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Comptes Rendus
Mathématique

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Volume 360 (2022), p. 1193-1204

Published online: 8 December 2022

https://doi.org/10.5802/crmath.287

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
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https://doi.org/10.5802/crmath.287

Harmonic analysis, Geometry and Topology / *Analyse harmonique, Géométrie et Topologie*

Harmonic vector fields and the Hodge Laplacian operator on Finsler geometry

Behroz Bidabad[*, a, b] **and Mir Ahmad Mirshafeazadeh**[c]

[a] Department of Mathematics and Computer Sciences, Amirkabir University of Technology (Tehran Polytechnic), 424 Hafez Ave. 15914 Tehran, Iran

[b] Institut de Mathematique de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France

[c] Department of Mathematics, Shabestar Branch, Islamic Azad University, Shabestar, Iran

E-mails: bidabad@aut.ac.ir, behroz.bidabad@math.univ-toulouse.fr (B. Bidabad), ah.mirshafeazadeh@gmail.com (M. A. Mirshafeazadeh)

Abstract. We first present the natural definitions of the horizontal differential, the divergence (as an adjoint operator) and a p-harmonic form on a Finsler manifold. Next, we prove a Hodge-type theorem for a Finsler manifold in the sense that a horizontal p-form is harmonic if and only if the horizontal Laplacian vanishes. This viewpoint provides a new appropriate natural definition of harmonic vector fields in Finsler geometry. This approach leads to a Bochner–Yano type classification theorem based on the harmonic Ricci scalar. Finally, we show that a closed orientable Finsler manifold with a positive harmonic Ricci scalar has zero Betti number.

Résumé. Nous présentons d'abord les définitions naturelles de la différentielle horizontale, de la divergence (comme opérateur adjoint) et d'une forme p-harmonique sur une variété finslérienne. Ensuite, nous prouvons un théorème de type Hodge pour une variété finslérienne dans le sens où une p-forme horizontale est harmonique si et seulement si le Laplacien horizontal est nul. Ce point de vue fournit une nouvelle définition naturelle appropriée des champs de vecteurs harmoniques en géométrie finslérienne. Cette méthode conduit à un théorème de classification de type Bochner–Yano basé sur le scalaire de Ricci harmonique. Enfin, nous montrons qu'une variété finslérienne fermée et orientable, avec un scalaire de Ricci harmonique positif, a un nombre de Betti nul.

2020 Mathematics Subject Classification. 58B20.

Manuscript received 10 June 2021, revised 28 September 2021, accepted 21 October 2021.

* Corresponding author.

1. Introduction

The existence of harmonic vector fields on the Riemannian manifolds is directly related to the sign of the Ricci tensor. Bochner and Yano have studied the non-existence of harmonic vector fields on the compact Riemannian manifolds with positive Ricci curvature based on the Laplace–Beltrami operator. Next, Bochner proved that if the Ricci curvature on a Riemannian manifold is positive-definite, then all harmonic vector fields vanish [6]. Yano proved that a vector field X is harmonic, if and only if the Laplacian of its corresponding 1-form vanishes [12, 13].

In Finsler geometry, Akbar-Zadeh introduced the divergence of horizontal and vertical 1-forms on SM without defining the harmonic forms on a Finsler manifold, where $SM := \bigcup_{x\in M} S_xM$ and $S_xM := \{y \in T_xM \mid F(y) = 1\}$, [1].

Harmonic forms in Finsler geometry are studied in [3, 4, 8, 14]. Recently, the second author introduced a definition of harmonic vector fields on a Finsler manifold, which is slightly modified here in the present work, see [9, 10], and Remark 10 in this article. Moreover some natural extensions of Riemannian results, more or less linked to this question are studied in [5].

In the present work, the horizontal differential operator $\mathrm{d_H}$ and the horizontal co-differential operator δ_{H}, are defined as adjoint operators. The above operators provide a Finslerian version of a well-known Hodge theorem on the Riemannian manifolds in the following sense.

Theorem 1. *Let (M,F) be a closed Finsler manifold. If ω is a horizontal p-form on SM, then*

$$\Delta_H\omega = 0 \quad \textit{if and only if} \quad \mathrm{d}_H\omega = 0,\ and \quad \delta_H\omega = 0. \tag{1}$$

We can thus define harmonic p-forms naturally on a Finsler manifold in the sense that, a horizontal p-form is harmonic if and only if the horizontal Laplacian vanishes.

The definition of harmonic p-forms on SM will provide a new definition of a harmonic vector field on a Finsler manifold in the sense that, a vector field on (M,F) is harmonic if and only if the horizontal Laplacian vanishes.

Finally, we obtain a classification of harmonic vector fields based on the *harmonic Ricci scalar* $\widetilde{\mathrm{Ric}}$ defined by the equation (32).

Theorem 2. *Let (M,F) be a closed Finsler manifold and X a harmonic vector field on M.*

- *If $\widetilde{\mathrm{Ric}} = 0$, then X is parallel.*
- *If $\widetilde{\mathrm{Ric}} > 0$, then X vanishes.*

This theorem is an extension of a well-known result obtained by Bochner and Yano, see Theorem 16. Finally, this brings us to the following fundamental results.

Theorem 3. *Let (M,F) be a Finsler manifold. Every cohomology class $H^1(M)$ contains a unique harmonic representative.*

Corollary 4. *In a closed orientable Finsler manifold with a positive harmonic Ricci scalar $\widetilde{\mathrm{Ric}} > 0$, the first Betti number vanishes.*

In Section 2, the necessary tools, concepts and definitions of Finsler geometry using the Cartan connection are stated. In Section 3, the definition of $\Lambda_p^H(SM)$ the space of horizontal p-forms and the definition of d_H the horizontal divergence operator on the unit fiber bundle SM with an inner product $(\cdot,\cdot)$ on $\Lambda_p^H(SM)$ are expressed. In Section 4, the definition of the horizontal (co-differential) divergence, a horizontal Laplacian and a new type of harmonic p-form are introduced using the horizontal Laplacian. Section 5 deals with harmonic vector fields on Finsler manifolds where the proof of Theorem 2 is presented. In Section 6, we prove that a closed orientable Finsler manifold with a positive harmonic Ricci scalar has zero Betti number.

