



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

Fatih Say

First-order general differential equation for multi-level asymptotics at higher levels and recurrence relationship of singulants

Volume 359, issue 10 (2021), p. 1267-1278

Published online: 4 January 2022

<https://doi.org/10.5802/crmath.264>



This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569



Partial differential equations, Mathematical physics / *Équations aux dérivées partielles, Physique mathématique*

First-order general differential equation for multi-level asymptotics at higher levels and recurrence relationship of singulants

Fatih Say^{® a}

^a Department of Mathematics, Faculty of Arts and Sciences, Ordu University, Ordu, Turkey

E-mail: fatihsay@odu.edu.tr

Abstract. We construct a relation between the leading pre-factor function $A(z)$ and the singulants $u_0(z)$, $u_1(z)$, and recurrence relation of the singulants at higher levels for the solution of singularly-perturbed first-order ordinary general differential equation with a small parameter via the method of multi-level asymptotics. The particular equation is chosen due to its appearance at every level of multi-level asymptotic approach for the first-order differential equations. By the relations derived by the asymptotic analysis from the equation, Stokes and anti-Stokes lines can be extracted more quickly and so which exponentials of the expansions are actually contributed in each sector of the complex plane can be deduced faster. Multi-level asymptotic analysis of the first-order singular equations and the Stokes phenomenon may be done straightaway from the higher levels of the analysis.

2020 Mathematics Subject Classification. 34M40, 34E20.

Funding. This work was supported by an abroad Ph.D. scholarship of Ministry of National Education, Turkey.

Manuscript received 19th April 2021, accepted 23rd August 2021.

1. Introduction

In this paper, the method of multi-level asymptotics introduced by Say in [32] will be considered by analysing a general singularly perturbed linear ordinary differential equation of the first-order. Multi-level asymptotics is the method for extracting the exponentially small terms; unlike the usual perturbation expansion, it enables us to study the asymptotic behaviour of the exponentially small terms which can be quite large as certain lines are crossed. Therefore, it constitutes “asymptotics beyond all orders”. Extracting the exponentially small terms via the growing subject of exponential asymptotics has been studied by many, for example, [5, 6, 8, 9, 14, 24, 25, 27, 29]. The basis of this work is similar to the methods of hyperasymptotics [6]. However, the way it approaches the resultant remainders and the related truncation points of each level is different, and this will be addressed in the paper in more detail. In this paper, we are interested in the behaviour

of the following equation for small ϵ in an asymptotic sense and aim to locate and investigate its Stokes and anti-Stokes lines

$$\epsilon \frac{dy(z)}{dz} + A(z)y(z) = f(z), \quad (1)$$

where $0 < \epsilon \ll 1$. Because this equation presents in each level of the multi-level asymptotic expansion after the zeroth-level for first-order singular equations, it is a rather special differential equation. Through studying this equation, some key features of the multi-level asymptotics arising from each stage of the analysis will be demonstrated and more information on the behaviour of the differential equation will be gathered. In light of the link between the singulant introduced by Dingle [19, p. viii] and the pre-factor $A(z)$ and as well as the recursion relationships between the successive singulants, asymptotic behaviour of the equations and local analysis of the Stokes phenomenon near the singularities will be extracted much quicker at any higher level. With the development of exponential asymptotics, the presence of exponentially small terms encoded in the divergent tails is no longer negligible upon their optimal truncation [3]. Deriving the exact general form of the perturbation coefficients of the asymptotic expansions is usually intractable. However, those coefficients mostly occur at the standard factorial-over-a-power form [14, 19], which will be discussed later. Under the consideration of this form along with the recursive relationships of the singulants obtained in the paper, one can directly find the optimal truncation points of each level and analyse the associated exponentially small remainders straightforwardly.

The present paper is organised as follows: We first naively expand the problem in powers of ϵ as explained by Hinch in [21] and determine the leading order term. Because it is a singular perturbation problem, the series will be divergent and has to be truncated to obtain the minimum error. In order to do so, we first need to find out the general representation of the late-order terms. Hence, the derivation of the general term is our second step in Section 2. We will provide a connection formula between the singulant of the expansion and $A(z)$ function of the equation (1). The optimal truncation point is the point at which the series changes its form from decreasing to increasing. Section 3 is devoted to optimally truncate the expansion. Substitution of this truncated expansion into the original equation will provide an inhomogeneous differential equation for the remainder term, which is expected to be exponentially small as indicated by Berry [5]; this will be specifically demonstrated there. In Section 4, we will introduce various choices of $A(z)$ functions. The third step is to inspect the behaviour of the resultant remainder near its singularities, which means that the singulant of the asymptotic expansion in this section will be derived and so the relevant Stokes and anti-Stokes lines based on the various pre-factors will be located. Therein, detailed prescriptions existing in the literature which are required for the Stokes phenomenon will be given. In Section 5, we will derive the general first-level differential equation of the multi-level asymptotics generated by the remainder for general $A(z)$ together with the recursive relation of the singulants. Finally, the findings of the paper will be briefly summarised in Section 6 along with a detailed discussion.

