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Translated sums of primitive sets

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Abstract. The Erdős primitive set conjecture states that the sum $f(A) = \sum_{a\in A} \frac{1}{a\log a}$, ranging over any primitive set A of positive integers, is maximized by the set of prime numbers. Recently Laib, Derbal, and Mechik proved that the translated Erdős conjecture for the sum $f(A,h) = \sum_{a\in A} \frac{1}{a(\log a+h)}$ is false starting at $h = 81$, by comparison with semiprimes. In this note we prove that such falsehood occurs already at $h = 1.04\cdots$, and show this translate is best possible for semiprimes. We also obtain results for translated sums of k-almost primes with larger k.

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1. Introduction

A set $A \subset \mathbb{Z}_{>1}$ of positive integers is called *primitive* if no member divides another (we trivially exclude the singleton {1}). An important family of examples is the set $\mathbb{N}_k$ of *k-almost primes*, that is, numbers with exactly k prime factors counted with multiplicity. For example, $k = 1,2$ correspond to the sets of primes and semiprimes, respectively.

In 1935 Erdős [3] proved that $f(A) = \sum_{a\in A} \frac{1}{a\log a}$ converges uniformly for any primitive A. In 1988 he further conjectured that the maximum of $f(A)$ is attained by the primes $A = \mathbb{N}_1$. One may directly compute $f(\mathbb{N}_1) = 1.636\cdots$, whereas the best known bound is $f(A) < e^\gamma = 1.781\cdots$ for any primitive A [8]. As a special case, Zhang [9] proved that the primes maximize $f(\mathbb{N}_k)$, that is, $f(\mathbb{N}_1) > f(\mathbb{N}_k)$ for all $k > 1$.

One may pose a translated analogue of the Erdős conjecture, namely, the maximum of $f(A,h) = \sum_{a\in A} \frac{1}{a(\log a+h)}$ is attained by the primes $A = \mathbb{N}_1$. Recently Laib, Derbal, and Mechik [5] proved that this translated conjecture is false, by showing $f(\mathbb{N}_1,h) < f(\mathbb{N}_2,h)$ for all $h \ge 81$. Their proof method is direct, by studying partial sum truncations of $f(A,h)$. (Laib [4] very recently announced a bound $h \ge 60$, as a refinement of the same method.)

By different methods, we extend the range of such falsehood down to $h > 1.04\cdots$, and show this translate is best possible for semiprimes.

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Theorem 1. *Let $P(s) = \sum_p p^{-s}$ denote the prime zeta function. We have $f(\mathbb{N}_1, h) < f(\mathbb{N}_2, h)$ if and only if $h > h_2$, where $t = h_2 = 1.04\cdots$ is the unique real solution to*

$$\int_1^\infty \left[P(s) - \frac{1}{2}\left(P(s)^2 + P(2s)\right)\right] e^{(1-s)t}\,\mathrm{d}s = 0,$$

Moreover $f(\mathbb{N}_1, h) < f(\mathbb{N}_k, h)$ for all k sufficiently large, if and only if $h > 0.277\cdots$.

This suggests that the Erdős conjecture, if true, is only barely so. Moreover, for the same value $h = h_2 = 1.04\cdots$ we show the primes *minimize* $f(\mathbb{N}_k, h)$, which may be viewed as the inverse analogue of Zhang's maximization result.

Theorem 2. *For $h_2 = 1.04\cdots$, we have $f(\mathbb{N}_1, h_2) < f(\mathbb{N}_k, h_2)$ for all $k > 1$.*

2. Proof of Theorem 1

We introduce the zeta function for k-almost primes $P_k(s) = \sum_{n\in\mathbb{N}_k} n^{-s}$, whose relevance to us arises from the identity,

$$\begin{aligned} f(\mathbb{N}_k, h) &= \sum_{n\in\mathbb{N}_k} \frac{1}{n(\log n + h)} = \sum_{n\in\mathbb{N}_k} \frac{1}{n\log\left(ne^h\right)} \\ &= \sum_{n\in\mathbb{N}_k} e^h \int_1^\infty \left(ne^h\right)^{-s}\mathrm{d}s = \int_1^\infty P_k(s)e^{(1-s)h}\,\mathrm{d}s. \end{aligned} \tag{1}$$

Here the interchange of sum and integral holds by Tonelli's theorem, since $f(\mathbb{N}_k, h) \le f(\mathbb{N}_k)$ converges uniformly after Erdős. The significance of the identity (1) was first observed when $k = 1$, $h = 0$ by H. Cohen [2, p. 6], who rapidly computed $f(\mathbb{N}_1) = 1.636616\cdots$ to 50 digits accuracy. By comparison, the direct approach by partial sums $\sum_{p\le x} 1/(p\log p)$ converge too slowly, i.e. $O(1/\log x)$. Similarly for $k > 1$, we shall see (1) leads to sharper results.