2. Preliminaries and notations

We first recall some Riemannian definitions of harmonic analysis. Let (M,g) be a compact and orientable Riemannian manifold of dimension n. A p-form on (M,g) for $1 \le p \le n$ is given by

$$\varphi = \frac{1}{p!}\varphi_{i_1\dots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

where the indices $i_1,\dots,i_p$ run over the range $1,\dots,n$ and the coefficients are components of the skew-symmetric tensor fields of type $(0,p)$. The differential $d\varphi$ is a $(p+1)-$form given by

$$d\varphi = \frac{1}{(p+1)!}(\nabla_i\varphi_{i_1\dots i_p} - \nabla_{i_1}\varphi_{ii_2\dots i_p} - \cdots - \nabla_{i_p}\varphi_{i_1\dots i_{p-1}i})dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

where the coefficients are components of the skew-symmetric tensor fields of type $(0,p+1)$ and ∇_j are the components of Levi-Civita covariant derivative. The co-differential $\delta\varphi$ is a $(p-1)-$form given by

$$\delta\varphi = -\frac{1}{(p-1)!}g^{ji}\nabla_j\varphi_{ii_2\dots i_p} dx^{i_2} \wedge \cdots \wedge dx^{i_p},$$

where the coefficients are components of the skew-symmetric tensor fields of type $(0,p-1)$. The co-differential of a scalar function is defined to be zero. It is easy to verify that $d(d\varphi) = 0$ and $\delta(\delta\varphi) = 0$, see for instance [13]. In Riemannian geometry a differential form φ is called *harmonic* if it satisfies $d\varphi = 0$ and $\delta\varphi = 0$. A vector field X is said to be *harmonic* if its associated 1-form is harmonic. It is well known that a necessary and sufficient condition for a p-form φ to be harmonic is

$$\Delta\varphi = (\delta d + d\delta)\varphi = 0, \tag{2}$$

where Δ is called Laplacian, see [13] for more details.

We then turn to the more general cases of Finsler manifolds. Let M be a connected differentiable manifold, $\pi : TM_0 \to M$ the bundle of non-zero tangent vector where $TM_0 = TM\backslash 0$ is the entire slit tangent bundle. A point of TM is denoted by $z = (x,y)$, where $x \in M$ and $y \in T_xM$. Let (x^i) be a local chart with the domain $U \subseteq M$ and (x^i, y^i) the induced local coordinates on $\pi^{-1}(U)$, where $\mathbf{y} = y^i\frac{\partial}{\partial x^i} \in T_{\pi z}M$, and i running over the range $1,2,\dots,n$. A (globally defined) *Finsler structure* on M is a function $F : TM \longrightarrow [0,\infty)$ with the following properties; F is C^∞ on the entire slit tangent bundle $TM\backslash 0$; $F(x,\lambda y) = \lambda F(x,y)$ $\forall\ \lambda > 0$; the $n \times n$ *Hessian matrix* $(g_{ij}) = \frac{1}{2}([F^2]_{y^iy^j})$ is positive-definite at every point of TM_0. The pair (M,g) is called a *Finsler manifold*, cf. [2]. Denote by TTM_0 and SM the tangent bundle of TM_0 and the sphere bundle respectively, where $SM := \bigcup_{x\in M} S_xM$ and $S_xM := \{y \in T_xM \,|\, F(y) = 1\}$.

Let us consider the natural projection $p : SM \to M$ which pulls back the tangent bundle TM to an n-dimensional vector bundle p^*TM over the $(2n-1)-$dimensional base SM. Given the natural induced coordinates (x_i, y_i) on TM, the coefficients of spray vector field are defined by (cf. [11, p. 32])

$$G^i := \frac{1}{4}g^{ih}\left(\frac{\partial^2 F^2}{\partial y^h \partial x^j}y^j - \frac{\partial F^2}{\partial x^h}\right). \tag{3}$$

The pair $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ forms a *horizontal* and *vertical* frame for TTM, where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j}$, and $N^j_i := \frac{\partial G^j}{\partial y^i}$ are called the coefficients of nonlinear connection. The tangent bundle TTM_0 of TM_0 can be split into the direct sum of the horizontal part HTM spanned by $\{\frac{\delta}{\delta x^i}\}$ and the vertical part VTM spanned by $\{\frac{\partial}{\partial y^i}\}$. The dual basis of $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ is $\{dx^i, \delta y^i\}$, where

$$\delta y^i := dy^i + N^i_j dx^j, \tag{4}$$

and we have the following Whitney sum cf. [11, p. 29].

$$TTM_0 = HTM \oplus VTM = \text{span}\left\{\frac{\delta}{\delta x^i}\right\} \oplus \text{span}\left\{\frac{\partial}{\partial y^i}\right\},$$
$$T^*TM_0 = H^*TM \oplus V^*TM = \text{span}\{\mathrm{d}x^i\} \oplus \text{span}\{\delta y^i\}. \tag{5}$$

The *Cartan connection* is a natural extension of the Riemannian connection, which is metric compatible and semi-torsion free. For a global approach to the Cartan connection one can refer to [1]. According to the definition, the 1-forms of Cartan connection with respect to the dual basis $\{\mathrm{d}x^i, \delta y^i\}$ are given by

$$\omega^i_j := \Gamma^i_{jk}\mathrm{d}x^k + C^i_{jk}\delta y^k,$$

where, Γ^i_{jk} and C^i_{jk} are the *horizontal* and *vertical coefficients* of Cartan connection respectively defined by

$$\Gamma^i_{jk} := \frac{1}{2}g^{il}(\delta_j g_{lk} + \delta_k g_{jl} - \delta_l g_{jk}), \quad C^i_{jk} := \frac{1}{2}g^{il}\dot{\partial}_l g_{jk},$$

and $\delta_i := \frac{\delta}{\delta x^i}$, $\dot{\partial}_i := \frac{\partial}{\partial y^i}$. In local coordinates we have

$$\nabla_k\dot{\partial}_j = \Gamma^i_{jk}\dot{\partial}_j, \quad \dot{\nabla}_k\dot{\partial}_j = C^i_{jk}\dot{\partial}_j,$$
$$\nabla_k\delta_j = \Gamma^i_{jk}\delta_i, \quad \dot{\nabla}_k\delta_j = C^i_{jk}\delta_i,$$

where in, $\nabla_k := \nabla_{\frac{\delta}{\delta x^k}}$, $\dot{\nabla}_k := \nabla_{\frac{\partial}{\partial y^k}}$.

