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A quick proof of the regularity of the flow of analytic vector fields

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Abstract. We offer a new and elementary proof of the convergence of the Lie series giving the flow of an analytic vector field as well as a natural deduction of such series.

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1. Introduction

Consider a germ X of an analytic vector field defined at a point p on a n-complex analytic manifold M. In local coordinates $z = (z_1, \dots, z_n)$ such that $z(p) = 0 \in \mathbb{C}^n$, X is a vector field in $(\mathbb{C}^n, 0)$, i.e., a n-tuple of convergent power series

$$X = (X_1, \dots, X_n) \in \mathbb{C}\{z\}^n,$$

or the derivation

$$X(f)(z) = \sum_{j=1}^{n} X_j(z) \frac{\partial f}{\partial z_j}(z), \quad \text{where } f \in \mathbb{C}\{z\}.$$

The dynamics of X is determined through its flow $\phi_X(t, z)$, which is the solution of the differential equation

$$\partial_t \phi_X(t, z) = X(\phi_X(t, z)), \qquad \phi_X(0, z) = z. \tag{1}$$

Geometrically, for $z_0 \in \mathbb{C}^n$ near 0, the map $t \in (\mathbb{C}, 0) \mapsto \phi_X(t, z_0)$ is a curve passing through z_0 at $t = 0$, having at the point $\phi_X(t, z_0)$ the tangent vector $X(\phi_X(t, z_0))$.

A classical result in the general theory of ordinary differential equations is that (1) has a unique solution which is as regular as X is. In our case, if X is analytic, the flow will be analytic in both variables (t, z). More specifically, we have:

Theorem 1. *Given a germ X of an analytic vector field at $0 \in \mathbb{C}^n$, the differential equation* (1) *admits a unique analytic solution $\phi_X(t,z) \in \mathbb{C}\{t,z\}^n$ which is given by the Lie series*

$$\phi_X(t,z) = \sum_{m=0}^{\infty} (X^m(z_1),\dots,X^m(z_n))\frac{t^m}{m!}. \tag{2}$$

Here X^m denotes the iterations of X as a derivation, i.e.,

$$X^0(f) = f, \quad X^{m+1}(f) := X(X^m(f)), \quad m \geq 0, \quad \text{for } f \in \mathbb{C}\{z\}. \tag{3}$$

The proof of the analyticity of ϕ_X is usually done using the Cauchy mayorant series technique [3, Chapter VII] or by the contraction map principle [5, Chapter I.1] which extends to less regular systems. Then, the expansion (2) is obtained as a consequence of analyticity writing ϕ_X as its Taylor series at $t = 0$. In particular, once we have an analytic solution of (1) at $(t,z) = (0,0)$, it must be series (2). On the other hand, the series (2) can be shown to be convergent employing mayorizing series as explained in Gröbner's book [4, Satz 2]. Due to his contribution this series is also known as a Gröbner–Lie series, having applications in the inversion of a mapping of power series [9, Section 2], in the resolution of systems of autonomous and non-autonomous differential equations [4, 8], and in the classification of tangent to the identity biholomorphisms [1, 6].

We will prove Theorem 1 finding the series (2) as a solution to an auxiliary *linear* partial differential equation, proving its convergence with the aid of Nagumo norms, and finally checking that (2) solves (1) using the analyticity of this map. This proof is even more elementary than previous approaches due to the linear character of the auxiliary problem. In fact, it only uses Cauchy's formulas and Taylor expansions.

2. The proof of the theorem

We need some preliminaries. First, we extend (3) to maps $F = (f_1,\dots,f_n) \in \mathbb{C}\{z\}^n$ by

$$X^m(F) := (X^m(f_1),\dots,X^m(f_n)), \qquad m \geq 0.$$

In particular, $X^m(\mathrm{id}) = (X^m(z_1),\dots,X^m(z_n))$ where $\mathrm{id}(z) = z$. Moreover,

$$X^{m+1}(F)(z) = D_z[X^m(F)](z) \cdot X(z), \tag{4}$$

where D_z denotes the Jacobian matrix with respect to z and $\cdot$ stands for the usual product of a matrix and a vector.

Second, if $\phi_X(s,\phi_X(\tau,z_0))$ and $\phi_X(\tau+s,z_0)$ are defined, the 1-parametric group law asserts that

$$\phi_X(s,\phi_X(\tau,z_0)) = \phi_X(\tau+s,z_0),$$

because both sides of the equality are solutions of (1) at the point $\phi_X(\tau,z_0)$, but such solution is unique. In particular, $\phi_X(t,\phi_X(-t,z_0)) = z_0$, for small z_0 and t. If we differentiate this equation with respect to t, we find that

$$\partial_t\phi_X(t,\phi_X(-t,z_0)) + D_z\phi_X(t,\phi_X(-t,z_0))\partial_t\phi_X(-t,z_0)(-1) = 0.$$

Using (1) at $-t$ and setting $z = \phi_X(-t,z_0)$ it turns out that $w = \phi_X$ satisfies

$$\partial_t w(t,z) = D_z w(t,z) \cdot X(z), \tag{5}$$

see [1, p. 546]. This is a linear PDE in $w \in \mathbb{C}^n$ that will lead to the series (2), as we shall see.