Note one has $P_1(s) = P(s)$ and $P_2(s) = \frac{1}{2}P(s)^2 + \frac{1}{2}P(2s)$, as well as

$$\begin{aligned} P_3(s) &= \frac{1}{6}P(s)^3 + \frac{1}{2}P(s)P(2s) + \frac{1}{3}P(3s), \\ P_4(s) &= \frac{1}{24}P(s)^4 + \frac{1}{4}P(s)^2P(2s) + \frac{1}{8}P(2s)^2 + \frac{1}{3}P(s)P(3s) + \frac{1}{4}P(4s). \end{aligned}$$

In general for $k \ge 1$, [6, Proposition 3.1] gives an explicit formula for P_k in terms of P,

$$P_k(s) = \sum_{n_1+2n_2+\cdots=k} \prod_{j\ge1} \frac{1}{n_j!}\left(P\left(js\right)/j\right)^{n_j}. \tag{2}$$

Here the above sum ranges over all partitions of k. Also see [7, Proposition 2.1].

In practice, we may rapidly compute P_k (and P_k') using recursion relations.

Lemma 3. *For $k \ge 1$ let $P_k(s) = \sum_{\Omega(n)=k} n^{-s}$ and $P_1(s) = P(s) = \sum_p p^{-s}$. We have*

$$P_k(s) = \frac{1}{k}\sum_{j=1}^{k} P_{k-j}(s)P\left(js\right) \quad \text{and} \quad P_k'(s) = \sum_{j=1}^{k} P_{k-j}(s)P'\left(js\right). \tag{3}$$

Proof. The recursion (3) for P_k is given in [6, Proposition 3.1], and is equivalent to (2).

The recursion (3) for P_k' is obtained by differentiating (2),

$$\begin{aligned} P_k'(s) &= \sum_{n_1+2n_2+\cdots=k}\sum_{i\le k} P'(is)\frac{(P(is)/i)^{n_i-1}}{(n_i-1)!}\prod_{j\ne i}\frac{1}{n_j!}\left(P\left(js\right)/j\right)^{n_j} \\ &= \sum_{i\le k} P'(is) \sum_{n_1+2n_2+\cdots=k-i}\prod_{j\ge1}\frac{1}{n_j!}\left(P\left(js\right)/j\right)^{n_j} = \sum_{i\le k} P'(is)P_{k-i}(s). \end{aligned}$$

□

As observed empirically in [1], the Dirichlet series $P_2(s)-P(s)=\frac{1}{2}[P(s)^2+P(2s)]-P(s)$ has a unique root at $s=\sigma=1.1403\cdots$, through which it passes from positive to negative. We prove this more generally for $k\le 20$.

Lemma 4. *For $2\le k\le 20$, the Dirichlet series $P_k(s)-P(s)$ has a unique root at $s=\sigma_k>1$, through which it passes from positive to negative.*

Proof. For each $k\ge 1$, $P_k(s)$ is monotonically decreasing in $s>1$. As such there is a unique $s_k>1$ for which $P(s)$ passes through $(k!)^{1/(k-1)}$ from above. Using the main term in (2), $P_k(s)>P(s)^k/k!$ which is larger than $P(s)$ iff $P(s)^{k-1}>k!$ iff $s<s_k$ by definition. That is,

$$P_k(s)>P(s)>0 \qquad \text{for} \quad s\in(1,s_k). \tag{4}$$

Also there is a unique $s_k'>1$ for which $P_{k-1}(s)$ passes through 1 from above. Using the first term in the recursion (3), $-P_k'(s)>-P'(s)P_{k-1}(s)$ which is larger than $-P'(s)>0$ iff $P_{k-1}(s)>1$ iff $s<s_k'$ by definition. That is,

