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Yuan Shen and Yang Guo

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The Calabi–Yau property of Ore extensions of two-dimensional Artin–Schelter regular algebras and their PBW deformations

Yuan Shen[*, a] **and Yang Guo**[a]

[a] Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China
E-mails: yuanshen@zstu.edu.cn, guoyangxyz@163.com

Abstract. Let A be a noncommutative Artin–Schelter regular algebra of dimension 2 with the Nakayama automorphism μ_A and U a PBW deformation of A with the Nakayama automorphism μ_U. We prove that any graded Ore extension $A[z;\mu_A,\delta]$ and any filtered Ore extension $U[z;\mu_U,\widetilde{\delta}]$ are 3-Calabi–Yau.

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Introduction

Many known 3-Calabi–Yau algebras can be constructed by Ore extensions (see [1, 5, 9, 10, 18, 20]). Actually, this fact is due to the twisted Calabi–Yau (also called skew Calabi–Yau) property, which is a generalization of the Calabi–Yau property, is preserved and may be untwisted under Ore extensions (see [7, 12]). Then one may ask when the Ore extension $A[z;\sigma,\delta]$ is Calabi–Yau for a twisted Calabi–Yau algebra A. Fortunately, we have known the automorphism σ must be a Nakayama automorphism μ_A of A (see [12]). So there is a natural question what role of the μ_A-derivation δ plays in the construction of Calabi–Yau algebras by Ore extensions. Recently, the papers [11, 18, 21] made contribution to this problem in abstract terms.

For certain classes of algebras, there are some specific results to this problem in the connected graded case, which is exactly the case of Artin–Schelter regular (AS-regular, for short) algebras. He, Van Oystaeyen and Zhang in [10] showed that graded Ore extensions of graded 2-Calabi–Yau algebras are graded 3-Calabi–Yau if the values of the generating relation acting by derivations vanish. The authors in [18] showed that the μ_A-derivation does not make any influence on the graded Ore extensions of noncommutative noetherian AS-regular algebras of dimension 2 being graded 3-Calabi–Yau. In fact, these two results do not occur incidentally. In this note, we make a step further to show the same conclusion in [18] holds without the noetherian assumption.

[*] Corresponding author.

Theorem 1. *Let A be a noncommutative AS-regular algebra of dimension 2. Then the graded Ore extension $A[z;\sigma,\delta]$ is graded 3-Calabi–Yau if and only if σ is the Nakayama automorphism of A.*

The proof of Theorem 1 is given in the first half of Section 1. Then we discuss the superpotential and the coherence of such graded 3-Calabi–Yau algebras (see Proposition 5 and Theorem 6). In Section 2, we consider ungraded algebras. We are interested in Ore extensions of Poincaré–Birkhoff–Witt (PBW) deformations of AS-regular algebras, which are always twisted Calabi–Yau (see [19]). Roughly speaking, a PBW deformation U of a graded algebra A is an algebra equipped with a canonical filtration such that the associate graded algebra is isomorphic to A. Restricted on the 2-dimensional case, we have a similar result to the graded one.

Theorem 2. *Let A be a noncommutative AS-regular algebra of dimension 2 and U a nontrivial PBW deformation of A. Then the filtered Ore extension $U[z;\widetilde{\sigma},\widetilde{\delta}]$ is 3-Calabi–Yau if and only if $\widetilde{\sigma}$ is the Nakayama automorphism of U.*

Throughout k is an algebraically closed field of characteristic 0. All vector spaces and algebras are over k. Unless otherwise stated, the tensor product $\otimes$ means $\otimes_k$.

An algebra A is called *twisted Calabi–Yau* of dimension d, if

(1) A is homologically smooth, that is, A has a finite length projective resolution as a left A^e-module (or equivalently, an A-bimodule) such that each term is finitely generated, where $A^e = A \otimes A^{op}$ is the enveloping algebra of A;
(2) $\mathrm{Ext}^i_{A^e}(A, A^e) = 0$ if $i \neq d$ and $\mathrm{Ext}^d_{A^e}(A, A^e) \cong A^{\mu_A}$ as A-bimodules for some automorphism μ_A of A, where the right A-action on A^{μ_A} is twisted by μ_A. We call μ_A is a *Nakayama automorphism* of A, and it is unique up to inner automorphisms.

If a Nakayama automorphism of a twisted Calabi–Yau algebra of dimension d is inner, it is called a *d-Calabi–Yau algebra* (in the sense of [6]). Graded twisted Calabi–Yau algebras and graded Calabi–Yau algebras are defined similarly in categories of graded modules.

Let A be a connected graded algebra. Then A is a graded twisted Calabi–Yau algebra of dimension d is equivalent to A is an AS-regular algebra of dimension d, that is, the global dimension of A is d, $\underline{\mathrm{Ext}}^i_A(k, A) = 0$ if $i \neq d$ and $\underline{\mathrm{Ext}}^d_A(k, A)$ is a 1-dimensional vector space (see [17, Lemma 1.2]). In this case, the Nakayama automorphism of A is unique.

