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Decreasing properties of two ratios defined by three and four polygamma functions

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Dedicated to my elder brother, Can-Long Qi, and his family

Abstract. In the paper, by virtue of the convolution theorem for the Laplace transforms, with the aid of three monotonicity rules for the ratios of two functions, of two definite integrals, and of two Laplace transforms, in terms of the majorization, and in the light of other analytic techniques, the author presents decreasing properties of two ratios defined by three and four polygamma functions.

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1. Motivations

In the literature [1, Section 6.4], the function

$$\Gamma(z)=\int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t,\quad \Re(z)>0$$

and its logarithmic derivative $\psi(z)=[\ln\Gamma(z)]'=\frac{\Gamma'(z)}{\Gamma(z)}$ are called Euler's gamma function and digamma function respectively. Moreover, the functions $\psi'(z)$, $\psi''(z)$, $\psi'''(z)$, and $\psi^{(4)}(z)$ are known as the trigamma, tetragamma, pentagamma, and hexagamma functions respectively. As a set, all the derivatives $\psi^{(k)}(z)$ for $k\in\mathbb{N}_0=\{0\}\cup\mathbb{N}$ are known as polygamma functions.

Recall from Chapter XIII in [13], Chapter 1 in [36], and Chapter IV in [40] that, if a function $f(x)$ on an interval I has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x)\ge 0$ for $x\in I$ and $n\in\mathbb{N}_0$, then we call $f(x)$ a completely monotonic function on I.

In [28, Theorem 1.1] and [27, Theorem 3], among other things, the function

$$[\psi'(x)]^2+\lambda\psi''(x)$$

was proved to be completely monotonic on $(0,\infty)$ if and only if $\lambda\le 1$. In [9, Theorem 1], it was proved that, among the functions

$$f_{m,n}(x)=\left[\psi^{(m)}(x)\right]^2+\psi^{(n)}(x),\quad m,n\in\mathbb{N},\quad x\in(0,\infty),$$

(1) the functions
$$f_{1,2}(x)=[\psi'(x)]^2+\psi''(x)$$
and
$$f_{m,2n-1}(x)=\left[\psi^{(m)}(x)\right]^2+\psi^{(2n-1)}(x)$$
are completely monotonic on $(0,\infty)$, but the complete monotonicity of $f_{m,2n-1}(x)$ is trivial;

(2) the functions
$$f_{m,2n}(x)=\left[\psi^{(m)}(x)\right]^2+\psi^{(2n)}(x)$$
for $(m,n)\ne(1,1)$ are not monotonic and does not keep the same sign on $(0,\infty)$.

For $k\in\mathbb{N}$ and $x\in(0,\infty)$, let
$$\mathscr{F}_{k,\eta_k}(x)=\psi^{(2k)}(x)+\eta_k\left[\psi^{(k)}(x)\right]^2 \quad\text{and}\quad \mathfrak{F}_{k,\vartheta_k}(x)=\frac{\psi^{(2k)}(x)}{[(-1)^{k+1}\psi^{(k)}(x)]^{\vartheta_k}}.$$
In [20, Theorem 3.2], the author proved the following conclusions:

(1) if and only if $\eta_k\ge\frac{1}{2}\frac{(2k)!}{(k-1)!k!}$, the function $\mathscr{F}_{k,\eta_k}(x)$ is completely monotonic on $(0,\infty)$;
(2) if and only if $\eta_k\le0$, the function $-\mathscr{F}_{k,\eta_k}(x)$ is completely monotonic on $(0,\infty)$;
(3) if and only if $\vartheta_k\ge2$, the function $\mathfrak{F}_{k,\vartheta_k}(x)$ is decreasing on $(0,\infty)$;
(4) if and only if $\vartheta_k\le\frac{2k+1}{k+1}$, the function $\mathfrak{F}_{k,\vartheta_k}(x)$ is increasing on $(0,\infty)$;
(5) the following limits are valid:
$$\lim_{x\to0^+}\mathfrak{F}_{k,\vartheta_k}(x)=\begin{cases}-\dfrac{(2k)!}{[(k)!]^{(2k+1)/(k+1)}}, & \vartheta_k=\dfrac{2k+1}{k+1}\\ 0, & \vartheta_k>\dfrac{2k+1}{k+1}\\ -\infty, & \vartheta_k<\dfrac{2k+1}{k+1}\end{cases}$$
and
$$\lim_{x\to\infty}\mathfrak{F}_{k,\vartheta_k}(x)=\begin{cases}-\dfrac{(2k-1)!}{[(k-1)!]^2}, & \vartheta_k=2\\ -\infty, & \vartheta_k>2\\ 0, & \vartheta_k<2;\end{cases}$$
(6) the double inequality
$$-\frac{1}{2}\frac{(2k)!}{(k-1)!k!}<\frac{\psi^{(2k)}(x)}{[(-1)^{k+1}\psi^{(k)}(x)]^2}<0$$
is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any greater and less numbers respectively.

Let $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\dots,\beta_n)\in\mathbb{R}^n$. A n-tuple α is said to strictly majorize β (in symbols $\alpha\succ\beta$) if $\left(\alpha_{[1]},\alpha_{[2]},\dots,\alpha_{[n]}\right)\ne\left(\beta_{[1]},\beta_{[2]},\dots,\beta_{[n]}\right)$, $\sum_{i=1}^k\alpha_{[i]}\ge\sum_{i=1}^k\beta_{[i]}$ for $1\le k\le n-1$, and $\sum_{i=1}^n\alpha_i=\sum_{i=1}^n\beta_i$, where $\alpha_{[1]}\ge\alpha_{[2]}\ge\dots\ge\alpha_{[n]}$ and $\beta_{[1]}\ge\beta_{[2]}\ge\dots\ge\beta_{[n]}$ are rearrangements of α and β in a descending order. See [12, p. 8, Definition A.1] or closely related texts and references in the papers [5,35,37,47].