$$P_k'(s)<P'(s)<0 \qquad \text{for} \quad s\in\left(1,s_k'\right). \tag{5}$$

For $c<2^k$, $P_k(s)c^s$ is monotonically decreasing in $s>1$, and so is $P_k(s)/(2^{-s}+3^{-s})=\{1/[P_k(s)2^s]+1/[P_k(s)3^s]\}^{-1}$. Thus there is a unique $t_k>1$ for which $P_k(s)/(2^{-s}+3^{-s})$ passes through 1 from above. Now by definition $P(s)/(2^{-s}+3^{-s})>1=P_k(t_k)/(2^{-t_k}+3^{-t_k})$, which is larger than $P_k(s)/(2^{-s}+3^{-s})$ iff $s>t_k$ by monotonicity. That is,

$$0<P_k(s)<P(s) \qquad \text{for} \quad s\in(t_k,\infty). \tag{6}$$

In summary P_k-P has at most one root in $(1,s_k')$, and no roots in $(1,s_k)\cup(t_k,\infty)$. We directly compute that $s_k<t_k<s_k'$ for $2\le k\le 20$, and so $P_k(s)-P(s)$ has a unique root $\sigma_k\in(s_k,t_k)$ as claimed. □

We deduce the following corollary, which gives (the first part of) Theorem 1 when $k=2$.

Corollary 5. *For $2\le k\le 20$, $f(\mathbb{N}_k,h)-f(\mathbb{N}_1,h)$ has a unique root at $h_k>0$, through which it passes from negative to positive.*

Proof. For $h\ge 0$, recall $f(\mathbb{N}_k,h)=\int_1^\infty P_k(s)e^{(1-s)h}\,\mathrm{d}s$ by (1). Now by Lemma 4, $P_k(s)-P(s)$ passes from positive to negative at $s=\sigma_k>1$. Thus

$$\begin{aligned}\left[f\left(\mathbb{N}_k,h\right)-f\left(\mathbb{N}_1,h\right)\right]e^{(\sigma_k-1)h} &= \int_1^\infty \left[P_k(s)-P(s)\right]e^{(\sigma_k-s)h}\,\mathrm{d}s \\ &= \int_1^{\sigma_k}\left[P_k(s)-P(s)\right]e^{(\sigma_k-s)h}\,\mathrm{d}s-\int_{\sigma_k}^\infty \left[P(s)-P_k(s)\right]e^{(\sigma_k-s)h}\,\mathrm{d}s\end{aligned}$$

is difference of two integrals with positive integrands, which are mononotically increasing and decreasing in $h\ge 0$, respectively. Hence the difference is mononotically increasing in $h\ge 0$. And $f(\mathbb{N}_k,0)-f(\mathbb{N}_1,0)<0$ by Zhang [9], so the result follows. □

For the second part of Theorem 1, note for any fixed $h\ge 0$ we have $\log n+h\sim\log n$ for $n\in\mathbb{N}_k$ as $k\to\infty$, since $n\ge 2^k$. Thus $f(\mathbb{N}_k,h)\sim f(\mathbb{N}_k,0)$ as $k\to\infty$. Hence by [6, Theorem 2.2],

$$\lim_{k\to\infty} f\left(\mathbb{N}_k,h\right)=\lim_{k\to\infty} f\left(\mathbb{N}_k\right)=1.$$

Note $1-f(\mathbb{N}_1,h)$ passes from negative to positive at a unique root $h_\infty=0.277\cdots$. So for each $h>h_\infty$, we have $f(\mathbb{N}_k,h)-f(\mathbb{N}_1,h)>0$ for k sufficiently large (and similarly for the converse). This completes the proof of Theorem 1.

k	s_k	t_k	s'_k	σ_k	h_k
2	1.11313	1.40678	1.39943	1.14037	1.04466
3	1.06861	1.23367	1.25922	1.09224	0.98213
4	1.04306	1.15231	1.17696	1.06206	0.93018
5	1.02761	1.104	1.12386	1.04231	0.89038
6	1.01795	1.07259	1.08784	1.02907	0.86146
7	1.01179	1.05125	1.06272	1.02007	0.84126
8	1.00779	1.0364	1.04493	1.0139	0.8276
9	1.00518	1.02594	1.03223	1.00964	0.8186
10	1.00346	1.0185	1.02312	1.0067	0.8128
11	1.00231	1.0132	1.01658	1.00466	0.80915
12	1.00155	1.00942	1.01187	1.00325	0.80689
13	1.00105	1.00672	1.00849	1.00226	0.80551
14	1.0007	1.00479	1.00607	1.00158	0.8047
15	1.00048	1.00341	1.00433	1.0011	0.8042
16	1.00032	1.00243	1.00309	1.00077	0.80391
17	1.00022	1.00173	1.0022	1.00053	0.80374
18	1.00015	1.00123	1.00157	1.00037	0.80365
19	1.0001	1.00087	1.00112	1.00026	0.80359
20	1.00007	1.00062	1.00079	1.00018	0.80356