Let us consider the components of an arbitrary (2,2)-tensor field T^{jk}_{is} on TM. The *horizontal* and *vertical* components of the Cartan connection of T^{jk}_{is} in a local coordinates are given respectively by

$$\nabla_h T^{jk}_{is} = \delta_h T^{jk}_{is} - T^{jk}_{ps}\Gamma^p_{ih} - T^{jk}_{ip}\Gamma^p_{sh} + T^{pk}_{is}\Gamma^j_{ph} + T^{jp}_{is}\Gamma^k_{ph},$$
$$\dot{\nabla}_h T^{jk}_{is} = \dot{\partial}_h T^{jk}_{is} - T^{jk}_{ps}C^p_{ih} - T^{jk}_{ip}C^p_{sh} + T^{pk}_{is}C^j_{ph} + T^{jp}_{is}C^k_{ph}.$$

The *curvature tensor* in Cartan connection is given by the *hh-curvature*, *hv-curvature* and *vv-curvature* with the following components, cf. [1];

$$R^h_{kij} = \delta_i\Gamma^h_{jk} - \delta_j\Gamma^h_{ik} + \Gamma^l_{jk}\Gamma^h_{il} - \Gamma^l_{ik}\Gamma^h_{jl} + R^l_{ij}C^h_{lk},$$
$$P^h_{kij} = \dot{\partial}_k\Gamma^h_{ki} - \delta_i C^h_{kj} + \Gamma^r_{ki}C^h_{rj} - C^r_{kj}\Gamma^h_{rj} + \dot{\partial}_j N^r_i C^h_{kr},$$
$$Q^h_{kij} = C^h_{rj}C^r_{ki} - C^h_{ri}C^r_{kj},$$

respectively where

$$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j} = y^m R^i_{mjk}. \tag{6}$$

Trace of the *hh-curvature* of Cartan connection is denoted by $R_{ij} := R^l_{ilj}$, which is not symmetric in general.

Let (M, F) be a Finsler manifold, $\pi : TM_0 \to M$ the bundle of non-zero tangent vectors and π^*TM the pullback bundle. The tangent space T_xM, $x \in M$ can be considered as a fiber of the pullback bundle π^*TM. Therefore a section X on π^*TM is denoted by $X = X^i(x, y)\frac{\partial}{\partial x^i}$. The *Ricci identity* for Cartan connection is given by the following equation

$$\nabla_k\nabla_h X^i - \nabla_h\nabla_k X^i = X^r R^i_{rkh} - \dot{\nabla}_r X^i R^r_{kh}, \tag{7}$$

cf. [1]. Now we are in a position to define some basic notions on harmonic forms on Finsler manifolds.

3. The p-forms and horizontal operators

Here and everywhere in this paper, we assume the differential manifold M is compact and without boundary or simply closed. Let (M,F) be a closed Finsler manifold, $u: M \to SM$ a unitary vector field and $\omega = u_i \mathrm{d}x^i$ the corresponding 1-form on M. A *volume element* on SM is given by $\eta = \frac{(-1)^{\frac{n(n-1)}{2}}}{(n-1)!}\omega \wedge (\mathrm{d}\omega)^{n-1}$, cf. [1]. We denote the space of all *horizontal p-forms* on SM by $\Lambda_p^{\mathrm{H}}(SM)$ or simply Λ_p^{H},

$$\Lambda_p^{\mathrm{H}}(SM) := \left\{\varphi_{i_1 i_2 \ldots i_p}(z)\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_p} \,\middle|\, \varphi_{i_1 i_2 \ldots i_p} \in C^{\infty}(SM)\right\}. \tag{8}$$

Let $\pi = a_i(z)\mathrm{d}x^i$ be a horizontal 1-form on SM. The *co-differential* or *divergence* of π concerning the Cartan connection is defined by

$$\delta\pi = -(\nabla^j a_j - a_j \nabla_0 T^j), \tag{9}$$

where, $T_{kij} = C_{kij} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$, are the components of Cartan tensors and $\nabla_0 = y^i \nabla_i$ cf. [1, p. 223]. Also, we have

$$\int_{SM} \delta\pi\, \eta = -\int_{SM} (\nabla^j a_j - a_j \nabla_0 T^j)\eta = -\int_{SM} (\nabla_j a^j - a^j \nabla_0 T_j)\eta = 0, \tag{10}$$

where $a^i = g^{ij} a_j$, cf. [1, p. 67]. Let us denote the horizontal part of the differential $\mathrm{d}\pi$ by

$$\mathrm{Hd}\pi := \frac{1}{2}(\nabla_i a_j - \nabla_j a_i)(z)\, \mathrm{d}x^i \wedge \mathrm{d}x^j,$$

cf. [1, p. 224]. According to the above discussion, we are in a position to define a horizontal differential operator in the following sense.

