times M$ and A is a symmetric matrix of order N, both with constant entries.

We observe that Kalman's rank condition

$$\operatorname{rank}(D, AD, \dots, A^{N-1}D) = N \tag{7}$$

is necessary for this uniqueness of solution (see Theorem 7). However, since a matrix D satisfying Kalman's rank condition (7) is not invertible in general, the partial observation (6) cannot imply the nullity of the full observation:

$$\gamma\phi^{(i)} = 0, \quad 1 \leqslant i \leqslant N, \tag{8}$$

so the uniqueness of solution to the over-determined system (5)-(6) cannot be obtained by the standard Carleman's theorem of uniqueness of continuation (see [3,4]). Since only Kalman's rank condition is not sufficient, some additional conditions should be required for the uniqueness of continuation.

Definition 1. *Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of A. The matrix A satisfies the ϵ-closing condition if there exists a number a such that*

$$\sup_{1\leqslant l\leqslant m} |\lambda_l - a| \leqslant \epsilon. \tag{9}$$

Definition 2. *The operator L satisfies the c-gap condition if there exists a number $c > 0$, such that*

$$|\alpha_n - \alpha_{n'}| \geqslant c \tag{10}$$

holds true for all distinct eigenvalues $\alpha_n \neq \alpha_{n'}$ of L.

Definition 3. *The pair (L, γ) is observable if there exists a constant $c > 0$, independent of $\beta \in \mathbb{R}$ and $f \in \mathscr{H}$, such that the observability inequality*

$$c\|\phi\|_{\mathscr{H}} \leqslant \|f\|_{\mathscr{H}} \tag{11}$$

holds for any given solution ϕ to the following over-determined scalar problem

$$\beta^2\phi - L\phi = f \quad \textit{with} \quad \gamma\phi = 0. \tag{12}$$

Let us recall the following generalized rank condition of Kalman's type (see [8]).

Proposition 4. *The rank condition*

$$\operatorname{rank}(D, AD, \dots, A^{N-1}D) = N - d \tag{13}$$

holds for one integer $d \geqslant 0$ *if and only if* d *is the largest dimension of the subspaces which are invariant for* A *and contained in* $\operatorname{Ker}(D^T)$.

Now we can give the main result on the uniqueness of continuation.

Theorem 5. *Assume that the pair* (A, D) *satisfies Kalman's rank condition* (7). *Then, the over-determined system* (5)-(6) *has only the trivial solution* $\Phi \equiv 0$ *in any one of the following situations:*

(i) *The operator* γ *is global, namely,*

$$\gamma\phi = 0 \Longrightarrow \phi = 0. \tag{14}$$

(ii) *The matrix* A *satisfies the* ϵ*-closing condition* (9) *with* $\epsilon > 0$ *small enough, the operator* L *satisfies the* c*-gap condition* (10), *and the over-determined scalar problem*

$$\beta^2\phi - L\phi = 0 \quad \textit{and} \quad \gamma\phi = 0 \tag{15}$$

has only the trivial solution $\phi \equiv 0$.

(iii) *The matrix* A *satisfies the* ϵ*-closing condition* (9) *with* $\epsilon > 0$ *small enough and the pair* (L, γ) *is observable.*

In order to simplify the proof, we make some necessary preparations.

First, under a suitable basis, A can be written as

$$A = diag(\overbrace{\lambda_1, \dots, \lambda_1}^{\sigma_1}, \dots\dots, \overbrace{\lambda_m, \dots, \lambda_m}^{\sigma_m}),$$

where λ_l are the eigenvalues of A of multiplicity σ_l $(1 \leqslant l \leqslant m)$.

Accordingly, we regroup the state variable $\Phi = (\phi^{(1)}, \dots, \phi^{(N)})^T$ as

$$\Phi = (\phi^{(1)}, \dots, \phi^{(\mu_1)}, \dots\dots, \phi^{(\mu_{m-1}+1)}, \dots, \phi^{(\mu_m)})^T,$$

where the integers μ_r are defined by

$$\mu_0 = 0, \quad \mu_r = \mu_{r-1} + \sigma_r, \quad r = 1, \dots, m, \text{ and } \mu_m = N.$$

On the other hand, if we replace A by $A + bI$, and β^2 by $\beta^2 + b$ for any given $b > 0$ in (5), it will not modify anything in Theorem 5. So, without loss of generality, we may assume that the eigenvalues of A are strictly positive.