Theorem 1 ([20, Theorem 3.1]). *Let $p,q,m,n\in\mathbb{N}_0$ satisfying $(p,q)\succ(m,n)$ and let*
$$F_{p,m,n,q;c}(x)=\begin{cases}\left|\psi^{(m)}(x)\right|\left|\psi^{(n)}(x)\right|-c\left|\psi^{(p)}(x)\right|, & q=0\\ \left|\psi^{(m)}(x)\right|\left|\psi^{(n)}(x)\right|-c\left|\psi^{(p)}(x)\right|\left|\psi^{(q)}(x)\right|, & q\ge1\end{cases}$$
for $c\in\mathbb{R}$ and $x\in(0,\infty)$. Then

(1) *for* $q \geq 0$, *if and only if*

$$c \leq \begin{cases} \dfrac{(m-1)!(n-1)!}{(p-1)!}, & q = 0 \\ \dfrac{(m-1)!(n-1)!}{(p-1)!(q-1)!}, & q \geq 1, \end{cases}$$

the function $F_{p,m,n,q;c}(x)$ *is completely monotonic in* $x \in (0,\infty)$;

(2) *for* $q \geq 1$, *if and only if* $c \geq \frac{m!n!}{p!q!}$, *the function* $-F_{p,m,n,q;c}(x)$ *is completely monotonic in* $x \in (0,\infty)$;

(3) *the double inequality*

$$-\frac{(m+n-1)!}{(m-1)!(n-1)!} < \frac{\psi^{(m+n)}(x)}{\psi^{(m)}(x)\psi^{(n)}(x)} < 0 \tag{1}$$

for $m, n \in \mathbb{N}$ *and the double inequality*

$$\frac{(m-1)!(n-1)!}{(p-1)!(q-1)!} < \frac{\psi^{(m)}(x)\psi^{(n)}(x)}{\psi^{(p)}(x)\psi^{(q)}(x)} < \frac{m!n!}{p!q!} \tag{2}$$

for $m, n, p, q \in \mathbb{N}$ *with* $p > m \geq n > q \geq 1$ *and* $m + n = p + q$ *are valid on* $(0,\infty)$ *and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller scalars respectively.*

In [16, Remark 6.2], the preprint of the formally published paper [20], the author guessed that,

(1) for $m, n \in \mathbb{N}$, the function

$$Q_{m,n}(x) = \frac{\psi^{(m+n)}(x)}{\psi^{(m)}(x)\psi^{(n)}(x)} \tag{3}$$

should be decreasing on $(0,\infty)$;

(2) for $m, n, p, q \in \mathbb{N}$ such that $(p,q) \succ (m,n)$, the function

$$\mathcal{Q}_{m,n;p,q}(x) = \frac{\psi^{(m)}(x)\psi^{(n)}(x)}{\psi^{(p)}(x)\psi^{(q)}(x)} \tag{4}$$

should be decreasing on $(0,\infty)$.

It is clear that $Q_{k,k}(x) = \mathfrak{F}_{k,2}(x)$ for $k \in \mathbb{N}$, which is decreasing on $(0,\infty)$.

In this paper, we aim to confirm these two guesses. We also supply an alternative proof of Theorem 1.

2. Lemmas

The following lemmas are necessary in this paper.

Lemma 2 ([3, p. 10–11, Theorem 1.25]). *For* $a, b \in \mathbb{R}$ *with* $a < b$, *let* $f(x)$ *and* $g(x)$ *be continuous on* $[a,b]$, *differentiable on* (a,b), *and* $g'(x) \neq 0$ *on* (a,b). *If the ratio* $\frac{f'(x)}{g'(x)}$ *is increasing on* (a,b), *then both* $\frac{f(x)-f(a)}{g(x)-g(a)}$ *and* $\frac{f(x)-f(b)}{g(x)-g(b)}$ *are increasing in* $x \in (a,b)$.

Lemma 3 ([1, p. 260, 6.4.1]). *The integral representation*

$$\psi^{(n)}(z) = (-1)^{n+1}\int_0^\infty \frac{t^n}{1-\mathrm{e}^{-t}}\mathrm{e}^{-zt}\,\mathrm{d}t \tag{5}$$

is valid for $\Re(z) > 0$ *and* $n \geq 1$.

Lemma 4 (Convolution theorem for the Laplace transforms [40, p. 91–92]). *Let the functions $f_k(t)$ for $k = 1,2$ be piecewise continuous in arbitrary finite intervals included in $(0,\infty)$. If there exist some constants $M_k > 0$ and $c_k \ge 0$ such that the inequalities $|f_k(t)| \le M_k e^{c_k t}$ for $k = 1,2$ are valid, then*

$$\int_0^\infty \left[\int_0^t f_1(u) f_2(t-u)\,\mathrm{d}u\right] \mathrm{e}^{-st}\,\mathrm{d}t = \int_0^\infty f_1(u)\,\mathrm{e}^{-su}\,\mathrm{d}u \int_0^\infty f_2(v)\,\mathrm{e}^{-sv}\,\mathrm{d}v.$$

Lemma 5 ([43, Lemma 4] and [45, Section 3]). *Let the functions $A(x)$ and $B(x) \ne 0$ be defined on $(0,\infty)$ such that their Laplace transforms exist. If the ratio $\frac{A(x)}{B(x)}$ is increasing, then the ratio $\frac{\int_0^\infty A(x)\,\mathrm{e}^{-xt}\,\mathrm{d}x}{\int_0^\infty B(x)\,\mathrm{e}^{-xt}\,\mathrm{d}x}$ is decreasing on $(0,\infty)$.*

Lemma 6. *Let*

$$g(t) = \begin{cases} \dfrac{t}{1-\mathrm{e}^{-t}}, & t \ne 0; \\ 1, & t = 0. \end{cases}$$

Then the following conclusions are valid.

(1) *The function $g(t)$ is infinitely differentiable on $(-\infty,\infty)$, increasing from $(-\infty,\infty)$ onto $(0,\infty)$, convex on $(-\infty,\infty)$, and logarithmically concave on $(-\infty,\infty)$.*
(2) *For fixed $s \in (0,1)$, the ratio $\frac{g^s(t)}{g(st)}$ is decreasing in t from $(0,\infty)$ onto $(0,1)$.*
(3) *For $s \in \left(0, \frac{1}{2}\right)$ and $t \in (0,\infty)$, the mixed second-order partial derivative*

$$\frac{\partial^2 \ln[g(st)g((1-s)t)]}{\partial s \partial t} > 0. \tag{6}$$

Proof. The differentiability, monotonicity, and convexity of $g(t)$ come from utilization of [22, Lemma 2.3].

