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Area minimizing unit vector fields on antipodally punctured unit 2-sphere

Fabiano G. B. Brito [a], Jackeline Conrado [b], Icaro Gonçalves [a] and Adriana V. Nicoli [b]

[a] Centro de Matemática, Computação e Cognição, Universidade Federal do ABC, Santo André, 09210-170, Brazil

[b] Dpto. de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, R. do Matão 1010, São Paulo-SP, 05508-900, Brazil

E-mails: fabiano.brito@ufabc.edu.br, avnicoli@ime.usp.br, icaro.goncalves@ufabc.edu.br, avnicoli@ime.usp.br

In memory of Amine Fawaz

Abstract. We provide a lower value for the volume of a unit vector field tangent to an antipodally punctured Euclidean sphere $\mathbb{S}^2$ depending on the length of an ellipse determined by the indexes of its singularities. We also exhibit minimizing vector fields $\vec{v}_k$ within each index class and show that they are the only ones that are sharp for the volume. These fields have areas given essentially by the length of ellipses depending just on the indexes in N and S.

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1. Introduction and main results

For a compact Riemannian manifold (M, g), the volume of a smooth vector field $\vec{v}: M \to TM$ is the volume of its image $\vec{v}(M) \hookrightarrow (TM; g^{Sas})$, where g^{Sas} is called Sasaki metric and it is defined by declaring the orthogonal complement of the vertical distribution to be the horizontal distribution given by the Levi-Civita connection ∇. In terms of ∇ and g,

$$\mathrm{vol}(\vec{v}) = \int_M \sqrt{\det(I + (\nabla\vec{v})(\nabla\vec{v})^*)}\,\nu, \tag{1}$$

where I is the identity, $(\nabla\vec{v})^*$ is the adjoint operator an ν is the volume form of M.

Back in 1986, Gluck and Ziller ([5]) proved that Hopf flows are the unit vector fields of minimum volume in $M = \mathbb{S}^3$. The theorem reads

Theorem 1 (Gluck and Ziller). *Hopf unit vector fields ($\vec{v}_H$) are the minimum for the volume on $\mathbb{S}^3$ and no others.*

However, in 1988 Johnson proved that Hopf vector fields are unstable in higher dimensions (see [6]):

Theorem 2 (Johnson). *The Hopf vector fields on* $\mathbb{S}^{2n+1}$ *are unstable for* $n > 1$.

Later, in 2008 Brito, Chacón and Johnson ([2]) established a relationship between the volume of unit vector fields and their indexes around isolated singularities. More precisely:

Theorem 3 (Brito, Chacón and Johnson). *Let* $\mathbb{S}^2$ *or* $\mathbb{S}^3$ *be the standard Euclidean sphere with two antipodal points N and S are removed. Let* $\vec{v}$ *be a unit smooth vector field defined in those manifolds and* $I_{\vec{v}}(P)$ *the Poincaré index of* $\vec{v}$ *around P. Then,*

- *for* $\mathbb{S}^2\backslash\{N,S\}$, $\mathrm{vol}(\vec{v}) \geq \frac{1}{2}(\pi + |I_{\vec{v}}(N)| + |I_{\vec{v}}(S)| - 2)\,\mathrm{vol}(\mathbb{S}^2)$
- *for* $\mathbb{S}^3\backslash\{N,S\}$, $\mathrm{vol}(\vec{v}) \geq (|I_{\vec{v}}(N)| + |I_{\vec{v}}(S)|)\,\mathrm{vol}(\mathbb{S}^3)$

Let $\vec{v}_R$ be the north-south field, then it will achieve both equalities in the theorem. In this case, for $\mathbb{S}^2\backslash\{N,S\}$, $\mathrm{vol}(\vec{v}_R) = \frac{1}{2}\pi\,\mathrm{vol}(\mathbb{S}^2)$ and in $\mathbb{S}^3\backslash\{N,S\}$, $\mathrm{vol}(\vec{v}_R) = 2\,\mathrm{vol}(\mathbb{S}^3) = \mathrm{vol}(\vec{v}_H)$

