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Quasiconformal extension for harmonic mappings on finitely connected domains

Iason Efraimidis *, [a]

[a] Department of Mathematics and Statistics, Texas Tech University, Box 41042, Lubbock, TX 79409, United States.
E-mail: iason.efraimidis@ttu.edu

Abstract. We prove that a harmonic quasiconformal mapping defined on a finitely connected domain in the plane, all of whose boundary components are either points or quasicircles, admits a quasiconformal extension to the whole plane if its Schwarzian derivative is small. We also make the observation that a univalence criterion for harmonic mappings holds on uniform domains.

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1. Introduction

Let f be a harmonic mapping in a planar domain D and let $\omega = \overline{f_{\overline{z}}}/f_z$ be its dilatation. According to Lewy's theorem the mapping f is locally univalent if and only if its Jacobian $J_f = |f_z|^2 - |f_{\overline{z}}|^2$ does not vanish. Duren's book [4] contains valuable information about the theory of planar harmonic mappings.

The Schwarzian derivative of f was defined by Hernández and Martín [9] as

$$S_f = \rho_{zz} - \frac{1}{2}(\rho_z)^2, \qquad \text{where} \qquad \rho = \log J_f. \tag{1}$$

When f is holomorphic this reduces to the classical Schwarzian derivative. Another definition, introduced by Chuaqui, Duren and Osgood [3], applies to harmonic mappings which admit a lift to a minimal surface via the Weierstrass–Enneper formulas. However, focusing on the planar theory in this note we adopt the definition (1).

We assume that $\overline{\mathbb{C}}\backslash D$ contains at least three points, so that D is equipped with the hyperbolic metric, defined by

$$\lambda_D(\pi(z))\left|\pi'(z)\right||dz| = \lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1-|z|^2}, \qquad z \in \mathbb{D},$$

* Corresponding author.

where $\mathbb{D}$ is the unit disk and $\pi : \mathbb{D} \to D$ is a universal covering map. The size of the Schwarzian derivative of a mapping f in D is measured by the norm

$$\left\|S_f\right\|_D = \sup_{z\in D} \lambda_D(z)^{-2} \left|S_f(z)\right|.$$

A domain D in $\overline{\mathbb{C}}$ is a K-quasidisk if it is the image of the unit disk under a K-quasiconformal self-map of $\overline{\mathbb{C}}$, for some $K \geq 1$. The boundary of a quasidisk is called a quasicircle.

According to a theorem of Ahlfors [1], if D is a K-quasidisk then there exists a constant $c > 0$, depending only on K, such that if f is analytic in D with $\|S_f\|_D \leq c$ then f is univalent in D and has a quasiconformal extension to $\overline{\mathbb{C}}$. This has been generalized by Osgood [12] to the case when D is a finitely connected domain whose boundary components are either points or quasicircles. Further, the univalence criterion was generalized to uniform domains (see Section 4 for a definition) by Gehring and Osgood [7] and, subsequently, the quasiconformal extension criterion was generalized to uniform domains by Astala and Heinonen [2].

For harmonic mappings and the definition (1) of the Schwarzian derivative, a univalence and quasiconformal extension criterion in the unit disk $\mathbb{D}$ was proved by Hernández and Martín [8]. This was recently generalized to quasidisks by the present author in [5]. Moreover, in [5] it was shown that if all boundary components of a finitely connected domain D are either points or quasicircles then any harmonic mapping in D with sufficiently small Schwarzian derivative is injective. The main purpose of this note is to prove the following theorem.

Theorem 1. *Let D be a finitely connected domain whose boundary components are either points or quasicircles and let also $d \in [0,1)$. Then there exists a constant $c > 0$, depending only on the domain D and the constant d, such that if f is harmonic in D with $\|S_f\|_D \leq c$ and with dilatation ω satisfying $|\omega(z)| \leq d$ for all $z \in D$ then f admits a quasiconformal extension to $\overline{\mathbb{C}}$.*

As mentioned above, for the case when D is a (simply connected) quasidisk this was shown in [5] while, on the other hand, for the case $d = 0$ (when f is analytic) this was proved by Osgood in [12]. Osgood's proof amounts to proving a univalence criterion in $f(D)$. Such an approach does not seem to work here since for a holomorphic ϕ on $f(D)$ the composition $\phi \circ f$ is not, in general, harmonic.