Definition 5. *Let (M,F) be a Finsler manifold and $\varphi = \frac{1}{p!}\varphi_{i_1\ldots i_p}(z)\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} \in \Lambda_p^H$ a horizontal p-form on SM. A* horizontal differential operator *is a differential operator on SM given by*

$$\begin{aligned} \mathrm{d}_H : \Lambda_p^H &\to \Lambda_{p+1}^H \\ \varphi &\to \mathrm{d}_H \varphi, \end{aligned} \tag{11}$$

where, for $1 \le i, i_k \le n$ and $1 \le k \le p$, we have

$$\mathrm{d}_H \varphi = \frac{1}{(p+1)!}(\nabla_i \varphi_{i_1\ldots i_p} - \nabla_{i_1}\varphi_{i i_2 \ldots i_p} - \cdots - \nabla_{i_p}\varphi_{i_1\ldots i_{p-1} i})\mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}. \tag{12}$$

Let φ and π be the two arbitraries horizontal p-forms on SM with the components $\varphi_{i_1\ldots i_p}$ and $\pi_{i_1\ldots i_p}$, respectively. We consider an inner product $(\cdot,\cdot)$ on Λ_p^{H} as follows

$$(\varphi, \pi) := \int_{SM} \frac{1}{p!}\, \varphi^{i_1\ldots i_p}\, \pi_{i_1\ldots i_p}\, \eta, \tag{13}$$

where, $\varphi^{i_1\ldots i_p} = g^{i_1 j_1}\ldots g^{i_p j_p}\varphi_{j_1\ldots j_p}$.

4. The horizontal Laplacian and harmonic p-forms

Using the above concepts, we define the horizontal Laplacian. This definition of Laplacian is different from those given in [1,4] and [11].

Let (M,F) be a Finsler manifold and ψ a horizontal (p+1)-form on SM, given by

$$\psi = \frac{1}{(p+1)!}\psi_{i i_1\ldots i_p}\mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}.$$

We define the *horizontal divergence* (co-differential) of ψ by

$$(\delta_{\mathrm{H}}\,\psi)_{j_1\ldots j_p} := -\frac{1}{p!}g^{ij}(\nabla_i \psi_{j j_1\ldots j_p} - \psi_{j j_1\ldots j_p}\nabla_0 T_i). \tag{14}$$

Remark 6. If φ is a horizontal 1-form on SM, then δ_{H} reduces to δ, and we have

$$\delta_{\mathrm{H}}\,\varphi = \delta\varphi = -(\nabla^j \varphi_j - \varphi_j \nabla_0 T^j). \tag{15}$$

Definition 7. *Let* (M,F) *be a Finsler manifold. A* horizontal Laplacian *on* SM *is defined by*

$$\Delta_H := \mathrm{d}_H \delta_H + \delta_H \mathrm{d}_H, \tag{16}$$

where d_H *and* δ_H *are horizontal differential and horizontal co-differential operators on* SM*, respectively.*

Now we are able to show the basic equivalence relation

$$\Delta_{\mathrm{H}}\,\omega = 0 \quad \text{if and only if} \quad \mathrm{d}_{\mathrm{H}}\,\omega = 0,\ \mathit{and} \quad \delta_{\mathrm{H}}\,\omega = 0, \tag{17}$$

in the following theorem.

Proof of Theorem 1. It is clear that if $\delta_{\mathrm{H}} = 0$ and $\mathrm{d}_{\mathrm{H}}\omega = 0$, then we have $\Delta_{\mathrm{H}}\,\omega = 0$. Conversely, Let $\varphi = \frac{1}{p!}\varphi_{i_1\dots i_p}(z)\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p} \in \Lambda^H_p$ be a horizontal p-form on SM and ψ a horizontal (p+1)-form on SM, given by

$$\psi = \frac{1}{(p+1)!}\psi_{ii_1\dots i_p}\mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_p}.$$

Antisymmetric property of p-forms yield

$$\begin{aligned}
\nabla_{i_k}\varphi_{i_1\dots i_{k-1} i i_{k+1}\dots i_p}\psi^{ii_1\dots i_p} &= \nabla_i \varphi_{i_1\dots i_{k-1} i_k i_{k+1}\dots i_p}\psi^{i_k i_1\dots i_{k-1} i i_{k+1}\dots i_p}\\
&= (-1)^{k+(k-1)}\nabla_i \varphi_{i_1\dots i_{k-1} i_k i_{k+1}\dots i_p}\psi^{ii_1\dots i_{k-1} i_k i_{k+1}\dots i_p}\\
&= -\nabla_i \varphi_{i_1\dots i_{k-1} i_k i_{k+1}\dots i_p}\psi^{ii_1\dots i_{k-1} i_k i_{k+1}\dots i_p}.
\end{aligned}$$

Using the last equation and the inner product (13) we have

$$\begin{aligned}
(\mathrm{d}_{\mathrm{H}}\,\varphi,\psi) &= \int_{SM}\frac{1}{(p+1)!}(\nabla_i \varphi_{i_1\dots i_p} - \dots - \nabla_{i_p}\varphi_{i_1\dots i_{p-1} i})\,\psi^{ii_1\dots i_p}\,\eta\\
&= \int_{SM}\frac{1}{(p+1)!}(\nabla_i \varphi_{i_1\dots i_p} + \dots + \nabla_i \varphi_{i_1\dots i_p})\,\psi^{ii_1\dots i_p}\,\eta\\
&= \int_{SM}\frac{1}{p!}\nabla_i \varphi_{i_1\dots i_p}\,\psi^{ii_1\dots i_p}\,\eta.
\end{aligned} \tag{18}$$

Letting $a^i = \varphi_{i_1\dots i_p}\psi^{ii_1\dots i_p}$, equation (10) yields

$$\int_{SM}\nabla_i(\varphi_{i_1\dots i_p}\psi^{ii_1\dots i_p})\eta = \int_{SM}\varphi_{i_1\dots i_p}\psi^{ii_1\dots i_p}\nabla_0 T_i \eta. \tag{19}$$