2. Asymptotic power series

In this section, our motivation is to express the relationship between the late-order term of an asymptotic expansion and the pre-factor $A(z)$, by which Stokes and anti-Stokes lines of the equation can be identified. Particularly, deriving this relationship will play an essential role in understanding the Stokes structure of the higher levels of multi-level asymptotics. Hence, the first step of the investigation of obtaining an accurate approximation is principally to expand

$y(z)$ in powers of ϵ . Let us assume the corresponding asymptotic series solution of $y(z)$ we seek in (1) in the limit $\epsilon \rightarrow 0$ is

$$y(z) \sim \sum_{n=0}^{\infty} \epsilon^n y_n(z). \tag{2}$$

Substitution of this solution back into the original differential equation (1) and equating like powers of ϵ generate a leading order equation for $y_0(z)$ at $O(1)$ and a recurrence relation for $y_n(z)$ at $O(\epsilon^n)$ in the following sense

$$y_0(z) = \frac{f(z)}{A(z)},$$

$$\frac{dy_{n-1}(z)}{dz} + A(z)y_n(z) = 0 \quad \text{for } n \geq 1. \tag{3}$$

This recurrence relation in (3) permits us to compute the general term $y_n(z)$ in terms of $y_0(z)$. It may be appropriate to point out herein that zeros of $A(z)$ and singularities of $f(z)$ are the singularities of the leading order term $y_0(z)$, which means they are the singularities of the expansion for all n . The strength of the singularity or singularities of the expansion increases as n increases. As is known, if an asymptotic series solution has a singularity or singularities, then the solution is divergent and this has to be of the form factorial-over-a-power [11, 14, 19], which is a generic feature of exponential asymptotics. In fact, this diverging nature of the series occurs due to Darboux's theorem since the asymptotic behaviour of the leading order terms is determined by the closest relevant singularity in the expansion as the singular points control the whole expression, see Darboux's theorem [10, 11, 18, 19]. Moreover, while extracting the late-order terms, it is also essential to ensure that the terms of an expansion are in the expected factorial-over-a-power nature. Therefore, let us assume the late-order terms of the expansion take the form

$$y_n(z) \sim \frac{\Gamma(n + \gamma)}{u_0(z)^{n+\gamma}} C \text{ as } n \rightarrow \infty, \tag{4}$$

in which $\Gamma(\cdot)$ is the gamma function described in [1, 30], and C is a constant which could be a function of ϵ and could be derived by going to the next order of the matching; when this method is employed, we observe that it is indeed a constant. The constant γ is the number of differentiations added as a power of $u_0(z)$ to have an easy cancellation while obtaining $y_n(z)$ from $y_{n-1}(z)$ upon its substitution into the recurrence relation in (3). The denominator function $u_0(z)$ is the singulant subject to singular points of the expansion. Henceforth, the subscript(s) of $u_0(z)$ refers to the level of the multi-level asymptotics addressed herein and throughout the paper. In order to further study the behaviour of the expansion and the Stokes phenomenon, the late-order terms have to be generated for large n . Therefore, substituting this ansatz back into the recurrence relation given in (3) provides us with the following equation after some computations

$$\frac{\Gamma(n + \gamma + 1)}{u_0(z)^{n+\gamma+1}} (-u_0'(z) + A(z)) C = 0, \tag{5}$$

which requires

$$u_0'(z) = A(z), \tag{6}$$

where prime ' represents the derivative d/dz . Due to the relation between $A(z)$ and $y_0(z)$, the integration constant can be derived from the singularities of $y_0(z)$, whilst completing the determination of $u_0(z)$. Nevertheless, the constant γ in the denominator could be obtained via the repeated use of the recurrence relation from which the degree of the singulant arises, which will be more thoroughly demonstrated later. The link mentioned earlier between the denominator of the expansion and the pre-factor $A(z)$ of the equation (1) is finally established in (6); as a consequence, this will be used while locating the Stokes phenomenon of the equation.

2.1. Complex ϵ

This subsection briefly addresses what happens and what difference it would make if ϵ is complex for the equation (1); particularly, it discusses whether this methodology of multi-level asymptotics can be further extended for complex ϵ . For this purpose, we introduce the rescaled ϵ without loss of generality as

$$\epsilon = e^{i\theta} \epsilon_1, \quad (7)$$

where ϵ_1 is real, and $0 < \epsilon_1 \ll 1$. Back substitution of the complex ϵ in (7) into the original differential equation (1) gives

$$e^{i\theta} \epsilon_1 y'(z) + A(z)y(z) = f(z). \quad (8)$$

Doing the usual expansion for $y(z)$ of the form $y(z) \sim \sum_{n=0}^{\infty} \epsilon_1^n y_n(z)$ and substituting the expression back into (8) itself, we match the coefficients of the like powers of ϵ_1 , yielding at $O(1)$ and $O(\epsilon^n)$ respectively

$$y_0(z) = \frac{f(z)}{A(z)}, \quad \text{and} \quad e^{i\theta} y'_{n-1}(z) + A(z)y_n(z) = 0 \quad \text{for } n \geq 1. \quad (9)$$

In order to find the late-order terms of the expansion, we employ the factorial-over-a-power method as in (4). In doing this, after some computations, we establish the link between the singulant and the pre-factor $A(z)$ as

$$u'_0(z) = e^{-i\theta} A(z). \quad (10)$$

In comparison with (6), we express the derivative of the singulant as a multiplication of the pre-factor $A(z)$ and $e^{-i\theta}$, as a result of rescaling ϵ . Hence, the method can be extended to complex ϵ . We succinctly discuss the Stokes phenomenon for complex ϵ in the Example 1 of Section 4.

We are interested in determining the general form of the optimal truncation point for this particular equation and then demonstrating whether the remainder is exponentially small. In essence, we further study the Stokes and anti-Stokes lines based on various choices of $A(z)$ functions via (6) in the forthcoming sections.