Finally, denote by ϵ_i $(i = 1, \dots, N)$ the canonical basis vectors in $\mathbb{R}^N$, and by d_i the i-th column-vector of the matrix D^T. Because $D^T\epsilon_i = d_i$ $(i = 1, \dots, N)$ and the subspace $V = Span\{\epsilon_{\mu_{l-1}+1}, \dots, \epsilon_{\mu_l}\}$ is invariant for A $(1 \leqslant l \leqslant m)$, by Proposition 4, Kalman's rank condition (7) implies that $V \cap \operatorname{Ker}(D^T) = \{0\}$, namely,

$$\sum_{i=\mu_{l-1}+1}^{\mu_l} \alpha_i d_i = 0 \text{ if and only if } \alpha_i = 0 \text{ for all } \mu_{l-1} + 1 \leqslant i \leqslant \mu_l.$$

Therefore, for each $1 \leqslant l \leqslant m$, the vectors $d_{\mu_{l-1}+1}, \dots, d_{\mu_l}$ are linearly independent.

Proof. Now we give the proof of Theorem 5.

Case (i)**.** From (6) and (14) we have

$$D^T\mathcal{G}\Phi = \gamma(D^T\Phi) = 0 \Longrightarrow D^T\Phi \equiv 0.$$

Then, applying D^T to (5) and noting the first formula of (4), it follows that

$$D^T A\Phi \equiv 0,$$

namely,

$$\sum_{l=1}^{m}\sum_{i=\mu_{l-1}+1}^{\mu_l}\lambda_l\phi^{(i)}d_i=0. \tag{16}$$

We write (5) as

$$L\phi^{(i)}=(\beta^2+\lambda_l)\phi^{(i)},\quad \mu_{l-1}+1\leqslant i\leqslant\mu_l,\quad 1\leqslant l\leqslant m. \tag{17}$$

Since L is self-adjoint, the eigenspaces

$$\mathrm{Span}\{\phi^{(\mu_{l-1}+1)},\dots,\phi^{(\mu_l)}\} \tag{18}$$

associated with the different eigenvalues λ_l are mutually orthogonal. Then it follows from (16) that

$$\sum_{i=\mu_{l-1}+1}^{\mu_l}\lambda_l\phi^{(i)}d_i=0,\quad 1\leqslant l\leqslant m.$$

Since for each $1\leqslant l\leqslant m$, the eigenvalue $\lambda_l\neq 0$ and the vectors $d_{\mu_{l-1}+1},\dots,d_{\mu_l}$ are linearly independent, it follows that

$$\phi^{(i)}=0,\quad \mu_{l-1}+1\leqslant i\leqslant\mu_l,\quad 1\leqslant l\leqslant m,$$

namely $\Phi\equiv 0$.

Case (ii)**.** Assume that there exist two integers l,k with $1\leqslant l,k\leqslant m$ and $l\neq k$, such that $\phi^{(i)}\not\equiv 0$ for some i with $\mu_{l-1}+1\leqslant i\leqslant\mu_l$, and $\phi^{(i')}\not\equiv 0$ for some i' with $\mu_{k-1}+1\leqslant i'\leqslant\mu_k$. Then $\phi^{(i)}$ and $\phi^{(i')}$ will be eigenfunctions of L, so there exist the corresponding eigenvalues α_{n_l} and α_{n_k} such that

$$\beta^2+\lambda_l=\alpha_{n_l}\quad\text{and}\quad\beta^2+\lambda_k=\alpha_{n_k};$$

then

$$\lambda_l-\lambda_k=\alpha_{n_l}-\alpha_{n_k}.$$

However, because of the ϵ-closing condition (9) and the c-gap condition (10), the above equality cannot be satisfied for $\epsilon>0$ small enough. Therefore, there exists at most one integer k with $1\leqslant k\leqslant m$, such that

$$\phi^{(i)}\equiv 0\quad\text{for } i=\mu_{l-1}+1,\dots,\mu_l \text{ and } l\neq k. \tag{19}$$