1. Calabi–Yau porperty and Coherence of Graded Ore extensions

Let A be an AS-regular algebra of dimension 2. By [22, Theorem 0.1], there is an n-dimensional vector space V with a basis $\{x_1, x_2, \dots, x_n\}$ and an $n \times n$ invertible matrix $Q = (q_{ij})$ such that $A = T(V)/(r)$, where $r = \mathbf{x}^T Q\mathbf{x}$ and $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$. It is well known that A is Koszul and the Nakayama automorphism μ_A of A (for instance, see [9, Scetion 3]) satisfies

$$\mu_A(\mathbf{x}) = -(Q^T)^{-1}Q\mathbf{x}.$$

It is clear that A is graded Calabi–Yau if and only if Q is an invertible anti-symmetric matrix.

Let $B = A[z;\mu_A,\delta]$ be a graded Ore extension of A with $\deg z = 1$. Clearly, B is a Koszul AS-regular algebra of dimension 3. Write μ for the graded automorphism of the tensor algebra $T(V)$ such that

$$\mu(\mathbf{x}) = -(Q^T)^{-1}Q\mathbf{x},$$

which lifts μ_A. We also lift the μ_A-derivation δ to a μ-derivation of the tensor algebra $T(V)$, still denoted by δ. In the sequel, we write the

$$\delta(x_i) = \sum_{s,t=1}^{n} \gamma_{ist} x_s \otimes x_t,$$

for some $\gamma_{ist} \in k$, where $i, s, t = 1, \dots, n$. Since μ-derivation δ is lifted from a map of A and A is a Koszul algebra of dimension 2, one obtains that there is a unique pair (δ_r, δ_l) of elements in V such that

$$\delta(r) = r \otimes \delta_r + \delta_l \otimes r. \tag{1}$$

1.1. *Proof of Theorem 1*

Firstly, we study the Calabi–Yau property of B. By [18, Theorem 0.1] and [14, Theorem 4.11], one obtains the Nakayama automorphism of B.

Proposition 3. *The Nakayama automorphism μ_B of B satisfies*

$$\mu_{B|A} = \mathrm{id}_A, \qquad \mu_B(z) = z + \delta_r + \delta_l.$$

Write $\delta_r = \sum_{h=1}^n a_h x_h, \delta_l = \sum_{h=1}^n b_h x_h$, where $a_1, \dots, a_n, b_1, \dots, b_n \in k$. Then we give some explicit description of $\delta(r) \in V^{\otimes 3}$. By (1), one obtains

$$\begin{aligned}\delta(r) &= r \otimes \left(\sum_{h=1}^n a_h x_h\right) + \left(\sum_{h=1}^n b_h x_h\right) \otimes r = \sum_{h,i,j=1}^n (a_h q_{ij} x_i \otimes x_j \otimes x_h + b_h q_{ij} x_h \otimes x_i \otimes x_j)\\ &= \sum_{h,i,j=1}^n (a_h q_{ij} + b_i q_{jh}) x_i \otimes x_j \otimes x_h.\end{aligned} \tag{2}$$

On the other hand, we use the fact that δ is a μ-derivation to get

$$\begin{aligned}\delta(r) &= \delta\left(\sum_{i,j=1}^n q_{ij} x_i \otimes x_j\right) = \sum_{i,j=1}^n q_{ij}(\mu(x_i) \otimes \delta(x_j) + \delta(x_i) \otimes x_j)\\ &= \sum_{i,j=1}^n q_{ij}\left(\left(-(\det Q)^{-1} \sum_{u,v=1}^n Q_{iu} q_{uv} x_v\right) \otimes \sum_{s,t=1}^n \gamma_{jst} x_s \otimes x_t + \sum_{s,t=1}^n \gamma_{ist} x_s \otimes x_t \otimes x_j\right)\\ &= -(\det Q)^{-1} \sum_{j,s,t,u,v=1}^n \left(\sum_{i=1}^n q_{ij} Q_{iu}\right) q_{uv} \gamma_{jst} x_v \otimes x_s \otimes x_t + \sum_{i,j,s,t=1}^n q_{ij} \gamma_{ist} x_s \otimes x_t \otimes x_j\\ &= -\sum_{j,s,t,v=1}^n q_{jv} \gamma_{jst} x_v \otimes x_s \otimes x_t + \sum_{i,j,s,t=1}^n q_{ij} \gamma_{ist} x_s \otimes x_t \otimes x_j\\ &= \sum_{j,s,t=1}^n \left(\sum_{i=1}^n \gamma_{ist} q_{ij} - \gamma_{itj} q_{is}\right) x_s \otimes x_t \otimes x_j,\end{aligned} \tag{3}$$

where Q_{iu} is the cofactor of q_{iu} in the matrix Q for any $i, u = 1, \dots, n$. Comparing with (2) and (3), we have

$$a_s q_{ij} + b_i q_{js} = \sum_{t=1}^n (\gamma_{tij} q_{ts} - \gamma_{tjs} q_{ti}), \tag{4}$$

for any $i, j, s = 1, \dots, n$.

All Ore extensions $A[z; \mu_A, \delta]$ of A and all pairs (δ_r, δ_l) can be determined by the system of equations (4). However, we can read off an interesting result from (4) without solving it.