3. Optimal truncation

The next step in the calculation is to asymptotically analyse the equation (1) by truncating the asymptotic series (2) after N_0 terms, that is

$$y(z) = \sum_{n=0}^{N_0-1} \epsilon^n y_n(z) + R_{N_0}(z), \quad (11)$$

wherein the resultant remainder $R_{N_0}(z)$ is of $O(\epsilon^{N_0})$. The subscript 0 in the truncation point, N_0 , similar to that of the singulant $u_0(z)$, indicates the level of the multi-level asymptotics for the rest of the paper. To proceed, substituting the truncated series back into the differential equation (1) produces an inhomogeneous remainder differential equation such that

$$\begin{aligned} \epsilon R'_{N_0}(z) + A(z)R_{N_0}(z) &= -\epsilon^{N_0} y'_{N_0-1}(z) \\ &= \epsilon^{N_0} A(z)y_{N_0}(z). \end{aligned} \quad (12)$$

Before proceeding, it is worth observing here that the remainder equation (12) has the same form as the original equation (1), besides their right-hand side (RHS). The optimal truncation point of the expansion, in its simplest definition, can be found by equating the successive ratio of the RHS of the remainder equation (12) in magnitude to 1 [16, 25], particularly,

$$\left| \frac{\epsilon^{N_0+1} A(z)y_{N_0+1}(z)}{\epsilon^{N_0} A(z)y_{N_0}(z)} \right| \sim 1,$$

whereupon it is furnished in the form of a function divided by ϵ by

$$N_0 \sim \frac{|u_0(z)|}{\epsilon}. \tag{13}$$

This means that the minimum number of terms, N_0 , needed in the expansion depends on the arbitrariness of ϵ and z . For any given z , as ϵ goes to zero, the optimal truncation point clearly increases. Note that α needs to be added to ensure N_0 is an integer,

$$N_0 = \frac{|u_0(z)|}{\epsilon} + \alpha, \tag{14}$$

where $0 \leq \alpha < 1$. Determination of the optimal truncation point means that the series solution (2) does not start to grow until this point. After having derived the truncation point, we find out that the remainder changes its nature from algebraically small as such

$$O(\epsilon^{N_0}),$$

to exponentially small. More precisely, employing this value of the truncation point in (14) shows that the remainder is asymptotic to

$$\begin{aligned} R_{N_0}(z) &\sim \epsilon^{N_0} \frac{\Gamma(N_0 + \gamma)}{u_0(z)^{N_0 + \gamma}} C \\ &\sim \frac{\exp\left(-\frac{|u_0(z)|}{\epsilon} - \alpha - \gamma + 1\right)}{\sqrt{u_0(z)} \epsilon^{\gamma - \frac{1}{2}}} \left[\frac{|u_0(z)|}{u_0(z)} \right]^{\frac{|u_0(z)|}{\epsilon} - \frac{1}{2} + \alpha + \gamma} \sqrt{2\pi} C. \end{aligned} \tag{15}$$

Upon performing Stirling's approximation of gamma function [30], we verify above (15) by exponential factor $\exp(-u_0(z)/\epsilon)$ that the remainder turns out to be exponentially small as expected. Therefore, examining the truncated remainder equation in its least value enables us to investigate the behaviour of the exponentially small terms. It is also readily deducible from (15) that this exponentially small term controls the order of the error. Meanwhile, we further observe that the optimal truncation point obtained in (14) occurred exactly the same as that in the power of the exponential factor of (15).

4. Stokes structure

In exponential asymptotics, critical points (singularities or turning points) where Stokes and anti-Stokes lines sprout from play a key role. A turning point of an asymptotic expansion is a point where the pre-factor $A(z)$ vanishes in which the expansion breaks down since its zeros appear to be the singularities of such an expansion. It could also occur when the RHS of the differential equation has a singularity. In the paper, we approximate a linear combination of two asymptotic expansions whereby they have the leading behaviours as $\exp(-u_0(z)/\epsilon)$ and $\exp(0)$; particularly, for this inhomogeneous equation, the particular integral is the solution which contains an asymptotic expansion. This expansion may generate a Stokes line and switch on and off the exact homogeneous solution. However, the complementary function does not know anything about a particular integral since there is not a divergent tail from the exact homogeneous solution; therefore, it can not turn the particular integral on. Stokes in [34] pioneered the idea that across certain rays, known as Stokes lines, in the complex plane subdominant solutions change abruptly, which has been much studied in the literature, for example, [2–4, 12, 14, 20, 28, 29]. There are several prescriptions for locating these lines:

- (i) Stokes lines may be present when singulants or the powers in the leading exponentials satisfy [3, 14, 16, 17, 19, 33],

$$\Im(u_0(z)) = \Im(0), \quad \text{and} \quad \Re(u_0(z)) > \Re(0).$$

- (ii) Stokes lines occur where the dominant factor of the RHS of the residual equation (15) is maximal at $ph(z) = \theta$ [19, 29] [34, p. 105].
- (iii) Stokes lines can be determined by minimising the magnitude of the RHS of (15) [26].
- (iv) Stokes lines may sometimes be located when the RHS of (15) does not depend on α in Olde Daalhuis et al [29].
- (v) Stokes lines arise when the phases of the higher order terms in an expansion are the same [15, 19].
- (vi) Anti-Stokes lines arise where the singulants are purely imaginary or the exponentials are of the same order [3, 15, 17, 23, 28, 33], i.e.