Then (6) reduces to

$$D^T\mathscr{G}\Phi=\sum_{l=1}^{m}\sum_{i=\mu_{l-1}+1}^{\mu_l}\gamma\phi^{(i)}d_i=\sum_{i=\mu_{k-1}+1}^{\mu_k}\gamma\phi^{(i)}d_i=0.$$

Since the vectors $d_{\mu_{k-1}+1},\dots,d_{\mu_k}$ are linearly independent, it follows from (17) that

$$L\phi^{(i)}=(\beta^2+\lambda_k)\phi^{(i)}\quad\text{with}\quad\gamma\phi^{(i)}=0,\quad\mu_{k-1}+1\leqslant i\leqslant\mu_k. \tag{20}$$

Then the uniqueness of continuation of the scalar problem (15) implies that

$$\phi^{(i)}=0,\quad\mu_{k-1}+1\leqslant i\leqslant\mu_k,$$

which, combined with (19), leads to

$$\phi^{(i)}=0,\quad\mu_{l-1}+1\leqslant i\leqslant\mu_l,\quad 1\leqslant l\leqslant m,$$

namely, $\Phi\equiv 0$.

Case (iii)**.** Applying D^T to (5) and noting $W=D^T\Phi$, we get

$$(\beta^2-a)W-\mathscr{L}W=D^TA\Phi-aW. \tag{21}$$

Setting

$$W=(w_j),\quad D^TA\Phi-aW=(f_j)\quad\text{and}\quad D^T=(d_{ji})$$

for $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant M$, we have

$$w_j = \sum_{i=1}^{N} d_{ji}\phi^{(i)} = \sum_{l=1}^{m} \sum_{i=\mu_{l-1}+1}^{\mu_l} d_{ji}\phi^{(i)}$$

and

$$f_j = \sum_{l=1}^{m} (\lambda_l - a) \sum_{i=\mu_{l-1}+1}^{\mu_l} d_{ji}\phi^{(i)}. \tag{22}$$

On the other hand, by the definition of $\mathscr{G}$ in (4), condition (6) leads to

$$\mathscr{G}W = \mathscr{G}D^T\Phi = D^T\mathscr{G}\Phi = 0. \tag{23}$$

Then, taking the j-th component of (21) and (23), we get

$$(\beta^2 - a)w_j - Lw_j = f_j \tag{24}$$

with the additional condition

$$\gamma w_j = 0. \tag{25}$$

If $\beta^2 - a \leqslant 0$, multiplying (24) by w_j, we get

$$-(\beta^2 - a)\|w_j\|^2_{\mathscr{H}} + \|w_j\|^2_V = -(f_j, w_j)_{\mathscr{H}}.$$

It follows that

$$c\|w_j\|_{\mathscr{H}} \leqslant \|f_j\|_{\mathscr{H}}. \tag{26}$$

If $\beta^2 - a > 0$, then w_j satisfies the scalar problem (12). Since (L, γ) is observable, we get again (26).

By the orthogonality of the eigenspaces (18), it follows from (22) that

$$\|f_j\|^2_{\mathscr{H}} \leqslant \sup_{1\leqslant l\leqslant m} |a-\lambda_l|^2 \sum_{l=1}^{m} \left\| \sum_{i=\mu_{l-1}+1}^{\mu_l} d_{ji}\phi^{(i)} \right\|^2_{\mathscr{H}} = \sup_{1\leqslant l\leqslant m} |a-\lambda_l|^2 \|w_j\|^2_{\mathscr{H}}.$$

By the ϵ-closing condition (9), we get

$$\|f_j\|_{\mathscr{H}} \leqslant \sup_{1\leqslant l\leqslant m} |a-\lambda_l| \|w_j\|_{\mathscr{H}} \leqslant \epsilon \|w_j\|_{\mathscr{H}}.$$

Thus, it follows from (26) that

$$\sum_{i=\mu_{l-1}+1}^{\mu_l} d_{ji}\phi^{(i)} = 0, \quad 1 \leqslant j \leqslant M, \quad 1 \leqslant l \leqslant m,$$

provided that $\epsilon < c$; namely, we have

$$\sum_{i=\mu_{l-1}+1}^{\mu_l} d_i\phi^{(i)} = 0, \quad 1 \leqslant l \leqslant m.$$