Finally, to establish the convergence of (2) we use the Nagumo norms. These were introduced by M. Nagumo in his work [7] on power series solutions of analytic PDEs. These norms have numerous applications, for instance, in singularly perturbed problems, see [2].

Let $D_r^n = \{z \in \mathbb{C}^n : |z_j| < r, j = 1,\dots,n\}$ be the polydisc centered at the origin with polyradius $(r,\dots,r)$, for a common $r > 0$. We also write $\mathscr{O}(D_r^n)$ for the set of complex-valued holomorphic functions on D_r^n.

Definition 2. *If $r>0$, $f\in\mathscr{O}(D_r^n)$, and $m\in\mathbb{N}$ the mth* Nagumo norm *of f is*

$$\|f\|_m := \sup_{z\in D_r^n} |f(z)|(r-|z_1|)^m\cdots(r-|z_n|)^m.$$

If $m=0$ this is just the supremum norm which is finite in case f is bounded on D_r^n. In this situation $\|f\|_m\le r^{mn}\|f\|_0$. Moreover, we have:

Lemma 3. *If $f,g\in\mathscr{O}(D_r^n)$ and $m,k\in\mathbb{N}$, then*

(1) $\|f+g\|_m\le\|f\|_m+\|g\|_m$ *and* $\|fg\|_{m+k}\le\|f\|_m\|g\|_k$.
(2) $\|\frac{\partial f}{\partial z_j}\|_{m+1}\le e(m+1)r^{n-1}\|f\|_m$, $j=1,\dots,n$.

For the sake of completeness we include the proof at the end of the paper. Since we work with vector- and matrix-valued maps, we extend these norms for $F=(f_1,\dots,f_n)\in\mathscr{O}(D_r^n)^n$ and $A=(A_{ij})_{i,j=1,\dots,n}\in\mathscr{O}(D_r^n)^{n\times n}$ by the rules

$$\|F\|_m := \max_{1\le j\le n}\|f_j\|_m \qquad \text{and} \qquad \|A\|_m := \max_{1\le i\le n}\sum_{j=1}^n\|A_{ij}\|_m.$$

Then Lemma 3 shows that

$$\|A\cdot F\|_{m+k}\le\|A\|_m\|F\|_k,\qquad \|D_zF\|_{m+1}\le ner^{n-1}(m+1)\|F\|_m. \tag{6}$$

We are now ready to prove the result.

Proof of Theorem 1. We assume X is not identically zero. Let us fix a closed polydisc $\overline{D_r^n}$ where each X_j is bounded and holomorphic in a neighborhood of this domain. Taking into account equation (5), we search for a solution $w(t,z)$ of the initial-value problem

$$\partial_t w(t,z)=D_z w(t,z)\cdot X(z),\quad w(0,z)=z. \tag{7}$$

Setting $w(t,z)=\sum_{m=0}^\infty \varphi_m(z)t^m/m!$, after equating coefficients in common powers of t^m, we find

$$\varphi_0(z)=w(0,z)=z,\quad \varphi_{m+1}(z)=D_z\varphi_m(z)\cdot X(z),\quad m\ge 0. \tag{8}$$

Due to (4), $\varphi_m(z)=X^m(\mathrm{id})(z)$, and thus φ_m is holomorphic and bounded on D_r^n, for all $m\ge0$. To prove that $w(t,z)$ converges we study the series

$$W(\tau):=\sum_{m=0}^\infty\|\varphi_m\|_m\frac{\tau^m}{m!},$$

where $\|\cdot\|_m$ denotes the vector mth Nagumo norm on the polydisc D_r^n. First, since X is bounded, then $\|X\|_0>0$ is finite. Now, thanks to (6) we find that

$$\frac{\|\varphi_{m+1}\|_{m+1}}{(m+1)!}\le\frac{\|D_z\varphi_m\|_{m+1}}{(m+1)!}\|X\|_0\le ner^{n-1}\|X\|_0\frac{\|\varphi_m\|_m}{m!}.$$

Since $\|\varphi_0\|_0=\|z\|_0=r$, the previous inequality shows that

$$\frac{\|\varphi_m\|_m}{m!}\le r\alpha^m,\qquad \alpha:=ner^{n-1}\|X\|_0.$$

This proves that $W(\tau)$ is convergent. Finally, if $|z_1|,\dots,|z_n|\le\rho<r$, then

$$\max_{1\le j\le n}\frac{|X^m(z_j)|}{m!}\le\frac{\|\varphi_m\|_m/m!}{(r-|z_1|)^m\cdots(r-|z_n|)^m}\le r\alpha^m(r-\rho)^{-mn},$$

showing that $w(t,z)$ converges in the polydisc $D_{(r-\rho)^n/\alpha}\times D_\rho^n$.