$$\Re(u_0(z)) = \Re(0).$$

Across anti-Stokes lines, exponentials change their behaviour from being dominant to subdominant or vice-versa. This means that anti-Stokes lines indicate the region of the complex plane where the exponential is dominant; therefore, anti-Stokes lines matters. In this paper, inactive Stokes lines correspond to the irrelevant Stokes lines in which the subdominant exponential can not switch the dominant exponential on; particularly, in our case, they present when the powers in the leading exponentials satisfy $\Im(u_0(z)) = \Im(0)$ and $\Re(u_0(z)) < \Re(0)$. For more details regarding the irrelevant Stokes lines, we refer to [13, 22]. Generally, a differential equation of the remainder appears as in (12). In order to examine this type of equation after the zeroth-level via using methods of multi-level asymptotics [32], one needs to rescale the remainder based on its RHS whereupon the RHS becomes 1 in the new scaled equation, which will become apparent in Section 5. Therefore, for the following instructive examples, the RHS of (1) is set out as $f(z) = 1$. The rest of the section discusses the Stokes and anti-Stokes lines of equation (1) according to various choices of pre-factor $A(z)$ taking into account the link derived in (6).

Example 1. Assume in (1) that $A(z)$ is given as $A(z) = z$.

As indicated earlier, since the Stokes phenomenon of (1) is associated with the pre-factor $A(z)$, we begin the determination of the Stokes and anti-Stokes lines of (1) for this particular $A(z)$ by first deriving its turning point. The turning point of the expansion (2) lies at the origin whereby $A(z) = z$ vanishes for $z = 0$. In order to derive the singulant $u_0(z)$ of the late-order terms given in (4), we solve the differential equation (6) to find

$$u_0(z) = \frac{z^2}{2} + C_1,$$

in which C_1 is a constant of integration. As explained above, because $u_0(z)$ has to vanish at the singular point, it requires C_1 to be zero, whence the singulant of the late-order term is completely derived as

$$u_0(z) = \frac{z^2}{2}. \quad (16)$$

We now have sufficient information to locate the Stokes and anti-Stokes lines. Upon introducing the polar coordinates $z = r \exp(i\theta)$ and directly implementing them into the singulant enable us to write down $u_0(z)$ in terms of fast variable θ and r , yielding

$$u_0(z) = \frac{r^2 \exp(2i\theta)}{2} = \frac{r^2}{2} (\cos(2\theta) + i \sin(2\theta)). \quad (17)$$

Adopting one of the definitions given earlier such as (i) for (17) gives that Stokes lines occur whenever θ fulfils the condition

$$\sin(2\theta) = 0 \quad \text{while} \quad \cos(2\theta) > 0. \quad (18)$$

Therefore, Stokes lines in this example are $\theta = 0$ and π . Inactive Stokes lines lie along $\pi/2$ and $3\pi/2$ in which the RHS of (18) is $\cos(2\theta) < 0$. Likewise, the set of points satisfying the condition (vi)

is an anti-Stokes line. Considering (17) under this condition reads that anti-Stokes lines appear whenever θ satisfies $\cos(2\theta) = 0$. This means they occur along $\theta = \pi/4, 3\pi/4, 5\pi/4$ and $7\pi/4$. Both Stokes and anti-Stokes lines are illustrated in Figure 1.

On the other hand, employing the link in (10) when ϵ is complex, we derive the associated singulant for $A(z) = z$ is

$$u_0(z) = \frac{\exp(-i\theta)z^2}{2}. \tag{19}$$

After performing the Stokes lines analysis as before, we do not repeat here for the sake of brevity, we find that the associated Stokes and anti-Stokes lines are rotated in the counterclockwise direction by $\theta = \pi/4$ as mapped in the Figure 1b for selected $\epsilon = -i\epsilon_1$. Hence, the method can be used when ϵ is complex. Moreover, it is perhaps worth commenting that the solution of the equation (1) based on Example 1, which could be derived via the integration factor method, comes from the same family of the error function [3,31].

We are, mainly, considering two exponential solutions that have the leading behaviour of a first-order differential equation: one can be obtained from the homogeneous solution as $\exp(-z^2/\epsilon^2)$ while the other is derived from the RHS of the differential equation as $1 = \exp(0)$. Once the conditions (i) and (vi) of the above Stokes and anti-Stokes lines prescriptions are applied to these two exponentials, one can see that this will yield the same Stokes lines results as above. Stokes and anti-Stokes lines can be determined through these exponential terms of the equation (1) that are all mapped in Figure 1a, in which $\exp(-z^2/\epsilon^2)$ and $\exp(0)$ are labelled by ① and ②, respectively. This figure additionally illustrates the sectors in the z -plane where these exponential terms switch on and off, and they become present. This switching on and off behaviour of the solutions when Stokes lines are crossed is not the main focus here, but will be discussed in more detail in a future paper; for more details, see [2, 7, 13, 22, 23, 32]. In order to see it, we need to make a full circuit around the complex plane divided into 4 sectors by the active and inactive Stokes lines. Assume we are in Sector IV of Figure 1a, and ② is present there. We travel through Sector IV to I in a counterclockwise direction. As we cross the Stokes line from Sector IV to Sector I, ① gets turned on by ② and it must be added to the solution. Thus, ② and ① are both present in Sector I. As the Stokes line from Sector I to Sector II is inactive, no change is observed as we move leftward meaning that ② and ① are also present in Sector II. However, crossing through Sector III, ① is switched off by ② across the Stokes line. Therefore, ① is not present in Sector III. Since the Stokes line is inactive while travelling rightward from Sector III toward to Sector IV, again nothing gets turned on and off; hence, ② stays present in Sector IV. As we arrive at the point of initiation, we conclude that we return back to where we started wherein we began with an assumption that ② was present. In other words, through this circuit around the plane, we have observed that the resultant asymptotic contribution is identical to the initial contribution. Present exponentials of the associated sectors bounded by the Stokes lines are illustrated in boxes in the figure.