Since $d_{\mu_{l-1}+1}, \ldots, d_{\mu_l}$ are linearly independent, we obtain that

$$\phi^{(i)} = 0, \quad \mu_{l-1} + 1 \leqslant i \leqslant \mu_l, \quad 1 \leqslant l \leqslant m,$$

namely, $\Phi \equiv 0$. The proof is then complete. □

The above theorem can be read as *"under Kalman's rank condition on the coupling matrices A and D, the observability of a scalar equation implies the uniqueness of solution to a complex system"*. Thus, it provides a simple and efficient approach to solve a seemingly difficult problem of uniqueness for a complex system.

Case (i) of global observation is similar to the finite-dimensional case. In this case, without any additional conditions on the matrix A or on the operator L, only Kalman's rank condition is sufficient for the uniqueness of solution to the over-determined system (see Theorem 16). In case (ii), thanks to the c-gap condition, the unique continuation of a scalar problem implies the uniqueness of solution to the over-determined system (see Theorem 14). In the general case (iii),

the observability inequality is required and can be established under suitable geometrical control conditions (see Theorem 15). Moreover, the necessity of ϵ-closing condition is shown by an example in [8].

In the previous discussion, we have assumed that the matrix A is symmetric, which is actually essential for the stability of the evolution systems that we will study in the next section. A generalization of Theorem 5 can be found in the complete version [6].

2. Asymptotic synchronization by groups

Recall that the operators L and γ satisfy the conditions (1)-(3). In what follows, we assume furthermore that the operator γ is compact from $\mathcal{V}$ into $\mathcal{V}'$.

Let A and D be symmetric and positive semi-definite matrices. Consider the following second order evolution system with variable $U=(u^{(1)},\dots,u^{(N)})^T$:

$$U_{tt}+\mathscr{L}U+AU+D\mathscr{G}U_t=0 \tag{27}$$

associated with the initial data

$$t=0:\quad U=U_0,\ U_t=U_1. \tag{28}$$

Defining the linear operator $\mathscr{A}$ by

$$\mathscr{A}(U,V)=(V,-\mathscr{L}U-AU-D\mathscr{G}V), \tag{29}$$

we formulate (27) into a first-order evolution system

$$(U,V)_t=\mathscr{A}(U,V). \tag{30}$$

Clearly (see [11]), the operator $\mathscr{A}$ generates a semi-group of contractions in the space $(\mathcal{V}\times\mathscr{H})^N$.

Definition 6. *The system* (27) *is asymptotically (strongly) stable if for any given initial data* $(U_0,U_1)\in(\mathcal{V}\times\mathscr{H})^N$, *the corresponding solution* U *to problem* (27)-(28) *satisfies*

$$(U(t),U_t(t))\to(0,0)\quad \text{in } (\mathcal{V}\times\mathscr{H})^N \text{ as } t\to+\infty. \tag{31}$$

Theorem 7. *If the system* (27) *is asymptotically stable in* $(\mathcal{V}\times\mathscr{H})^N$, *then Kalman's rank condition* (7) *holds. Inversely, assume that the pair* (A,D) *satisfies Kalman's rank condition* (7), *then the system* (27) *is asymptotically stable in any one of the three situations described in Theorem 5.*

Proof. Assume that the rank condition (7) fails. By Proposition 4, $\mathrm{Ker}(D)$ contains at least an eigenvector $E\in\mathbb{R}^N$ associated with an eigenvalue $\lambda\in\mathbb{R}$:

$$AE=\lambda E\quad\text{and}\quad DE=0.$$

Then, applying E to (27) and setting $\phi=(E,U)$, we get

$$\phi_{tt}+L\phi+\lambda\phi=0,$$

which is conservative, so never asymptotically stable.

Inversely, because the resolvent of $\mathscr{A}$ is compact in the space $(\mathcal{V}\times\mathscr{H})^N$, by the classic theory of semi-groups (see [1, 2]), the system (27) is asymptotically stable if and only if $\mathscr{A}$ has no pure imaginary eigenvalues.