Replacing (19) in (18) and using the metric compatibility of Cartan connection yields

$$\begin{aligned}
p!(\mathrm{d}_{\mathrm{H}}\,\varphi,\psi) &= \int_{SM}\nabla_i(\varphi_{i_1\dots i_p}\psi^{ii_1\dots i_p})\eta - \int_{SM}\varphi_{i_1\dots i_p}\nabla_i \psi^{ii_1\dots i_p}\eta\\
&= \int_{SM}\varphi_{i_1\dots i_p}\psi^{ii_1\dots i_p}\nabla_0 T_i \eta - \int_{SM}\varphi_{i_1\dots i_p}\nabla_i \psi^{ii_1\dots i_p}\eta\\
&= -\int_{SM}(\nabla_i \psi^{ii_1\dots i_p} - \psi^{ii_1\dots i_p}\nabla_0 T_i)\varphi_{i_1\dots i_p}\eta\\
&= -\int_{SM} g^{ij} g^{i_1 j_1}\dots g^{i_p j_p}(\nabla_i \psi_{jj_1\dots j_p} - \psi_{jj_1\dots j_p}\nabla_0 T_i)\varphi_{i_1\dots i_p}\eta.
\end{aligned} \tag{20}$$

Therefore (18) becomes

$$\begin{aligned}
p!(\mathrm{d}_{\mathrm{H}}\,\varphi,\psi) &= \int_{SM} g^{i_1 j_1}\dots g^{i_p j_p}(\delta_{\mathrm{H}}\psi)_{j_1\dots j_p}\,\varphi_{i_1\dots i_p}\,\eta\\
&= p!(\delta_{\mathrm{H}}\,\psi,\varphi),
\end{aligned}$$

which yields

$$(\mathrm{d_H}\,\varphi,\psi)=(\varphi,\delta_{\mathrm{H}}\,\psi). \tag{21}$$

If $\varphi=\omega$ is a p-form and $\psi=\mathrm{d}_H\omega$, then the equation (21) yields

$$(\mathrm{d_H}\,\omega,\mathrm{d_H}\,\omega)=(\omega,\delta_{\mathrm{H}}\mathrm{d_H}\,\omega). \tag{22}$$

If $\varphi=\delta_{\mathrm{H}}\,\omega$ and $\psi=\omega$, using (21) we have

$$(\mathrm{d_H}\delta_{\mathrm{H}}\,\omega,\omega)=(\delta_{\mathrm{H}}\,\omega,\delta_{\mathrm{H}}\,\omega). \tag{23}$$

Through the equations (22), (23) and (16) we have

$$\begin{aligned}(\Delta_{\mathrm{H}}\omega,\omega)&=(\mathrm{d_H}\delta_{\mathrm{H}}\,\omega,\omega)+(\delta_{\mathrm{H}}\mathrm{d_H}\omega,\omega)\\&=(\delta_{\mathrm{H}}\,\omega,\delta_{\mathrm{H}}\,\omega)+(\mathrm{d_H}\,\omega,\mathrm{d_H}\,\omega)\geq 0.\end{aligned}$$

If $\Delta_{\mathrm{H}}\,\omega=0$, we conclude that $\delta_{\mathrm{H}}\,\omega=0$ and $\mathrm{d_H}\,\omega=0$ which completes the proof. □

4.1. *Horizontal Laplacian of p-forms*

Let φ be a horizontal p-form on SM, by definitions of horizontal differential and co-differential we can easily see that

$$\begin{aligned}\delta_{\mathrm{H}}\,\mathrm{d_H}\,\varphi=-\frac{1}{p!}\Big[&(g^{rs}(\nabla_r\nabla_s\varphi_{i_1\dots i_p}-\nabla_s\varphi_{i_1\dots i_p}\nabla_0T_r)\\&-g^{rs}(\nabla_r\nabla_{i_1}\varphi_{si_2\dots i_p}-\nabla_{i_1}\varphi_{si_2\dots i_p}\nabla_0T_r)\\&-g^{rs}(\nabla_r\nabla_{i_2}\varphi_{i_1si_3\dots i_p}-\nabla_{i_2}\varphi_{i_1si_3\dots i_p}\nabla_0T_r)-\dots\\&-g^{rs}(\nabla_r\nabla_{i_p}\varphi_{i_1\dots i_{p-1}s}-\nabla_{i_p}\varphi_{i_1\dots i_{p-1}s}\nabla_0T_r)\Big]\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p},\end{aligned} \tag{24}$$

and

$$\delta_{\mathrm{H}}\,\varphi=-\frac{1}{(p-1)!}g^{rs}(\nabla_r\varphi_{si_2\dots i_p}-\varphi_{si_2\dots i_p}\nabla_0T_r)\mathrm{d}x^{i_2}\wedge\cdots\wedge\mathrm{d}x^{i_p}.$$

On the other hand, by definition we have

$$\begin{aligned}\mathrm{d_H}\,\delta_{\mathrm{H}}\,\varphi=-\frac{1}{p!}\Big[&g^{rs}(\nabla_{i_1}\nabla_r\varphi_{si_2\dots i_p}-\nabla_{i_1}(\varphi_{si_2\dots i_p}\nabla_0T_r))\\&-g^{rs}(\nabla_{i_2}\nabla_r\varphi_{si_1i_3\dots i_p}-\nabla_{i_2}(\varphi_{si_1i_3\dots i_p}\nabla_0T_r))-\dots\\&-g^{rs}(\nabla_{i_p}\nabla_r\varphi_{si_2\dots i_{p-1}i_1}-\nabla_{i_p}(\varphi_{si_2\dots i_{p-1}i_1}\nabla_0T_r))\Big]\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}.\end{aligned} \tag{25}$$