Direct computation yields

$$\begin{aligned}[\ln g(t)]'' = [\ln g(-t)]'' &= -\frac{\mathrm{e}^{2t} - \mathrm{e}^t\left(t^2+2\right)+1}{(\mathrm{e}^t-1)^2 t^2} \\ &= -\frac{1}{(\mathrm{e}^t-1)^2 t^2} \sum_{k=4}^\infty [2^k - (k-1)k - 2] \frac{t^k}{k!} < 0\end{aligned}$$

on $(0,\infty)$. Hence, the function $g(t)$ is logarithmically concave on $(-\infty,\infty)$. See also the proof of [16, Lemma 2.3].

It is straightforward that

$$\lim_{t\to 0} \frac{g^s(t)}{g(st)} = \frac{\lim_{t\to 0} g^s(t)}{\lim_{t\to 0} g(st)} = \frac{1^s}{1} = 1$$

and

$$\lim_{t\to\infty} \frac{g^s(t)}{g(st)} = \frac{\lim_{t\to\infty}[g(t)/t]^s}{\lim_{t\to\infty}[g(st)/st]} \lim_{t\to\infty} \frac{t^s}{st} = \frac{1}{s} \lim_{t\to\infty} \frac{1}{t^{1-s}} = 0.$$

The first derivative of the ratio $\frac{g^s(t)}{g(st)}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{g^s(t)}{g(st)}\right] = \frac{s g^s(t)}{g(st)}\left[\frac{g'(t)}{g(t)} - \frac{g'(st)}{g(st)}\right].$$

Hence, for arriving at decreasing property of the ratio $\frac{g^s(t)}{g(st)}$, it is sufficient to show that the ratio $\frac{g'(t)}{g(t)}$ is decreasing on $(0,\infty)$. For this, it is sufficient to show that the function $g(t)$ is logarithmically concave on $(0,\infty)$. This requirement has been verified in last paragraph.

By Lemma 2, straightforward differentiation gives

$$\frac{\partial \ln[g(st)g((1-s)t)]}{\partial t} = \frac{1}{t}\frac{t+2\,\mathrm{e}^{t}+2-\mathrm{e}^{st}[(1-s)t+2]-\mathrm{e}^{(1-s)t}(st+2)}{(\mathrm{e}^{st}-1)[\mathrm{e}^{(1-s)t}-1]},$$

$$\lim_{s\to0}\bigl[t+2\,\mathrm{e}^{t}+2-\mathrm{e}^{st}[(1-s)t+2]-\mathrm{e}^{(1-s)t}(st+2)\bigr]=\lim_{s\to0}\bigl((\mathrm{e}^{st}-1)[\mathrm{e}^{(1-s)t}-1]\bigr)=0,$$

$$\frac{[t+2\,\mathrm{e}^{t}+2-\mathrm{e}^{st}[(1-s)t+2]-\mathrm{e}^{(1-s)t}(st+2)]'_s}{[(\mathrm{e}^{st}-1)(\mathrm{e}^{(1-s)t}-1)]'_s}=\frac{\mathrm{e}^{t}(st+1)-\mathrm{e}^{2st}[(1-s)t+1]}{\mathrm{e}^{t}-\mathrm{e}^{2st}},$$

$$\lim_{s\to1/2}\bigl(\mathrm{e}^{t}(st+1)-\mathrm{e}^{2st}[(1-s)t+1]\bigr)=\lim_{s\to1/2}\bigl(\mathrm{e}^{t}-\mathrm{e}^{2st}\bigr)=0,$$

and

$$\frac{\bigl(\mathrm{e}^{t}(st+1)-\mathrm{e}^{2st}[(1-s)t+1]\bigr)'_s}{(\mathrm{e}^{t}-\mathrm{e}^{2st})'_s}=\frac{1}{2}-\frac{\mathrm{e}^{(1-2s)t}-(1-2s)t-1}{2t}$$

is increasing in $s\in\left[0,\frac{1}{2}\right]$. This means that the partial derivative $\frac{\partial \ln[g(st)g((1-s)t)]}{\partial t}$ is increasing in $s\in\left[0,\frac{1}{2}\right]$ for fixed $t>0$. As a result, the inequality (6) is valid. The proof of Lemma 6 is complete. □

Lemma 7 ([20, Lemma 2.1]). *For $k\in\mathbb{N}$, we have the limits*

$$\lim_{x\to0^+}\bigl[x^k\psi^{(k-1)}(x)\bigr]=(-1)^k(k-1)! \tag{7}$$

and

$$\lim_{x\to\infty}\bigl[x^k\psi^{(k)}(x)\bigr]=(-1)^{k-1}(k-1)!. \tag{8}$$

Lemma 8. *For $m,n,p,q\in\mathbb{N}$ such that $(p,q)>(m,n)$, the function*

$$\frac{s^{m-1}(1-s)^{n-1}+(1-s)^{m-1}s^{n-1}}{s^{p-1}(1-s)^{q-1}+(1-s)^{p-1}s^{q-1}}$$

is increasing in $s\in\left(0,\frac{1}{2}\right)$.