In terms of foliations on $\mathbb{S}^2$, Fawaz (see [4]) studied the minimal value of meromorphic foliations. The minimum value is achieved by taking the foliation of the sphere $\mathbb{S}^2$ by parallels. Formally:

Theorem 4 (Fawaz). *Let* $\mathscr{F}$ *be a foliation on the Riemann sphere* $\mathbb{S}^2$ *given by the real part of a meromorphic or holomorphic vector field, then* $\mathrm{vol}(\mathscr{F}) \geq 2\pi^2$.

In 2010, Borrelli and Gil-Medrano proved (see [1]) that the Pontryagin fields are area-minimizing of the unit 2-sphere. Pontryagin fields of $\mathbb{S}^n$ are any unit vector field $\vec{v}_{Ptry}$ defined in a dense open subset U such that the closure of $\vec{v}_{Ptry}(U)$ is the n-dimensional generalized Pontryagin cycle of the unit vector bundle of the n-sphere $(T^1\mathbb{S}^n)$.

Theorem 5 (Borrelli and Gil-Medrano). *Among unit vector fields without boundary of* $\mathbb{S}^2(1)\backslash\{P\}$ *those of least area are Pontryagin fields (*$\vec{v}_{Ptry}$*) and no others.*

Recently, in 2019, Theorem 3 was extended to odd dimensional spheres $\mathbb{S}^{2n+1}\backslash\{\pm P\}$, see [3].

Theorem 6 (Brito, Gomes and Gonçalves). *If* $\vec{v}$ *is a unit vector field on* $\mathbb{S}^{2n+1}\backslash\{\pm P\}$*, then*

$$\mathrm{vol}(\vec{v}) \geq \frac{\pi}{4}\,(|I_{\vec{v}}(P)| + |I_{\vec{v}}(-P)|)\,\mathrm{vol}(\mathbb{S}^{2n}).$$

In this article, we establish sharp lower bounds for the total area of unit vector fields on antipodally punctured Euclidean sphere $\mathbb{S}^2$, and these values depend on the indexes of their singularities. We show that this lower bound is sharp and describe the vector fields $\vec{v}_k$ that achieve the minimum for each positive index k, we also show that these unit vector fields are the only ones with this property. In fact, given an index k, we informally declare a unit vector field $\vec{v}_k$ with one or two singularities satisfying:

(1) $\vec{v}_k$ is parallel along meridians
(2) $\vec{v}_k$ turns $k-1$ times along each parallel at a constant angle speed
(3) we establish an "initial meridian" along which $\vec{v}_k$ makes an angle θ with each parallel

Theorem. *Let* $\vec{v}$ *be a unit vector field defined on* $M = \mathbb{S}^2\setminus\{N,S\}$*. If* $k = \max\{I_{\vec{v}}(N), I_{\vec{v}}(S)\}$*, then*

$$\mathrm{vol}(\vec{v}) \geq \pi L(\varepsilon_k),$$

where $L(\varepsilon_k)$ *is the length of the ellipse* $\frac{x^2}{k^2} + \frac{y^2}{(k-2)^2} = 1$*, with a positive index k and* $I_{\vec{v}}(P)$ *stands for the Poincaré index of* $\vec{v}$ *around P.*

This is a natural extension of Theorems 3 and 5 aforementioned. We also exhibit vector fields $\vec{v}_k$ achieving the minimum volume for each index k, i.e. the lower bound is sharp. These results, as long as we know, completely solve the Gluck-Ziller problem for the antipodally punctured unit 2-sphere. Borreli and Gil-Medrano solved the case where $k = 2$, [1].