In fact, let $\beta\in\mathbb{R}$ and $(\Phi,\Psi)\in\mathcal{V}\times\mathscr{H}$, such that

$$\mathscr{A}(\Phi,\Psi)=i\beta(\Phi,\Psi).$$

It follows that

$$\mathscr{L}\Phi+A\Phi+i\beta D\mathscr{G}\Phi=\beta^2\Phi. \tag{32}$$

Since $\mathscr{L}+A$ is positive definite, we have $\beta\neq0$. Since $\mathscr{L}+A$ and $D\mathscr{G}$ are symmetric, equation (32) is equivalent to the over-determined system (5)-(6), which has only the trivial solution $\Phi\equiv0$ by virtue of Theorem 5. □

By Theorem 7, when the pair (A,D) does not satisfy Kalman's rank condition (7), the system (27) is not asymptotically stable. We then return to consider a weakened notion, the asymptotic synchronization.

Let $p \geqslant 1$ be one integer and

$$0 = n_0 < n_1 < n_2 < \cdots < n_p = N$$

with $n_r - n_{r-1} \geqslant 2$ for all $1 \leqslant r \leqslant p$. We re-arrange the components of the state variable U into p groups as

$$(u^{(1)},\dots,u^{(n_1)}),\ (u^{(n_1+1)},\dots,u^{(n_2)}),\dots\dots,(u^{(n_{p-1}+1)},\dots,u^{(n_p)}).$$

Definition 8. *The system* (27) *is asymptotically synchronizable by p-groups if for any given initial data $(U_0,U_1)\in(\mathscr{V}\times\mathscr{H})^N$, the corresponding solution U to problem* (27)-(28) *satisfies*

$$u^{(k)}(t)-u^{(l)}(t)\to 0,\quad u_t^{(k)}(t)-u_t^{(l)}(t)\to 0\quad \text{in } \mathscr{V}\times\mathscr{H} \text{ as } t\to+\infty \tag{33}$$

for all $n_{r-1}+1\leqslant k,l\leqslant n_r$ and $1\leqslant r\leqslant p$.

Let S_r be the full row-rank matrix of order $(n_r-n_{r-1}-1)\times(n_r-n_{r-1})$:

$$S_r=\begin{pmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & \ddots & \ddots \\ & & & 1 \quad -1 \end{pmatrix}\quad \text{for } 1\leqslant r\leqslant p.$$

Define the $(N-p)\times N$ matrix C_p of synchronization by p-groups as

$$C_p=\begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}.$$

Then the asymptotic synchronization by p-groups (33) can be equivalently rewritten as

$$C_p(U(t),U_t(t))\to(0,0)\quad \text{in } (\mathscr{V}\times\mathscr{H})^{N-p} \text{ as } t\to+\infty. \tag{34}$$

Since the above asymptotic synchronization by p-groups investigates the behaviour of the solutions on the *infinite horizon*, the notion of synchronizable state by p-groups is no longer available as for the synchronization on a *finite interval* considered in [7–10]; therefore the corresponding asymptotic synchronization by p-groups certainly proposes interesting questions and needs new effective methods. This is the second topic to be developed in this paper.

Before starting the study on the asymptotic synchronization, we first give some algebraic preliminaries.

Definition 9. *The matrix A satisfies the condition of C_p-compatibility if*

$$A\,\mathrm{Ker}(C_p)\subseteq\mathrm{Ker}(C_p), \tag{35}$$

or equivalently, there exists a positive semi-definite matrix $\overline{A}_p$ of order $(N-p)$, such that

$$(C_pC_p^T)^{-1/2}C_pA=\overline{A}_p(C_pC_p^T)^{-1/2}C_p. \tag{36}$$

Definition 10. *The matrix D satisfies the condition of strong C_p-compatibility if*

$$\mathrm{Ker}(C_p)\subseteq\mathrm{Ker}(D), \tag{37}$$

or equivalently, there exists a positive semi-definite matrix R of order $(N-p)$, such that

$$D=C_p^TRC_p. \tag{38}$$

In this case, D satisfies also the condition of C_p-compatibility. Moreover, setting