2.1. *Computations*

For $k \le 20$, we compute the unique roots σ_k and h_k of $P_k(s) - P(s)$ and $f(\mathbb{N}_k, h) - f(\mathbb{N}_1, h)$, respectively, as well as verify that the auxiliary parameters (as defined in Lemma 4) satisfy $s_k < \sigma_k < t_k < s'_k$. We similarly compute the root of $1 - f(\mathbb{N}_1, h)$ as $h_\infty = 0.277\cdots$.

In our computations, we express P_k in terms of P using the recursion in (3). In turn by Möbius inversion $P(s) = \sum_{m\ge 1}(\mu(m)/m)\log\zeta(ms)$, so P is obtained via well-known rapid computation of ζ. Finally, we compute $f(\mathbb{N}_k, h)$ from its integral form (1). The data are displayed in the table below, obtained using Mathematica (for technical convenience, we first compute $y_k = \log h_k$) [1].

We believe a unique of root h_k exists as in Corollary 5 for all $k > 20$ as well. This would enable a strengthening of Theorem 2 to $f(\mathbb{N}_k, h) > f(\mathbb{N}_1, h)$ for all values of $k > 1$, $h \ge h_2$ (so far we only establish this for $\{1 < k \le 20,\ h \ge h_2\}$ or $\{k > 1, h = h_2\}$). Uniqueness of h_k would follow if σ_k is unique, as in Lemma 4, for all k. In turn it would suffice to show $t_k < s'_k$ for all k (note $s_k < t_k$ holds automatically by (4), (6)), though it is not clear how to establish such an inequality in general.

Moreover, it appears both h_k and σ_k are monotonically decreasing in k. This may be related to some empirical trends for $f(\mathbb{N}_k)$, found in a recent disproof of a conjecture of Banks–Martin, see [1, 6].

---

[1]
```
P[k_Integer,s_]:= If[k==1,PrimeZetaP[s],
Expand[(Sum[P[1,j*s]*P[k-j,s],j,1,k-1]+P[1,k*s])/k]]
   FindRoot[P[1, s] == (k!)^(1/(k - 1)), s, 1 + 1/k^3]
   FindRoot[P[k, t]/(2^(-t) + 3^(-t)) == 1, t, 1 + 1/k^3]
   FindRoot[P[k - 1, s1] == 1, s1, 1 + 1/k^3]
   FindRoot[P[k, sigma] == P[1, sigma], sigma, 1 + 1/k^3]]
   FindRoot[NIntegrate[(P[1, s] - P[k, s])/y^s,s,1,Infinity, WorkingPrecision->30,
AccuracyGoal->13, PrecisionGoal->13], y, 1 + 1/k^3]
```

3. Proof of Theorem 2

Proof. We have already verified the claim directly for $k \le 20$, since in this case $h_k \le h_2 = 1.04\cdots$.

For $k > 20$, the proof strategy is similar to that of [6, Theorem 5.5]. That is, the integral $f(\mathbb{N}_k, h) = \int_1^\infty P_k(s) e^{(1-s)h}\, \mathrm{d}s$ has its mass concentrated near 1 as $k \to \infty$, so it suffices to truncate the integration to $[1, 1.01]$ say, as a lower bound. Thus by (1),

$$f\big(\mathbb{N}_k, h_2\big) = \int_1^\infty P_k(s) e^{(1-s)h_2}\, \mathrm{d}s > e^{-.01 h_2} \int_1^{1.01} P_k(s)\, \mathrm{d}s. \tag{7}$$

Next, we may lower bound $P_k(s)$ by $P(s)^k/k!$, which constitutes the first of the terms in the identity (2), one per partition of k. Note the terms of partitions built from small parts contribute the most mass. So by also including the terms for the partitions $k = 1\cdot(k-j) + j$ and $k = 1\cdot(k-j-2) + 2 + j$ for $j \le 6$, we shall obtain a sufficiently tight lower bound to deduce the result. Indeed, we have

$$\begin{aligned}\int_1^{1.01} P_k(s)\, \mathrm{d}s > \frac{1}{k!}\int_1^{1.01} P(s)^k\, \mathrm{d}s + \sum_{j=2}^{6} \frac{\int_1^{1.01} P(s)^{k-j} P\big(js\big)\, \mathrm{d}s}{j\big(k-j\big)!} \\ + \frac{\int_1^{1.01} P(s)^{k-4} P(2s)^2\, \mathrm{d}s}{2!2^2(k-4)!} + \sum_{j=3}^{6} \frac{\int_1^{1.01} P(s)^{k-j-2} P(2s) P\big(js\big)\, \mathrm{d}s}{2j\big(k-j-2\big)!}.\end{aligned} \tag{8}$$