2. Existence and uniqueness of vector fields with total area $\pi L(\varepsilon)$

Let $M = \mathbb{S}^2 \setminus \{N, S\}$ be the Euclidean sphere in which two antipodal points N and S are removed. Denote by g the usual metric of $\mathbb{S}^2$ induced from $\mathbb{R}^3$, and by ∇ the Levi-Civita connection associated to g. Consider the oriented orthonormal local frame $\{e_1, e_2\}$ on M, where e_1 is tangent to the parallels and e_2 to the meridians. Let $\vec{v}$ be a unit vector field tangent to M and consider another oriented orthonormal local frame $\{\vec{v}, \vec{v}^{\perp}\}$ on M and its dual basis $\{\omega_1, \omega_2\}$ compatible with the orientation of $\{e_1, e_2\}$.

In dimension 2, the volume of $\vec{v}$ given in equation (1) reduces to

$$\operatorname{vol}(\vec{v}) = \int_{\mathbb{S}^2} \sqrt{1+\gamma^2+\delta^2}\, \nu, \tag{2}$$

where $\gamma = g(\nabla_{\vec{v}}\vec{v}, \vec{v}^{\perp})$ and $\delta = g(\nabla_{v^{\perp}}\vec{v}^{\perp}, \vec{v})$ are the geodesic curvatures associated to $\vec{v}$ and $\vec{v}^{\perp}$, respectively, and ν is the volume form.

Let $\mathbb{S}^1_{\alpha}$ be the parallel of $\mathbb{S}^2$ at latitude $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and S^1_{β} be the meridian of $\mathbb{S}^2$ at longitude $\beta \in (0, 2\pi)$.

We formally define a family of unit vector fields achieving the volume given by the Theorem.

Definition 7. *Let k be a positive integer and define:*

(1) $\vec{v}_1(p) = \vec{e}_2(p)$, *if* $k = 1$;

(2) $\vec{v}_k(p) = \cos\theta(p)\vec{e}_1(p) + \sin\theta(p)\vec{e}_2(p)$, *if* $k > 1$, *where* $\theta : \mathbb{S}^2 \setminus \{N, S\} \to \mathbb{R}$ *is given by* $\theta(\alpha, \beta) = (k-1)\beta + \theta_0$, *(where θ_0 is a constant), in that way,*

$$\theta_1(p) = \frac{k-1}{\sqrt{x^2+y^2}} \quad \textit{and} \quad \theta_2(p) = 0,$$

where $\theta_1 = \mathrm{d}\theta(e_1)$ and $\theta_2 = \mathrm{d}\theta(e_2)$ are the derivatives of θ on e_1 and e_2, respectively.

Notice that θ has constant variation along the parallel $x^2 + y^2 = \cos\alpha$, with $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ constant, and this includes the case where $k = 1$. Remember that the case $k = 2$ has one singularity (see [1]).

If we use spherical coordinates (α, β) so that $p = (\cos\alpha\cos\beta, \cos\alpha\sin\beta, \sin\alpha)$, we can say that the vector $\vec{v}_k$ spins around a point P at a constant speed of rotation along the parallel α. Moreover, $\vec{v}_k$ gives exactly $k-1$ turns when it passes the α parallel, with respect to the referential $\{\vec{e}_1, \vec{e}_2\}$, and it gives k turns with respect to a fixed polar referential, in this case, $\theta_1(p) = \frac{k-1}{\cos\alpha}$. In Figure 1 is given a visual representation about the behavior of $\vec{v}_k$s and in Figure 2 we have a unit vector field with $k = 4$.

Lemma. *Let $\theta \in [0, \pi/2]$ be the oriented angle from e_1 to $\vec{v}$. If $\vec{v} = (\cos\theta)e_1 + (\sin\theta)e_2$ and $\vec{v}^{\perp} = (-\sin\theta)e_1 + (\cos\theta)e_2$, then*