Lemma 4. *If $n \geq 3$, $\delta_r + \delta_l = 0$.*

Proof. It suffices to show that $a_u + b_u = 0$ for any $u = 1, \dots, n$. We divide it into two cases. Let u be an element of the set $\{1, \dots, n\}$.

Case 1: A is not graded Calabi–Yau, that is, Q is not anti-symmetric.

(1) If $q_{uu} \neq 0$, one obtains that

$$a_u + b_u = 0,$$

by taking $i = j = s = u$ in (4) and the result follows.

(2) If $q_{uu} = 0$, we have

$$a_u q_{vu} = \sum_{t=1}^{n} (\gamma_{tvu} q_{tu} - \gamma_{tuu} q_{tv}),$$
$$b_u q_{uv} = \sum_{t=1}^{n} (\gamma_{tuu} q_{tv} - \gamma_{tuv} q_{tu}),$$
$$a_u q_{uv} + b_u q_{vu} = \sum_{t=1}^{n} (\gamma_{tuv} q_{tu} - \gamma_{tvu} q_{tu}),$$

by taking two elements of i, j, s to be u and the other one to be v in (4) for any $v = 1, \dots, n$. Then

$$(a_u + b_u)(q_{vu} + q_{uv}) = 0.$$

If $q_{vu} + q_{uv} \neq 0$ for some $v = 1, \dots, n$, the result follows.

Otherwise, by taking one element of i, j, s to be u in (4), one obtains that

$$(a_u + b_u)(q_{ij} + q_{ji}) = (a_u + b_u)(q_{ij} + q_{ji}) + (a_i + b_i)(q_{uj} + q_{ju}) + (a_j + b_j)(q_{iu} + q_{ui}) = 0,$$

for any $i, j = 1, \dots, n$. Since Q is not anti-symmetric, $q_{ij} + q_{ji} \neq 0$ holds for some $i, j = 1, \dots, n$ and $a_u + b_u = 0$.

Case 2: A is graded Calabi–Yau, that is, Q is an invertible anti-symmetric matrix. In this case, n is even, say $n = 2l \geq 4$. Each invertible anti-symmetric matrix is congruent to the following form

$$\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 1 & \dots & 0 \\ 0 & \dots & -1 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ -1 & \dots & 0 & 0 & \dots & 0 \end{pmatrix},$$

which is called a standard anti-symmetric matrix in [10]. In other words, there exists an $n \times n$ invertible matrix P such that the matrix $P^T QP$ is standard. The linear transformation of $\mathbf{x}$ defined by an $n \times n$ invertible matrix P provides an algebra isomorphism between A and the AS-regular algebras of dimension 2 defined by $P^T QP$. So we assume Q is standard, that is,

$$q_{ij} = \begin{cases} 1, & i + j = n + 1,\ j > l, \\ -1, & i + j = n + 1,\ j \leq l, \\ 0, & \text{otherwise}. \end{cases}$$

The condition (4) becomes

$$a_s q_{ij} + b_i q_{js} = \begin{cases} \gamma_{n-s+1,i,j} - \gamma_{n-i+1,j,s}, & s > l,\ i > l, \\ \gamma_{n-s+1,i,j} + \gamma_{n-i+1,j,s}, & s > l,\ i \leq l, \\ -\gamma_{n-s+1,i,j} - \gamma_{n-i+1,j,s}, & s \leq l,\ i > l, \\ -\gamma_{n-s+1,i,j} + \gamma_{n-i+1,j,s}, & s \leq l,\ i \leq l, \end{cases}$$

for any $i, j, s = 1, \dots, n$. More precisely, we have

$$\gamma_{n-s+1,i,j} + \gamma_{n-i+1,j,s} = \begin{cases} a_s & j = n-i+1, \\ b_i & j = n-s+1, \quad \text{if } s > l,\ i \le l; \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

$$-\gamma_{n-s+1,i,j} - \gamma_{n-i+1,j,s} = \begin{cases} -a_s & j = n-i+1, \\ -b_i & j = n-s+1, \quad \text{if } s \le l,\ i > l; \\ 0 & \text{otherwise}, \end{cases} \tag{6}$$

$$\gamma_{n-s+1,i,j} - \gamma_{n-i+1,j,s} = \begin{cases} b_s - a_s & j = n-i+1,\ i = s, \\ -a_s & j = n-i+1,\ i \ne s, \\ b_i & j = n-s+1,\ i \ne s, \\ 0 & \text{otherwise}, \end{cases} \quad \text{if } i, s > l; \tag{7}$$

$$-\gamma_{n-s+1,i,j} + \gamma_{n-i+1,j,s} = \begin{cases} a_s - b_s & j = n-i+1,\ i = s, \\ a_s & j = n-i+1,\ i \ne s, \\ -b_i & j = n-s+1,\ i \ne s, \\ 0 & \text{otherwise}, \end{cases} \quad \text{if } i, s \le l, \tag{8}$$

for any $i, j, s = 1, \dots, n$.