Example 2. Assume in (1) that $A(z)$ is given as $A(z) = z^2$.

In order to give the location of the Stokes and anti-Stokes lines of Example 2, one can now directly start the investigation by looking at the turning points of the expansion (2) for given $A(z)$. The turning point explicitly occurs at the origin, that is, $A(z) = z^2$ vanishes for $z = 0$. Equation (6) implies that the relevant singulant $u_0(z)$ of the example turns out to be

$$u_0(z) = \frac{z^3}{3} + C_2,$$

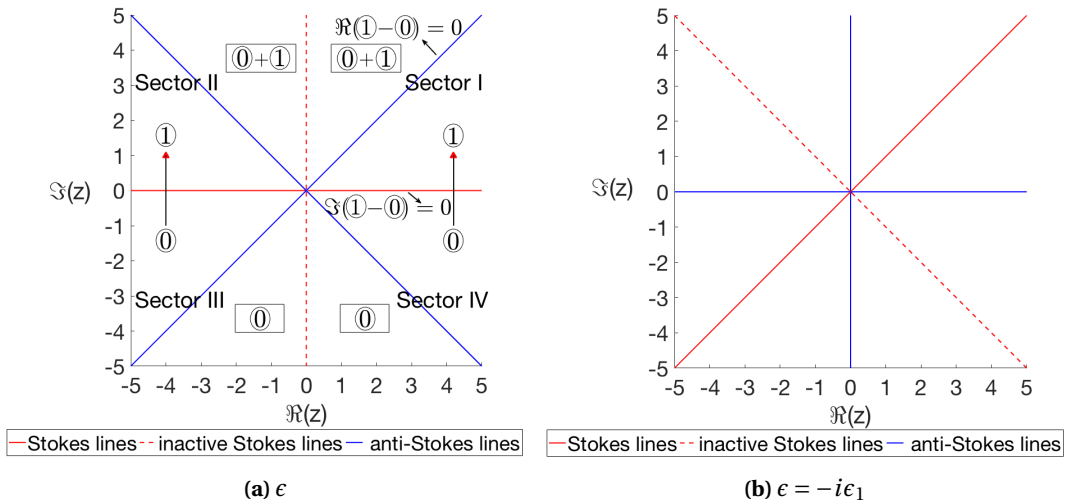


Figure 1. The red and blue lines represent, respectively, the Stokes and anti-Stokes lines of the equation (1) for chosen $A(z) = z$ in Example 1. Figure 1a demonstrates the sudden appearance and disappearance of the exponential terms $\exp(-z^2/2\epsilon)$ and $\exp(0)$ labelled as ① and ②, respectively. The direction of red arrows represents dominant exponential switches the subdominant exponential on across the Stokes lines. Present exponentials of the sectors are shown in the boxes. When ϵ is complex, Stokes and anti-Stokes lines picture of (1) rotates counterclockwise by $\pi/4$ from Figure 1a to Figure 1b.

where C_2 is a constant. Since $u_0(z)$ has to vanish at the singular point, C_2 needs to be zero. Similar to the previous example, applying the polar coordinates, i.e. $z = r \exp(i\theta)$, into the above singulant produces

$$u_0(z) = \frac{r^3 \exp(3i\theta)}{3} = \frac{r^3}{3} (\cos(3\theta) + i \sin(3\theta)). \tag{20}$$

As already discussed, singulant of the expansion presents all the putative Stokes and anti-Stokes lines, whence applying the conditions of (i) among the definitions of Stokes lines for (20) demonstrates that they appear when

$$\sin(3\theta) = 0 \quad \text{while} \quad \cos(3\theta) > 0,$$

which implies Stokes lines follow the curve along $0, 2\pi/3$ and $4\pi/3$ whilst the inactive Stokes lines lie along $\pi/3, \pi$ and $5\pi/3$. In a similar manner, anti-Stokes lines by the condition (vi) may present when θ satisfies $\cos(3\theta) = 0$, which implies they occur along $\pi/6, \pi/2, 5\pi/6, 7\pi/6, 3\pi/2$ and $11\pi/6$. Stokes and anti-Stokes lines of the equation (1) according to this $A(z)$ are shown in Figure 2.

In addition to Stokes lines analysis, this particular example is multivalued at the origin. In particular, solving the differential equation (1) for $A(z) = z^2$ with the integration factor method, one easily verifies that the solution can be written out in terms of an incomplete gamma function $\Gamma(a, z)$ which is a generalization of an exponential integral, a complementary error function and a hypergeometric function, and all have branch cuts [1, 30, 32]. Therefore, this example is an analytic function in the z -plane, except the branch cut so that the branch cut must be introduced in order to make the function single-valued or, in other words, to avoid the multivaluedness.