Since isolated boundary points are removable for quasiconformal mappings (see [10, Ch. I, § 8.1]), we may assume for the proof of Theorem 1 that ∂D consists of n non-degenerate quasicircles. Our proof will be based on the following theorem of Springer [13] (see also [10, Ch.II, § 8.3]).

Theorem 2 ([13]). *Let D and D' be two n-tuply connected domains whose boundary curves are quasicircles. Then every quasiconformal mapping of D onto D' can be extended to a quasiconformal mapping of the whole plane.*

Hence, to prove Theorem 1 it suffices that we show that the boundary components of $f(D)$ are quasicircles. We prove this in Section 3. It relies on Osgood's [12] quasiconformal decomposition, which we briefly present in Section 2. In Section 4 we give a univalence criterion on uniform domains.

2. Quasiconformal Decomposition

Let D be a domain in $\overline{\mathbb{C}}$. A collection $\mathfrak{D}$ of domains $\Delta \subset D$ is called a K-quasiconformal decomposition of D if each Δ is a K-quasidisk and any two points $z_1, z_2 \in D$ lie in the closure of some $\Delta \in \mathfrak{D}$. This definition was introduced by Osgood in [12], along with the following lemma.

Lemma 3 ([12]). *If D is a finitely connected domain and each component of ∂D is either a point or a quasicircle then D is quasiconformally decomposable.*

We now present, almost verbatim, the construction proving Lemma 3. We focus on the parts of the construction we will be needing, maintaining the notation of [12] and skipping all the relevant proofs. The interested reader should consult [12] for further details.

As we mentioned earlier, we may assume that ∂D consists of non-degenerate quasicircles $C_0, C_1, \ldots, C_{n-1}$, for $n \geq 2$. Let F be a conformal mapping of D onto a circle domain D'. Then, with an application of Theorem 2 to F^{-1}, it will be sufficient to find a quasiconformal decomposition of D'. Hence we may assume that D itself is a circle domain with boundary circles C_j, $j = 0, \ldots, n-1$.

If $n = 2$ then we may assume that D is the annulus $1 < |z| < R$. Then the domains

$$\Delta_1 = \left\{ z \in D : 0 < \arg(z) < \frac{4\pi}{3} \right\}, \qquad \Delta_2 = e^{2\pi i/3}\Delta_1, \qquad \Delta_3 = e^{4\pi i/3}\Delta_1$$

make a quasiconformal decomposition of D.

Let $n \geq 3$. Then there exists a conformal mapping Ψ of the circle domain D onto a domain D' consisting of the entire plane minus n finite rectilinear slits lying on rays emanating from the origin. The mapping can be chosen so that no two distinct slits lie on the same ray. The boundary behavior of Ψ is the following: it can be analytically extended to $\overline{D}$, and the two endpoints of the slit $C'_j = \Psi(C_j)$ correspond to two points on the circle C_j which partition C_j into two arcs, each of which is mapped onto C'_j in a one-to-one fashion.

Let ξ_j be the endpoint of C'_j furthest from the origin and let Q'_j be the part of the ray that joins ξ_j to infinity. Let also S'_j be the sector between C'_j and C'_{j+1}. Let ω'_j be the midpoint of C'_j and let P'_j be a polygonal arc joining ω'_j to ω'_{j+1} that, except for its endpoints, lies completely in S'_j. Then

$$P' = \bigcup_{j=0}^{n-1} P'_j$$

is a closed polygon separating 0 from ∞ that does not intersect any of the Q'_j. Let G'_0 and G'_1 be the components of $D' \backslash P'$ that contain 0 and ∞, respectively. Now define

$$\Delta'_{0j} = G'_0 \cup S'_j, \qquad \Delta'_j = D' \backslash \Delta'_{0j}$$

and

$$\mathfrak{D}' = \left\{ \Delta'_{0j} : j = 0, 1, \ldots, n-1 \right\} \cup \left\{ \Delta'_j : j = 0, 1, \ldots, n-1 \right\}.$$

This collection has the covering property for D'.