The equations (24) and (25) yield

$$\begin{aligned}(\delta_{\mathrm{H}}\,\mathrm{d_H}+\mathrm{d_H}\,\delta_{\mathrm{H}})\varphi=-\frac{1}{p!}\Big[&g^{rs}(\nabla_r\nabla_s\varphi_{i_1\dots i_p}-\nabla_s\varphi_{i_1\dots i_p}\nabla_0T_r)\\&-g^{rs}(\nabla_r\nabla_{i_1}\varphi_{si_2\dots i_p}-\nabla_{i_1}\nabla_r\varphi_{si_2\dots i_p})\\&-g^{rs}(\nabla_r\nabla_{i_2}\varphi_{i_1si_3\dots i_p}-\nabla_{i_2}\nabla_r\varphi_{i_1si_3\dots i_p})-\dots\\&-g^{rs}(\nabla_r\nabla_{i_p}\varphi_{i_1\dots i_{p-1}s}-\nabla_{i_p}\nabla_r\varphi_{i_1\dots i_{p-1}s})\\&-g^{rs}(\varphi_{si_2\dots i_p}\nabla_{i_1}\nabla_0T_r+\varphi_{i_1si_3\dots i_p}\nabla_{i_2}\nabla_0T_r\\&+\cdots+\varphi_{i_1\dots i_{p-1}s}\nabla_{i_p}\nabla_0T_r)\Big]\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}.\end{aligned} \tag{26}$$

In particular for an arbitrary horizontal 1-form $\varphi=\varphi_i(z)\mathrm{d}x^i$ on SM, the above equation reduces to

$$\begin{aligned}(\delta_{\mathrm{H}}\,\mathrm{d_H}+\mathrm{d_H}\,\delta_{\mathrm{H}})\varphi=-\big[&g^{rs}(\nabla_r\nabla_s\varphi_i-\nabla_s\varphi_i\nabla_0T_r)\\&-g^{rs}(\nabla_r\nabla_i\varphi_s-\nabla_i\nabla_r\varphi_s)\\&-g^{rs}(\varphi_s\nabla_i\nabla_0T_r)\big]\mathrm{d}x^i.\end{aligned} \tag{27}$$

This fact gives rise to a new definition of horizontal harmonic vector fields on Finsler manifolds.

Definition 8. *A horizontal p-form φ on SM is called* horizontally harmonic *if we have*

$$\Delta_H \varphi = 0.$$

The horizontal harmonic p-forms will be referred to in the following as h-harmonic p-forms or simply h-harmonic.

Remark 9. C. Bertrand and A. Rauzy, using a horizontal lift of a p-form on M to SM have defined the Laplacian on a Finsler manifold which is different from our point of view. More intuitively, they construct a sub-elliptic operator on the associated unitary bundle and give a lower bound for the first eigenvalue of this operator by using the horizontal Ricci tensor of the Berwald connection, see [4].

5. The harmonic vector fields on Finsler manifolds

Recently, one of the present authors has introduced in a joint work a definition for harmonic vector fields on Finsler manifolds using the Cartan and Berwald connections in the following sense.

Remark 10. Let (M,F) be a closed Finsler manifold. A vector field $X = X^i \frac{\partial}{\partial X^i}$ on M is called harmonic if its corresponding horizontal 1-form $X = X_i(z)\mathrm{d}x^i$ on SM satisfies $\Delta X = 0$ or $\mathrm{d}X = 0$ and $\delta X = 0$, where

$$\begin{aligned} \mathrm{d}X &= \frac{1}{2}(D_i X_j - D_j X_i)\mathrm{d}x^i \wedge \mathrm{d}x^j - \frac{\partial X_i}{\partial y^j}\mathrm{d}x^i \wedge \mathrm{d}y^j, \\ \delta X &= -(\nabla^j X_j - X_j \nabla_0 T^j) = -g^{ij} D_i X_j, \end{aligned} \tag{28}$$

and ∇ and D are the covariant derivatives of Cartan and Berwald connections, respectively, cf. [9, 10].

The above definition of harmonic vector fields and the corresponding harmonic 1-forms have some inconveniences. First, it could not be easily extended to the harmonic p-forms on Finsler manifolds. In particular, the occurrence of the mixed terms of differential and co-differential could not be readily established in the Finsler setting. Second, the both Berwald's and Cartan's covariant derivatives must be considered in this calculations which needs more preliminaries for this definition. Finally, contrary to the definition of harmonic vector fields on the Riemannian manifolds, we do not have the following proper bilateral relation in general;

$$\Delta\varphi = \mathrm{d}\delta\varphi + \delta\mathrm{d}\varphi = 0 \quad \Longleftrightarrow \quad \mathrm{d}\varphi = 0 \quad \text{and} \quad \delta\varphi = 0. \tag{29}$$

The remedy lies in a slight modification of definition in the following sense. Let $X = X^i(x)\frac{\partial}{\partial x^i}$ be a vector field on M. One can associate to X a 1-form $\widetilde{X}$ on SM defined by

$$\widetilde{X} = X_i(z)\mathrm{d}x^i + \dot{X}_i \frac{\delta y^i}{F},$$

where $\dot{X}_i = \frac{1}{F}(\nabla_0 X_i - y_i \nabla_0 (y^j X_j) F^{-2})$, and $z \in SM$ [1]. The horizontal part of the associated 1-form $\widetilde{X}$ on SM is called *associate horizontal 1-form* and denoted by $X = X_i(z)\mathrm{d}x^i$.

Definition 11. *Let (M,F) be a Finsler manifold. A vector field $X = X^i(x)\frac{\partial}{\partial x^i}$ on M is called* harmonic *related to the Finsler structure F if the associate horizontal 1-form $X = X_i(z)\mathrm{d}x^i$ is h-harmonic on SM.*

Remark 12. According to this definition of the Finslerian harmonic vector field, if X is a harmonic vector field concerning the Finsler structure F, then the associate horizontal 1-form $X = X_i(z)\mathrm{d}x^i$, is h-harmonic on SM, where $X_i(z)$ is a real function on SM and $z = (x, y) \in SM$.