If $l < u \le 2l$, we choose an integer $l < v \le n$ such that $u \ne v$. Since $l \ge 2$, such v exists. One obtains

$$\begin{aligned} a_u + b_u &= -\gamma_{n-u+1,v,n-v+1} + \gamma_{n-v+1,n-v+1,u} + \gamma_{v,u,v} + \gamma_{n-u+1,v,n-v+1} \\ &= \gamma_{n-v+1,n-v+1,u} + \gamma_{v,u,v} \\ &= 0, \end{aligned}$$

where the first equation holds by taking $j = n-v+1, s = u, i = v$ in (7) and taking $j = v, s = n-v+1, i = u$ in (6), and the final equation holds by taking $j = u$, $s = v$ and $i = n-v+1$ in (5). Similarly, one can obtain $a_u + b_u = 0$ for any $1 \le u \le l$ by (5), (6) and (8). □

Proof of Theorem 1. If $A[z;\sigma,\delta]$ is graded Calabi–Yau, σ is the Nakayama automorphism of A by [18, Theorem 0.1] (or [12, Theorem 0.2]).

Assume σ is the Nakayama automorphism μ_A of A. If A is generated by two elements, then A is noetherian and the conclusion is true by [18, Theorem 0.2 (b)]. Suppose the minimal generating set of A contains more than two elements. Since A is AS-regular of dimension 2, the graded Ore extension $A[z;\mu_A,\delta]$ is AS-regular of dimension 3. The Nakayama automorphism of $A[z;\mu_A,\delta]$ is the identity map by Proposition 3 and Lemma 4. □

It is well known that any graded 3-Calabi–Yau algebra is determined by a superpotential (see [3, Theorem 3.1]). Write $\widehat{V} = V \oplus kz$. Since B is graded 3-Calabi–Yau, there is a superpotential $\omega \in \widehat{V}^{\otimes 3}$, that is, $\omega = (\mathrm{id} \otimes \tau)(\tau \otimes \mathrm{id})(\omega)$ where τ is the flipping map, such that $B \cong T(\widehat{V})/(\partial_{x_i}(\omega), i = 1, \dots, n+1)$ where $\partial_{x_i}(\omega) = (x_i^* \otimes \mathrm{id}^{\otimes 2})(\omega)$ for any $i = 1, \dots, n+1$ and $x_{n+1} = z$. There is an explicit description of the superpotential for B which is a generalization of [10, Theorem 0.1 (ii)] and a special case of [18, Theorem 3.11].

Proposition 5. *Suppose A is noncommutative. The element*

$$\begin{aligned} \omega &= z \otimes r - (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \mu \otimes \mathrm{id})(z \otimes r) + r \otimes z - (\delta \otimes \mathrm{id})(r) + r \otimes \delta_r \\ &= z \otimes r - (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \mu \otimes \mathrm{id})(z \otimes r) + r \otimes z + (\mu_A \otimes \delta)(r) + \delta_r \otimes r \end{aligned}$$

is a superpotential, and $B = T(\widehat{V})/(\partial_{x_i}(\omega), i = 1, \dots, n+1)$.

Proof. It is straightforward to check that $(\mu \otimes \mu)(r) = r$. Then the conclusion is an immediate result by Theorem 1, [18, Theorem 3.11] and Lemma 4. □

1.2. *Coherence*

Although AS-regular algebras of dimension 2 are not noetherian if $n \geq 3$, they are graded coherent (see [15, Theorem 4.1] or [13, Theorem 4.16]). The paper [10] showed that graded Ore extensions of graded 2-Calabi–Yau algebras are also coherent under some assumptions. We follow the idea in [10] to give a generalized version about the coherence for graded Ore extensions of 2-dimensional AS-regular algebras.

By some basic techniques of linear algebra, one may prove that each invertible matrix is congruent to an upper anti-triangular matrix, whose pattern is $(\diagup\!\!\!\!\square)$. Then we can assume the matrix Q in the relation r of A is an upper anti-triangular matrix by a linear transformation of the basis of V. Now we show a generalization of [10, Theorem 0.1 (iii)].

Theorem 6. *Let $A = T(V)/(r)$ be an AS-regular algebra of dimension 2, where V is an n-dimensional vector space with a basis $\{x_1, \dots, x_n\}$ and Q is an $n \times n$ invertible upper anti-triangular matrix such that $r = (x_1, \dots, x_n)Q(x_1, \dots, x_n)^T$. Let σ be an automorphism of A such that $\sigma(x_i) = \sum_{j=1}^{i} \alpha_{ij} x_j$ and δ a degree-one σ-derivation of A such that $\delta(x_i) = \sum_{s,t=1}^{n} \gamma_{ist} x_s \otimes x_t$, where $\alpha_{ij}, \gamma_{ist} \in k$ for any $i, j, s, t = 1, \dots, n$. If $\gamma_{inn} = 0$ for any $i = 1, 2, \dots, n-1$, then $A[z; \sigma, \delta]$ is graded coherent.*

As a consequence, if the μ_A-derivation δ satisfies the conditions above, then $A[z; \mu_A, \delta]$ is graded coherent.

Proof. This result may be proved by a slight modification on the proofs of [10, Lemma 2.2, Theorem 0.1 (iii)]. For completeness, we give the whole proof here.