We denote the various parts of D corresponding under Ψ^{-1} to those of D' by the same symbol without the prime. Then

$$\mathfrak{D} = \left\{ \Delta_{0j} : j = 0, 1, \ldots, n-1 \right\} \cup \left\{ \Delta_j : j = 0, 1, \ldots, n-1 \right\}$$

is a quasiconformal decomposition of D.

3. Proof of Theorem 1

Let f be a mapping in D as in Theorem 1. By [5, Theorem 2], f is injective if c is sufficiently small. Also, f extents continuously to ∂D since every boundary point of D belongs to $\partial\Delta$ for some Δ in the collection $\mathfrak{D}$ and, by [5, Theorem 1], the restriction of f on Δ admits a homeomorphic extension to $\overline{\mathbb{C}}$.

Let Ψ be a conformal mapping of D onto the slit domain D' of the previous section. Let C_j be a boundary quasicircle of D. We first prove that $f(C_j)$ is a Jordan curve. The slit C'_j is divided by its midpoint ω'_j into two line segments, which we denote by $\Sigma'_j(m)$, $m = 1, 2$, so that

$$\Sigma'_j(1) = \left\{ z \in C'_j : |z| \leq \left|\omega'_j\right| \right\} \qquad \text{and} \qquad \Sigma'_j(2) = \left\{ z \in C'_j : |z| \geq \left|\omega'_j\right| \right\}.$$

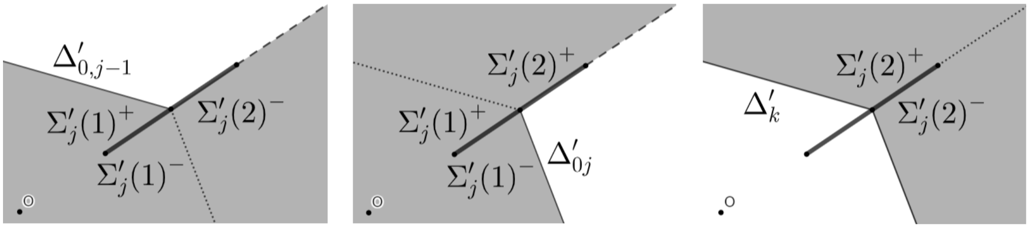

Figure 1. The slit C'_j and the three distinguished domains.

Let $\Sigma'_j(m)^{\pm}$ denote the two sides of $\Sigma'_j(m)$, so that a point z_0 on $\Sigma'_j(m)^-$ is reached only by points $z \in S'_{j-1}$, meaning that $\arg z \to (\arg z_0)^-$ when $z \to z_0$. Similarly, a point z_0 on $\Sigma'_j(m)^+$ is reached only by points $z \in S'_j$, so that $\arg z \to (\arg z_0)^+$ when $z \to z_0$. Corresponding under Ψ^{-1} are four disjoint -except for their endpoints- arcs on the quasicircle C_j, denoted without the prime by $\Sigma_j(m)^{\pm}, m = 1,2$. Now consider the domains $\Delta_{0,j-1}, \Delta_{0j}$ and Δ_k in the collection $\mathfrak{D}$, for some $k \neq j-1, j$; see Figure 1 for their images under Ψ. By [5, Theorem 1] f is injective up to the boundary of each $\Delta \in \mathfrak{D}$. Note that the arcs $\Sigma_j(1)^-, \Sigma_j(1)^+$ and $\Sigma_j(2)^-$ are subsets of $\partial\Delta_{0,j-1}$, so that their images under f, except for their endpoints, are disjoint. It remains to show that the images of these three arcs under f are not intersected by the remaining image $f(\Sigma_j(2)^+)$. Note that the arcs $\Sigma_j(1)^-, \Sigma_j(1)^+$ and $\Sigma_j(2)^+$ are subsets of $\partial\Delta_{0j}$, so that $f(\Sigma_j(2)^+)$ does not intersect $f(\Sigma_j(1)^-)$ nor $f(\Sigma_j(1)^+)$. What remains to be seen is that $f(\Sigma_j(2)^-)$ and $f(\Sigma_j(2)^+)$ are disjoint and this follows from the fact that the arcs $\Sigma_j(2)^-$ and $\Sigma_j(2)^+$ are subsets of $\partial\Delta_k$.