Proof. Direct computation yields

$$\begin{aligned}\frac{s^{m-1}(1-s)^{n-1}+(1-s)^{m-1}s^{n-1}}{s^{p-1}(1-s)^{q-1}+(1-s)^{p-1}s^{q-1}}&=\frac{\left(\frac{1}{s}-1\right)^{n-1}+\left(\frac{1}{s}-1\right)^{m-1}}{\left(\frac{1}{s}-1\right)^{q-1}+\left(\frac{1}{s}-1\right)^{p-1}}\\&=\frac{z^{n-1}+z^{m-1}}{z^{q-1}+z^{p-1}}=\frac{z^{n-q}+z^{m-q}}{1+z^{p-q}}\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z^b+z^c}{1+z^a}\right)&=\frac{\bigl(bz^b+cz^c\bigr)(1+z^a)-az^a\bigl(z^b+z^c\bigr)}{z(1+z^a)^2}\\&=\frac{bz^b\bigl(1-z^{2c}\bigr)+cz^c\bigl(1-z^{2b}\bigr)}{z(1+z^a)^2}\\&<0,\end{aligned}$$

where $z=\frac{1}{s}-1\in(1,\infty)$ and $a=p-q>b=m-q\ge c=n-q>0$ with $a=b+c$. The proof of Lemma 8 is complete. □

Lemma 9. *Let the functions $U(x)$, $V(x)>0$, and $W(x,t)>0$ be integrable in $x\in(a,b)$. If the ratios $\frac{\partial W(x,t)/\partial t}{W(x,t)}$ and $\frac{U(x)}{V(x)}$ are both increasing or both decreasing in $x\in(a,b)$, then the ratio*

$$R(t)=\frac{\int_a^b U(x)W(x,t)\,\mathrm{d}x}{\int_a^b V(x)W(x,t)\,\mathrm{d}x}$$

is increasing in t; if one of the ratios $\frac{\partial W(x,t)/\partial t}{W(x,t)}$ and $\frac{U(x)}{V(x)}$ are increasing and the other is decreasing in $x\in(a,b)$, then the ratio $R(t)$ is decreasing in t.

Proof. Direct differentiation gives

$$
\begin{aligned}
R'(t) &= \frac{\begin{bmatrix} \int_a^b U(x)\frac{\partial W(x,t)}{\partial t}\mathrm{d}x \int_a^b V(x)W(x,t)\mathrm{d}x \\ -\int_a^b U(x)W(x,t)\mathrm{d}x \int_a^b V(x)\frac{\partial W(x,t)}{\partial t}\mathrm{d}x \end{bmatrix}}{\left[\int_a^b V(x)W(x,t)\mathrm{d}x\right]^2} \\
&= \frac{\begin{bmatrix} \int_a^b\int_a^b U(x)\frac{\partial W(x,t)}{\partial t} V(y)W(y,t)\mathrm{d}x\mathrm{d}y \\ -\int_a^b\int_a^b U(x)W(x,t)V(y)\frac{\partial W(y,t)}{\partial t}\mathrm{d}x\mathrm{d}y \end{bmatrix}}{\left[\int_a^b V(x)W(x,t)\mathrm{d}x\right]^2} \\
&= \frac{\int_a^b\int_a^b U(x)V(y)W(x,t)W(y,t)\left[\frac{\partial W(x,t)/\partial t}{W(x,t)} - \frac{\partial W(y,t)/\partial t}{W(y,t)}\right]\mathrm{d}x\mathrm{d}y}{\left[\int_a^b V(x)W(x,t)\mathrm{d}x\right]^2} \\
&= \frac{\int_a^b\int_a^b \begin{pmatrix} \left[\frac{U(x)}{V(x)} - \frac{U(y)}{V(y)}\right]\left[\frac{\partial W(x,t)/\partial t}{W(x,t)} - \frac{\partial W(y,t)/\partial t}{W(y,t)}\right] \\ \times V(x)V(y)W(x,t)W(y,t) \end{pmatrix}\mathrm{d}x\mathrm{d}y}{2\left[\int_a^b V(x)W(x,t)\mathrm{d}x\right]^2}.
\end{aligned}
$$

The proof of Lemma 9 is complete. □

Lemma 10 ([40, p. 161, Theorem 12b]). *A function $f(x)$ is completely monotonic on $(0,\infty)$ if and only if*

$$
f(x) = \int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}\sigma(t), \quad x \in (0,\infty), \tag{9}
$$

where $\sigma(s)$ is non-decreasing and the integral in (9) *converges for $x \in (0,\infty)$.*

3. Decreasing property of a ratio defined by three polygamma functions

In this section, we prove that the function $Q_{m,n}(x)$ defined in (3) is decreasing.

Theorem 11. *For $m,n \in \mathbb{N}$, the function $Q_{m,n}(x)$ defined in* (3) *is decreasing from $(0,\infty)$ onto $\left(-\frac{(m+n-1)!}{(m-1)!(n-1)!}, 0\right)$. Consequently, the double inequality* (1)*, that is,*

$$
-\frac{(m+n-1)!}{(m-1)!(n-1)!} < Q_{m,n} < 0, \quad m,n \in \mathbb{N},
$$

is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller numbers respectively.

Proof. By virtue of the integral representation (5), we can rearranged $Q_{m,n}(x)$ as

$$
Q_{m,n}(x) = -\frac{\int_0^\infty t^{m+n-1} g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t}{\int_0^\infty t^{m-1} g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t \int_0^\infty t^{n-1} g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t}.
$$

By Lemma 4, we obtain

$$
Q_{m,n}(x) = -\frac{\int_0^\infty t^{m+n-1} g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t}{\int_0^\infty \left[\int_0^t u^{m-1}(t-u)^{n-1} g(u)g(t-u)\mathrm{d}u\right]\mathrm{e}^{-xt}\,\mathrm{d}t}.
$$

By Lemma 5, we only need to prove the ratio

$$
\begin{aligned}
P_{m,n}(t) &= \frac{t^{m+n-1} g(t)}{\int_0^t u^{m-1}(t-u)^{n-1} g(u)g(t-u)\mathrm{d}u} \\
&= \frac{t^{m+n-1} g(t)}{t^{m+n-1}\int_0^1 s^{m-1}(1-s)^{n-1} g(st)g((1-s)t)\mathrm{d}s} \\
&= \frac{1}{\int_0^1 s^{m-1}(1-s)^{n-1}\frac{g(st)}{g^s(t)}\frac{g((1-s)t)}{g^{1-s}(t)}\mathrm{d}s}
\end{aligned}
$$

is decreasing on $(0,\infty)$. Hence, it suffices to show that the ratio $\frac{g(st)}{g^s(t)}$ for fixed $s\in(0,1)$ is increasing in $t\in(0,\infty)$. This increasing property of $\frac{g(st)}{g^s(t)}$ has been proved in Lemma 6 in this paper. As a result, the function $Q_{m,n}(x)$ defined in (3) is decreasing on $(0,\infty)$.