$$1+\gamma^2+\delta^2 = 1 + (\tan\alpha + \theta_1)^2 + \theta_2^2.$$

Proof. We write γ and δ as the following sums

$$\gamma = A + B + C + D \quad \text{and} \quad \delta = A' + B' + C' + D',$$

with

$$A = g\left(\nabla_{(\cos\theta)e_1}(\cos\theta)e_1, \vec{v}^{\perp}\right), \qquad B = g\left(\nabla_{(\sin\theta)e_2}(\cos\theta)e_1, \vec{v}^{\perp}\right),$$
$$C = g\left(\nabla_{(\cos\theta)e_1}(\sin\theta)e_2, \vec{v}^{\perp}\right), \qquad D = g\left(\nabla_{(\sin\theta)e_2}(\sin\theta)e_2, \vec{v}^{\perp}\right)$$

and

$$A' = g\left(\nabla_{(-\sin\theta)e_1}(-\sin\theta)e_1, \vec{v}\right), \qquad B' = g\left(\nabla_{(\cos\theta)e_2}(-\sin\theta)e_1, \vec{v}\right)$$
$$C' = g\left(\nabla_{(-\sin\theta)e_1}(\cos\theta)e_2, \vec{v}\right), \qquad D' = g\left(\nabla_{(\cos\theta)e_2}(\cos\theta)e_2, \vec{v}\right).$$

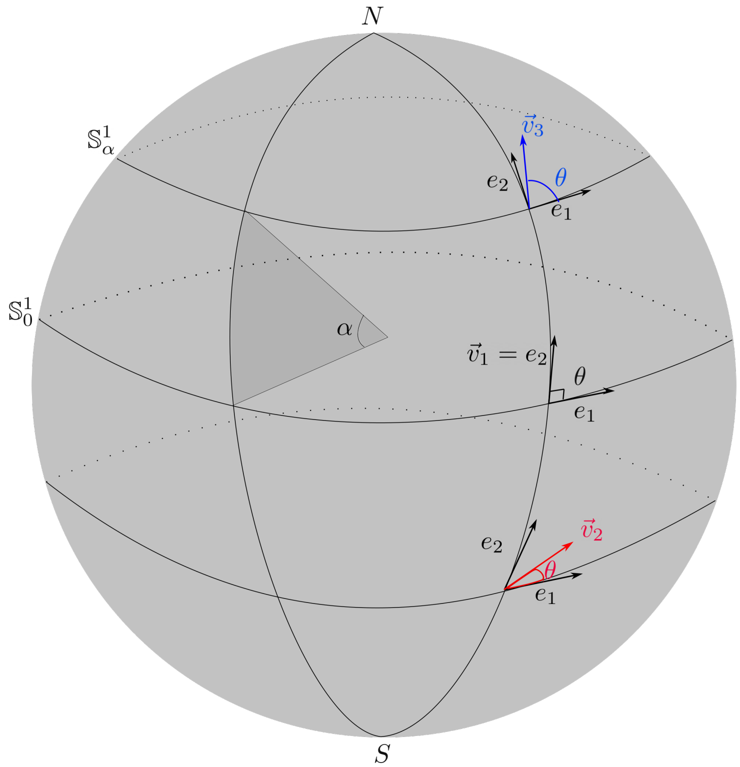

Figure 1. Visual representation of $\vec{v}_k$s: The angle θ, between the parallels and the field $\vec{v}_k$, changes as k changes. For example, the angle θ between $\vec{v}_3$ and parallels is twice the angle θ between parallels and $\vec{v}_2$. This is to be expected, since $\vec{v}_3$ has a singularity with an index equal to 3 (with four "petals") and $\vec{v}_2$ has a singularity with an index equal to 2 (with two "petals"). Also, $\vec{v}_1$ is the south-north vector field, which forms an angle $\theta = \frac{\pi}{2}$ with the parallels. See [1] and [7] for more details.

Observe that $\tan\alpha = g\big(\nabla_{e_1} e_1, e_2\big)$ and $\nabla_{e_2} e_2 = 0$. Indeed, define $\psi : U \to \mathbb{R}^3$ as $\psi(\alpha,\beta) = (\cos\alpha\sin\beta, \cos\alpha\cos\beta, \sin\alpha)$, where $\big(-\frac{\pi}{2}, \frac{\pi}{2}\big) \times \big(0, 2\pi\big) := U \subset \mathbb{R}^2$. In this case, $\psi_\alpha = e_2$, $\psi_\beta = e_1\cos(\alpha)$ and $\psi_{\alpha\alpha} = \nabla_{e_2} e_2$. Therefore, $\tan\alpha = g\big(\nabla_{e_1} e_1, e_2\big)$ and since the e_2 is tangent to the meridians, $\nabla_{e_2} e_2 = 0$ in $\mathbb{S}^2$.