Next, we concisely travel around the complex plane to see the present exponentials of the sectors defined by dividing the complex plane via the active Stokes lines and the branch cut. Likewise in Figure 1, the homogeneous solution of the equation (1) for $A(z) = z^2$ is $\exp(-u_0(z)/\epsilon) = \exp(-z^3/3\epsilon)$ represented by $\textcircled{1}$ while its RHS $1 = \exp(0)$ is represented by $\textcircled{0}$ which does the switching on and off of the subdominant exponential term across the Stokes lines. Regardless of the region we begin travelling, we will come back to where we started due to crossing three active Stokes lines and one branch cut which leads to removal of $\textcircled{1}$ by discontinuity. Hence, which exponentials are present in which sectors of the complex plane are summarised in boxes in Figure 2 through circuiting along the plane.

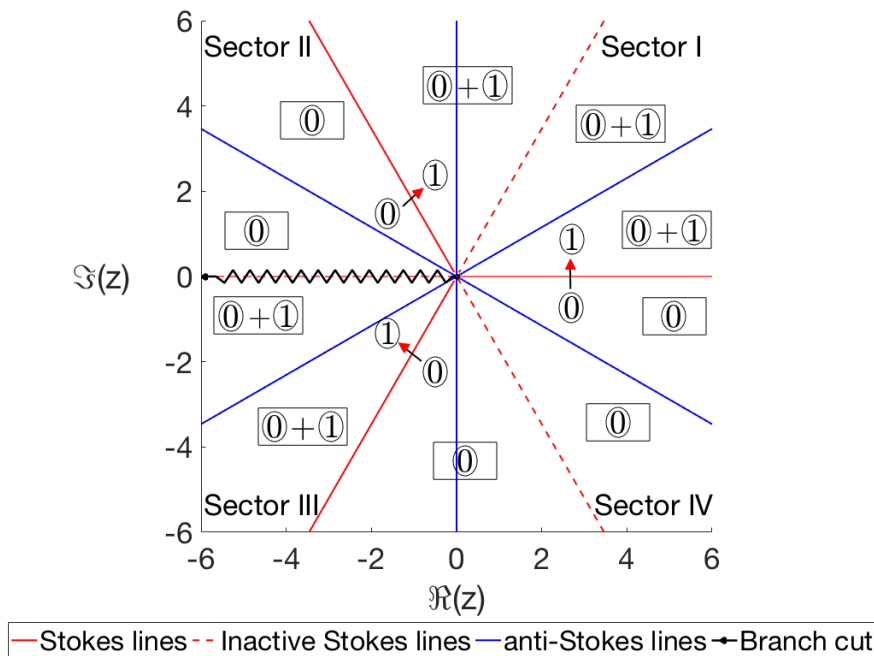


Figure 2. Active and inactive Stokes lines of the equation (1) are illustrated by the lines in red, while anti-Stokes lines are demonstrated in blue for chosen pre-factor $A(z) = z^2$ in Example 2. Zigzag in black represents the branch cut. The direction of the red arrows represents dominant exponential $\textcircled{0}$ turns the subdominant exponential $\textcircled{1}$ on across the Stokes lines. Present exponentials of the sectors generated by the active Stokes lines and the branch cut are shown in boxes.

5. First-level of multi-level asymptotics for general $A(z)$

We next go beyond the zeroth-level of multi-level asymptotics as it is complete. More generally, we are now in a position to discuss the first-level of multi-level asymptotics for general $A(z)$ and what is likely to happen at higher levels of the method. This section aims to establish the inhomogeneous differential equation of the multi-level asymptotic approach in the first-level, which is the main objective of the method, based on the relation obtained in (6) for general $A(z)$ generated by the remainder. To do so, our strategy herein and in every level of the multi-level

asymptotics is first to rescale the resultant remainder $R_{N_0}(z)$ of level zero according to the RHS of the differential equation (12) such that

$$R_{N_0}(z) = \epsilon^{N_0} A(z) y_{N_0}(z) Q(z), \quad (21)$$

in which $Q(z)$ will be addressed shortly. However, before proceeding, let us define ν_0 by $\nu_0 = \epsilon N_0$ which is influenced by the truncation point (13). The straightforward substitution of (21) into (12), under the consideration of ν_0 , leads to a singular differential equation as

$$\epsilon Q'(z) + \left(u_0'(z) - \nu_0 \frac{u_0'(z)}{u_0(z)} + \epsilon \frac{u_0''(z)}{u_0'(z)} \right) Q(z) \sim 1. \quad (22)$$

For ϵ sufficiently small, the leading parts of pre-factor of $Q(z)$ in (22) turn out to be

$$\epsilon Q'(z) + u_0'(z) \left(1 - \frac{\nu_0}{u_0(z)} \right) Q(z) \sim 1. \quad (23)$$