Making use of the limits (7) and (8) in Lemma 7 yields

$$\lim_{x\to0^+} Q_{m,n}(x)=\frac{(\lim_{x\to0^+}x)\lim_{x\to0^+}\left[x^{m+n+1}\psi^{(m+n)}(x)\right]}{\lim_{x\to0^+}[x^{m+1}\psi^{(m)}(x)]\lim_{x\to0^+}[x^{n+1}\psi^{(n)}(x)]}=0$$

and

$$\begin{aligned}\lim_{x\to\infty} Q_{m,n}(x)&=\frac{\lim_{x\to\infty}\left[x^{m+n}\psi^{(m+n)}(x)\right]}{\lim_{x\to\infty}[x^{m}\psi^{(m)}(x)]\lim_{x\to\infty}[x^{n}\psi^{(n)}(x)]}\\&=\frac{(-1)^{m+n-1}(m+n-1)!}{(-1)^{m-1}(m-1)!(-1)^{n-1}(n-1)!}\\&=-\frac{(m+n-1)!}{(m-1)!(n-1)!}.\end{aligned}$$

The proof of Theorem 11 is complete. □

4. Decreasing property of a ratio defined by four polygamma functions

In this section, we prove that the function $\mathcal{Q}_{m,n;p,q}$ defined in (4) is decreasing.

Theorem 12. *For $m,n,p,q\in\mathbb{N}$ with the majorizing relation $(p,q)\succ(m,n)$, the ratio $\mathcal{Q}_{m,n;p,q}(x)$ defined in* (4) *is decreasing from $(0,\infty)$ onto the interval $\left(\frac{(m-1)!(n-1)!}{(p-1)!(q-1)!},\frac{m!n!}{p!q!}\right)$. Consequently, for $m,n,p,q\in\mathbb{N}$ with $(p,q)\succ(m,n)$, the double inequality* (2)*, that is,*

$$\frac{(m-1)!(n-1)!}{(p-1)!(q-1)!}<\mathcal{Q}_{m,n;p,q}(x)<\frac{m!n!}{p!q!},$$

is valid on $(0,\infty)$ and sharp in the sense that the lower and upper bounds cannot be replaced by any larger and smaller scalars respectively.

Proof. By the limits (7) and (8) in Lemma 7, we obtain

$$\begin{aligned}\lim_{x\to0^+}\mathcal{Q}_{m,n;p,q}(x)&=\frac{\lim_{x\to0^+}\left[x^{m+1}\psi^{(m)}(x)\right]\lim_{x\to0^+}\left[x^{n+1}\psi^{(n)}(x)\right]}{\lim_{x\to0^+}\left[x^{p+1}\psi^{(p)}(x)\right]\lim_{x\to0^+}\left[x^{q+1}\psi^{(q)}(x)\right]}\\&=\frac{(-1)^{m+1}m!(-1)^{n+1}n!}{(-1)^{p+1}p!(-1)^{q+1}q!}\\&=\frac{m!n!}{p!q!}\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to\infty}\mathcal{Q}_{m,n;p,q}(x)&=\frac{\lim_{x\to\infty}\left[x^{m}\psi^{(m)}(x)\right]\lim_{x\to\infty}\left[x^{n}\psi^{(n)}(x)\right]}{\lim_{x\to\infty}\left[x^{p}\psi^{(p)}(x)\right]\lim_{x\to\infty}\left[x^{q}\psi^{(q)}(x)\right]}\\&=\frac{(-1)^{m-1}(m-1)!(-1)^{n-1}(n-1)!}{(-1)^{p-1}(p-1)!(-1)^{q-1}(q-1)!}\\&=\frac{(q-1)!(n-1)!}{(p-1)!(q-1)!}.\end{aligned}$$

Making use of the integral representation (5) yields

$$\mathcal{Q}_{m,n;p,q}(x)=\frac{\int_0^\infty t^{m-1}g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t\int_0^\infty t^{n-1}g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t}{\int_0^\infty t^{p-1}g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t\int_0^\infty t^{q-1}g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t}.$$

Utilizing Lemma 4 gives

$$\mathcal{Q}_{m,n;p,q}(x)=\frac{\int_0^\infty\left[\int_0^t u^{m-1}(t-u)^{n-1}g(u)g(t-u)\,\mathrm{d}u\right]\mathrm{e}^{-xt}\,\mathrm{d}t}{\int_0^\infty\left[\int_0^t u^{p-1}(t-u)^{q-1}g(u)g(t-u)\,\mathrm{d}u\right]\mathrm{e}^{-xt}\,\mathrm{d}t}.$$

Employing Lemma 5 tells us that, it suffices to prove the increasing property in t of the ratio

$$\begin{aligned}\frac{\int_0^t u^{m-1}(t-u)^{n-1}g(u)g(t-u)\,\mathrm{d}u}{\int_0^t u^{p-1}(t-u)^{q-1}g(u)g(t-u)\,\mathrm{d}u}&=\frac{\int_0^1 s^{m-1}(1-s)^{n-1}g(st)g((1-s)t)\,\mathrm{d}s}{\int_0^1 s^{p-1}(1-s)^{q-1}g(st)g((1-s)t)\,\mathrm{d}s}\\&=\frac{\int_0^{1/2}\left[s^{m-1}(1-s)^{n-1}+(1-s)^{m-1}s^{n-1}\right]g(st)g((1-s)t)\,\mathrm{d}s}{\int_0^{1/2}\left[s^{p-1}(1-s)^{q-1}+(1-s)^{p-1}s^{q-1}\right]g(st)g((1-s)t)\,\mathrm{d}s}\\&=\frac{\int_0^{1/2}\phi_{m,n}(s)\varphi(s,t)\,\mathrm{d}s}{\int_0^{1/2}\phi_{p,q}(s)\varphi(s,t)\,\mathrm{d}s},\end{aligned}$$

where

$$\phi_{i,j}(s)=s^{i-1}(1-s)^{j-1}+(1-s)^{i-1}s^{j-1}\quad\text{and}\quad\varphi(s,t)=g(st)g((1-s)t).\tag{10}$$