From [6, (5.10)], we have

$$0 < P(s) - \log\Big(\frac{\alpha}{s-1}\Big) < 1.4(s-1), \quad \text{for } s \in [1,2], \tag{9}$$

where $\alpha = \exp\big(-\sum_{m\ge 2} P(m)/m\big) = .7292\cdots$. Thus for every $k \ge 1$, since $\log(\alpha/.01) > 4$,

$$\int_1^{1.01} P(s)^k\, \mathrm{d}s > \int_0^{.01} \log\Big(\frac{\alpha}{s}\Big)^k \mathrm{d}s = \alpha \int_{\log(\alpha/.01)}^\infty u^k e^{-u}\, du > \alpha\Gamma(k+1, 4) > .729\, k!, \tag{10}$$

where $\Gamma(k+1, 4)$ the incomplete Gamma function, and noting $\Gamma(k+1,4)/k!$ is monotonically increasing in k. Also note

$$\int_0^1 s \log\Big(\frac{\alpha}{s}\Big)^k \mathrm{d}s = \alpha^2 \int_0^\infty u^k e^{-2u}\, du = \frac{\alpha^2}{2^{k+1}} k!.$$

Using the first order Taylor approximation $P(js) > P(j) + P'(j)(s-1)$ for $j \ge 2$,

$$\begin{aligned}\int_1^{1.01} P(s)^{k-j} P\big(js\big)\, \mathrm{d}s &> P\big(j\big) \int_0^{.01} \log\Big(\frac{\alpha}{s}\Big)^{k-j} \mathrm{d}s + P'\big(j\big) \int_0^1 s \log\Big(\frac{\alpha}{s}\Big)^{k-j} \mathrm{d}s \\ &> .729\big(k-j\big)! \Big(P\big(j\big) + \frac{\alpha P'\big(j\big)}{2^{k-j}}\Big)\end{aligned}$$

by (10). Similarly,

$$\begin{aligned}&\int_1^{1.01} P(s)^{k-j-2} P(2s) P\big(js\big)\, \mathrm{d}s \\ &\quad > P(2)P\big(j\big) \int_0^{.01} \log\Big(\frac{\alpha}{s}\Big)^{k-j-2} \mathrm{d}s + \Big[P'(2)P\big(j\big) + P(2)P'\big(j\big)\Big] \int_0^1 s\log\Big(\frac{\alpha}{s}\Big)^{k-j-2} \mathrm{d}s \\ &\quad > .729\big(k-j-2\big)! \Big(P(2)P\big(j\big) + \frac{\alpha}{2^{k-j-1}}\Big[P'(2)P\big(j\big) + P(2)P'\big(j\big)\Big]\Big).\end{aligned}$$

Hence plugging back into (8),

$$f\big(\mathbb{N}_k, h_2\big) e^{.01 h_2} > \int_1^{1.01} P(s)^k\, \mathrm{d}s > \ell_k,$$

for the explicit lower bound

$$\ell_k := .729\left[1+\sum_{j=2}^{6}\frac{P(j)+\alpha P'\big(j\big)/2^{k-j}}{j}+\frac{1}{8}\left(P(2)^2+\frac{\alpha P(2)P'(2)}{2^{k-4}}\right)\right.$$
$$\left.+\sum_{j=3}^{6}\frac{1}{2j}\left(P(2)P\big(j\big)+\frac{\alpha}{2^{k-j-1}}\Big[P'(2)P\big(j\big)+P(2)P'\big(j\big)\Big]\right)\right].$$

Note ℓ_k is clearly increasing in k (recall $P'(s)<0$). Hence for $k>20$ we have

$$f\big(\mathbb{N}_k,h_2\big) > e^{-.01h_2}\ell_k > e^{-.01h_2}\ell_{20} > .98 > 0.91 > f\big(\mathbb{N}_1,h_2\big). \tag{11}$$

Here we compute $\ell_{20} = 0.99069\cdots$ and $f(\mathbb{N}_1,h_2) = 0.908599\cdots$. This completes the proof of Theorem 2. □

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