The differential equation of the first-level in the analysis is finally established. Notice that the equation is in the form of the original equation (1). The analysis of this paper clearly indicates that proceeding to the higher levels of multi-level asymptotics, one can easily see that the remainder equation of every level is actually in the nature of the equation (1). Therefore, for given $A(z)$ and its associated relation with the singulant, we can execute the systematic analysis of multi-level asymptotics in the first-level (and higher levels) upon expansion of $Q(z)$, or directly begin the examination from the first-level for either the optimal or arbitrary selection of the truncation point in the zeroth-level for the first-level differential equations. Although we do not conduct first-level analysis in this paper since it is not the principal concern here, we could determine the generic form of the first-level singulant and the related optimal truncation point for the first-order singular differential equations with the formula derived in (6). However, before doing this, it is important to pinpoint at this stage that the first-level equation has a singularity or singularities when z is a solution of

$$u_0(z) - \nu_0 = 0. \quad (24)$$

Let us assume the first-level singulant of $Q(z)$ when expanded is $u_1(z)$ which must satisfy (24). Applying the formula (6) in the pre-factor of (23) gives the first-level singulant by means of

$$u_1(z) = u_0(z) - \nu_0 \ln(u_0(z)) + c_1, \quad (25)$$

in which c_1 is an integration constant and can be derived via the singularity of the expansion. Thence, after substituting (24) into (25), one completes the general description of the first-level singulant in terms of a zeroth-level singulant as

$$u_1(z) = u_0(z) - \nu_0 \ln(u_0(z)) + \nu_0 \ln(\nu_0) - \nu_0. \quad (26)$$

It is crucial to emphasise herein that the singulant (26) has a logarithmic branch cut at the singularities of level zero. Despite the fact that turning points of the zeroth-level are not the turning points of the first-level of multi-level asymptotics, zeroth-level singularities arise in the logarithmic function of the singulant of the first-level. In fact, our analysis indicates that the logarithmic branch cut occurs in order to preserve the level zero singularities in its succeeding level. Hence, the preceding level singularities disappear in the succeeding level; particularly, the singularities approach to infinity and lead to a logarithmic branch cut. For instance, in the case of $u_0(z) = z$, the first-level equation (23) and the singulant (26) are equivalent to the singulant of exponential integral [32]. Therefore, this (23) and the associated singulant is indeed observed for each level of the exponential integral in [32].

As a final remark without loss of generality, it is pertinent to stress that because a remainder differential equation of every level of the multi-level asymptotics turns out to be in the nature of the equation (1), we can extract a recursive relationship between a singulant and its succeeding

level singulant or vice-versa. Particularly, in light of (26), because any two successive level singulants correspond to the first-level of its preceding level at any higher levels in multi-level asymptotic approach, the recursive relationship of the singulants can be succinctly generalized as

$$u_{i+1}(z) = u_i(z) - v_i - v_i \ln(u_i(z)) + v_i \ln(v_i), \quad \text{for } i \geq 0, \quad (27)$$

where v_i can either be defined via the truncation point of i^{th} level of multi-level asymptotics or via its relation with the singulant, i.e., $|u_i(z)| = v_i$. Notice that when z is a solution of $u_i(z) = v_i$, singulant of the related level becomes zero so that the expansion breaks down at that point(s). If this recursive relationship is iterated, singulants of every single level can be extracted so that every putative Stokes and anti-Stokes lines can be found out at higher levels without going through the higher-level analysis of multi-level asymptotics in detail. Likewise, our analysis indicates that since the optimal truncation points are defined via $N_i \sim |u_i(z)|/\epsilon$ like in (13), one can find out the optimal truncation points for any levels in light of this recursion relation. Nevertheless, it is possible to state that all the results regarding the singulant in (26) apply for every singulant at higher levels, i.e., for (27).

6. Summary and conclusion

To summarise, through scrutinising the equation in (1), we have found the general representation of the optimal truncation point and expressed the relationship obtained in (6) between the singulants of the expansion and $A(z)$ function for the first-order singular perturbation problem. This relationship can be implemented in the higher levels of the multi-level asymptotics and clarify the understanding of the Stokes phenomenon. We have determined the Stokes and anti-Stokes lines of the equation emerging from the singular points of the expansion by examination of these points in light of various values of pre-factor $A(z)$. In Section 5, the general representation of the inhomogeneous singular differential equation of the resultant remainder (23) along with the singulant (26) was determined so that the first-level investigation of any differential equation in the form of (1) can be done straightaway. Moreover, Stokes and anti-Stokes lines investigation can be extracted via the recursive relation of the singulants without the need for further asymptotic analysis at higher levels. It is important to highlight that a singular differential equation generated by the remainder function in higher levels is a generic feature of the multi-level asymptotic approach for the first-order differential equations. The use of the multi-level asymptotic procedure is central to these conclusions.

Acknowledgments

Part of this research is studied during my PhD. I would like to thank Prof John R King and Dr Paul C Matthews for their guidance and constructive comments on the paper.

References

- [1] M. Abramowitz, I. A. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications, 1992, Reprint of the 1972 edition, xiv+1046 pages.
- [2] M. V. Berry, "Stokes' phenomenon; smoothing a Victorian discontinuity", *Publ. Math., Inst. Hautes Étud. Sci.* (1988), no. 68, p. 211-221.
- [3] ———, "Uniform asymptotic smoothing of Stokes's discontinuities", *Proc. R. Soc. Lond., Ser. A* **422** (1989), no. 1862, p. 7-21.
- [4] ———, "Waves near Stokes lines", *Proc. R. Soc. Lond., Ser. A* **427** (1990), no. 1873, p. 265-280.
- [5] ———, "Stokes's phenomenon for superfactorial asymptotic series", *Proc. R. Soc. Lond., Ser. A* **435** (1991), no. 1894, p. 437-444.