Lemma 8 implies that the ratio $\frac{\phi_{m,n}(s)}{\phi_{p,q}(s)}$ is increasing in $s\in\left(0,\frac{1}{2}\right)$ for $(p,q)\succ(m,n)$. Further making use of the inequality (6) in Lemma 6 and utilizing Lemma 9 reveal that the function $\mathcal{Q}_{m,n;p,q}(x)$ is decreasing in $x\in(0,\infty)$. The proof of Theorem 12 is complete. $\square$

5. An alternative proof of Theorem 1

In this section, we supply an alternative proof of Theorem 1.

For $q=0$, we have

$$\begin{aligned}F_{p,m,n,0;c}(x)&=\left|\psi^{(m)}(x)\right|\left|\psi^{(n)}(x)\right|-c\left|\psi^{(p)}(x)\right|\\&=\int_0^\infty\frac{t^m}{1-\mathrm{e}^{-t}}\mathrm{e}^{-xt}\,\mathrm{d}t\int_0^\infty\frac{t^n}{1-\mathrm{e}^{-t}}\mathrm{e}^{-xt}\,\mathrm{d}t-c\int_0^\infty\frac{t^p}{1-\mathrm{e}^{-t}}\mathrm{e}^{-xt}\,\mathrm{d}t\\&=\int_0^\infty\left[\int_0^t u^{m-1}(t-u)^{n-1}g(u)g(t-u)\,\mathrm{d}u-ct^{m+n-1}g(t)\right]\mathrm{e}^{-xt}\,\mathrm{d}t\\&=\int_0^\infty\left[\int_0^1 s^{m-1}(1-s)^{n-1}\frac{g(st)g((1-s)t)}{g(t)}\,\mathrm{d}s-c\right]t^{m+n-1}g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t,\end{aligned}$$

where we used the integral representation (5) and Lemma 4. From the second property in Lemma 6, it follows that the function

$$\int_0^1 s^{m-1}(1-s)^{n-1}\frac{g(st)g((1-s)t)}{g(t)}\,\mathrm{d}s$$

is decreasing in $t\in(0,\infty)$ and has the limits

$$\int_0^1 s^{m-1}(1-s)^{n-1}\frac{g(st)g((1-s)t)}{g(t)}\,\mathrm{d}s\to\begin{cases}\int_0^1 s^{m-1}(1-s)^{n-1}\,\mathrm{d}s, & t\to0;\\ \infty, & t\to\infty.\end{cases}$$

Consequently, basing on Lemma 10, we see that, if and only if

$$c\le\int_0^1 s^{m-1}(1-s)^{n-1}\,\mathrm{d}s=B(m,n)=\frac{(m-1)!(n-1)!}{(m+n-1)!},$$

the function $F_{p,m,n,0;c}(x)$ is completely monotonic on $(0,\infty)$.

For $q \ge 1$, we have

$$
\begin{aligned}
F_{p,m,n,q;c}(x) &= \left|\psi^{(m)}(x)\right|\left|\psi^{(n)}(x)\right| - c\left|\psi^{(p)}(x)\right|\left|\psi^{(q)}(x)\right| \\
&= \int_0^\infty t^{m-1} g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t \int_0^\infty t^{n-1} g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t - c\int_0^\infty t^{p-1} g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t \int_0^\infty t^{q-1} g(t)\,\mathrm{e}^{-xt}\,\mathrm{d}t \\
&= \int_0^\infty \left[\int_0^t u^{m-1}(t-u)^{n-1} g(u)g(t-u)\,\mathrm{d}u\right]\mathrm{e}^{-xt}\,\mathrm{d}t \\
&\quad - c\int_0^\infty \left[\int_0^t u^{p-1}(t-u)^{q-1} g(u)g(t-u)\,\mathrm{d}u\right]\mathrm{e}^{-xt}\,\mathrm{d}t \\
&= \int_0^\infty \left[t^{m+n-1}\int_0^1 s^{m-1}(1-s)^{n-1} g(st)g((1-s)t)\,\mathrm{d}s\right. \\
&\qquad \left. - ct^{p+q-1}\int_0^1 s^{p-1}(1-s)^{q-1} g(st)g((1-s)t)\,\mathrm{d}s\right]\mathrm{e}^{-xt}\,\mathrm{d}t \\
&= \int_0^\infty \left[\frac{\int_0^1 s^{m-1}(1-s)^{n-1} g(st)g((1-s)t)\,\mathrm{d}s}{\int_0^1 s^{p-1}(1-s)^{q-1} g(st)g((1-s)t)\,\mathrm{d}s} - c\right] \\
&\qquad \times \left[t^{m+n-1}\int_0^1 s^{p-1}(1-s)^{q-1} g(st)g((1-s)t)\,\mathrm{d}s\right]\mathrm{e}^{-xt}\,\mathrm{d}t \\
&= \int_0^\infty \left[\frac{\int_0^{1/2} [s^{m-1}(1-s)^{n-1} + s^{n-1}(1-s)^{m-1}] g(st)g((1-s)t)\,\mathrm{d}s}{\int_0^{1/2} [s^{p-1}(1-s)^{q-1} + s^{q-1}(1-s)^{p-1}] g(st)g((1-s)t)\,\mathrm{d}s} - c\right] \\
&\qquad \times \left[t^{m+n-1}\int_0^1 s^{p-1}(1-s)^{q-1} g(st)g((1-s)t)\,\mathrm{d}s\right]\mathrm{e}^{-xt}\,\mathrm{d}t,
\end{aligned}
$$

where we used the integral representation (5) and Lemma 4. Employing the inequality (6) in Lemma 6 and applying Lemmas 8 and 9 reveal that the function