Write $C = A[z; \sigma, \delta]$ and I for the ideal of C generated by $x_1, x_2, \dots, x_{n-1}$. The connected graded algebra C is generated by $x_1, \dots, x_{n-1}, x_n, z$ and subject to the relations

$$r = \sum_{i=1}^{n} \sum_{j=1}^{n-i+1} q_{ij} x_i \otimes x_j,$$

$$z \otimes x_i - \sum_{j=1}^{i} \alpha_{ij} x_j \otimes z - \sum_{s=1}^{n-1} \sum_{t=1}^{n} \gamma_{ist} x_s \otimes x_t - \sum_{t=1}^{n-1} \gamma_{int} x_n \otimes x_t, \quad \text{for any } i = 1, \dots, n-1,$$

$$z \otimes x_n - \alpha_{nn} x_n \otimes z - \gamma_{nnn} x_n \otimes x_n - \sum_{j=1}^{n-1} \alpha_{nj} x_j \otimes z - \sum_{s=1}^{n-1} \sum_{t=1}^{n} \gamma_{nst} x_s \otimes x_t - \sum_{t=1}^{n-1} \gamma_{nnt} x_n \otimes x_t.$$

Then one obtains that

$$C/I \cong k\langle x_n, z \rangle / (z x_n - \alpha_{nn} x_n z - \gamma_{nnn} x_n^2).$$

Since σ is an automorphism, α_{nn} is nonzero and C/I is a noetherian AS-regular algebra of dimension 2. By [15, Proposition 3.2] and the exact sequence

$$0 \longrightarrow I \longrightarrow C \longrightarrow C/I \longrightarrow 0,$$

it suffices to show that I is a graded free C-module on both sides.

Write the vector space $Y = kx_1 \oplus \left(\bigoplus_{i=2}^{n-1} \bigoplus_{j=0}^{\infty} k x_i x_n^j \right)$, CY (resp. CYC) for the multiplication of C and Y (resp. C, Y and C) by the multiplication of C. Firstly, we show $CY = CYC$, that is, CY is an ideal of C. It suffices to check that $Y x_i \subseteq CY$ and $Yz \subseteq CY$ for any $i = 1, 2, \dots, n$. Obviously, $Y x_i \subseteq CY$ for $i = 1, \dots, n-1$. Since Q is upper anti-triangular, one obtains

$$x_1 x_n = -q_{1n}^{-1} \big((q_{11} x_1 + \cdots + q_{n1} x_n) x_1 + (q_{12} x_1 + \cdots + q_{n-1,2} x_{n-1}) x_2 + \cdots + (q_{1,n-1} x_1 + q_{2,n-1} x_2) x_{n-1} \big) \in CY.$$

So $Yx_n \subseteq CY$.

Clearly, $\sigma^{-1}(x_i) = \sum_{j=1}^{i} \beta_{ij} x_j$ for some $\beta_{ij} \in k$ and any $i = 1, \dots, n$. We have

$$x_i z = z\sigma^{-1}(x_i) - \delta(\sigma^{-1}(x_i)) = \sum_{j=1}^{i} \beta_{ij} z x_j - \sum_{j=1}^{i} \beta_{ij} \left(\sum_{s=1}^{n-1} \sum_{t=1}^{n} \gamma_{jst} x_s x_t + \sum_{t=1}^{n-1} \gamma_{jnt} x_n x_t \right) \in CY,$$

for any $i = 1, \dots, n-1$. Let p be a positive integer. Now assume $x_i x_n^j z \in CY$ for any $j = 0, 1, \dots, p-1$ and $i = 2, \dots, n-1$. Then

$$\begin{aligned} x_i x_n^p z &= x_i x_n^{p-1} z\sigma^{-1}(x_n) - x_i x_n^{p-1} \delta(\sigma^{-1}(x_n)) \\ &= \sum_{j=1}^{n} \beta_{nj} x_i x_n^{p-1} z x_j - \sum_{j=1}^{n} \beta_{nj} x_i x_n^{p-1} \left(\sum_{s=1}^{n-1} \sum_{t=1}^{n} \gamma_{jst} x_s x_t + \sum_{t=1}^{n-1} \gamma_{jnt} x_n x_t \right) - \beta_{nn} \gamma_{nnn} x_i x_n^{p+1} \\ &\in \sum_{j=1}^{n} CY x_j + CY \subseteq CY. \end{aligned}$$

Hence, CY is an ideal of C. Since $CY \subset I$ and $x_1, \dots, x_{n-1} \in CY$, $I = CY$. It implies the graded C-module homomorphism $\varphi : C \otimes Y \to I$ defined by the multiplication of C is surjective. We claim that the Hilbert series of $C \otimes Y$ and I are the same. Then φ is bijective, and $I \cong C \otimes Y$ is a graded free left C-module.

As vector spaces, $C \cong A \otimes k[z]$. So

$$\begin{aligned} H_C(t) &= H_A(t) H_{k[z]}(t) = (1 - nt + t^2)^{-1}(1-t)^{-1}, \\ H_I(t) &= H_C(t) - H_{C/I}(t) = ((n-1)t - t^2)(1 - nt + t^2)^{-1}(1-t)^{-2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} H_Y(t) &= (n-1)t + (n-2)t^2 + (n-2)t^3 + \cdots = ((n-1)t - t^2)(1-t)^{-1}, \\ H_{C\otimes Y}(t) &= H_C(t) H_Y(t) = ((n-1)t - t^2)(1 - nt + t^2)^{-1}(1-t)^{-2}. \end{aligned}$$

The claim holds. Similarly, we can prove I is a graded free right C-module.