By a straightforward computation,

$$\gamma = \cos\theta(\tan\alpha + \theta_1) + (\sin\theta)\theta_2, \tag{3}$$

$$\delta = \sin\theta(\tan\alpha + \theta_1) - (\cos\theta)\theta_2. \tag{4}$$

From equations (3) and (4), we have

$$\begin{aligned} 1 + \gamma^2 + \delta^2 &= 1 + (\cos\theta(\tan\alpha + \theta_1) + (\sin\theta)\theta_2)^2 + (\sin\theta(\tan\alpha + \theta_1) - (\cos\theta)\theta_2)^2 \\ &= 1 + \cos^2\theta(\tan\alpha + \theta_1)^2 + (\sin\theta)^2\theta_2^2 + \sin^2\theta(\tan\alpha + \theta_1)^2 + (\cos^2\theta)\theta_2^2 \\ &= 1 + (\tan\alpha + \theta_1)^2 + \theta_2^2. \end{aligned}$$

Finally,

$$1 + \gamma^2 + \delta^2 = 1 + (\tan\alpha + \theta_1)^2 + \theta_2^2. \qquad \square$$

This Lemma allows us to rewrite the volume functional as an integral depending on the latitude α and the derivatives of θ

$$\mathrm{vol}(\vec{v}) = \int_M \sqrt{1+(\tan\alpha+\theta_1)^2+\theta_2^2}\,\nu. \tag{5}$$

Proposition 8. *Let $\vec{v}_k$ be unit vector field on $M = \mathbb{S}^2\backslash\{N,S\}$. Then,*

$$\mathrm{vol}(\vec{v}_k) = \pi L(\varepsilon_k),$$

if and only if $\vec{v}_k$ satisfies the aforementioned definition.

Proof. Using the Lemma we have

$$\mathrm{vol}(\vec{v}_k) = \int_M \sqrt{1+(\tan\alpha+\theta_1)^2+\theta_2^2}\,\nu.$$

Assuming $\theta_1 = \frac{k-1}{\cos\alpha}$ and $\theta_2 = 0$, we obtain

$$\begin{aligned}
\mathrm{vol}(\vec{v}_k) &= \int_M \sqrt{1+\left(\tan\alpha+\frac{k-1}{\cos\alpha}\right)^2}\,\nu \\
&= \int_M \sqrt{1+\left(\frac{\sin\alpha+k-1}{\cos\alpha}\right)^2}\,\nu \\
&= \int_M \frac{\sqrt{1+(k-1)^2+2(k-1)\sin\alpha}}{\cos\alpha}\,\nu \\
&= \lim_{\alpha_0\to-\frac{\pi}{2}} \int_{\alpha_0}^{\frac{\pi}{2}}\int_0^{2\pi} \frac{\sqrt{1+(k-1)^2+2(k-1)\sin\alpha}}{\cos\alpha}\cos\alpha\,\mathrm{d}\beta\,\mathrm{d}\alpha \\
&= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1+(k-1)^2+2(k-1)\sin\alpha}\,\mathrm{d}\alpha.
\end{aligned}$$

Taking $t = \frac{\alpha}{2}+\frac{\pi}{4}$ we have

$$\begin{aligned}
\mathrm{vol}(\vec{v}_k) &= 4\pi\int_0^{\frac{\pi}{2}} \sqrt{(k-2)^2+4(k-1)\sin^2 t}\,\mathrm{d}t \\
&= \pi L(\varepsilon_k).
\end{aligned}$$