$$
\frac{\int_0^{1/2} [s^{m-1}(1-s)^{n-1} + s^{n-1}(1-s)^{m-1}] g(st)g((1-s)t)\,\mathrm{d}s}{\int_0^{1/2} [s^{p-1}(1-s)^{q-1} + s^{q-1}(1-s)^{p-1}] g(st)g((1-s)t)\,\mathrm{d}s} = \frac{\int_0^{1/2} \phi_{m,n}(s)\varphi(s,t)\,\mathrm{d}s}{\int_0^{1/2} \phi_{p,q}(s)\varphi(s,t)\,\mathrm{d}s}
$$

is increasing in $t \in (0,\infty)$, where $\phi_{i,j}(s)$ and $\varphi(s,t)$ are defined in (10). It is easy to see that

$$
\begin{aligned}
\lim_{t\to 0} \frac{\int_0^{1/2} \phi_{m,n}(s)\varphi(s,t)\,\mathrm{d}s}{\int_0^{1/2} \phi_{p,q}(s)\varphi(s,t)\,\mathrm{d}s} &= \frac{\int_0^{1/2} [s^{m-1}(1-s)^{n-1} + s^{n-1}(1-s)^{m-1}]\,\mathrm{d}s}{\int_0^{1/2} [s^{p-1}(1-s)^{q-1} + s^{q-1}(1-s)^{p-1}]\,\mathrm{d}s} \\
&= \frac{\int_0^1 s^{m-1}(1-s)^{n-1}\,\mathrm{d}s}{\int_0^1 s^{p-1}(1-s)^{q-1}\,\mathrm{d}s} = \frac{B(m,n)}{B(p,q)} = \frac{(m-1)!(n-1)!}{(p-1)!(q-1)!}.
\end{aligned}
$$

Since $\lim_{t\to\infty} \frac{g(t)}{t} = 1$, we acquire

$$
\begin{aligned}
\lim_{t\to \infty} \frac{\int_0^{1/2} \phi_{m,n}(s)\varphi(s,t)\,\mathrm{d}s}{\int_0^{1/2} \phi_{p,q}(s)\varphi(s,t)\,\mathrm{d}s} &= \frac{\int_0^{1/2} [s^{m-1}(1-s)^{n-1} + s^{n-1}(1-s)^{m-1}]s(1-s)\,\mathrm{d}s}{\int_0^{1/2} [s^{p-1}(1-s)^{q-1} + s^{q-1}(1-s)^{p-1}]s(1-s)\,\mathrm{d}s} \\
&= \frac{\int_0^1 s^{m}(1-s)^{n}\,\mathrm{d}s}{\int_0^1 s^{p}(1-s)^{q}\,\mathrm{d}s} = \frac{B(m+1,n+1)}{B(p+1,q+1)} = \frac{m!n!}{p!q!}.
\end{aligned}
$$

Combining these with Lemma 10 concludes that,

(1) if and only if

$$
c \le \frac{(m-1)!(n-1)!}{(p-1)!(q-1)!},
$$

the function $F_{p,m,n,q;c}(x)$ is completely monotonic in $x \in (0,\infty)$;

(2) if and only if $c \ge \frac{m!n!}{p!q!}$, the function $-F_{p,m,n,q;c}(x)$ is completely monotonic in $x \in (0,\infty)$.

The proof of Theorem 1 is complete.

6. Remarks

Finally, we list several remarks on our main results and their proofs.

Remark 13. The papers [2, 4, 6, 7, 42] are related to Theorem 1. Theorems 11 and 12 in this paper are related to some results reviewed and surveyed in [14,26] and closely related references therein.

Remark 14. Lemma 6 in this paper generalizes the second item in [16, Lemma 2.3], which reads that the function $\frac{g(2t)}{g^2(t)}$ is decreasing from $(0,\infty)$ onto $(0,1)$.

Remark 15. Taking $W(x,t)=\mathrm{e}^{-xt}$ in Lemma 9 gives

$$\frac{\partial W(x,t)/\partial t}{W(x,t)}=\frac{\partial \mathrm{e}^{-xt}/\partial t}{\mathrm{e}^{-xt}}=-x,$$

which is decreasing in $x\in(-\infty,\infty)$. Further setting $U(x)=A(x)$, $V(x)=B(x)$, and $(a,b)=(0,\infty)$ in Lemma 9 leads to Lemma 5, which was established in [43, Lemma 4]. This means that Lemma 9 in this paper is a generalization of [43, Lemma 4]. Lemma 9 has been announced in [30, Remark 7.2].

Remark 16. From the majorizing relation $(n+2,n)\succ(n+1,n+1)$, we see that Theorem 12 in this paper generalizes a conclusion in [42, Theorem 2], which states that the function $\frac{[\psi^{(n+1)}(x)]^2}{\psi^{(n)}(x)\psi^{(n+2)}(x)}$ for $n\ge1$ is decreasing from $(0,\infty)$ onto the interval $\left(\frac{n}{n+1},\frac{n+1}{n+2}\right)$.

Remark 17. Direct differentiation gives

$$Q'_{m,n}(x)=\frac{\psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x)-\psi^{(m+n)}(x)\big[\psi^{(m)}(x)\psi^{(n)}(x)\big]'}{[\psi^{(m)}(x)\psi^{(n)}(x)]^2}.$$

The decreasing property of $Q_{m,n}(x)$ in Theorem 11 implies that the inequality

$$\psi^{(m+n)}(x)\big[\psi^{(m)}(x)\psi^{(n)}(x)\big]'-\psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x)>0,$$

equivalently,

$$\frac{\big[\psi^{(m)}(x)\psi^{(n)}(x)\big]'}{\psi^{(m)}(x)\psi^{(n)}(x)}>\frac{\psi^{(m+n+1)}(x)}{\psi^{(m+n)}(x)},$$

is valid on $(0,\infty)$ for $m,n\in\mathbb{N}$.

We guess that, for $m,n\in\mathbb{N}$, the function

$$\psi^{(m+n)}(x)\big[\psi^{(m)}(x)\psi^{(n)}(x)\big]'-\psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x)$$

should be completely monotonic in $x\in(0,\infty)$.