$$\overline{D}_p = (C_pC_p^T)^{1/2}R(C_pC_p^T)^{1/2},$$

we have

$$(C_pC_p^T)^{-1/2}C_pD = \overline{D}_p(C_pC_p^T)^{-1/2}C_p.$$

Now we give the basic idea of asymptotic synchronization by p-groups. Assume that A and D satisfy the conditions of compatibility. Applying the matrix $(C_pC_p^T)^{-1/2}C_p$ to the system (27), and setting $W = (C_pC_p^T)^{-1/2}C_pU$, we get a reduced self-closed system

$$W_{tt} + \mathscr{L}W + \overline{A}_pW + \overline{D}_p\mathscr{G}W_t = 0. \tag{39}$$

Then the asymptotic synchronization by p-groups of the system (27) is equivalent to the asymptotic stability of the reduced system (39). Since the reduced matrices $\overline{A}_p$ and $\overline{D}_p$ are still symmetric, the asymptotic stability of the reduced system (39) is equivalent to the uniqueness of the over-determined system with the reduced variable $\Psi = (\psi^{(1)}, \dots, \psi^{(N-p)})^T$:

$$\mathscr{L}\Psi + \overline{A}_p\Psi = \beta^2\Psi \tag{40}$$

associated with the condition of observation

$$\overline{D}_p\mathscr{G}\Psi = 0. \tag{41}$$

The previous approach is direct and efficient. However, the necessity of the conditions of compatibility is a delicate question. The following theorem shows that it is in fact the consequence of the minimality of the rank of Kalman's matrix. This makes the theory of asymptotic synchronization by p-groups more complete for systems of partial differential equations.

Theorem 11. *If the system* (27) *is asymptotically synchronizable by p-groups, then necessarily*

$$\operatorname{rank}(D, AD, \dots, A^{N-1}D) \geqslant N - p. \tag{42}$$

Furthermore, if the system (27) *is asymptotically synchronizable by p-groups under the minimal rank condition*

$$\operatorname{rank}(D, AD, \dots, A^{N-1}D) = N - p, \tag{43}$$

then, necessarily, A satisfies the condition of C_p-compatibility (36) *and D is given by* (38).

Theorem 12. *Assume that the pair (A, D) satisfies Kalman's rank condition* (43). *Assume furthermore that the matrix A satisfies the condition of C_p-compatibility* (36) *and that the matrix D is given by* (38). *Then the system* (27) *is asymptotically synchronizable by p-groups in any one of the three situations described in Theorem 5.*

Proof. As explained above, it suffices to show the asymptotic stability of the reduced system (39), or the uniqueness of the reduced over-determined problem (40)-(41). By Theorem 7, this is true under the condition

$$\operatorname{rank}(\overline{D}_p, \overline{A}_p\overline{D}_p, \dots, \overline{A}_p^{N-p-1}\overline{D}_p) = N - p,$$

which is equivalent to (43) (see [6] for details). □

Theorem 13. *Assume that the matrix A satisfies the condition of C_p-compatibility* (36) *and that the matrix D is given by* (38). *If the system* (27) *is asymptotically synchronizable by p-groups, then for any given initial data $(U_0, U_1) \in (\mathscr{V} \times \mathscr{H})^N$, there exist some scalar functions $u_1, \dots, u_p$ such that*

$$u^{(k)}(t) - u_r(t) \to 0, \quad u_t^{(k)}(t) - u_{rt}(t) \to 0 \quad \text{in } \mathscr{V} \times \mathscr{H} \text{ as } t \to +\infty$$

for all $n_{r-1} + 1 \leqslant k \leqslant n_r$ and $1 \leqslant r \leqslant p$.

In this case, the system (27) is called asymptotically synchronizable by p-groups in the pinning sense with the asymptotically synchronizable state by p-groups $u = (u_1, \dots, u_p)^T$. However, unlike the approximately synchronizable state by p-groups (see [8, 9]), the asymptotically synchronizable state by p-groups u is not uniquely determined.