Theorem 13. *Let (M,F) be a closed Finsler manifold. A vector field $\varphi = \varphi^i \frac{\partial}{\partial x^i}$ on M is harmonic if and only if*

$$g^{rs}(\nabla_r \nabla_s \varphi_i - \nabla_s \varphi_i \nabla_0 T_r) = \varphi^t R_{ti} - \dot{\nabla}_t \varphi^r R^t_{ri} + \varphi^r \nabla_i \nabla_0 T_r. \tag{30}$$

Proof. The Ricci identity (7) yields

$$\begin{aligned} g^{rs}(\nabla_r \nabla_i \varphi_s - \nabla_i \nabla_r \varphi_s) &= \nabla_r \nabla_i \varphi^r - \nabla_i \nabla_r \varphi^r \\ &= \varphi^t R^r_{tri} - \dot{\nabla}_t \varphi^r R^t_{ri} \\ &= \varphi^t R_{ti} - \dot{\nabla}_t \varphi^r R^t_{ri}. \end{aligned} \tag{31}$$

Substituting the last equation in (27) we get the result. □

A Finsler manifold (M,F) is called a *Landsberg manifold* if $\nabla_0 T = 0$. We have the following corollary.

Corollary 14. *Let (M,F) be a closed Landsberg manifold. A vector field $\varphi = \varphi^i \frac{\partial}{\partial x^i}$ on M is harmonic if and only if*

$$g^{rs} \nabla_r \nabla_s \varphi_i = \varphi^t R_{ti} - \dot{\nabla}_t \varphi^r R^t_{ri}.$$

If (M,F) is Riemannian, then the above equation reduces to the following well known form.

$$g^{rs} \nabla_r \nabla_s \varphi_i = \varphi^t R_{ti}.$$

Let $X = X^i(x)\frac{\partial}{\partial x^i}$ be a vector field on (M,F). Inspired by [9] and [10] and based on the Ricci tensor, we define the *harmonic Ricci scalar* $\widetilde{\mathrm{Ric}}$ as follows

$$\widetilde{\mathrm{Ric}}(X,X) := X^k X^t R_{tk} - X^k \dot{\nabla}_r X^j R^r_{jk} - X^k \nabla_k X^j \nabla_0 T_j. \tag{32}$$

Furthermore, we obtain a classification result given in Theorem 2.

Proof of Theorem 2. Let $X = X^i(x)\frac{\partial}{\partial x^i}$ be a vector field on (M,F) and Y and Z two 1-forms on SM defined at $z \in SM$ by $Y = (X^k \nabla_k X_i)(z) \mathrm{d}x^i$ and $Z = (X_i \nabla_j X^j)(z) \mathrm{d}x^i$, respectively. Using (9) we have

$$\begin{aligned} \delta Y &= -\nabla_j (X^k \nabla_k X^j) + X^k \nabla_k X^j \nabla_0 T_j \\ &= -\nabla_j X^k \nabla_k X^j - X^k \nabla_j \nabla_k X^j + X^k \nabla_k X^j \nabla_0 T_j, \end{aligned} \tag{33}$$

and similarly

$$\begin{aligned} \delta Z &= -\nabla_k X^k \nabla_j X^j - X^k \nabla_k \nabla_j X^j + X^k \nabla_j X^j \nabla_0 T_k \\ &= -\nabla_k X^k (\nabla_j X^j - X^k \nabla_0 T_k) - X^k \nabla_k \nabla_j X^j \\ &= \nabla_k X^k \delta X - X^k \nabla_k \nabla_j X^j. \end{aligned} \tag{34}$$

The difference of δZ and δY yields

$$\delta Z - \delta Y = \nabla_k X^k \delta X + X^k (\nabla_j \nabla_k X^j - \nabla_k \nabla_j X^j) + \nabla_j X^k \nabla_k X^j - X^k \nabla_k X^j \nabla_0 T_j. \tag{35}$$

On the other hand we have

$$\mathrm{d_H}\, X = \frac{1}{2}(\nabla_i X_j - \nabla_j X_i) dx^i \wedge dx^j,$$

from which

$$\begin{aligned} \|\mathrm{d_H}\, X\|^2 &= \frac{1}{4}(\nabla_i X_j - \nabla_j X_i)(\nabla^i X^j - \nabla^j X^i) \\ &= \frac{1}{4}[(\nabla_i X_j)(\nabla^i X^j) - (\nabla_i X_j)(\nabla^j X^i) - (\nabla_j X_i)(\nabla^i X^j) + (\nabla_j X_i)(\nabla^j X^i)] \\ &= \frac{1}{2}[\|\nabla X\|^2 - (\nabla_i X_j)(\nabla^j X^i)]. \end{aligned}$$

Therefore

$$\nabla_j X^k \nabla_k X^j = \|\nabla X\|^2 - 2\|\mathrm{d_H}\, X\|^2. \tag{36}$$

Replacing (36) and (7) in (35) we obtain

$$\delta Z - \delta Y = \nabla_k X^k \delta X + X^k X^t R_{tk} - X^k \dot{\nabla}_r X^j R^r_{jk} + \|\nabla X\|^2 - 2\|\mathrm{d_H}\, X\|^2 - X^k \nabla_k X^j \nabla_0 T_j. \tag{37}$$

If X is a harmonic vector field, then by definition of $\widetilde{\mathrm{Ric}}$ given by (32) the last equation becomes

$$\delta Z - \delta Y = \|\nabla X\|^2 + \widetilde{\mathrm{Ric}}.$$

By integration over SM and using (10), we obtain

$$\int_{SM} (\widetilde{\mathrm{Ric}} + \|\nabla X\|^2)\eta = 0. \tag{38}$$

If $\widetilde{\mathrm{Ric}} = 0$, or

$$X^k X^t R_{tk} = X^k \dot{\nabla}_r X^j R^r_{jk} + X^k \nabla_k X^j \nabla_0 T_j,$$

then (38) yields the first assertion. If $\widetilde{\mathrm{Ric}} > 0$, that is, if we have

$$X^k X^t R_{tk} > X^k \dot{\nabla}_r X^j R^r_{jk} + X^k \nabla_k X^j \nabla_0 T_j,$$

then using the equation (38) we get the second assertion. □

Remark 15. For a closed Landsberg manifold and a harmonic vector field X on M, Theorem 2 reads

(1) If $X^k X^t R_{tk} = X^k \dot{\nabla}_r X^j R^r_{jk}$, then X is parallel.
(2) If $X^k X^t R_{tk} > X^k \dot{\nabla}_r X^j R^r_{jk}$, then X vanishes.