On the other hand, if $\mathrm{vol}(\vec{v}_k) = \pi L(\varepsilon_k)$,

$$\begin{aligned}
\mathrm{vol}(\vec{v}_k) &= \int \sqrt{1+(\tan\alpha+\theta_1)^2+\theta_2^2}\,\nu \\
&\geq \int \sqrt{1+(\tan\alpha+\theta_1)^2}\,\nu \\
&\geq \int |\cos\varphi+\sin\varphi(\tan\alpha+\theta_1)|\,\nu \\
&= \pi L(\varepsilon_k),
\end{aligned}$$

then $\theta_2 = 0$ and $\cos\varphi(\tan\alpha+\theta_1) = \sin\varphi$, where $\varphi\in\mathbb{R}$. We conclude that $\theta_1 = \tan\varphi - \tan\alpha$ and $\varphi = \varphi(\alpha) = \arctan\left(\tan\alpha+\frac{k-1}{\cos\alpha}\right)$, which implies $\theta_1 = \frac{k-1}{\cos\alpha}$. □

Example (Poincaré Index 4 and −2). Let $\vec{v}$ be a unit vector field in $\mathbb{S}^2$ with two singularities with indexes 4 and −2. Thus the volume of $\vec{v}$ is bounded by

$$\mathrm{vol}(\vec{v}) \geq 4\pi\int_0^{2\pi} \sqrt{4+12\sin^2 t}\,\mathrm{d}t$$

Figures 2 and 3 provide a visual representation for a unit vector field with two singularities with Poincaré indexes 4 and −2.

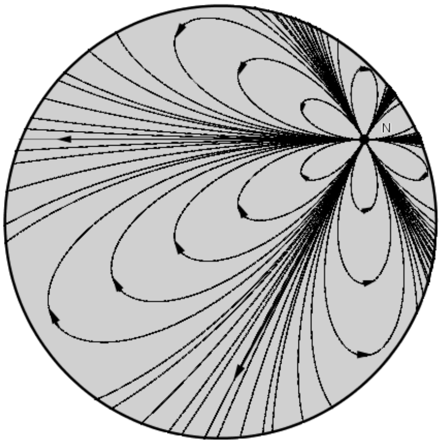

Figure 2. Singularity of index 4 in north pole

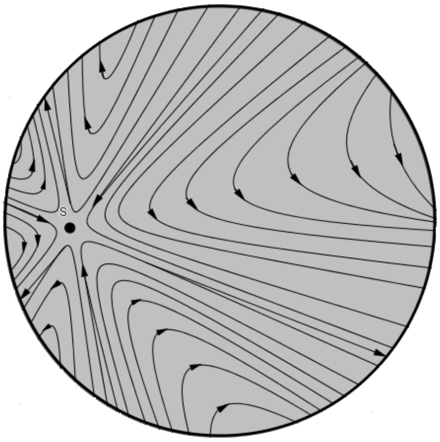

Figure 3. Singularity of index −2 in south pole

3. Proof of the Theorem

We will follow most of the arguments found in the proof of the main Theorem of [2].

Proof of the Theorem. Given a, b, $\varphi \in \mathbb{R}$ we have a general inequality $\sqrt{a^2+b^2} \geq |a\cos\varphi + b\sin\varphi|$, which implies

$$\sqrt{1+(\tan\alpha+\theta_1)^2} \geq \left|\cos\varphi + \sin\varphi(\tan(\alpha)+\theta_1)\right|.$$

Therefore,

$$\operatorname{vol}(\vec{v}) \geq \int_M \left(\cos\varphi + \sin\varphi|\tan\alpha+\theta_1|\right)\nu. \tag{6}$$

This inequality is valid for all φ such that $0 \leq \varphi \leq 2\pi$, therefore we may use φ obtained in the previous Proposition:

$$\varphi_k(\alpha) = \arctan\left(\tan\alpha + \frac{k-1}{\cos\alpha}\right);\text{ where } \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{7}$$