If Q is an upper anti-triangular matrix, then the matrix $(Q^{-1})^T Q$ is a lower triangular matrix. Since $\mu_A(\mathbf{x}) = -(Q^{-1})^T Q\mathbf{x}$, it satisfies the condition for σ. The last assertion follows. □

Then we have a skewed version of [10, Proposition 2.4].

Corollary 7. *Let A be an AS-regular algebra of dimension 2 with the Nakayama automorphism μ_A. Then $A[z;\mu_A]$ is always graded coherent.*

Proof. By the argument above Theorem 6, we assume A is defined by an upper anti-triangular matrix. Then all the conditions of Theorem 6 are satisfied and the result follows. □

If the graded Ore extension $B = A[z;\mu_A,\delta]$ is graded coherent, we can follow Polishchuk to obtain a noncommutative projective space constructed in [16], which is an abelian category

$$\operatorname{cohproj} B = \operatorname{coh} B / \operatorname{coh}^b B,$$

where $\operatorname{coh} B$ is the category of all finitely presented graded left B-modules and $\operatorname{coh}^b B$ is the category of all finite dimensional graded left B-modules. Up to derived equivalence, the study of noncommutative projective space $\operatorname{cohproj} B$ is equivalent to the study of some finite dimensional algebra (see [13, Theorem 4.14]). To be specific, there is an equivalence of triangulated categories

$$D^b(\operatorname{cohproj} B) \cong D^b(\operatorname{mod} \nabla B),$$

where $\nabla B = \begin{pmatrix} k & B_1 & B_2 \\ 0 & k & B_1 \\ 0 & 0 & k \end{pmatrix}$ is called the Beilinson algebra of B and $\operatorname{mod} \nabla B$ is the category of finite dimensional left ∇B-modules.

2. Proof of Theorem 2

For a Koszul algebra $A = T(V)/(R)$, a *PBW deformation* of A is an algebra $U = T(V)/(f - \kappa(f) - \theta(f), f \in R)$, where $\kappa : R \to V$ and $\theta : R \to k$ are two linear maps, with a filtration $\{F_i U\}_{i\geq 0}$ defined by $\left\{F_i\left(T(V)\right) = \bigoplus_{j\leq i} V^{\otimes j}\right\}_{i\geq 0}$ such that the canonical algebra homomorphism from A to the associated graded algebra $\operatorname{gr} U = \bigoplus_{i\geq 0} F_i U/F_{i-1}U$ is an isomorphism. By [4, Lemma 3.3], there are some Jacobian type conditions for linear maps κ and θ in the PBW deformation:

(1) $(\kappa\otimes \mathrm{id} - \mathrm{id}\otimes\kappa)(R\otimes V\cap V\otimes R)\subseteq R$;
(2) $(\kappa(\kappa\otimes \mathrm{id} - \mathrm{id}\otimes\kappa) + (\theta\otimes \mathrm{id} - \mathrm{id}\otimes\theta))\,(R\otimes V\cap V\otimes R) = 0$;
(3) $\theta(\kappa\otimes \mathrm{id} - \mathrm{id}\otimes\kappa)(R\otimes V\cap V\otimes R) = 0$.

In general, a PBW deformation U of a d-dimensional Koszul AS-regular algebra A is ungraded, but it is always a twisted Calabi–Yau algebra of dimension d and equipped with a filtered Nakayama automorphism μ_U by [19, Theorem 3.3], that is, $\mu_U(F_iU)\subseteq F_iU$ for any $i\geq 0$. We consider the Calabi–Yau property of the filtered Ore extension $U[z;\mu_U,\widetilde{\delta}]$, where $\widetilde{\delta}$ is a degree-one filtered μ_U-derivation, that is, $\widetilde{\delta}$ is a μ_U-derivation such that $\widetilde{\delta}(F_iU)\subseteq F_{i+1}U$ for any $i\geq 0$.

Now assume $A = T(V)/(r)$ is an AS-regular algebra of dimension 2, where V is an n-dimensional vector space with a basis $\{x_1,\dots,x_n\}$ and Q is an $n\times n$ invertible matrix such that $r = (x_1,\dots,x_n)Q(x_1,\dots,x_n)^T$. For any $v\in V$ and $\lambda\in k$, the algebra $U = T(V)/(r - v - \lambda)$ is a PBW deformation of A. By [22, Theorem 0.2 (2)], A is a domain. Then the inner isomorphism of U is unique, and so is the Nakayama automorphism of U (see also [19, Lemma 2.2]). The Nakayama automorphism of U has been described explicitly. Write $\mathbf{x} = (x_1, x_2,\dots,x_n)^T$ and $\mathbf{c} = (c_1,c_2,\dots,c_n)^T\in k^n$ such that $v = \mathbf{x}^T\mathbf{c}$.