- [6] M. V. Berry, C. J. Howls, "Hyperasymptotics", *Proc. R. Soc. Lond., Ser. A* **430** (1990), no. 1880, p. 653-668.
- [7] G. L. Body, J. R. King, R. H. Tew, "Exponential asymptotics of a fifth-order partial differential equation", *Eur. J. Appl. Math.* **16** (2005), no. 5, p. 647-681.
- [8] J. P. Boyd, "A comparison of numerical and analytical methods for the reduced wave equation with multiple spatial scales", *Appl. Numer. Math.* **7** (1991), no. 6, p. 453-479.
- [9] ———, "A hyperasymptotic perturbative method for computing the radiation coefficient for weakly nonlocal solitary waves", *J. Comput. Phys.* **120** (1995), no. 1, p. 15-32.
- [10] ———, "The devil's invention: asymptotic, superasymptotic and hyperasymptotic series", *Acta Appl. Math.* **56** (1999), no. 1, p. 1-98.
- [11] ———, "Hyperasymptotics and the linear boundary layer problem: why asymptotic series diverge", *SIAM Rev.* **47** (2005), no. 3, p. 553-575.
- [12] S. J. Chapman, "On the non-universality of the error function in the smoothing of Stokes discontinuities", *Proc. R. Soc. Lond., Ser. A* **452** (1996), no. 1953, p. 2225-2230.
- [13] S. J. Chapman, C. J. Howls, J. R. King, A. B. Olde Daalhuis, "Why is a shock not a caustic? The higher-order Stokes phenomenon and smoothed shock formation", *Nonlinearity* **20** (2007), no. 10, p. 2425-2452.
- [14] S. J. Chapman, J. R. King, K. L. Adams, "Exponential asymptotics and Stokes lines in nonlinear ordinary differential equations", *Proc. R. Soc. Lond., Ser. A* **454** (1998), no. 1978, p. 2733-2755.
- [15] S. J. Chapman, J. M. H. Lawry, J. R. Ockendon, R. H. Tew, "On the theory of complex rays", *SIAM Rev.* **41** (1999), no. 3, p. 417-509.
- [16] S. J. Chapman, D. B. Mortimer, "Exponential asymptotics and Stokes lines in a partial differential equation", *Proc. R. Soc. Lond., Ser. A* **461** (2005), no. 2060, p. 2385-2421.
- [17] S. J. Chapman, P. H. Trinh, T. P. Witelski, "Exponential asymptotics for thin film rupture", *SIAM J. Appl. Math.* **73** (2013), no. 1, p. 232-253.
- [18] G. Darboux, "Mémoire sur l'approximation des fonctions de très-grands nombres, et sur une classe étendue de développements en série.", *J. Math. Pures Appl.* **4** (1878), p. 5-56.
- [19] R. B. Dingle, *Asymptotic expansions: their derivation and interpretation*, Academic Press Inc., 1973, xv+521 pages.
- [20] J. Heading, "The Stokes phenomenon and certain n th order differential equations. II. The Stokes phenomenon", *Proc. Camb. Philos. Soc.* **53** (1957), p. 419-441.
- [21] E. J. Hinch, *Perturbation methods*, Cambridge Texts in Applied Mathematics, Cambridge University Press, 1991, xii+160 pages.
- [22] C. J. Howls, P. J. Langman, A. B. Olde Daalhuis, "On the higher-order Stokes phenomenon", *Proc. R. Soc. Lond., Ser. A* **460** (2004), no. 2048, p. 2285-2303.
- [23] C. J. Howls, A. B. Olde Daalhuis, "Exponentially accurate solution tracking for nonlinear ODEs, the higher order Stokes phenomenon and double transseries resummation", *Nonlinearity* **25** (2012), no. 6, p. 1559-1584.
- [24] N. Joshi, C. J. Lustri, S. Luu, "Nonlinear q -Stokes phenomena for q -Painlevé I", *J. Phys. A, Math. Theor.* **52** (2019), no. 6, article no. 065204 (30 pages).
- [25] J. R. King, S. J. Chapman, "Asymptotics beyond all orders and Stokes lines in nonlinear differential-difference equations", *Eur. J. Appl. Math.* **12** (2001), no. 4, p. 433-463.
- [26] C. J. Lustri, "Exponential asymptotics in unsteady and three-dimensional flows", PhD Thesis, University of Oxford, 2013.
- [27] J. J. Mahony, "Error estimates for asymptotic representations of integrals", *J. Aust. Math. Soc.* **13** (1972), p. 395-410.
- [28] J. B. McLeod, "Smoothing of Stokes discontinuities", *Proc. R. Soc. Lond., Ser. A* **437** (1992), no. 1900, p. 343-354.
- [29] A. B. Olde Daalhuis, S. J. Chapman, J. R. King, J. R. Ockendon, R. H. Tew, "Stokes phenomenon and matched asymptotic expansions", *SIAM J. Appl. Math.* **55** (1995), no. 6, p. 1469-1483.
- [30] F. W. J. Olver, *Asymptotics and special functions*, A K Peters, 1997, Reprint of the 1974 original, xviii+572 pages.
- [31] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, M. A. McClain (eds.), *NIST Digital Library of Mathematical Functions*, <http://dlmf.nist.gov/>, Release 1.1.3 of 2021-09-15.
- [32] F. Say, "Exponential asymptotics: multi-level asymptotics of model problems", PhD Thesis, University of Nottingham, 2016.
- [33] W. P. Schleich, "E: Airy Function", in *Quantum Optics in Phase Space*, Wiley Online Library, 2001, p. 621-627.
- [34] G. G. Stokes, "On the discontinuity of arbitrary constants that appear in divergent developments", *Trans. Camb. Philos. Soc.* **10** (1864), p. 106-128.