Generally, one can discuss necessary and sufficient conditions on $\Omega_{m,n}\in\mathbb{R}$ such that the function

$$\psi^{(m+n)}(x)\big[\psi^{(m)}(x)\psi^{(n)}(x)\big]'-\Omega_{m,n}\psi^{(m+n+1)}(x)\psi^{(m)}(x)\psi^{(n)}(x)$$

and its opposite are respectively completely monotonic on $(0,\infty)$.

Remark 18. It is immediate that

$$\mathscr{Q}'_{m,n;p,q}(x)=\frac{\begin{pmatrix}\big[\psi^{(m)}(x)\psi^{(n)}(x)\big]'[\psi^{(p)}(x)\psi^{(q)}(x)]\\-\big[\psi^{(m)}(x)\psi^{(n)}(x)\big][\psi^{(p)}(x)\psi^{(q)}(x)]'\end{pmatrix}}{[\psi^{(p)}(x)\psi^{(q)}(x)]^2}.$$

The decreasing property of $\mathscr{Q}_{m,n;p,q}(x)$ in Theorem 12 implies that the inequality

$$\big[\psi^{(m)}(x)\psi^{(n)}(x)\big][\psi^{(p)}(x)\psi^{(q)}(x)]'-\big[\psi^{(m)}(x)\psi^{(n)}(x)\big]'[\psi^{(p)}(x)\psi^{(q)}(x)]>0,$$

equivalently,

$$\frac{[\psi^{(p)}(x)\psi^{(q)}(x)]'}{\psi^{(p)}(x)\psi^{(q)}(x)}>\frac{\big[\psi^{(m)}(x)\psi^{(n)}(x)\big]'}{\psi^{(m)}(x)\psi^{(n)}(x)},$$

is valid on $(0,\infty)$ for $(p,q) \succ (m,n)$.

We guess that, for $(p,q) \succ (m,n)$, the function

$$\big[\psi^{(m)}(x)\psi^{(n)}(x)\big][\psi^{(p)}(x)\psi^{(q)}(x)]' - \big[\psi^{(m)}(x)\psi^{(n)}(x)\big]'[\psi^{(p)}(x)\psi^{(q)}(x)]$$

should be completely monotonic in $x \in (0,\infty)$.

Generally, for $(p,q) \succ (m,n)$, one can discuss necessary and sufficient conditions on $\Omega_{m,n;p,q} \in \mathbb{R}$ such that the function

$$\big[\psi^{(m)}(x)\psi^{(n)}(x)\big][\psi^{(p)}(x)\psi^{(q)}(x)]' - \Omega_{m,n;p,q}\big[\psi^{(m)}(x)\psi^{(n)}(x)\big]'[\psi^{(p)}(x)\psi^{(q)}(x)]$$

and its opposite are respectively completely monotonic on $(0,\infty)$.

Remark 19. For $n \ge 2$ and two nonnegative integer tuples $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n) \in \mathbb{N}_0^n$ and $\beta = (\beta_1,\beta_2,\dots,\beta_n) \in \mathbb{N}^n$, let

$$P_{\alpha,\beta;C_{\alpha,\beta}}(x) = \prod_{r=1}^{n}\psi^{(\alpha_r)}(x) - C_{\alpha,\beta}\prod_{r=1}^{n}\psi^{(\beta_r)}(x)$$

and

$$Q_{\alpha,\beta}(x) = \frac{\prod_{r=1}^{n}\psi^{(\alpha_r)}(x)}{\prod_{r=1}^{n}\psi^{(\beta_r)}(x)}$$

on $(0,\infty)$, where we denote $\psi^{(0)}(x) = -1$ for our own convenience. It is clear that

$$P_{(2k,0),(k,k);C_{(2k,0),(k,k)}}(x) = \mathscr{F}_{k,-C_{(2k,0),(k,k)}}(x), \qquad Q_{(2k,0),(k,k)}(x) = \mathfrak{F}_{k,2}(x),$$
$$Q_{(m+n,0),(m,n)}(x) = Q_{m,n}(x), \qquad Q_{(m,n),(p,q)}(x) = Q_{m,n;p,q}(x).$$

We guess that, if $\alpha \succ \beta$, the function $Q_{\alpha,\beta}(x)$ is increasing from $(0,\infty)$ onto the interval

$$\left(\prod_{r=1}^{n}\frac{\alpha_r!}{\beta_r!}, \prod_{r=1}^{n}\frac{(\alpha_r-1)!}{(\beta_r-1)!}\right).$$

Generally, for $\alpha \succ \beta$, one can discuss necessary and sufficient conditions on $C_{\alpha,\beta} \in \mathbb{R}$ such that the function $P_{\alpha,\beta;C_{\alpha,\beta}}(x)$ and its opposite are respectively completely monotonic on $(0,\infty)$.

Remark 20. Gurland's ratio

$$T(s,t) = \frac{\Gamma(s)\Gamma(t)}{[\Gamma((s+t)/2)]^2}$$

was firstly defined in [10]. In appearance, we can regard the functions $Q_{m,n}(x)$ and $\mathscr{Q}_{m,n;p,q}(x)$ defined in (3) and (4) as analogues of Gurland's ratio $T(s,t)$. In [32, 38], there existed a detailed survey and review of Gurland's ratio $T(s,t)$ and related results. In [46], the functions $T\big(\frac{1}{p},\frac{3}{p}\big)$ and $T\big(\frac{1}{p},\frac{5}{p}\big)$ with their statistical backgrounds were mentioned.

Remark 21. The ratios of finitely many gamma functions and polygamma functions have applications in differential geometry, manifolds, statistics, probability, and their intersections. See, for example, the papers [8, 11, 33, 34, 48].

Remark 22. As a generalization of decreasing property of real functions of one variable, one can consider (logarithmically) complete monotonicity and completely monotonic degrees. For details, please refer to [14, 19, 31, 39, 41, 44, 49] and the review article [26].

Remark 23. This paper is a revised version of the electronic preprint [15] and is the eighth one in a series of articles including [17, 18, 20–25, 29].

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