Replacing this condition in equation (6) we find

$$\operatorname{vol}(\vec{v}) \geq \int_M \left(\cos\left(\varphi_k(\alpha)\right) + \sin\left(\varphi_k(\alpha)\right)|\tan\alpha+\theta_1|\right)\nu. \tag{8}$$

On the other hand, 7 provides that

$$\cos\left(\varphi_k(\alpha)\right) = \frac{\cos\alpha}{\sqrt{1+(k-1)^2+2(k-1)\sin\alpha}}$$

and

$$\sin\left(\varphi_k(\alpha)\right) = \frac{k-1+\sin\alpha}{\sqrt{1+(k-1)^2+2(k-1)\sin\alpha}}.$$

Thus, the second part of the inequality (8) is equal to

$$\lim_{\alpha_0\to-\frac{\pi}{2}} \int_{\alpha_0}^{\frac{\pi}{2}} \int_0^{2\pi} \left(\frac{\cos\alpha}{\sqrt{1+(k-1)^2+2(k-1)\sin\alpha}} + \frac{k-1+\sin\alpha}{\sqrt{1+(k-1)^2+2(k-1)\sin\alpha}}|\tan\alpha+\theta_1|\right)\cos\alpha\,\mathrm{d}\beta\mathrm{d}\alpha. \tag{9}$$

Remember that Cartan's connection form ω_{12} is given by

$$\omega_{12} = \delta\omega_1 - \gamma\omega_2,$$

where $\{\omega_1, \omega_2\}$ is dual basis of $\{\vec{v}, \vec{v}^{\perp}\}$. If $i : \mathbb{S}^1_\alpha \hookrightarrow \mathbb{S}^2$ is the inclusion map, and $e_1 = \sin\theta\,\vec{v}^{\perp} + \cos\theta\,\vec{v}$, then

$$i^*(\omega_{12})(e_1) = \delta\sin\theta - \gamma\cos\theta,$$

where i^* is the pullback of i.

From equations (3) and (4), we have

$$\begin{aligned} i^*(\omega_{12})(e_1) &= \sin\theta\left[\sin\theta\big(\tan\alpha + \theta_1\big) - \cos\theta(\theta_2)\right] - \cos\theta\left[\cos\theta\big(\tan\alpha + \theta_1\big) + \sin\theta(\theta_2)\right] \\ &= \tan\alpha + \theta_1. \end{aligned}$$

Thus, from (9)

$$\operatorname{vol}(\vec{v}) \geq \lim_{\alpha_0 \to -\frac{\pi}{2}} \int_{\alpha_0}^{\frac{\pi}{2}} \int_0^{2\pi} \left(\frac{\cos\alpha + \big((k-1) + \sin\alpha\big) i^*(\omega_{12})(e_1)}{\sqrt{1 + (k-1)^2 + 2(k-1)\sin\alpha}} \right) \cos\alpha \, \mathrm{d}\beta \, \mathrm{d}\alpha. \tag{10}$$

In order to compute the integral of $i^*\omega_{12}$ over the parallel of $\mathbb{S}^2$ at constant latitude α, we follow the same arguments in the proof Theorem 1.1 of [2].

$$\mathbb{S}^2_\alpha = \{(x, y, z) \in \mathbb{R}^3; z \geq \sin\alpha\}, \quad \alpha_0 \leq \alpha \leq \frac{\pi}{2}.$$

The 2-form $\mathrm{d}\omega_{12}$ is given by

$$\mathrm{d}\omega_{12} = \omega_1 \wedge \omega_2.$$

A simple application of Stokes' theorem implies that

$$\int_{\mathbb{S}^2_\alpha} \mathrm{d}\omega_{12} = 2\pi\big(I_N(\vec{v})\big) - \int_{\mathbb{S}^1_\alpha} i^*\omega_{12}.$$

Suppose that $I_N(\vec{v}) = \sup\{I_N(\vec{v}), I_S(\vec{v})\} = k$. We obtain

$$\int_{\mathbb{S}^1_\alpha} i^*\omega_{12} = 2\pi k - \operatorname{Area}\left(\mathbb{S}^2_\alpha\right) = 2\pi k - 2\pi\big(1 - \sin\alpha\big) = 2\pi\big(k - 1 + \sin\alpha\big). \tag{11}$$