To see that the Jordan curve $f(C_j)$ is actually a quasicircle note that each point of $f(C_j)$ belongs to some open subarc of $f(C_j)$ which is entirely included in the boundary of either $f(\Delta_{0,j-1}), f(\Delta_{0j})$ or $f(\Delta_k)$. These three domains are quasidisks by [5, Theorem 3]. Now the assertion that $f(C_j)$ is a quasicircle follows by an application of Theorem 8.7 in [10, Ch. II, § 8.9].

4. Remarks on uniform domains

A domain D in $\mathbb{C}$ is called uniform if there exist positive constants a and b such that each pair of points $z_1, z_2 \in D$ can be joined by an arc $\gamma \subset D$ so that for each $z \in \gamma$ it holds

$$\ell(\gamma) \leq a\,|z_1 - z_2|$$

and

$$\min_{j=1,2} \ell(\gamma_j) \leq b \operatorname{dist}(z, \partial D),$$

where γ_1, γ_2 are the components of $\gamma \backslash \{z\}$, $\operatorname{dist}(z, \partial D)$ denotes the euclidean distance from z to the boundary of D and $\ell(\cdot)$ denotes euclidean length. Uniform domains were introduced by Martio and Sarvas [11]; see also, e.g., [7] for this equivalent definition. In [11] it was shown that all boundary components of a uniform domain are either points or quasicircles. The converse of this is also true for finitely connected domains, but not, in general, for domains of infinite connectivity; see [6, § 3.5]. The following univalence criterion was proved in [11].

Theorem 4 ([7, 11]). *If D is a uniform domain then there exists a constant $c > 0$ such that every analytic function f in D with $\|S_f\|_D \leq c$ is injective.*

Gehring and Osgood [7] gave a different proof of Theorem 4 by providing a characterization of uniform domains. They showed that a domain D is uniform if and only if it is quasiconformally decomposable in the following weaker (than the one we saw in Section 2) sense: there exists a constant K with the property that for each $z_1, z_2 \in D$ there exists a K-quasidisk $\Delta \subset D$ for which

$z_1, z_2 \in \overline{\Delta}$. Note that, in contrast to Osgood's [12] decomposition, here Δ depends on the points z_1, z_2. However, this can readily be used to generalize the implication (i) $\Rightarrow$ (iii) of [5, Theorem 2], according to which a univalence criterion for harmonic mappings holds on finitely connected uniform domains. The following theorem extends it to all uniform domains.

Theorem 5. *Let D be a uniform domain in $\mathbb{C}$. Then there exists a constant $c > 0$ such that if f is harmonic in D with $\|S_f\|_D \le c$ then f is injective.*

Proof. Assume that there exist distinct points $z_1, z_2 \in D$ for which $f(z_1) = f(z_2)$. By [7], there exists a K-quasidisk $\Delta \subset D$ for which $z_1, z_2 \in \overline{\Delta}$. The domain monotonicity for the hyperbolic metric shows that

$$\left\| S_f \right\|_\Delta \le \left\| S_f \right\|_D \le c.$$

But the homeomorphic extension of [5, Theorem 1] shows that if c is sufficiently small then f is injective up to the boundary of Δ, a contradiction. □

Regarding quasiconformal extension, Astala and Heinonen [2] proved the following theorem.

Theorem 6 ([2]). *If D is a uniform domain then there exists a constant $c > 0$ such that every analytic function f in D with $\|S_f\|_D \le c$ admits a quasiconformal extension to $\overline{\mathbb{C}}$.*

This evidently implies Theorem 4 and was also proved in substantially greater generality, but we omit it here. It is not clear how to generalize Theorem 6 to the setting of harmonic mappings. Therefore, we propose the following problem.

Problem. *Let D be a uniform domain. Does there exist a constant $c > 0$ such that if f is harmonic in D with $\|S_f\|_D \le c$ and with dilatation ω satisfying $\sup_{z\in D} |\omega(z)| < 1$ then f admits a quasiconformal extension to $\overline{\mathbb{C}}$?*

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