Lemma 8 ([8, Theorem 3.5]). *The Nakayama automorphism μ_U of $U = T(V)/(r - v - \lambda)$ is filtered and satisfies*

$$\mu_U(\mathbf{x}) = -(Q^{-1})^T Q\mathbf{x} + (Q^{-1})^T\mathbf{c}.$$

Let $\widetilde{\delta}$ be a degree-one filtered μ_U-derivation of U. Since the associated graded automorphism of μ_U is the Nakayama automorphism μ_A of A, one can check that the associated linear map $\operatorname{gr}\widetilde{\delta}$ is a degree-one μ_A-derivation of A, denoted by δ. Then we have a filtered Ore extension $D = U[z;\mu_U,\widetilde{\delta}]$ of U and a graded Ore extension $B = A[z;\mu_A,\delta]$.

We lift μ_A (resp. μ_U) to an automorphism of $T(V)$ denoted by μ (resp. $\widetilde{\mu}$) and δ (resp. $\widetilde{\delta}$) to a map of $T(V)$, still denoted by δ (resp. $\widetilde{\delta}$). By Lemma 8, write $\widetilde{\mu}(x_i) = \mu(x_i) + \widetilde{c}_i$ and $\widetilde{\delta}(x_i) = \delta(x_i) + \sum_{j=1}^n d_{ij}x_j + l_i$, where $(\widetilde{c}_1,\dots,\widetilde{c}_n)^T = (Q^{-1})^T\mathbf{c}$ and $d_{ij}, l_i\in k$ for any $i,j = 1,\dots,n$. More precisely, $\widetilde{c}_i = (\det Q)^{-1}\sum_{j=1}^n Q_{ij}c_j$ where Q_{ij} is the cofactor of q_{ij} in the matrix Q for any $i,j = 1,\dots,n$.

Write $\widehat{V} = V\oplus kz$, $\widehat{R}$ for the vector space spanned by $\{r, f_1,\dots,f_n\}$ where $f_i = z\otimes x_i - \mu(x_i)\otimes z - \delta(x_i)$ for any $i = 1,\dots,n$, and $\widetilde{R}$ for the vector space spanned by $r - v - \lambda$ and

$$z\otimes x_i - \widetilde{\mu}(x_i)\otimes z - \widetilde{\delta}(x_i) = f_i - \widetilde{c}_i z - \sum_{j=1}^n d_{ij}x_j - l_i,\quad i = 1,\dots,n.$$

Clearly, $B = T(\widehat{V})/(\widehat{R})$ and $D = T(\widehat{V})/(\widetilde{R})$. Define two linear maps by

$$\begin{aligned} \kappa : \widehat{R} &\longrightarrow \widehat{V} & \theta : \widehat{R} &\longrightarrow k\\ r &\longmapsto v, & r &\longmapsto \lambda,\\ f_i &\longmapsto \textstyle\sum_{j=1}^n d_{ij}x_j + \widetilde{c}_i z, & f_i &\longmapsto l_i. \end{aligned}$$

Then $D = T(\widehat{V})/(\widehat{r} - \kappa(\widehat{r}) - \theta(\widehat{r}), \widehat{r}\in\widehat{R})$.

Lemma 9. *The algebra D is a PBW deformation of B.*

Proof. The canonical filtration on D is defined by the filtration $F_i(T(\widehat{V})) = \bigoplus_{j\le i} \widehat{V}^{\otimes i}$ for any $i \ge 0$. Since $\widehat{R} \subseteq \widehat{V}\otimes\widehat{V} \subseteq F_2(T(\widehat{V}))$, $\kappa(\widehat{R}) \subseteq F_1(T(\widehat{V}))$ and $\theta(\widehat{R}) \subseteq F_0(T(\widehat{V}))$, one obtains that the associated graded algebra $\operatorname{gr} D$ is isomorphic to B. □

By Theorem 1, we have known that B is a graded 3-Calabi–Yau algebra. We use the conditions discovered in [2, 8] to discuss the Calabi–Yau property of PBW deformations of graded 3-Calabi–Yau.

Proof of Theorem 2. Keep the notations above. Since A is a domain by [22, Theorem 0.2 (2)], so is the graded Ore extension B. It implies the inner automorphism of PBW deformation D of B is unique, and so is the Nakayama automorphism of D.

The "only if" part follows by [12, Theorem 0.2]. Now assume $\widetilde{\sigma}$ is the Nakayama automorphism of U.

Since B is Koszul 3-Calabi–Yau by Theorem 1, the vector space $\widehat{R}\otimes\widehat{V}\cap\widehat{V}\otimes\widehat{R}$ is 1-dimensional and the superpotential ω in Proposition 5 is a basis of this vector space. By Lemma 9 and the Jacobian type conditions, there is a linear map $\kappa\otimes\mathrm{id}-\mathrm{id}\otimes\kappa : \widehat{R}\otimes\widehat{V}\cap\widehat{V}\otimes\widehat{R} \to \widehat{R}$.

Write $(\kappa\otimes\mathrm{id}-\mathrm{id}\otimes\kappa)(\omega) = h_0 r + h_1 f_1 + \cdots + h_n f_n$ for some $h_0,\ldots,h_n \in k$. By [8, Proposition 4.8], one obtains that the Nakayama automorphism μ_D of D satisfies

$$\mu_D(x_1,\ldots,x_n,z)^T = (x_1,\ldots,x_n,z)^T + M_\omega^{-1}(h_0,h_1,\ldots,h_n)^T,$$

where M_ω is the matrix over k such that $\omega = (r, f_1,\ldots,f_n)M_\omega(x_1,\ldots,x_n,z)^T$. It suffices to show $h_0,\ldots,h_n$ are all 0, or equivalently $(\kappa\otimes\mathrm{id}-\mathrm{id}\otimes\kappa)(\omega) = 0$ (see also [2]).