To conclude we prove that $w(t,z)$ solves (1). For this, since $X\circ w$ is analytic, it is equal to its Taylor series around $t=0$:

$$X(w(t,z))=\sum_{m=0}^\infty\partial_t^m(X\circ w)(0,z)\frac{t^m}{m!}.$$

Therefore, we need to check that

$$\partial_t^m(X\circ w)(0,z)=\varphi_{m+1}(z),\quad \text{for all } m\ge0, \tag{9}$$

since then

$$X(w(t,z)) = \sum_{m=0}^{\infty} \varphi_{m+1}(z)\frac{t^m}{m!} = \partial_t\left(\sum_{m=0}^{\infty}\varphi_m(z)\frac{t^m}{m!}\right) = \partial_t w(t,z),$$

as required. We proceed by induction on m. If $m=0$, $X(w(0,z)) = X(z) = \varphi_1(z)$ and (9) is valid. If we assume it holds for some m, note first that

$$\begin{aligned}\partial_t^{m+1}(X\circ w) = \partial_t^m\partial_t(X\circ w) = \partial_t^m\left((D_zX\circ w)\cdot\partial_t w\right) &= \partial_t^m\left((D_zX\circ w)\cdot D_z w\cdot X\right)\\ &= \partial_t^m D_z(X\circ w)\cdot X = D_z\partial_t^m(X\circ w)\cdot X.\end{aligned}$$

Setting $t=0$ and using the induction hypothesis we conclude that

$$\begin{aligned}\partial_t^{m+1}(X\circ w)(0,z) &= D_z\partial_t^m(X\circ w)(0,z)\cdot X(z)\\ &= D_z\left(\partial_t^m(X\circ w)(0,z)\right)\cdot X(z) = D_z\varphi_{m+1}(z)\cdot X(z) = \varphi_{m+2}(z),\end{aligned}$$

as desired. □

Theorem 1 gives an alternative to establish some elementary observations:

(i). If we change the initial condition in (7) and consider instead

$$\partial_t w(t,z) = D_z w(t,z)\cdot X(z), \qquad w(0,z) = F(z),$$

for $F\in\mathbb{C}\{z\}^n$, this system has as unique solution $\Phi(t,z) = \sum_{m=0}^{\infty} X^m(F)(z)\frac{t^m}{m!}$, whose form and convergence are determined in exactly the same way as before. But $\Psi(t,z) = F(\phi_X(t,z))$ also satisfies this system since $\Psi(0,z) = F(z)$ and

$$\begin{aligned}\partial_t\Psi(t,z) &= D_z[F](\phi_X(t,z))\partial_t\phi_X(t,z)\\ &= D_z[F](\phi_X(t,z))D_z[\phi_X](t,z)X(z) = D_z\Psi(t,z)X(z),\end{aligned}$$

where we used (5). In conclusion, we recover the classical formula

$$F(\phi_X(t,z)) = \sum_{m=0}^{\infty} X^m(F)(z)\frac{t^m}{m!},$$

for the Lie series associated to X and F.

(ii). If $X(0)=0$ and $X(z) = Az+\cdots$, where $A = D_zX(0)$, then

$$D_z[X^m(\mathrm{id})](0) = A^m.$$

In fact, differentiating the second equation in (8) and taking $z=0$ shows that $D_z\varphi_{m+1}(0) = D_z\varphi_m(0)\cdot A$, which proves the claim. Writing $\phi_X(t,z)$ as a power series in z with coefficients series in t we get

$$\phi_X(t,z) = e^{At}z+\cdots.$$

In particular, $\phi_X(t,\cdot):(\mathbb{C}^n,0)\to(\mathbb{C}^n,0)$ is a biholomorphism. Moreover, if $A=0$, then $\phi_X(t,\cdot)$ is tangent to the identity.

We conclude this note with the proof of Lemma 3.

Proof of Lemma 3. We prove (2) for $m>0$ and $j=1$. The remaining statements are easily supplied. Fix $z=(z_1,z')\in D_r^n$, $z'\in\mathbb{C}^{n-1}$, and take $0<\rho<r-|z_1|$ so that $D=\{y\in\mathbb{C}:|y-z_1|\le\rho\}$ is contained in D_r. Cauchy's formulas show that

$$\left|\frac{\partial f}{\partial z_1}(z)\right| \le \frac{1}{\rho}\sup_{|y-z_1|=\rho}|f(y,z')|, \tag{10}$$

By Definition 2 we have that

$$|f(y,z')| \le \frac{\|f\|_m}{(r-|y|)^m R}, \quad \text{where } R = \prod_{j=2}^{n}(r-|z_j|)^m,$$

and taking the supremum for y on the boundary of the disc D, we obtain

$$\sup_{|y-z_1|=\rho} |f(y,z')| \leq \frac{\|f\|_m}{(r-(\rho+|z_1|))^m R}. \tag{11}$$

Putting together (10) and (11) we find

$$\left| \frac{\partial f}{\partial z_1}(z) \prod_{j=1}^{n} (r-|z_j|)^{m+1} \right| \leq r^{n-1} \frac{\|f\|_m}{\rho} \frac{(r-|z_1|)^{m+1}}{(r-|z_1|-\rho)^m}.$$

The result follows taking $\rho = \frac{r-|z_1|}{m+1}$ and using the inequality $\left(1-\frac{1}{m+1}\right)^{-m} < e$. □

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