3. Applications to wave equations

We will give some classic examples to illustrate possible applications of the abstract theory mentioned above. However, the approach is quite flexible and can easily be applied to other types of wave equations with variable density, or outside a star-shaped domain (cf. [5, 12, 13]).

In what follows, Ω is a bounded domain in $\mathbb{R}^n$ with suitably smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_0$ such that $mes(\Gamma_1) > 0$. Let $H^1_{\Gamma_0}(\Omega)$ denote the subspace of $H^1(\Omega)$ of functions with vanishing trace on Γ_0.

Consider the following coupled system of wave equations with the state variable $U = (u^{(1)}, \dots, u^{(N)})^T$:

$$\begin{cases} U_{tt} - \Delta U + AU = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}^+ \times \Gamma_0, \\ \partial_\nu U + DU_t = 0 & \text{on } \mathbb{R}^+ \times \Gamma_1. \end{cases} \tag{44}$$

Multiplying the system (44) by $\Phi = (\phi^{(1)}, \dots, \phi^{(N)}) \in (H^1_{\Gamma_0}(\Omega))^N$ and integrating by parts, we get the following variational formulation:

$$\int_\Omega (U_{tt}, \Phi)\, \mathrm{d}x + \int_\Omega (\nabla U, \nabla \Phi)\, \mathrm{d}x + \int_\Omega (AU, \Phi)\, \mathrm{d}x + \int_{\Gamma_1} (DU_t, \Phi)\, \mathrm{d}\Gamma = 0. \tag{45}$$

Let L and γ be defined by

$$\langle Lv, \phi \rangle = \int_\Omega \nabla v \cdot \nabla \phi \, \mathrm{d}x \quad \text{and} \quad \langle \gamma v, \phi \rangle = \int_{\Gamma_1} v\phi \, \mathrm{d}\Gamma. \tag{46}$$

With $\mathscr{L}$ and $\mathscr{G}$ as in (4), the variational equation (45) can be rewritten as

$$U_{tt} + \mathscr{L}U + AU + D\mathscr{G}U_t = 0. \tag{47}$$

3.1. *Case with the gap condition*

We first consider a specific situation on a rectangular domain

$$\Omega = (0, \pi) \times (0, a\pi), \quad \Gamma_0 = \{(0, y) \cup (\pi, y),\ 0 < y < a\pi\}, \quad \Gamma_1 = \Gamma \setminus \Gamma_0, \tag{48}$$

where $a > 0$ is a parameter.

Theorem 14. *Let a^2 be a rational. Assume that the pair (A, D) satisfies the rank condition* (7) *and that A satisfies the ϵ-closing condition with $\epsilon > 0$ small enough. Then the system* (44) *is asymptotically stable.*

Proof. Consider the following eigensystem:

$$\begin{cases} -\Delta \phi = \alpha \phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma_0, \\ \partial_\nu \phi = 0 & \text{on } \Gamma_1. \end{cases}$$

By Carleman's uniqueness theorem (see [3, 4]), under the additional condition $\phi = 0$ on Γ_1, the above system has only the trivial solution. Moreover, a straightforward computation gives the eigenvalues and the associated eigenvectors as follows:

$$\alpha_{k,l} = k^2 + \frac{l^2}{a^2}, \quad \phi_{k,l} = \sin(kx) \cos\Big(\frac{ly}{a}\Big).$$

Since a^2 is a rational, the eigenvalues satisfy the c-gap condition:

$$\alpha_{k,l} - \alpha_{p,q} = k^2 - p^2 + \frac{l^2 - q^2}{a^2} \geqslant c > 0$$

for all $\alpha_{k,l} > \alpha_{p,q}$. Then, by Theorem 7, the system (44) is asymptotically stable, provided that A satisfies the ϵ-closing condition with $\epsilon > 0$ small enough.