From inequality (10),

$$\begin{aligned} \operatorname{vol}(\vec{v}) &\geq \lim_{\alpha_0 \to -\frac{\pi}{2}} \int_{\alpha_0}^{\frac{\pi}{2}} \int_0^{2\pi} \left(\frac{\cos\alpha + \big((k-1) + \sin\alpha\big) i^*(\omega_{12})(e_1)}{\sqrt{1 + (k-1)^2 + 2(k-1)\sin\alpha}} \right) \cos\alpha \, \mathrm{d}\beta \, \mathrm{d}\alpha \\ &= \lim_{\alpha_0 \to -\frac{\pi}{2}} \int_{\alpha_0}^{\frac{\pi}{2}} \left(\frac{\cos\alpha}{\sqrt{1 + (k-1)^2 + 2(k-1)\sin\alpha}} \int_0^{2\pi} \cos\alpha \, \mathrm{d}\beta \right. \\ &\qquad \left. + \frac{\big((k-1) + \sin\alpha\big)}{\sqrt{1 + (k-1)^2 + 2(k-1)\sin\alpha}} \int_{\mathbb{S}^1_\alpha} i^*\omega_{12} \right) \mathrm{d}\alpha \\ &= \lim_{\alpha_0 \to -\frac{\pi}{2}} \int_{\alpha_0}^{\frac{\pi}{2}} \left(\frac{2\pi\cos^2\alpha}{\sqrt{1 + (k-1)^2 + 2(k-1)\sin\alpha}} + \frac{2\pi\big((k-1) + \sin\alpha\big)^2}{\sqrt{1 + (k-1)^2 + 2(k-1)\sin\alpha}} \right) \mathrm{d}\alpha, \end{aligned}$$

where the last inequality is obtained from (11). Therefore,

$$\operatorname{vol}(\vec{v}) \geq 2\pi \lim_{\alpha_0 \to -\frac{\pi}{2}} \int_{\alpha_0}^{\frac{\pi}{2}} \left(\frac{\cos^2\alpha + \big((k-1) + \sin\alpha\big)^2}{\sqrt{1 + (k-1)^2 + 2(k-1)\sin\alpha}} \right) \mathrm{d}\alpha.$$

Analogously,

$$\operatorname{vol}(\vec{v}) \geq 2\pi \lim_{\alpha_0 \to -\frac{\pi}{2}} \int_{\alpha_0}^{\frac{\pi}{2}} \left(\sqrt{1 + (k-1)^2 + 2(k-1)\sin\alpha} \right) \mathrm{d}\alpha.$$

A trigonometrical identity give us

$$\operatorname{vol}(\vec{v}) \geq 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(k-2)^2 + 4(k-1)\sin^2\left(\frac{\alpha}{2} + \frac{\pi}{4}\right)} \, \mathrm{d}\alpha.$$

Assume that $t = \frac{\alpha}{2} + \frac{\pi}{4}$, then

$$\mathrm{vol}(\vec{v}) \geq 4\pi \int_0^{\frac{\pi}{2}} \sqrt{(k-2)^2 + 4(k-1)\sin^2 t}\,\mathrm{d}t. \tag{12}$$

Consider $k > 2$ and an ellipse ε_k given by

$$\frac{x^2}{k^2} + \frac{y^2}{(k-2)^2} = 1.$$

Let μ be a parametrization for ε_k defined by $\mu(t) = (k\cos t, (k-2)\sin t)$. Its length is

$$L(\varepsilon_k) = 4\int_0^{2\pi} \left(\sqrt{(k-2)^2 + 4(k-1)\sin^2 t}\right)\mathrm{d}t. \tag{13}$$

Therefore,

$$\mathrm{vol}(\vec{v}) \geq \pi L(\varepsilon_k).$$ □

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