By Proposition 5, one obtains that

$$\begin{aligned}
(\kappa\otimes\mathrm{id})(\omega) &= (\kappa\otimes\mathrm{id})\left(\sum_{i,j=1}^n q_{ij}\left(z\otimes x_i\otimes x_j - \mu(x_i)\otimes z\otimes x_j + x_i\otimes x_j\otimes z - \delta(x_i)\otimes x_j\right) + r\otimes\delta_r\right)\\
&= (\kappa\otimes\mathrm{id})\left(\sum_{i,j=1}^n q_{ij} f_i\otimes x_j + r\otimes z + r\otimes\delta_r\right)\\
&= \sum_{i,j,t=1}^n q_{ij} d_{it} x_t\otimes x_j + \sum_{i,j=1}^n q_{ij}\widetilde{c}_i z\otimes x_j + v\otimes z + v\otimes\delta_r\\
&= \sum_{i,j,t=1}^n q_{ij} d_{it} x_t\otimes x_j + \sum_{i,j,t=1}^n q_{ij}(\det Q)^{-1} Q_{it} c_t z\otimes x_t + \sum_{j=1}^n c_j x_j\otimes z + v\otimes\delta_r\\
&= \sum_{i,j,t=1}^n q_{ij} d_{it} x_t\otimes x_j + \sum_{j=1}^n c_j z\otimes x_j + \sum_{j=1}^n c_j x_j\otimes z + v\otimes\delta_r.
\end{aligned}$$

On the other hand, we have another form of ω by Proposition 5:

$$\omega = -\sum_{i,j=1}^n q_{ij}\mu(x_i)\otimes f_j + z\otimes r + \delta_r\otimes r.$$

By a straightforward computation, one obtains that

$$(\mathrm{id}\otimes\kappa)(\omega) = \sum_{i,j,t=1}^n q_{ij} d_{it} x_j\otimes x_t + \sum_{j=1}^n c_j x_j\otimes z + \sum_{j=1}^n c_j z\otimes x_j + \delta_r\otimes v.$$

Hence,

$$(\kappa\otimes\mathrm{id}-\mathrm{id}\otimes\kappa)(\omega) = \sum_{i,j,t=1}^n (q_{ij}d_{it} - q_{it}d_{ij}) x_t\otimes x_j + v\otimes\delta_r - \delta_r\otimes v \in V^{\otimes 2},$$

in which the element z does not occur. So $(\kappa\otimes\mathrm{id}-\mathrm{id}\otimes\kappa)(\omega) = h_0 r$. It remains to show $h_0 = 0$. There are two cases.

(1) $v \neq 0$. The Jacobian conditions for PBW deformations of Koszul algebras show that

$$\begin{aligned} h_0 v &= \kappa(\kappa\otimes \mathrm{id} - \mathrm{id}\otimes\kappa)(\omega) = (\mathrm{id}\otimes\theta - \theta\otimes\mathrm{id})(\omega) \\ &= (\mathrm{id}\otimes\theta)\left(\sum_{i,j=1}^{n} -q_{ij}\mu(x_i)\otimes f_j + z\otimes r + \delta_r\otimes r\right) - (\theta\otimes\mathrm{id})\left(\sum_{i,j=1}^{n} q_{ij} f_i\otimes x_j + r\otimes z + r\otimes\delta_r\right) \\ &= \sum_{i,j=1}^{n} q_{ij} l_j (\det Q)^{-1}\left(\sum_{s,t=1}^{n} Q_{is} q_{st} x_t\right) + \lambda(z+\delta_r) - \sum_{i,j=1}^{n} l_i q_{ij} x_j - \lambda(z+\delta_r) \\ &= \sum_{j,t=1}^{n} l_j q_{jt} x_t - \sum_{i,j=1}^{n} l_i q_{ij} x_j \\ &= 0, \end{aligned}$$

where Q_{is} is the cofactor of q_{is} in the matrix Q for any $i, s = 1,\dots,n$.

(2) $v = 0$. In this case, we assume $\lambda \neq 0$, otherwise U is just the graded algebra A. By the Jacobian conditions, one obtains

$$h_0\lambda = h_0\theta(r) = \theta(\kappa\otimes\mathrm{id} - \mathrm{id}\otimes\kappa)(\omega) = 0.$$

Both cases imply $h_0 = 0$. The proof is completed. □

The (ungraded) algebra D is a Calabi–Yau algebra and it is a PBW deformation of the Koszul 3-Calabi–Yau domain B, so it is defined by a (non-homogeneous) potential (see [2, 8] for details) $\widetilde{\omega} = \omega + (\kappa\otimes\mathrm{id})(\omega) + (\theta\otimes\mathrm{id})(\omega)$ by [8, Lemma 4.10, Theorem 4.11], where ω is the superpotential in Proposition 5.

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