If a^2 is an irrational, the c-gap condition is no longer valid. In this case, we don't know if the system (44) is asymptotically stable or not, even though the coupling matrix A satisfies the ϵ-closing condition with $\epsilon > 0$ small enough. □

3.2. *Case with the observability inequality*

In this subsection, we assume that there exists $x_0 \in R^n$, such that setting $m = x - x_0$, we have $(m \cdot \nu) \leqslant 0$ on Γ_0. Then there exists a constant $c > 0$ independent of β and f, such that any given solution ϕ to the over-determined scalar problem

$$\begin{cases} \beta^2\phi + \Delta\phi = f & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma, \\ \partial_\nu\phi = 0 & \text{on } \Gamma_1 \end{cases}$$

satisfies the following observability inequality

$$\int_\Omega (|\beta\phi|^2 + |\nabla\phi|^2)\,\mathrm{d}x \leqslant c\int_\Omega |f|^2\,\mathrm{d}x.$$

In other words, the pair (L, γ) defined by (46) is observable (see [6] for details). Noting that γ is compact from $H^1_{\Gamma_0}(\Omega)$ into $(H^1_{\Gamma_0}(\Omega))'$, by Theorem 12 we have the following

Theorem 15. *Assume that the pair* (A, D) *satisfies the rank condition* (43) *and that* A *satisfies the* ϵ*-closing condition* (9) *with* $\epsilon > 0$ *small enough. Assume furthermore that* A *satisfies the condition of* C_p*-compatibility* (36) *and that* D *is given by* (38). *Then the system* (44) *is asymptotically synchronizable by* p*-groups.*

3.3. *Case of global damping*

Finally, we investigate the asymptotic stability of a system of wave equations with globally distributed damping:

$$\begin{cases} U_{tt} - \Delta U + AU + DU_t = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{49}$$

As in the finite-dimensional case of ordinary differential equations, without any additional conditions, only Kalman's rank condition is sufficient for the asymptotic stability of wave equations. By Theorem 7, we have the following

Theorem 16. *Let the pair* (A, D) *satisfy the rank condition* (7). *Then the system* (49) *is asymptotically stable.*

References

[1] W. Arendt, C. J. K. Batty, "Tauberian theorems and stability of one-parameter semi-groups", *Trans. Am. Math. Soc.* **306** (1988), no. 2, p. 837-852.

[2] C. D. Benchimol, "A note on weak stabilization of contraction semi-groups", *SIAM J. Control Optimization* **16** (1978), p. 373-379.

[3] N. Garofalo, F.-H. Lin, "Unique continuation for elliptic operators: A geometric-variational approach", *Commun. Pure Appl. Math.* **40** (1987), no. 3, p. 347-366.

[4] H. Koch, D. Tataru, "Carleman estimates and uniqueness of solution for second-order elliptic equations with nonsmooth coefficients", *Commun. Pure Appl. Math.* **54** (2001), no. 3, p. 339-360.

[5] F. Li, Z. Jia, "Global existence and stability of a class of nonlinear evolution equations with hereditary memory and variable density", *Bound. Value Probl.* **2019** (2019), article ID 37.

[6] T. Li, B. Rao, "Uniqueness of solution for systems of elliptic operators and application to asymptotic synchronization of linear dissipative systems", to appear.
[7] ———, "Asymptotic controllability and asymptotic synchronization for a coupled system of wave equations with Dirichlet boundary controls", *Asymptotic Anal.* **86** (2014), no. 3-4, p. 199-226.
[8] ———, "Criteria of Kalman's type to the approximate controllability and the approximate synchronization for a coupled system of wave equations with Dirichlet boundary controls", *SIAM J. Control Optimization* **54** (2016), no. 1, p. 49-72.
[9] ———, "On the approximate boundary synchronization for a coupled system of wave equations: Direct and indirect controls", *ESAIM, Control Optim. Calc. Var.* **24** (2018), no. 4, p. 1675-1704.
[10] ———, *Boundary Synchronization for Hyperbolic Systems*, Progress in Nonlinear Differential Equations and their Applications, vol. 94, Birkhäuser, 2019.
[11] A. Pazy, *Semi-Groups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer, 1983.
[12] L. Ren, J. Xin, "Almost global existence for the Neumann problem of quasilinear wave equations outside star-shaped domains in 3D", *J. Differ. Equations* **2017** (2017), article ID 312 (22 pages).
[13] X. Zheng, J. Xin, X. Peng, "Orbital stability of periodic traveling wave solutions to the generalized long-short wave equations", *J. Appl. Anal. Comput.* **9** (2019), p. 2389-2408.