Recall that if the Finsler structure F is Riemannian, then Theorem 2 reduces to the following famous theorem of Bochner and Yano.

Theorem 16 ([12,13]). *Let (M,g) be a closed Riemannian manifold and X a harmonic vector field on M.*

(1) *If* $\mathrm{Ric}(X,X) = X^k X^t R_{tk} = 0$*, then X is parallel.*
(2) *If* $\mathrm{Ric}(X,X) = X^k X^t R_{tk} > 0$*, then X vanishes.*

6. Cohomology class and Betti number

On a smooth manifold M the de Rham cohomology $H^1_{\mathrm{dR}}(M) := Z^1(M)/B^1(M)$, is an equivalence class of the closed forms on M. The fact that a closed form is not exact indicates that the manifold has a certain global topological structure that prevents the existence of any hole or twist. The de Rham cohomology class is therefore, a way to understand, via the tangent bundle, the global topology of a manifold.

On a compact Riemannian manifold, every equivalence class in $H^k_{\mathrm{dR}}(M)$ contains exactly one harmonic form. That is, every member ω of a given equivalence class of closed forms can be written as $\omega = \alpha + \gamma$ where α is exact and γ is harmonic, i.e. $\Delta\gamma = 0$.

The dimension of the space of all harmonic forms of degree p on a manifold M is called the *pth Betti number* of the manifold.

Due to Hodge theory, the first Betti number is equal to the dimension of the space of harmonic 1-forms on M, and this space is isomorphic to $H^1_{\mathrm{dR}}(M)$.

As mentioned earlier, on a Finsler manifold (M,F), a vector field is harmonic if $X = X_i(x,y)\mathrm{d}x^i$, the associate horizontal 1-form on SM, is h-harmonic. Hence the definition of a harmonic form on (M,F) is closely related to the Finsler structure F.

The following theorem will be used in the sequel.

Theorem 17 ([7]). *If A is a closed, nonempty, convex subset of a Hilbert space B, then for every y in B there is a unique x in A that minimizes the distance from y to A.*

We are now able to prove Theorem 3.

Proof of Theorem 3. Uniqueness. Let (M,F) be a Finsler manifold, $\alpha^{(1)} = \alpha_i^{(1)}(x)\mathrm{d}x^i$ and $\alpha^{(2)} = \alpha_i^{(2)}(x)\mathrm{d}x^i$ the two 1-forms on M such that the associate horizontal 1-forms $\alpha^{(1)} = \alpha_i^{(1)}(x,y)\mathrm{d}x^i$ and $\alpha^{(2)} = \alpha_i^{(2)}(x,y)\mathrm{d}x^i$ on SM are h-harmonic and $\alpha_i^{(1)}(x,y)\mathrm{d}x^i - \alpha_i^{(2)}(x,y)\mathrm{d}x^i = \mathrm{d}_H f$ for some $f \in C^\infty(SM)$. Using the inner product (13) and the equation (21), we have

$$\begin{aligned}(\alpha_i^{(1)}(x,y)\mathrm{d}x^i - \alpha_i^{(2)}(x,y)\mathrm{d}x^i, &\alpha_i^{(1)}(x,y)\mathrm{d}x^i - \alpha_i^{(2)}(x,y)\mathrm{d}x^i) \\ &= (\alpha_i^{(1)}(x,y)\mathrm{d}x^i - \alpha_i^{(2)}(x,y)\mathrm{d}x^i, \mathrm{d}_H f) \\ &= (\delta_H(\alpha_i^{(1)}(x,y)\mathrm{d}x^i - \alpha_i^{(2)}(x,y)\mathrm{d}x^i), f) \\ &= (0,f) = 0,\end{aligned} \tag{39}$$

which yields $\alpha^{(1)} = \alpha^{(2)}$.

Existence. $B^1(SM)$ is closed in $Z^1(SM)$ and it is convex [7].

Let $\theta = \theta_i(x)\mathrm{d}x^i \in Z^1(M)$ such that $\theta = \theta_i(x,y)\mathrm{d}x^i \in Z^1(SM)$ is the associate 1-form on SM. Using Theorem 17, three is a unique minimizer, say $f_0 \in C^\infty(SM)$ such that $\|\theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0\|^2$ is minimized. For all $f \in C^\infty(SM)$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0 - t\mathrm{d}_H f\|^2 \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}(\theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0 - t\mathrm{d}_H f, \theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0 - t\mathrm{d}_H f) \\ &\quad = \frac{\mathrm{d}}{\mathrm{d}t}[(\theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0, \theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0) - 2t(\theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0, \mathrm{d}_H f) + t^2(\mathrm{d}_H f, \mathrm{d}_H f)].\end{aligned}$$

Since $\|\theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0 - t\mathrm{d}_H f\|^2$ has a unique minimum at $t = 0$, we deduce

$$(\theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0, \mathrm{d}_H f) = 0, \tag{40}$$

for all $f \in C^\infty(SM)$. On the other hand

$$(\theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0, \mathrm{d}_H f) = (\delta_H(\theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0), f). \tag{41}$$

The equations (40) and (41) yield $\delta_H(\theta_i(x,y)\mathrm{d}x^i - \mathrm{d}_H f_0) = 0$ and the proof is complete. □

We then prove the corollary.

Proof of Corollary 4. Let (M,F) be a closed orientable Finsler manifold and X a harmonic vector field related to F. Assuming $\widetilde{\mathrm{Ric}} > 0$, the second part of Theorem 2 asserts that the harmonic vector field X related to F vanishes identically. Theorem 3 yields that the dimension of the space of all harmonic forms of degree one is the first Betti number of the manifold. Hence the first Betti number is $b_1 = 0$. □

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